ON ESTIMATION OF POPULATION MEAN USING KNOWN AUXILIARY VARIABLE

Dissertation Submitted to

Babasaheb Bhimrao Ambedkar University

Established 1996

Accredited 'A' Grade by NAAC in 2015

(DEPARTMENT OF STATISTICS)

School of Physical and Decision Sciences

For the degree of

MASTER OF PHILOSOPHY

In

STATISTICS

By

Shanya Baghel, MSc.

Under the supervision of

Dr. Subhash Kumar Yadav

Assistant Professor

Department of Statistics

BBAU, Lucknow

SEPTEMBER 2019

Dedicated To

My Beloved Family

DECLARATION

The dissertation is submitted as a part of M.Phil. final semester completion entitled **On Estimation of Population Mean Using Known Auxiliary Variable** under the supervision of **Dr. Subhash Kumar Yadav,** Department of Statistics, Babasaheb Bhimrao Ambedkar University, Lucknow. I declare that this work is original and to the best of my knowledge has not been submitted in part or full to any university or institution for award of any other degree. I did not enlist unlawful assistance of someone else. The cited sources are marked and listed at the end of this dissertation.

Dated:...09/09/2019.....

Submitted By

Shanya Baghel

Shanya Baghel
Master of Science (M.Sc.)
Department of Statistics
BBAU,Lucknow

CERTIFICATE

This is to certify that the dissertation titled **On Estimation of Population Mean Using Known Auxiliary Variable** submitted by **Miss Shanya Baghel** is an original work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other university.

The dissertation submitted to **Babasaheb Bhimrao Ambedkar University, Lucknow** satisfies all the requirements as stipulated in the **Master of Philosophy (M.Phil.)** regulations-2015 as amended in 2016 and it is fit for submission and evaluation for the award of the **degree of Master of Philosophy.**

Dated: 09/09/2019

SK yadav

Supervisor

Head of the Department

ACKNOWLEDGEMENT

Starting with my thanking note, I first of all thank **God** for the spirit of this Dissertation Work in me and providing me the opportunity to pursue higher studies.

I feel privileged to express my sincere regards and gratitude to my **Supervisor Dr. Subhash Kumar Yadav**, Assistant Professor, Department of Statistics, BBAU who introduced me to the field of research; for his valuable guidance and constant encouragement throughout the course of my dissertation-work and being available for any help, what or whenever I needed. The critical comments, rendered by him during the discussion are deeply appreciated.

It gives me immense pleasure to express my profound gratefulness to **Prof. Madhulika Dubey**, Head, Department of Statistics, BBAU for inspiring me and boosting my morale with soothing and encouraging words. I owe a special debt of gratitude to **Prof. Surinder Kumar**, Professor, Department of Statistics, BBAU who provided all requisite support towards the completion of this dissertation work. I extend my heartfelt Thanks and gratitude to **Dr. Amit Kumar Misra**, Assistant Professor, Department of Statistics, BBAU for his academic support and encouragement and initiating us into the use of Latex. I am indebted to **Dr. Rahul Varshney**, Assistant Professor, Department of Statistics , BBAU who introduced us to the field of Sampling and taught us how to work patiently and efficiently. I am obliged to **Dr. Minakshi Mishra**, Assistant Professor, Department of Statistics, BBAU for motivating me to bring out the best in myself. I would like to give special thanks to **Research Scholars** and non-teaching staff specially **Miss Somya Trivedi** of the department and **friends** for their academic and non-academic support and encouragement.

I feel a deep sense of gratitude to **my parents** and my elder sister **Miss Samya Baghel** for their help, cooperation and inspiration extended to me in the course of present study and well wishers who helped me directly and indirectly.

Last but not the least; I am again thankful to the **Almighty** as it is because of His grace I am blessed with such people in my life.

Shanya Baghel
Master of Science (M.Sc.)
Department of Statistics
BBAU,Lucknow

Contents

Chapter 1

INTRODUCTION

Information about something can be obtained either as a whole where each and every detail is considered or as a summary where essence of overall details is provided. In definitive terms former is considered as the Complete Enumeration and later as the Sampling. Sampling is inevitable part of human life such as taking a handful of rice to check the quality of rice in the sack, meeting one or two person of any country to tell the nature of the people of that country, blood group testing etc.

Sampling is not just a partial coverage of whole; it is the Science of controlling and measuring reliability of useful Statistical information through the theory of Probability. Sample being part of the whole, its extrapolation leads to error called as Sampling Error. The aim of statistician is to reduce the sampling errors either by formulating suitable sampling schemes or by developing efficient estimators of the parameters or both. We have developed the efficient estimators for the population parameters in our dissertation. The simplest method of sampling is the Simple Random Sampling (SRS) Scheme where each unit of the population at the each draw has equal probability of being selected in the sample. SRS Scheme can be with replacement (WR) or without replacement (WOR). We have considered the SRSWOR Scheme in our dissertation.

Sampling is the cost effective method of drawing the valid inferences about the population parameters. These Population parameters are mean, variance, skewness, kurtosis etc. We have studied the estimation of population mean. If we denote the characteristic under study as y, simple random sample mean $\bar{y}$ is the basic estimator of the population mean. The auxiliary information which itself is independent but the characteristic under study has some degree of relationship with it; is used for improved estimation of parameters to increase the efficiencies of the estimators. For example to analyze the annual household expenditure we can take household size as auxiliary information.

The use of auxiliary information dates back to year 1926(Bowley), Newman (1934, 1938) who dealt with the stratification of finite population and putting forward a theoretical criticism of non-random (purposive) sampling. Watson (1937) gave the regression method of estimation in which there is reasonable degree of linear relationship between study variable (say y) and auxiliary variable (say x). Cochran

(1940, 1942) propounded the ratio method of estimation in which relationship between auxiliary variable and study variable is of rough proportionality. Hansen and Hurwitz (1943) were the first to suggest use of auxiliary information in the varying probability sampling scheme.

Ratio estimators are the one of the most widely used estimators for population mean or total when the auxiliary and study variable are positively correlated. When correlation is negative product estimator is used. Though the precision of regression estimator is higher than the ratio estimator yet for the large scale surveys ratio estimator would be the wiser choice because of its simplicity. Regression estimator is preferred when there is moderate or lower correlation or some kind of linear relationship between auxiliary variable and study variable.

Auxiliary information can be used at designing stage, sampling stage or estimation stage. The use of the auxiliary information at estimation stage is dealt in our present study. Several estimators have been developed till date to increase the efficiency using auxiliary variable. Parameters of the auxiliary variables such as mean, median, coefficient of variation etc have been used to increase the efficiency of the estimators. Many of such estimators are considered in the literature.

Goodman and Hartley (1958) were concerned with the modification of ratio estimator which would lead to an unbiased estimator. They considered both situations where sample size in each stratum was small and when large. Walsh (1970) modified the denominator of the ratio estimator and proposed the generalized form of the estimate of population total. Chakrabarty (1979) presented some ratio estimators. Sahai (1979) provided an efficient variant of the product and ratio estimator.

Ray, Sahai and Sahai (1979) suggested ratio and product type transformed estimators obtained through parametric linear combination of mean per unit estimator and ratio estimator; mean per unit estimator and product estimator respectively. Sahai and Ray (1980) proposed two-parameter families of ratio-type and product-type estimators for a finite population mean based on simple random samples of observations on the variable of interest and a concomitant variable. Using some prior information they showed that the families contain estimators which have in practical situations lower mean squared error than the usual ratio, product and sample mean estimators.

Sisodia and Dwivedi (1981) modified the ratio estimator using the coefficient of variation of auxiliary variable with increased efficiency. Srivastava (1983) provided the predictive estimation of finite population mean using the product estimator. Chauby, Singh and Dwivedi (1984) wrote down a note which points out the derivation of regression estimator through an optimality consideration over a class of estimators generating a Generalised Product and dual to ratio estimators. Bahl and Tuteja (1991) introduced new ratio and product type exponential estimators for estimating the mean of the finite population using information on single auxiliary variable. Upadhyaya and Singh (1999) by combining the coefficient of kurtosis and coefficient of variation of auxiliary variable proposed the estimator of the finite population mean with a greater precision. They also obtained the unbiased version of their suggested estimators using interpenetrating subsample design and Jack-knife technique.

Kadilar and Cingi (2003) studied on the chain ratio type estimators. Singh and Tailor (2003) used the known correlation coefficient for efficiently estimating the population mean. Singh, Tailor and Kakran (2004) improved the estimator of population mean using power transformation. Al-Omari, Jemain and Ibrahim (2009) suggested modified ratio estimators of the population mean of the variable of interest involving the first or third quartiles of an auxiliary variable that is correlated with the variable of interest. They also investigated the newly suggested estimators under simple random sampling (SRS) and ranked set sampling (RSS) methods. Yan and Tian (2010) obtained the ratio method to the mean estimation using coefficient of skewness of auxiliary variable.

Subramani and Kumarpandiyan (2012) modified the ratio estimators using function of quartiles, median and coefficient of skewness of auxiliary variable. Jeelani, Maqbool and Mir (2013) transformed the ratio estimators using linear combination of coefficient of skewness and quartile deviation. Singh, Solanki and Singh (2016) suggested the ratio-type and product-type exponential estimators of the population mean of a study variable through predictive approach using Bahl and Tuteja (1991) ratio-type and product-type exponential estimators as a predictor of the mean of the unobserved units of the population. Properties of the suggested estimators were also studied up to first order of approximation in simple random sampling using information on an auxiliary variable. Swain (2014) suggested an alternative ratio type exponential estimator with increased efficiency.

Singh and Pal (2015) advocated the problem of estimating the finite population mean using auxiliary information in sample surveys. They suggested a new chain ratio-ratio- type exponential estimator and its properties were studied up to first degree of approximation. Generalized version of the suggested chain ratio-ratio-type estimator was also given along with its properties. Yadav and Mishra (2015) developed an improved estimator using predictive method of estimation utilizing auxiliary information. Jerajuddin and Kishun (2016) improved the ratio estimator using size of the sample. Kadilar (2016) developed a new exponential type estimator which came out to be more efficient than the existing ones. Singh and Solanki (2016) provided a class of new estimators so as to gain the higher precision. Subramani (2016) utilized the median of study variable to obtain the much better estimator of the population mean.

Vishwakarma, Singh, Gupta, Pareek (2016) used the combination of determined constants to obtain the efficient estimator for the population mean. Soponviwatkul and Lawson (2017) derived new ratio estimators for estimating population mean in SRS using a coefficient of variation, correlation coefficient and a regression coefficient. Yadav and Dixit (2019) developed a class of ratio estimators combining various parameters of auxiliary variables and used it for the estimation of average yield of peppermint.

In practice almost all surveys suffer from non-response. The problem of non-response often happens due to the refusal of the subject, absenteeism and sometimes due to the lack of information. The pioneering work of Hansen and Hurwitz (1946), assumed that a sub sample of initial non-respondents is re-contacted with a more expensive method, suggesting the first attempt by mail questionnaire and the second attempt by a personal interview. In estimating population parameters such as the mean, total or ratio, sample survey experts sometimes use auxiliary information to improve precision of the estimates.

Rao (1986) proposed an improved ratio estimator with sub sampling the non respondent. Khare and Sinha (2007) suggested the estimation of the ratio of the two population means (response class and non response class) using multi auxiliary characteristics in the presence of non-response. Singh and Kumar (2008) provided a regression approach to the estimation of finite population mean in presence of non response. Singh and Kumar (2009) derived a general procedure for estimating the population mean in presence of non response under double sampling using auxiliary information. Kumar and Bhougal (2011) proposed a modified ratio-product type exponential estimator to estimate the finite population mean of the study variable in presence of non-response in different situations viz. (i) population mean of auxiliary variable is known, and (ii) population mean of auxiliary variable is unknown. The expressions of biases and mean squared error of the proposed estimators had been obtained under large sample approximation using single as well as double sampling. Khare and Sinha (2012) considered the problem of estimation of ratio of two population means using multivariate auxiliary characters with known population means under incomplete information. They proposed the general class of estimators and studied its properties. Chanu (2015) suggested an improved exponential ratio cum exponential dual ratio estimator of finite population mean in presence of non response. Pal and Singh (2016) derived a finite population mean estimation through a two parameter ratio estimator using auxiliary information in presence of non response. Zubir et al. (2018) proposed an efficient exponential estimator for population mean of study variable, with two auxiliary variables using two phase sampling scheme with sub sampling techniques in presence of non-response. The two situations with known population means of auxiliary variables, incomplete information on study variables and incomplete/complete information on auxiliary variables had been considered. Expressions for bias and mean square error had been derived. They proved their estimator to be considerably efficient.

The present thesis "On Estimation of Population Mean Using Known Auxiliary Variable" is confined in the use of auxiliary variable at the estimation stage. Inspired by many authors we have proposed estimators in different situations. The sampling scheme is taken as to be the Simple Random Sampling without replacement. Our interest lies in developing the ratio type of estimators because they are simpler to implement than the regression estimator. The data we used in the chapters to justify results are secondary data.

Chapter 2 consists in upgrading the estimator of population mean to a new generalized class of ratio estimators using the auxiliary information combined with the median of study variable. Since our estimator is biased we obtained its mean square error (MSE). Results are compared with many of the existing estimators theoretically. We verified our results with natural population.

Chapter 3 is concerned with the problem when we lack complete information on our study variable as well as the auxiliary variables. We have provided a remedy for improved estimation of population mean of study variable, using the information related to two auxiliary variables in the situation of non response under Simple Random Sampling Scheme. The problem of non response is removed through Double Sampling plan (Hansan and Hurwitz, 1946) and at each phase sample is selected through simple random sampling method. Bias and MSE are obtained and the results are verified through a natural population.

Chapter 4 takes the situation of developing a class of efficient estimator without the use of median of study variable. Bias and MSE are found and a comparative study with the conventional mean per unit unbiased estimator and existing competing estimators is provided. Empirical study showing the increased efficiency of proposed class of estimator over the mean per unit estimator and many existing estimators are also included as an illustration.

Chapter 5 is an improvement over the estimator of chapter 4. It provides a very useful class of estimators for estimation of population mean. The constants mentioned in the proposed class of estimator are chosen such that the mean squared error is minimum. By measuring the optimum value of the major constant we obtain a Bias estimator with minimum MSE. Results are compared theoretically as well as empirically.

The Notations and Formulae which would be used throughout the chapters are given in the following sections :

1.1 Notations

- N : Size of the population
- n : Size of the sample
- ${}^{N}C_n$: Number of possible samples of size n from the population of size N
- Y : Study Variable
- X,Z : Auxiliary Variables
- M : Median of the Study Variable
- ρ : Correlation Coefficient between X and Y
- β : Regression Coefficient of Y on X
- S_{yx} : Covariance between X and Y
- N_1 : Size of the response class of the population
- N_2 : Size of the non response class of the population
- r_1 : Number of units in the sample which responds in the first attempt
- $r_2 = n - r_1$: Number of units in the sample which do not respond in the first attempt
- k_2 : Size of the subsample from the r_2 units
- $h = \frac{r_2}{k_2}$: Inverse ratio to be sub sampled in non response class
- $\overline{Y}, \overline{X}, \overline{Z}$: Population means
- $\bar{y}, \bar{x}, \bar{z}$: Sample means
- $\bar{y}^*, \bar{x}^*, \bar{z}^*$: Sample means in the presence of non response
- M_y, M_x : Medians
- $\beta_{1(x)}$: Coffecient of Skewness
- $\beta_{2(x)}$: Coeffecient of Kurtosis
- $Q_{1(x)}$: First Quartile

- $Q_{3(x)}$: Third Quartile
- QD : Quartile Deviation
- $Q_{a(x)}$: Quartile Average
- $Q_{r(x)}$: Quartile Range
- TM : Tri Mean
- $S^2_{(\cdot)}$: Population Mean Square
- $C_{(\cdot)}$: Coefficient of Variation
- $Bias(\cdot)$: Bias of the estimator
- $V(\cdot)$: Variance of the estimator
- $MSE(\cdot)$: Mean Squared Error of the Estimator
- $PRE(\bar{y}, t)$: Percentage Relative Efficiency

1.2 Formulae

- $\lambda = \frac{1-f}{n}$
- $f = \frac{n}{N}$
- $\bar{y} = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$
- $\bar{x} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$
- $V(\bar{y}) = \lambda \overline{Y}^2 C_y^2$
- $V(\bar{x}) = \lambda \overline{X}^2 C_x^2$
- $C_y = \frac{S_y}{\overline{Y}}$
- $C_x = \frac{S_x}{\overline{X}}$
- $S_{yx} = \frac{1}{N-1}\sum\limits_{i=1}^{N} \left(Y_i - \overline{Y}\right)\left(X_i - \overline{X}\right)$
- $S_y^2 = \frac{1}{N-1}\sum\limits_{i=1}^{N} \left(Y_i - \overline{Y}\right)^2$
- $S_x^2 = \frac{1}{N-1}\sum\limits_{i=1}^{N} \left(X_i - \overline{X}\right)^2$
- $\rho = \frac{S_{yx}}{S_x S_y}$
- $g = \rho\frac{C_y}{C_x}$
- $d = \frac{C_y}{C_x}$
- $QD = \frac{Q_3 - Q_1}{2}$
- $Q_{a(x)} = \frac{Q_1 + M_x + Q_3}{3}$
- $Q_{r(x)} = Q_3 - Q_1$
- $TM = \frac{Q_1 + 2M_x + Q_3)}{4}$
- $w_2 = \frac{N_2}{N}$
- $\gamma = w_2\left(\frac{h-1}{n}\right)$
- $\bar{z} = \frac{1}{n}\sum\limits_{i=1}^{n} z_i$
- $V(\bar{z}) = \lambda \overline{Z}^2 C_x^2$
- $C_z = \frac{S_x}{\overline{Z}}$
- $S_z^2 = \frac{1}{N-1}\sum\limits_{i=1}^{N} \left(Z_i - \overline{Z}\right)^2$
- $C_{yx} = \rho_{yx} C_y C_x$
- $C_{yz} = \rho_{yz} C_y C_z$
- $C_{xz} = \rho_{xz} C_x C_z$
- $V_y = \lambda C_y^2$
- $V_x = \lambda C_x^2$
- $V_z = \lambda C_z^2$
- $V_{yx} = \lambda C_{yx}$
- $V_{yz} = \lambda C_{yz}$
- $V_{xz} = \lambda C_{xz}$

For Non Response Class

- $\bar{y}^* = \left(\frac{r_1}{n}\right)\bar{y}_{r_1} + \left(\frac{r_2}{n}\right)\bar{y}_{r_2}$
- $\bar{x}^* = \left(\frac{r_1}{n}\right)\bar{x}_{r_1} + \left(\frac{r_2}{n}\right)\bar{x}_{r_2}$
- $\bar{z}^* = \left(\frac{r_1}{n}\right)\bar{z}_{r_1} + \left(\frac{r_2}{n}\right)\bar{z}_{r_2}$
- $\bar{y}_{r_1} = \frac{1}{r_1}\sum_{i=1}^{r_1} y_i$
- $\bar{x}_{r_1} = \frac{1}{r_1}\sum_{i=1}^{r_1} x_i$
- $\bar{z}_{r_1} = \frac{1}{r_1}\sum_{i=1}^{r_1} z_i$
- $\bar{y}_{k_2} = \frac{1}{k_2}\sum_{i=1}^{k_2} y_i$
- $\bar{x}_{k_2} = \frac{1}{k_2}\sum_{i=1}^{k_2} x_i$
- $\bar{z}_{k_2} = \frac{1}{k_2}\sum_{i=1}^{k_2} z_i$
- $C_{y(2)} = \frac{S_{y(2)}}{\overline{Y}}$
- $C_{x(2)} = \frac{S_{x(2)}}{\overline{Y}}$
- $C_{z(2)} = \frac{S_{z(2)}}{\overline{Y}}$
- $C_{yx(2)} = \rho_{yx(2)} C_{y(2)} C_{x(2)}$
- $C_{yz(2)} = \rho_{yz(2)} C_{y(2)} C_{z(2)}$
- $C_{xz(2)} = \rho_{xz(2)} C_{x(2)} C_{z(2)}$
- $V_y^* = \lambda C_y^2 + \gamma C_{y(2)}^2$
- $V_x^* = \lambda C_x^2 + \gamma C_{x(2)}^2$
- $V_z^* = C_z^2 + \gamma C_{z(2)}^2$
- $V_{yx}^* = \lambda C_{yx} + \gamma C_{yx(2)}$
- $V_{yz}^* = \lambda C_{yz} + \gamma C_{yz(2)}$
- $V_{xz}^* = \lambda C_{xz} + \gamma C_{xz(2)}$

Chapter 2

UPGRADED FAMILY OF ESTIMATORS OF POPULATION MEAN UNDER SIMPLE RANDOM SAMPLING SCHEME USING KNOWN PARAMETERS OF AUXILIARY AND STUDY VARIABLE

2.1 Summary

The present chapter consists of an improved class of estimators for population mean of study variable using the informations related to an auxiliary variable combined with the median of the study variable under the Simple Random Sampling Scheme. We have shown that many of the existing estimators of population mean are the members of our proposed class. The Bias and MSE of our proposed class is derived upto the first order of approximation. Minimum values of Bias and MSE is obtained by optimizing the chracterizing scalar. Bias and MSE has also been compared with the existing estimators. Finally we have suggested some new members of our proposed class which are more efficient than the existing ones **numerically.**

2.2 Introduction

In case the size of the population is very large, we take a representative sample to estimate the population characteristics. The present paper is concerned with the estimation of population mean and we know well that the corresponding sample mean is the most suitable estimator. But it has the drawback of being largerly dispersed. A remedy for this is to use an auxiliary information of the auxiliarly variable which are positively or negatively correlated with the study variable. Let from the finite population (X, Y) of size N, a bivariate sample $(x_i, y_i); i = 1, 2, \ldots, n$ of size n is taken using SRSWOR scheme. It is already known that, sample means $\bar{x}$ and $\bar{y}$ are unbiased estimators of population means $\overline{X}$ and $\overline{Y}$ respectively. The ratio, product and regression methods of estimation are the conventional methods for population mean estimation.

The Less MSE, the better Results. So we have proposed an improved class of estimators for population mean of study variable which are more efficient than many of the existing estimators.

2.3 Existing Estimators

A summary of some existing estimators of population mean with their MSE and Bias upto the first order of approximation with which we have compared our estimators are given as follows :

Table 2.1: **Existing Estimators**

SNo	**Estimators**	**Bias**	**MSE/Variance**
1	$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$	unbiased	$\lambda\overline{Y}^2 C_y^2$
2	$t_1 = \bar{y} + \beta(\overline{X} - \bar{x})$ Watson(1937)	unbiased	$\lambda\overline{Y}^2 C_y^2(1-\rho^2)$
3	$t_2 = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)$ Cochran(1940)	$\lambda\overline{Y}\left(C_x^2 - \rho C_y C_x\right)$	$\lambda\overline{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_y C_x\right)$
4	$t_3 = \overline{X}\bar{r} + \frac{n(N-1)}{N(n-1)}(\bar{y} - \bar{r}\bar{x})$ Goodman and Hartley(1958)	unbiased	$\lambda\overline{Y}^2\left(C_y^2 + C_x^2 - 2\rho C_y C_x\right)$
5	$t_4 = \bar{y}\left(\frac{\bar{x}}{\overline{X}}\right)$ Goodman(1960)	$\frac{\lambda\rho C_y C_x}{\overline{X}^2\overline{Y}}$	$\lambda\overline{Y}^2\left(C_y^2 + C_x^2 + 2\rho C_y C_x\right)$

Existing Estimators

SNo	Estimators	Bias	MSE
6	$t_5 = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)^g$ Srivastava(1967)	$\frac{\lambda g}{2}C_x^2(1-g)$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
7	$t_6 = (1-g)\bar{y} + g\bar{y}\frac{\overline{X}}{\bar{x}}$ Chakrabarty(1979)	$\lambda\overline{Y}\left(\frac{g}{2}C_x^2 - g\rho C_yC_x\right)$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
8	$t_7 = \bar{y}\left[\frac{\overline{X}}{g\bar{x}+(1-g)\overline{X}}\right]$ Walsh(1970) Reddy(1974)	$\lambda C_x^2 g\overline{Y}\left(g - \frac{g}{\overline{Y}^2\overline{X}^2}\right)$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
9	$t_8 = \bar{y}\left[\frac{g\overline{X}+(1-g)\bar{x}}{g\bar{x}+(1-g)\overline{X}}\right]$ Sahai(1979)	$h\lambda\upsilon\overline{Y}C_x^2\left[\rho - \frac{h(1-\upsilon)}{2}\right]$	$\lambda\overline{Y}^2C_y^2(1+h^2+2h\rho\upsilon)$
10	$t_9 = \bar{y}\left[\frac{(1+g)\overline{X}-g\bar{x}}{\overline{X}}\right]$ $t_{10} = \bar{y}\left[\frac{(1+g)\bar{x}-g\overline{X}}{\bar{x}}\right]$ Ray et al.(1979)	$-g\overline{Y}\rho C_yC_x$ $g\overline{Y}(g-1)C_x^2$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2+2g\rho C_yC_x)$ $\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
11	$t_{11} = \bar{y}\left[2-\left(\frac{\bar{x}}{\overline{X}}\right)^g\right]$ Sahai and Ray(1980)	$\lambda\overline{Y}\left[\frac{g(g-1)}{2}C_x^2 - g\rho C_yC_x\right]$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
12	$t_{12} = \bar{y}\left(\frac{\overline{X}+C_x}{\bar{x}+C_x}\right)$ Sisodia and Dwivedi(1981)	$\lambda\overline{Y}\left(R_{12}^2C_x^2 - R_{12}\rho C_yC_x\right)$	$\lambda\overline{Y}^2(C_y^2 + R_{12}^2C_x^2 - 2R_{12}\rho C_yC_x)$
13	$t_{13} = \bar{y}\left[\frac{f\overline{X}+(1-2f)\bar{x}}{(\overline{X}-f\bar{x})}\right]$ Srivastava(1983)	$\lambda\overline{Y}\left(\rho C_yC_x + \frac{f}{1-f}C_x^2\right)$	$\lambda\overline{Y}^2(C_y^2 + C_x^2 + 2\rho C_yC_x)$

Existing Estimators

SNo	Estimators	Bias	MSE
14	$t_{14}=\bar{y}\left[\frac{g\bar{x}+(1-g)\overline{X}}{\overline{X}}\right]$ Chaubey et al.(1984)	-	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2+2g\rho C_yC_x)$
15	$t_{15}=\bar{y}\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Bahl and Tuteja(1991)	$\frac{\lambda}{8}\overline{Y}\left(3C_x^2-4\rho C_yC_x\right)$	$\lambda\overline{Y}^2\left(C_y^2+\frac{C_x^2}{4}-\rho C_yC_x\right)$
16	$t_{16}=\bar{y}\left(\frac{\overline{X}^2}{\bar{x}^2}\right)$ Kadilar and Cingi(2003)	$\lambda\overline{Y}(3C_x^2-2\rho C_yC_x)$	$\lambda\overline{Y}^2(C_y^2+4C_x^2-4\rho C_yC_x)$
17	$t_{17}=\bar{y}\left(\frac{\overline{X}+\rho}{\bar{x}+\rho}\right)$ Singh and Tailor(2003)	$\lambda\overline{Y}(R_{17}^2C_x^2-R_{17}\rho C_yC_x)$	$\lambda\overline{Y}^2(C_y^2+R_{17}^2C_x^2-2R_{17}\rho C_yC_x)$
18	$t_{18}=\bar{y}\frac{\overline{X}}{\bar{x}}\left(1-\frac{g\bar{x}s_x^2}{n\bar{x}^3}\right)^{-1}$ Pandey et al.(2011)	$\lambda\overline{Y}(g^2C_x^2-g\rho C_yC_x)$	$\lambda\overline{Y}^2(C_y^2+g^2C_x^2-2g\rho C_yC_x)$
19	$t_{19}=\bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)^{1/2}$ Swain(2014)	$\frac{\lambda}{8}\overline{Y}(3C_x^2-4\rho C_yC_x)$	$\lambda\overline{Y}^2\left(C_y^2+\frac{C_x^2}{4}-\rho C_yC_x\right)$
20	$t_{20}=f\bar{y}+(1-f)\bar{y}\exp\left[\frac{\overline{X}-\bar{x}}{(\overline{X}-\bar{x})-2f\bar{x}}\right]$ Singh et al. (2014)	$\frac{\lambda}{8}\overline{Y}\left(3C_x^2-4\rho C_yC_x\right)$	$\lambda\overline{Y}^2\left(C_y^2+\frac{C_x^2}{4}-\rho C_yC_x\right)$
21	$t_{21}=\alpha t_{Re}+(1-\alpha)t_{Pe}$ Yadav and Mishra(2015)	$\overline{Y}\left[\frac{1}{8}(4\alpha-1-7f+4f^2)C_x^2-\frac{(2\alpha-1)}{2(1-f)}\rho C_yC_x\right]$	$\lambda\overline{Y}^2\left(C_y^2+\alpha^2\frac{C_x^2}{4}-\alpha\rho C_yC_x\right)$

Existing Estimators

SNo	Estimators	Bias	MSE
22	$t_{22} = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Singh and Pal(2015)	$\frac{3}{8}\lambda\overline{Y}(5C_x^2 - 4\rho C_y C_x)$	$\lambda\overline{Y}^2\left(C_y^2 + \frac{9}{4}C_x^2 - 3\rho C_y C_x\right)$
23	$t_{23} = \bar{y}\left(\frac{\overline{X}+n}{\bar{x}+n}\right)$ Jerajuddin andKishun(2016)	$\lambda\overline{Y}(R_{23}^2 C_x^2 - R_{23}\rho C_y C_x)$	$\lambda\overline{Y}^2(C_y^2 + R_{23}^2 C_x^2 - 2R_{23}\rho C_y C_x)$
24	$t_{24} = \alpha_1\bar{y} + (1-\alpha_1)$ Vishwakarma et al.(2016)	$\lambda\overline{Y}C_x^2\left[\left(\frac{3}{8}-\frac{c}{2}\right) - \alpha_1\left(\frac{3}{8}-\frac{c}{2}\right)\right]$	$\lambda\overline{Y}^2\left[C_y^2 + \frac{C_x^2}{4}(1-\alpha_1)^2 - (1-\alpha_1)\rho C_y C_x\right]$
25	$t_{25} = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)^{\alpha_2}\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Kadilar(2016)	$\lambda\overline{Y}\left[\left(\frac{\alpha_2^2}{2}+\frac{3}{8}\right)C_x^2 - \left(\alpha_2+\frac{1}{2}\right)\rho C_y C_x\right]$	$\lambda\overline{Y}^2\left[C_y^2 + \left(\frac{1}{4}+\alpha_2^2+\alpha_2\right) C_x^2 + \rho C_y C_x(2\alpha_2+1)\right]$
26	$t_{26} = \bar{y}\left[\frac{a\overline{X}+b\bar{x}}{c\bar{x}+d\overline{X}}\right]$ Singh et al.(2016)	$\lambda\overline{Y}C_x^2\left[\Delta(\tau_1 - g)\right]$	$\lambda\overline{Y}^2\left[C_y^2 + \Delta C_x^2\{\Delta - 2g\}\right]$
27	$t_{27} = \bar{y}\left(\frac{\overline{X}+C_x}{\bar{x}+C_x}\right)^{b_1}$ $t_{28} = \bar{y}\left(\frac{\overline{X}+\rho}{\bar{x}+\rho}\right)^{b_2}$ Soponviwatkul and Lawson(2017)	$\lambda\overline{Y}\left[\frac{b_1(b_1+1)}{2}R_{12}^2C_x^2 - b_1R_{12}\rho C_y C_x\right]$ $\lambda\overline{Y}\left[\frac{b_2(b_2+1)}{2}R_{17}^2C_x^2 - b_2R_{17}\rho C_y C_x\right]$	$\lambda\overline{Y}^2\left(C_y^2 + b_1^2R_{12}^2C_x^2 - 2b_1R_{12}\rho C_y C_x\right)$ $\lambda\overline{Y}^2\left(C_y^2 + b_2^2R_{17}^2C_x^2 - 2b_2R_{17}\rho C_y C_x\right)$

where,

- $\bar{r} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}$
- $h = \frac{C_x}{C_y}$
- $b_1 = \frac{g}{R_{12}}$
- $b_2 = \frac{g}{R_{17}}$

- $\alpha = \frac{\rho C_y}{4C_x}$
- $\alpha_1 = 1 - 2c$
- $\alpha_2 = \frac{(C_x - 2\rho C_y)}{2C_x}$
- $\upsilon = \frac{1-g}{1+g}$
- $R_{12} = \frac{\overline{X}}{\overline{X}+C_x}$
- $R_{17} = \frac{\overline{X}}{\overline{X}+\rho}$
- $R_{23} = \frac{\overline{X}}{\overline{X}+n}$
- $t_{Re} = f\bar{y} + (1-f)\bar{y}\exp\left[\frac{(\overline{X}-\bar{x})}{(\overline{X}-\bar{x})-2f\bar{x}}\right]$
- $t_{Pe} = f\bar{y} + (1-f)\bar{y}\exp\left[\frac{(\bar{x}-\overline{X})}{\overline{X}-(1-2f)\bar{x}}\right]$

2.4 The Proposed Class of Estimators

If the Median **M** of the study variable is known, then based on the Singh, Pal and Solanki (2016), we suggest a class of Ratio Estimators of population mean $\overline{Y}$ in Simple Random Sampling as follows :

$$\mathbf{t} = \mathbf{k}\bar{\mathbf{y}}\left[\frac{\mathbf{a}\overline{\mathbf{X}} + \mathbf{b}\bar{\mathbf{x}}}{\mathbf{c}\bar{\mathbf{x}} + \mathbf{d}\overline{\mathbf{X}}}\right] \tag{2.4.1}$$

where (a, b, c, d) are either constants or parameters of auxiliary variable and k is a constant such that the Mean Squared Error(MSE) of our estimator is minimum. Here at least one of the constants (a, b, c, d). Also (a, b, c, d) are free to take those real and parametric values which makes the MSE minimum.

2.4.1 Bias and MSE

Biasedness and MSE upto the first order of approximation is given as follows :

$$B(t) = \left(\frac{A}{B} - 1\right) + \left(\frac{A}{B}\right)\overline{Y}\left[(\tau_1 - g)\Delta\lambda C_x^2\right] \tag{2.4.2}$$

$$MSE(t) = \overline{Y}^2\left(1 - \frac{A^2}{B}\right) \tag{2.4.3}$$

where
$$\begin{aligned}
\Delta &= \tau_1 - \tau_2 \\
\tau_1 &= c(c+d)^{-1} \\
\tau_2 &= b(a+b)^{-1} \\
g &= \rho\frac{C_y}{C_x} \\
\lambda &= \frac{1-f}{n} \\
A &= 1 - \Delta\lambda C_x^2(c - \tau_1) \\
B &= 1 + \lambda C_y^2 - \Delta\lambda C_x^2(4c - \Delta - 2\tau_1)
\end{aligned}$$

2.4.2 Bias and MSE : In-Depth Study

First let,

$$
\begin{aligned}
e_0 &= \frac{\bar{y}-\overline{Y}}{\overline{Y}} && \text{and} \quad e_1 &&= \frac{\bar{x}-\overline{X}}{\overline{X}} \\
\text{So,} \quad \bar{y} &= \overline{Y}(1+e_0) && \text{and} \quad \bar{x} &&= \overline{X}(1+e_1) \\
E(e_0) &= E(e_1) = 0 && \text{and} \quad E(e_0e_1) &&= \lambda\rho C_y C_x \\
E(e_0^2) &= \lambda C_y^2 && \text{and} \quad E(e_1^2) &&= \lambda C_x^2
\end{aligned}
$$

Now rewriting our proposed estimator from equation (2.4.1) as,

$$
\begin{aligned}
t &= k\overline{Y}(1+e_0)\left[\frac{a\overline{X}+b\overline{X}(1+e_1)}{c\overline{X}(1+e_1)+d\overline{X}}\right] \\
&= k\overline{Y}(1+e_0)\left[(a+b+be_1)\overline{X}\right]\left[(c+d+ce_1)\overline{X}\right]^{-1} \\
&= k\overline{Y}(1+e_0)(a+b)[1+b(a+b)^{-1}e_1](c+d)^{-1}[1+c(c+d)^{-1}e_1]^{-1} \\
&= k\overline{Y}(1+e_0)\left(\frac{a+b}{c+d}\right)\left[1+b(a+b)^{-1}e_1\right]\left[1+c(c+d)^{-1}e_1\right] \\
\implies t &= k\overline{Y}(1+e_0)\left(1+\tau_2e_1\right)\left(1+\tau_1e_1\right)^{-1} \qquad (2.4.4)
\end{aligned}
$$

where, $\dfrac{a+b}{c+d}=1, \quad c(c+d)^{-1}=\tau_1, \quad b(a+b)^{-1}=\tau_2$

We assume, $|\tau_1e_1|<1$ so that $(1+\tau_1e_1)^{-1}$is expendable.

Expanding the right hand side of equation (2.4.4) upto the first order of approximation after simplification we get,

$$
\begin{aligned}
t &= k\overline{Y}\left[1-(\tau_1-\tau_2)e_1+e_0+(\tau_1^2-\tau_1\tau_2)e_1^2-(\tau_1-\tau_2)e_0e_1\right] \\
\mathbf{t} &= \mathbf{k\overline{Y}}\left[\mathbf{1+e_0-\Delta e_1-\Delta e_0e_1+\Delta\tau_1e_1^2}\right] \qquad (2.4.5)
\end{aligned}
$$

Subtracting $\overline{Y}$ from both sides we get,

$$
(t-\overline{Y}) = k\overline{Y}\left[1+e_0-\Delta e_1-\Delta e_0e_1+\Delta\tau_1e_1^2\right]-\overline{Y} \qquad (2.4.6)
$$

Taking expectation on both sides of equation (2.4.6) we get the bias of t as follows,

$$
\begin{aligned}
B(t) &= k\overline{Y}\left[1-\Delta\lambda\rho C_yC_x+\Delta\tau_1\lambda C_x^2\right]-\overline{Y} \\
&= k\overline{Y}\left[1+\Delta\lambda C_x^2\left(\tau_1-\rho\frac{C_y}{C_x}\right)\right]-\overline{Y} \\
&= k\overline{Y}\left[1+\Delta\lambda C_x^2(\tau_1-g)\right]-\overline{Y} \qquad \text{where,} \quad g=\rho\frac{C_y}{C_x} \\
B(t) &= (k-1)\overline{Y}+k\overline{Y}\left[(\tau_1-g)\Delta\lambda C_x^2\right] \qquad (2.4.7)
\end{aligned}
$$

Squaring both sides of equation (2.4.6) and solving upto the first order of approximation we get,

$$
\begin{aligned}
(t-\overline{Y})^2 &= \left[k\overline{Y}(1+e_0-\Delta e_1-\Delta e_0 e_1+\Delta\tau_1 e_1^2)-\overline{Y}\right]^2 \\
&= \overline{Y}^2\left[k^2(1+e_0-\Delta e_1-\Delta e_0 e_1+\Delta\tau_1 e_1^2)^2+1\right. \\
&\quad \left.-2k(1+e_0-\Delta e_1-\Delta e_0 e_1+\Delta\tau_1 e_1^2)\right] \\
&= \overline{Y}^2\left[k^2\{1+2e_0-2\Delta e_1-4\Delta e_0 e_1+e_0^2+(\Delta^2+2\tau_1\Delta)e_1^2\}+1\right. \\
&\quad \left.-2k(1+e_0-\Delta e_1-\Delta e_0 e_1+\Delta\tau_1 e_1^2)\right]
\end{aligned}
$$

Taking expectation on both sides we obtain the MSE of t upto the first order of approximation as follows ,

$$
\begin{aligned}
\mathbf{M(t)} =& \mathbf{\overline{Y}^2}\left[\mathbf{k^2\{1-4\Delta\lambda\rho C_y C_x+\lambda C_y^2+(\Delta^2+2\tau_1\Delta)\lambda C_x^2\}+1}\right. \\
& \left.\mathbf{-2k(1-\Delta\lambda\rho C_y C_x+\Delta\lambda C_x^2)}\right]
\end{aligned} \tag{2.4.8}
$$

To obtain the value of k we differentiate $M(t)$ with respect to k and equate it to zero. The solution will be the value of k.

$$\frac{\partial M(t)}{\partial k}=0$$

$$\overline{Y}^2\left[2k\{1-4\Delta\lambda C_{yx}+\lambda C_y^2+(\Delta^2+2\tau_1\delta)\lambda C_x^2\}-2(1-\Delta\lambda C_{yx}+\Delta\lambda C_x^2)\right]=0$$

$$k\{1-4\Delta\lambda C_{yx}+\lambda C_y^2+(\Delta^2+2\tau_1\Delta)\lambda C_x^2\}=(1-\Delta\lambda C_{yx}+\Delta\lambda C_x^2)$$

$$\implies \quad \mathbf{k}=\frac{\mathbf{1-\Delta\lambda C_x^2(g-\tau_1)}}{\mathbf{1+\lambda C_y^2-\Delta\lambda C_x^2(4g-\Delta-2\tau_1)}}=\frac{\mathbf{A}}{\mathbf{B}} \tag{2.4.9}$$

and

$$\frac{\partial^2 MSE(t)}{\partial k^2}=1+\lambda C_y^2-\Delta\lambda C_x^2(4g-\Delta-2\tau_1)>0$$

Thus the minimum value of MSE of t is given as,

$$\mathbf{M(t)_{min}}=\mathbf{\overline{Y}^2}\left(\mathbf{1-\frac{A^2}{B}}\right) \tag{2.4.10}$$

Putting the value of k in equation (2.4.7) we get the Bias,

$$\mathbf{B(t)}=\left(\mathbf{\frac{A}{B}-1}\right)+\left(\mathbf{\frac{A}{B}}\right)\mathbf{\overline{Y}}\left[(\tau_1-\mathbf{g})\mathbf{\Delta\lambda C_x^2}\right] \tag{2.4.11}$$

2.4.3 Existing Members of Proposed Class

The proposed class of estimators reduce to some known estimators of population mean $\overline{Y}$ for different values of (a,b,c,d) in equation (2.4.1) which are given as follows,

Table 2.2: **Some Existing Members of Proposed Class of estimators t**

SNo.	Estimators	k	a	b	c	d
1	t_0	1	1	1	1	1
2	t_2	1	1	0	1	0
3	t_4	1	0	1	0	1
4	t_5	1	$\frac{1}{\overline{X}^{1-g}}$	0	$\frac{1}{\bar{x}^{1-g}}$	0
5	t_6	1	g	$1-g$	1	0
6	t_7	1	1	0	g	$1-g$
7	t_8	1	g	$1-g$	g	$1-g$
8	t_9	1	$1+g$	$-g$	0	1
9	t_{10}	1	$-g$	$1+g$	1	0
10	t_{11}	1	$\frac{2}{\overline{X}^{1-g}}$	$\frac{-1}{\bar{x}^{1-g}}$	$\frac{1}{\overline{X}^{1-g}}$	0
11	t_{12}	1	1	$\frac{C_x}{\bar{x}}$	1	$\frac{C_x}{\overline{X}}$
12	t_{13}	1	f	$1-2f$	$-f$	1
13	t_{14}	1	$1-g$	g	0	1
14	t_{16}	1	$\overline{X}$	0	$\bar{x}$	0
15	t_{17}	1	1	$\frac{\rho}{\bar{x}}$	1	$\frac{\rho}{\overline{X}}$
16	t_{18}	1	1	0	$\left(1-\frac{gs_x^2}{n\bar{x}^2}\right)$	0
17	t_{19}	1	$\frac{1}{\sqrt{\overline{X}}}$	0	$\frac{1}{\sqrt{\bar{x}}}$	0
18	t_{23}	1	1	$\frac{n}{\bar{x}}$	1	$\frac{n}{\overline{X}}$
19	$t_{26(1)}$	1	C_x^2	$-\rho$	C_x^2	$-\rho$
20	$t_{26(2)}$	1	g^2	$-C_x$	g^2	$-C_x$
21	$t_{26(3)}$	1	g^2	$-2f$	g^2	$-2f$
22	$t_{26(4)}$	1	C_y	$-C_x$	C_y	$-C_x$

2.4.4 New Members of the Proposed Class

We have suggested some new members of the proposed class of estimators which come out to be more efficient than the existing estimators of population mean. These are given as follows,

Table 2.3: **Some New Members of Proposed Class of estimators t**

SNo.	**Estimators**	k	**a**	b	c	d
1	$t_{(1)}$	k_1	M	$-\rho$	M	$-\rho$
2	$t_{(2)}$	k_2	$^{N}C_n$	$-M$	$^{N}C_n$	$-M$
3	$t_{(3)}$	k_3	M	$-(f\times\rho)$	M	$-(f\times\rho)$
4	$t_{(4)}$	k_4	M	$-C_x$	M	$-C_x$
5	$t_{(5)}$	k_5	S_x^2	$-M$	S_x^2	$-M$

2.5 Theoretical Efficiency Comparison

In this section the conditions are derived under which our estimator is proven more efficient than the many existing estimators.First let us take,

$$
\begin{aligned}
MSE(t) &= \overline{Y}^2\left[1-\frac{A^2}{B}\right]\\
&= \overline{Y}^2\left[1-\frac{\{1-\Delta\lambda C_x^2(c-\tau_1)\}^2}{1+\lambda C_y^2-\Delta C_x^2(4g-\Delta-2\tau_1)}\right]\\
&= \overline{Y}^2\left[\frac{1+\lambda C_y^2-\Delta C_x^2(4g-\Delta-2\tau_1)-\{1+\Delta^2\lambda^2C_x^4(c-\tau_1)^2-2\Delta\lambda C_x^2(c-\tau_1)\}}{B}\right]\\
&= \overline{Y}^2\left[\frac{1+\lambda C_y^2+\Delta\lambda C_x^2(-4g+\Delta+2\tau_1+2g-2\tau_1)-1-\Delta^2\lambda^2C_x^4(g-\tau_1)^2}{B}\right]\\
MSE(t) &= \lambda\overline{Y}^2\left[\frac{C_y^2+\Delta C_x^2(\Delta-2g)-\Delta\lambda C_x^4(c-\tau_1)^2}{B}\right]
\end{aligned}
$$

1. $\mathbf{MSE(t) < V(t_0)}$ **if**

$$
\begin{aligned}
&\frac{\lambda\overline{Y}^2[C_y^2+\Delta C_x^2(\Delta-2g)-\Delta\lambda C_x^4(g-\tau_1)^2]}{B} < \lambda\overline{Y}^2C_y^2\\
\Longrightarrow\quad & C_y^2+\Delta C_x^2(\Delta-2g)-\Delta\lambda C_x^4(g-\tau_1)^2 < BC_y^2\\
\Longrightarrow\quad & (1-B)C_y^2+\Delta C_x^2(\Delta-2g)-\Delta C_x^4(g-\tau_1)^2 < 0\\
\Longrightarrow\quad & \Delta C_x^4(g-\tau_1)^2 > (1-B)C_y^2+\Delta C_x^2(\Delta-2g)\\
\Longrightarrow\quad & C_x^2 > \frac{(1-B)d^2+\Delta(\Delta-2g)}{\Delta\lambda(g-\tau_1)^2}\\
\Longrightarrow\quad & \mathbf{C_x^2 > \frac{G}{G'}}
\end{aligned}
$$

2. $\mathbf{MSE(t) < MSE(t_1)}$ **if**

$$\begin{aligned}
& \lambda \overline{Y}^2 \frac{C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta C_x^4 (g-\tau_1)^2}{B} < \lambda \overline{Y}^2 C_y^2 (1-\rho^2) \\
\Longrightarrow \quad & C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2 < B C_y^2(1-\rho^2) \\
\Longrightarrow \quad & (1-B)C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2 + B C_y^2 \rho^2 < 0 \\
\Longrightarrow \quad & (1-B)d^2 + \Delta(\Delta - 2g) + Bd^2\rho^2 < \Delta\lambda C_x^2 (g-\tau_1)^2 \\
\Longrightarrow \quad & C_x^2 > \frac{(1-B)d^2 + \Delta(\Delta - 2g) + B(d^2\rho^2)}{\Delta\lambda(g-\tau_1)^2} \\
\Longrightarrow \quad & \mathbf{C_x^2} > \frac{\mathbf{G + B(d^2\rho^2)}}{\mathbf{G'}}
\end{aligned}$$

3. $\mathbf{MSE(t) < MSE(t_i)}$; $\mathbf{i = 2, 3}$ **if**

$$\begin{aligned}
& \lambda \overline{Y}^2 \left[\frac{C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2}{B} \right] < \lambda \overline{Y}^2 (C_y^2 + C_x^2 - 2\rho C_y C_x) \\
\Longrightarrow \quad & C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta C_x^4 (g-\tau_1)^2 < B C_y^2 + B C_x^2 - 2B\rho C_y C_x \\
\Longrightarrow \quad & (1-B)C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2 - B C_x^2 + 2B\rho C_y C_x < 0 \\
\Longrightarrow \quad & (1-B)d^2 + \Delta(\Delta - 2g) - B + 2B\rho g < \Delta\lambda C_x^2 (g-\tau_1)^2 \\
\Longrightarrow \quad & C_x^2 > \frac{(1-B)d^2 + \Delta(\Delta - 2g) + B(2\rho g - 1)}{\Delta\lambda(g-\tau_1)^2} \\
\Longrightarrow \quad & \mathbf{C_x^2} > \frac{\mathbf{G + B(2\rho g - 1)}}{\mathbf{G'}}
\end{aligned}$$

4. $\mathbf{MSE(t) < MSE(t_i)}$; $\mathbf{i = 4, 13}$ **if**

$$\begin{aligned}
& \lambda \overline{Y}^2 \left[\frac{C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2}{B} \right] < \lambda \overline{Y}^2 (C_y^2 + C_x^2 + 2\rho C_y C_x) \\
\Longrightarrow \quad & C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta C_x^4 (g-\tau_1)^2 < B C_y^2 + B C_x^2 + 2B\rho C_y C_x \\
\Longrightarrow \quad & G - B - 2B\rho g < G' C_x^2 \\
\Longrightarrow \quad & \mathbf{C_x^2} > \frac{\mathbf{G - B(1 + 2\rho g)}}{\mathbf{G'}}
\end{aligned}$$

5. $\mathbf{MSE(t) < MSE(t_i)}$; $\mathbf{i = 5, 6, 7, 9, 11, 18}$

$$\begin{aligned}
& C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4 (g-\tau_1)^2 < B C_y^2 + B g^2 C_x^2 - 2B\rho g C_y C_x \\
\Longrightarrow \quad & (1-B)C_y^2 + \Delta C_x^2(\Delta - 2g) - B g^2 C_x^2 + 2B\rho g C_y C_x < \Delta C_x^4 (g-\tau_1)^2 \\
\Longrightarrow \quad & (1-B)d^2 + \Delta(\Delta - 2g) - B g^2 + 2B\rho g^2 < G' C_x^2 \\
\Longrightarrow \quad & G + B g^2 (2\rho - 1) < G' C_x^2 \\
\Longrightarrow \quad & \mathbf{C_x^2} > \frac{\mathbf{G + Bg^2(2\rho - 1)}}{\mathbf{G'}}
\end{aligned}$$

6. $\mathbf{MSE(t) < MSE(t_8)}$ **if**

$$\begin{aligned}
&\frac{C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4(g-\tau_1)^2}{B} < C_y^2(1 + a^2 + 2a\rho\upsilon) \\
\Longrightarrow\quad & C_y^2 + \Delta C_x^2(\Delta - 2g) - \Delta\lambda C_x^4(g-\tau_1)^2 - BC_y^2 - Ba^2 - 2Ba\rho\upsilon < 0 \\
\Longrightarrow\quad & (1-B)C_y^2 + \Delta C_x^2(\Delta - 2g) - B(a^2 + 2a\rho\upsilon) < G^{'}C_x^4 \\
\Longrightarrow\quad & \mathbf{C_x^2 > \frac{G - Bh(h + 2\rho\upsilon)}{G'}}
\end{aligned}$$

7. $\mathbf{MSE(t) < MSE(t_i)}$; $i = 10, 14$ **if**

$$\mathbf{C_x^2 > \frac{G - Bg^2(1 + 2\rho)}{G'}}$$

8. $\mathbf{MSE(t) < MSE(t_i)}$; $i = 12, 17, 23$ **if**

$$\mathbf{C_x^2 > \frac{G + BR_i(2\rho g - R_i)}{G'}\ ;\ i = 12, 17, 23}$$

9. $\mathbf{MSE(t) < MSE(t_i)}$; $i = 15, 19, 20$ **if**

$$\begin{aligned}
& d^2(1-B) + \Delta(\Delta - 2g) - B\left(\frac{1}{4} - g\right) < \Delta\lambda C_x^2(c-g)^2 \\
\Longrightarrow\quad & \mathbf{C_x^2 > \frac{G + B\left(g - \frac{1}{4}\right)}{G'}}
\end{aligned}$$

10. $\mathbf{MSE(t) < MSE(t_{16})}$ **if**

$$\mathbf{C_x^2 > \frac{G + 4B(g-1)}{G'}}$$

11. $\mathbf{MSE(t) < MSE(t_{21})}$ **if**

$$\begin{aligned}
& C_x^2 > \frac{G + B\left(\alpha g - \frac{\alpha^2}{4}\right)}{G'} \\
\Longrightarrow\quad & C_x^2 > \frac{G + B\alpha\left(g - \frac{\alpha}{4}\right)}{G'} \\
\Longrightarrow\quad & \mathbf{C_x^2 > \frac{G + \frac{B\alpha}{4}(4g - \alpha)}{G'}}
\end{aligned}$$

12. $\mathbf{MSE(t) < MSE(t_{22})}$ **if**

$$\mathbf{C_x^2 > \frac{G + B\left(\frac{9}{4} - 3g\right)}{G'}}$$

13. $\mathbf{MSE(t) < MSE(t_{24})}$ **if**

$$\mathbf{C_x^2 > \frac{G + BH}{G'}}$$

14. $\mathbf{MSE(t) < MSE(t_{25})}$ **if**

$$\mathbf{C_x^2 > \frac{G + BH^{'}}{G^{'}}}$$

15. $\mathbf{MSE(t) < MSE(t_{26})}$ **if**

$$\mathbf{C_x^2 > \frac{G + B[\Delta(\Delta - 2g)]}{G^{'}}}$$

16. $\mathbf{MSE(t) < MSE(t_{27})}$ **if**

$$\mathbf{C_x^2 > \frac{G + Bb_1R_{12}(2\rho g - b_1R_{12})}{G^{'}}}$$

17. $\mathbf{MSE(t) < MSE(t_{28})}$ **if**

$$\mathbf{C_x^2 > \frac{G + Bb_2R_{17}(2\rho g - b_2R_{17})}{G^{'}}}$$

where,

$$d = \frac{C_y}{C_x}$$
$$G = (1 - B)d^2 + \Delta(\Delta - 2g) \quad , \quad G^{'} = \Delta\lambda(d - \tau_1)^2$$
$$H = \frac{(1 - \alpha_1)^2}{4} - (1 - \alpha_1)g \quad , \quad H^{'} = \left(\frac{1}{4} + \alpha_2^2 + \alpha_2\right) + (2\alpha_2 + 1)g$$

2.6 Bias Comparisons

Following are the conditions under which proposed class of estimators is less biased than the existing estimators,

Table 2.4: **Bias Comparisons**

SNo.	$\mathbf{Bias(t) < Bias(\bullet)}$	**Conditions**
1	$\lvert Bias(t)\rvert < \lvert Bias(t_2)\rvert$	$\lvert J\rvert < \lvert 1 - g\rvert$
2	$\lvert Bias(t)\rvert < \lvert Bias(t_4)\rvert$	$\lvert J\rvert < \left\lvert \frac{g}{\overline{X}^2\overline{Y}^2}\right\rvert$
3	$\lvert Bias(t)\rvert < \lvert Bias(t_5)\rvert$	$\lvert J\rvert < \left\lvert \frac{g(1-g)}{2\overline{Y}}\right\rvert$
4	$\lvert Bias(t)\rvert < \lvert Bias(t_6)\rvert$	$\lvert J\rvert < \left\lvert g\left(\frac{1}{2} - g\right)\right\rvert$
5	$\lvert Bias(t)\rvert < \lvert Bias(t_7)\rvert$	$\lvert J\rvert < \left\lvert g^2\left(1 - \frac{1}{\overline{X}^2\overline{Y}^2}\right)\right\rvert$

Bias Comparisons

SNo.	Bias(t) < Bias(•)	Conditions
6	$\lvert Bias(t)\rvert < \lvert Bias(t_8)\rvert$	$\lvert J\rvert < \left\lvert h\upsilon\left(\rho+\frac{h(\upsilon-1)}{2}\right)\right\rvert$
7	$\lvert Bias(t)\rvert < \lvert Bias(t_9)\rvert$	$\lvert J\rvert < \left\lvert\frac{g^2}{\lambda}\right\rvert$
8	$\lvert Bias(t)\rvert < \lvert Bias(t_{10})\rvert$	$\lvert J\rvert < \left\lvert\frac{g(g-1)}{\lambda}\right\rvert$
9	$\lvert Bias(t)\rvert < \lvert Bias(t_{11})\rvert$	$\lvert J\rvert < \left\lvert\frac{g(g+1)}{2}\right\rvert$
10	$\lvert Bias(t)\rvert < \lvert Bias(t_{12})\rvert$	$\lvert J\rvert < \lvert R_{12}(R_{12}-g)\rvert$
11	$\lvert Bias(t)\rvert < \lvert Bias(t_{13})\rvert$	$\lvert J\rvert < \left\lvert c+\frac{f}{1-f}\right\rvert$
12	$\lvert Bias(t)\rvert < \lvert Bias(t_{15})\rvert$	$\lvert J\rvert < \left\lvert\frac{3-4g}{8}\right\rvert$
13	$\lvert Bias(t)\rvert < \lvert Bias(t_{16})\rvert$	$\lvert J\rvert < \lvert 3-2g\rvert$
14	$\lvert Bias(t)\rvert < \lvert Bias(t_{17})\rvert$	$\lvert J\rvert < \lvert R_{17}(R_{17}-g)\rvert$
15	$\lvert Bias(t)\rvert < \lvert Bias(t_{19})\rvert$	$\lvert J\rvert < \left\lvert\frac{3-4g}{8}\right\rvert$
16	$\lvert Bias(t)\rvert < \lvert Bias(t_{20})\rvert$	$\lvert J\rvert < \left\lvert\frac{3-4g}{8}\right\rvert$
17	$\lvert Bias(t)\rvert < \lvert Bias(t_{21})\rvert$	$\lvert J\rvert < \left\lvert L-L^{'}\right\rvert$
18	$\lvert Bias(t)\rvert < \lvert Bias(t_{22})\rvert$	$\lvert J\rvert < \left\lvert\left(\frac{15}{8}-\frac{g}{2}\right)\right\rvert$
19	$\lvert Bias(t)\rvert < \lvert Bias(t_{23})\rvert$	$\lvert J\rvert < \lvert R_{23}(R_{23}-g)\rvert$
20	$\lvert Bias(t)\rvert < \lvert Bias(t_{24})\rvert$	$\lvert J\rvert < \left\lvert L^{'''}\right\rvert$
21	$\lvert Bias(t)\rvert < \lvert Bias(t_{25})\rvert$	$\lvert J\rvert < \lvert P\rvert$
22	$\lvert Bias(t)\rvert < \lvert Bias(t_{26})\rvert$	$\lvert J\rvert < \lvert\Delta(\tau_1-g)\rvert$
23	$\lvert Bias(t)\rvert < \lvert Bias(t_{27})\rvert$	$\lvert J\rvert < \left\lvert P^{'}\right\rvert$
24	$\lvert Bias(t)\rvert < \lvert Bias(t_{28})\rvert$	$\lvert J\rvert < \left\lvert P^{''}\right\rvert$

where,

$$J = \frac{AH^{''}-B}{\lambda BC_x^2} \quad , \quad H^{''} = 1+\Delta(\tau_1-g)\lambda C_x^2$$

$$L = \frac{1}{8\lambda}(4\alpha-1-7f+4f^2) \quad , \quad L^{'} = \frac{(2\alpha-1)}{2(1-f)}g$$

$$L^{'''} = \left(\frac{3}{8}-\frac{g}{2}\right)-\alpha_1\left(\frac{3}{8}-\frac{g}{2}\right) \quad , \quad P = \left(\frac{\alpha_2^2}{2}+\frac{3}{8}\right)-\left(\alpha_2+\frac{1}{2}\right)g$$

$$P^{'} = \frac{b_1(b_1+1)}{2}R_{12}^2-b_1R_{12}g \quad , \quad P^{''} = \frac{b_2(b_2+1)}{2}R_{17}^2-b_2R_{17}g$$

2.7 Computational Study

To prove the theoretical results numerically we have considered 2 Natural Populations each with sample sizes 3 and 5.

Natural Population Used	: From J. Subramani (2016)
Data Source	: Daroga Singh and F.S. Chaudhary (1986, Page-177)
Data Details	: **Study Variable (Populations 1 and 2) :**
	: Area under wheat in a region during year 1974
	: **Auxiliary Variable**
	: Population 1 : Cultivated Area under wheat in a region during year 1971
	: Population 2 : Cultivated Area under wheat in a region during year 1973

Table 2.5: **Parametric Values of the Population**

SNo	Parameters	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	N	34	34	34	34
2	n	3	3	5	5
3	${}^{N}C_{n}$	5984	5984	278256	278256
4	$\overline{Y}$	856.4118	856.4118	856.4118	856.4118
5	$\overline{X}$	208.8824	199.4412	208.8824	199.4412
6	M	767.5	767.5	767.5	767.5
7	ρ	0.4491	0.4453	0.4491	0.4453
8	$V(\bar{y})$	163356.4086	163356.4086	91690.3713	91690.3713
9	$V(\bar{x})$	6884.4455	6857.8555	3864.1726	3849.248
10	C_y^2	0.7328	0.7328	0.7328	0.7328
11	C_x^2	0.5192	0.5673	0.5192	0.5673
12	g	0.5336	0.5061	0.5336	0.5061

Table 2.6: **MSE of the Proposed Class of Estimators**

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	$t_{(1)}$	111571.2176	112546.9916	71666.7131	73145.5003
2	$t_{(2)}$	105792.2731	106029.0406	71595.4359	73065.0431
3	$t_{(3)}$	111538.9242	112512.1077	71635.7277	73111.0937
4	$t_{(4)}$	111592.6812	112573.5102	71688.7244	73173.4587
5	$t_{(5)}$	107321.8823	107787.8035	68813.7482	69904.5174

Table 2.7: **Biases of the Proposed Class of Estimators**

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	$t_{(1)}$	-130.2775	-131.4169	-83.6825	-85.4093
2	$t_{(2)}$	-123.5297	-123.8061	-83.5993	-85.3153
3	$t_{(3)}$	-130.2398	-131.3761	-83.6463	-85.3691
4	$t_{(4)}$	-130.3026	-131.4479	-83.7082	-85.4419
5	$t_{(5)}$	-125.3157	-125.8598	-80.3512	-81.6249

Table 2.8: **MSE of the Existing Estimators**

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	t_0	163356.4086	163356.4086	91690.3713	91690.3713
2	t_1	130408.92	130964.12	73197.27	73508.90
3	t_2	155585.30	161807.06	87328.52	90820.74
4	t_3	155585.30	161807.06	87328.52	90820.74
5	t_4	402579.3	417808.7	225963.9	234512.0
6	t_5	130408.92	130964.12	73197.27	73508.90
7	t_6	130408.92	130964.12	73197.27	73508.90
8	t_7	130408.92	130964.12	73197.27	73508.90
9	t_8	316642.9	331780.6	177728.6	186225.2
10	t_9	130408.92	130964.12	73197.27	73508.90
11	t_{10}	262198.9	260533.3	147169.7	146234.8
12	t_{11}	130408.92	130964.12	73197.27	73508.90
13	t_{12}	155215.57	161338.94	87120.99	90557.99
14	t_{13}	402579.3	417808.7	225963.9	234512.0
15	t_{14}	262198.9	260533.3	147169.7	146234.8
16	t_{15}	130539.38	130968.87	73270.49	73511.56
17	t_{16}	379266.0	413160.6	212878.3	231903.1
18	t_{17}	155354.23	161529.44	87198.83	90664.91
19	t_{18}	130408.92	130964.12	73197.27	73508.90
20	t_{19}	130539.38	130968.87	73270.49	73511.56
21	t_{20}	130539.38	130968.87	73270.49	73511.56
22	t_{21}	155634.34	155764.47	87356.05	87429.09
23	t_{22}	238494.2	255871.0	133864.5	143617.9
24	t_{23}	154079.99	159983.89	85947.50	89148.61

MSE of the Existing Estimators

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
25	t_{24}	130408.92	130964.12	73197.27	73508.90
26	t_{25}	130408.92	130964.12	73197.27	73508.90
27	$t_{26(1)}$	205590.49	781461.5	115395.97	438626.7
28	$t_{26(2)}$	1063854.0	944809.8	597130.9	530312.6
29	$t_{26(3)}$	243303.89	352614.99	233270.8	233501.6
30	$t_{26(4)}$	1438667.2	290991.99	807509.9	163331.02
31	t_{27}	130408.92	130964.12	73197.27	73508.90
32	t_{28}	130408.98	130964.20	73197.30	73508.94

Table 2.9: **Biases of the Existing Estimators**

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	t_0	-	-	-	-
2	t_1	-	-	-	-
3	t_2	63.0274	72.9217	35.3767	40.9303
4	t_3	-	-	-	-
5	t_4	0.0000000023	0.0000000027	0.0000000013	0.0000000014
6	t_5	0.0196	0.0215	0.0110	0.0121
7	t_6	-2.4208	-0.4578	-1.3588	-0.2570
8	t_7	0.0842	0.0873	0.0473	0.0490

Biases of the Existing Estimators

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
9	t_8	6534.214	7052.365	3667.591	3958.424
10	t_9	-126.5838	-124.4507	-126.5838	-124.4507
11	t_{10}	-110.6531	-121.4380	-110.6531	-121.4380
12	t_{11}	-69.9852	-74.2841	-39.2820	-41.6950
13	t_{12}	62.3478	72.0940	34.9952	40.4657
14	t_{13}	85.1784	89.0198	53.5468	56.2347
15	t_{14}	-	-	-	-
16	t_{15}	14.6226	18.0043	8.2075	10.1056
17	t_{16}	261.1836	293.4962	146.5998	164.7366
18	t_{17}	62.6029	72.4311	35.1384	40.6549
19	t_{18}	0	0	0	0
20	t_{19}	14.6226	18.0043	8.2075	10.1056
21	t_{20}	-20.65007	-20.5374	-12.1483	-12.1368
22	t_{21}	36.8703	35.1092	23.6411	20.4118
23	t_{22}	145.2144	164.7524	81.5074	92.4734
24	t_{23}	60.2489	69.6855	32.8180	37.9519
25	t_{24}	15.6045	18.2249	8.7587	10.2295
26	t_{25}	17.1196	18.4649	9.6091	10.3642
27	$t_{26(1)}$	12841.376	5079.894	7207.741	2851.295
28	$t_{26(2)}$	369.9017	306.3099	207.6223	171.9288
29	$t_{26(3)}$	15122.555	21759.287	1104.943	1096.790
30	$t_{26(4)}$	9089.971	18048.420	5102.11	3 10130.405
31	t_{27}	16.8813	18.5253	9.4753	10.3980
32	t_{28}	16.8563	18.4960	9.4613	10.3816

Table 2.10: **PRE of the Proposed Class of Estimators**

SNo	Estimators	**for n=3**		**for n=5**	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	$t_{(1)}$	146.4145	145.1451	127.9410	125.3534
2	$t_{(2)}$	154.4124	154.0676	128.0673	125.4914
3	$t_{(3)}$	146.4567	145.1901	127.9953	125.4124
4	$t_{(4)}$	146.3863	145.1109	127.9007	125.3055
5	$t_{(5)}$	152.2166	151.5537	133.2443	131.1652

Table 2.11: **PRE of the Existing Estimators**

SNo	Estimators	**for n=3**		**for n=5**	
		pop.1	**pop.2**	**pop.1**	**pop.2**
1	t_0	100.00	100.00	100.00	100.00
2	t_1	125.2647	124.7337	125.2647	124.7337
3	t_2	104.9948	100.9575	104.9948	100.9575
4	t_3	104.9948	100.9575	104.9948	100.9575
5	t_4	40.57745	39.09838	40.57745	39.09838
6	t_5	125.2647	124.7337	125.2647	124.7337
7	t_6	125.2647	124.7337	125.2647	124.7337
8	t_7	125.2647	124.7337	125.2647	124.7337
9	t_8	51.59010	49.23627	51.59010	49.23627
10	t_9	125.2647	124.7337	125.2647	124.7337
11	t_{10}	62.30248	62.70079	62.30248	62.70079

PRE of the Existing Estimators

SNo	Estimators	for n=3		for n=5	
		pop.1	**pop.2**	**pop.1**	**pop.2**
12	t_{11}	125.2647	124.7337	125.2647	124.7337
13	t_{12}	105.2449	101.2505	105.2449	101.2505
14	t_{13}	40.57745	39.09838	40.57745	39.09838
15	t_{14}	62.30248	62.70079	62.30248	62.70079
16	t_{15}	125.1396	124.7292	125.1396	124.7292
17	t_{16}	43.07173	39.53823	43.07173	39.53823
18	t_{17}	105.1509	101.1310	105.1509	101.1310
19	t_{18}	105.2449	101.2505	105.2449	101.2505
20	t_{19}	125.1396	124.7292	125.1396	124.7292
21	t_{20}	125.1396	124.7292	125.1396	124.7292
22	t_{21}	104.9617	104.8740	104.9617	104.8740
23	t_{22}	68.49493	63.84327	68.49493	63.84327
24	t_{23}	106.0205	102.1080	106.6818	102.8512
25	t_{24}	125.2647	124.7337	125.2647	124.7337
26	t_{25}	125.2647	124.7337	125.2647	124.7337
27	$t_{26(1)}$	79.4574	20.9040	79.4578	20.9040
28	$t_{26(2)}$	15.3552	17.2899	15.3552	17.2899
29	$t_{26(3)}$	67.1411	46.3271	39.3065	39.2675
30	$t_{26(4)}$	11.3547	56.1378	11.3547	56.1378
31	t_{27}	125.2647	124.7337	125.2647	124.7337
32	t_{28}	125.2647	124.7336	125.2647	124.7336

2.8 Results and Conclusion

1. Table 2.2 shows that many of the existing estimators are particular case of our suggested class of estimators for population mean. Table 2.3 suggests some

new members of proposed class. Conditions are also derived under which our proposed class is more efficient and less biased than the existing ones.

2. Tables 2.6,2.7 contain the MSE and Bias values of the new members of proposed class. Tables 2.8 and 2.9 show the PRE of proposed class and existing estimators with respect to the SRS mean. Here we can easily notice that proposed class have the greater PREs.

3. Since Efficiency is stronger property than the unbiasedness. Hence here we prefer the biased estimator with minimum MSE instead of unbiased estimator with higher MSE.

4. From table 2.10, for sample size $n = 3,\ t_{(2)}$ is most efficient and for sample size $n = 5,\ t_{(5)}$ is most efficient. So we can say that $t_{(2)}$ and $t_{(5)}$ are most efficient estimators for our proposed class.

5. Hence we have proved theoretically and numerically that our proposed class of estimator is better than the other given estimators.

6. Since our computations revolved around a natural population, therefore we can successfully recommed our class of estimators for practical problems.

References

Bahl, S. and Tuteja, R.K. (1991) : Ratio and product type exponential estimator.*Information and Optimization Sciences* , XII (I), 159-163.

Chakrabarty, R.P. (1979) : Some ratio estimators.*Journal of the Indian Society of Agricultural Statistics*, 31(1), 49–57.

Chauby, Y.P., Singh, M., Dwivedi, T.D. (1984) : A note on an optimality of the regression estimator.*Biometrical J.*, 26(4), 465-467.

Cochran, W.G. (1940): The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *The Journal of Agric. Science*, 30, 262-275.

Goodman, L.A. (1960) : On the exact variance of products *J. Amer. Stat. Ass.* , 55, 708-713.

Goodman, L. A. and Hartley, H.O. (1958): The Precision of Unbiased Ratio-Type Estimators. *Journal of the American Statistical Association*, 53, 282, 491-508.

H. Pandey, S.K. Yadav and A.K. Shukla (2011) : An Improved General Class of Estimators, Estimating Population Mean using Auxiliary Information.*International Journal of Statistics and Systems* ,Vol.6, 1, 1–7.

Jerajuddin, M. and Kishun, J. (2016) : Modified Ratio Estimators for Population Mean Using Size of the Sample, Selected From Population.*IJSRSET* , 2:2,10-16.

Kadilar, C., and Cingi, H. (2003) : A study on the chain ratio type estimator.*Hacettepe Journal of Mathematics and Statistics*, Vol.32, 105-108.

Kadilar, G.O. (2016) : A New Exponential Type Estimator for the Population Mean in Simple Random Sampling. *Journal of Modern Applied Statistical Methods* , 15:2 ,2017-2014 .

Ray, S.K., Sahai, A., Sahai, A. (1979) : A note on ratio and product type estimators.*Ann. Inst. Statist. Math.*, 31(1), 141-144.

Reddy, V.N. (1974) : On a transformed ratio method of estimation.*Sankhya C* , 36(1), 59-70.

Sahai, A. (1979) : An efficient variant of the product and ratio estimator.*Statistica Neerlandica*, 33(1),27-35.

SAHAI, A. AND RAY, S.K. (1980) : Efficient families of ratio and product type estimators.*Biometrika*, 67(1), 211–215.

SINGH, PAL AND SOLANKI (2016) : A new class of estimators of finite population mean in sample surveys *Communications in Statistics-Theory and Methods* , 46:6, 2630-2637 .

SRIVASTAVA, S.K. (1967) : An estimator using auxiuary information in sample surveys.*GkuftU Sfatist.Ass.Bull.*, 16,121-132.

SINGH, D., CHAUDHARY, F.S. (1986) : Sample Survey Designs.*New Age Int.(P) Ltd.*, 148, 182, 177.

SINGH, H.P. AND PAL, S.K. (2015) :A New Chain Ratio-Ratio-Type Exponential Estimator Using Auxiliary Information in Sample Surveys.*International Journal of Mathematics and its Applications*,3 ,4:B, 37-46.

SINGH, H. P., SOLANKI, R. S., SINGH, A. K. (2014) : Predictive Estimation of Finite Population Mean Using Exponential Estimators.*Statistika: Statistics and Economy Journal*, 94 (1), 41–53.

SINGH, H.P. AND TAILOR, R. (2003) : Use of known correlation coefficient in estimating the finite population mean.*Statistics in Transition*,6, 4, 555-560.

SISODIA, B.V.S. AND DWIVEDI, V.K (1981) : A modified ratio estimator using coefficient of variation of auxiliary variable., Journal of the Indian Society of Agricultural Statistics, 33, 13-18.

SOPONVIWATKUL, K AND LAWSON, N. (2017) : New ratio estimators for estimating population mean in simple random sampling using a coefficient of variation, correlation coefficient and a regression coefficient. *Gazi University Journal of Science* , 30(4) , 610-621.

SRIVASTAVA, S.K. (1983) : Predictive estimation of finite population mean using product estimator.*Metrica*, 30(1), 93-99.

SUBRAMANI (2016) : A new median based ratio estimator for estimation of the finite population mean. *Statistics in Transition new series* ,17:4 ,591-604.

SWAIN, A.K.P.C (2014) : On an improved ratio type estimator of finite population mean in sample surveys.*Revista Investigacion Operacional*, 35, 1, 49-57.

VISHWAKARMA, SINGH, GUPTA, PAREEK (2016) : Improved ratio and product type estimators of finite population mean in simple random sampling. *Revista Investigacion Operacional* , 36:3, 70-76.

WALSH, J.E. (1970) : Generalization of ratio estimator for population total.*Sankhya A*, 32(1), 99-106.

WATSON, D.J. (1937) : The Estimation of leaf areas.*Jour. Agri. Sci.*,27, 474.

YADAV, S.K. AND MISHRA, S.S. (2015) : Developing Improved Predictive Estimator for Finite Population Mean Using Auxiliary Information.*Statistika*, 95:1, 76-85.

Chapter 3

A REMEDY FOR ENHANCED ESTIMATION OF POPULATION MEAN USING EXPONENTIAL RATIO-CUM-PRODUCT TYPE ESTIMATOR UNDER NON-RESPONSE

3.1 Summary

The present chapter provides a remedy for improved estimation of population mean of study variable, using the information related to two auxiliary variables in the situation of non response under Simple Random Sampling Scheme. The Bias and MSE of the proposed estimator are derived upto the first order of approximation. The minimum value of the MSE is also obtained by optimizing the characterizing scalers. The Bias and MSE have also been compared with the considered existing competing estimators both theoretically and empirically. The theoretical conditions are verified for the proposed estimator to be more efficient than the considered existing estimators using the two natural populations.

3.2 Introduction

In case the size of the population is very large, we take a representative sample to estimate the population characteristics to save time and cost. For the estimation of population mean corresponding sample mean is the best suitable estimator. The ideal situation is that all measures are correctly and perfectly taken for the characterstic under study but practically it is not possible and occurrance of errors is inevitable. Apart from the sampling error there exists non sampling errors too.

Non sampling errors are of three type : Non Response Error, Response Error and Tabulation Error. We try to minimize all kinds of errors. This present paper is concerned with the minimization of Non Response error. Non response can occur under following situations :

- Failure to approach some units in the sample.
- Use of incomplete sampling frame.
- Respndent not availbale at the time of data collection.
- Lack of knowledge of the respondent for the concerned question.
- Refusal of the respondent to answer the question.

The only solution is to recontact the non respondents and try to gain as much information as possible. If the first attempt to get the answer from the respondents resulted in failure, we need to improve our questioning style. Instead of asking the information about the charcterstic under study directly, we can take the help of some additional information which is highly related to the variable under study. Thus we may know the answer indirectly. For example : To know the salary of a person we can also ask the type of A.C. or the type of car they posses. (As we know Rich people would surely have costlier car and A.C. than the middle class people.) This kind of additional information is termed as Auxiliary information and it is higly useful in obtaining the more precised results regarding the main variable.

We frequently come accross the situations where we lack the complete information on either the study variable or the auxiliary variable or sometimes on both. The Hansan and Horwitz (1946) were the first to address this problem and suggested a remedy using the double sampling plan using the following steps :

- A sample of size n units is drawn from the population of size N using Simple Random Sampling Without Replacement (SRSWOR) and approached via mail survey. Let r_2 resulted in non response.
- A subsample of size k_2 is again drawn from the class of r_2 non respondents and are approached through personal interviews.

Let the population is of size N containing size N_1 response class and size N_2 of non response class. Let from the sample of size n, r_1 units respond and $r_2 = (n - r_1)$ units do not respond in the first attempt. Let a subsample of size k_2 is drawn from r_2 such that $r_2 = k_2 h,\ (h > 1)$. Let y_i and $(x_i; z_i)$ be the values of the study variable (y) and the auxiliary variables (x, z) respectively. Let $\bar{y}$ and $(\bar{x}, \bar{z})$ be the sample means and $\overline{Y}$ and $(\overline{X}, \overline{Z})$ be the population means of the study and the auxiliary variables respectively.

3.3 Existing Estimators

There are a number estimators which have been developed till date for the estimation of population mean under various situations. The considered existing estimators under non response with there Mean Squared Errors and Biases are given in the Table-3.1.

Table 3.1: **Existing Estimators**

SNo	Estimators	Bias	MSE
1	Hansan and Horwitz(1946) $\bar{y}^* = \left(\frac{r_1}{n}\right)\bar{y}_{r_1} + \left(\frac{r_2}{n}\right)\bar{y}_{r_2}$	$-$	$\overline{Y}^2 V_y^*$
2	Ratio Estimator $\bar{y}_R^* = \bar{y}^*\left(\frac{\overline{X}}{\bar{x}^*}\right)$	$\overline{Y}\left(V_x^* - V_{yx}^*\right)$	$\overline{Y}^2\left(V_y^* + V_x^* - 2V_{yx}^*\right)$
3	Product Estimator $\bar{y}_P^* = \bar{y}^*\left(\frac{\bar{x}^*}{\overline{X}}\right)$	$\frac{V_{yx}^*}{\overline{X}^2\overline{Y}}$	$\overline{Y}^2\left(V_y^* + V_x^* + 2V_{yx}^*\right)$
4	Rao (1986) $\bar{y}_1^* = \bar{y}^*\left(\frac{\overline{X}}{\bar{x}}\right)$ $\bar{y}_2^* = \bar{y}^*\left(\frac{\bar{x}}{\overline{X}}\right)$	$\overline{Y}\left(V_x - V_{yx}\right)$ $\frac{V_{yx}}{\overline{X}^2\overline{Y}}$	$\overline{Y}^2\left(V_y^* + V_x - 2V_{yx}\right)$ $\overline{Y}^2\left(V_y^* + V_x + 2V_{yx}\right)$
6	Singh et al.(2009) $\bar{y}_7^* = \bar{y}^*\exp\left(\frac{\overline{X}-\bar{x}^*}{\overline{X}+\bar{x}^*}\right)$ $\bar{y}_8^* = \bar{y}^*\exp\left(\frac{\bar{x}^*-\overline{X}}{\bar{x}^*+\overline{X}}\right)$ $\bar{y}_9^* = \bar{y}^*\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ $\bar{y}_{10}^* = \bar{y}^*\exp\left(\frac{\bar{x}-\overline{X}}{\bar{x}+\overline{X}}\right)$	$\frac{\overline{Y}}{8}\left(3V_x^* - 4V_{yx}^*\right)$ $\frac{\overline{Y}}{8}\left(4V_{yx}^* - V_x^*\right)$ $\lambda\overline{Y}\left(\frac{3}{8}C_x^2 - \frac{1}{2}C_{yx}\right)$ $\lambda\overline{Y}\left(\frac{1}{2}C_{yx} - \frac{1}{8}C_x^2\right)$	$\overline{Y}^2\left(V_y^* + \frac{1}{4}V_x^* - V_{yx}^*\right)$ $\overline{Y}^2\left(V_y^* + \frac{1}{4}V_x^* + V_{yx}^*\right)$ $\overline{Y}^2\left[\left(\frac{1}{4}C_x^2 - C_{yx}\right)\lambda + V_y^*\right]$ $\overline{Y}^2\left[\left(\frac{1}{4}C_x^2 + C_{yx}\right)\lambda + V_y^*\right]$
7	Kumar and Bhougal (2011) $\bar{y}_{11}^* =$ $\bar{y}^*\left[\alpha_3\exp\left(\frac{\overline{X}-\bar{x}^*}{\overline{X}+\bar{x}^*}\right)\right.$ $\left.+(1-\alpha_3)\exp\left(\frac{\bar{x}^*-\overline{X}}{\bar{x}^*+\overline{X}}\right)\right]$	$\frac{1}{4}\overline{Y}\left[1 + 2(1-2\alpha_3)\right]V_{yx}^*$	$\overline{Y}^2\left(V_y^* - \frac{V_{yx}^{*2}}{V_x^*}\right)$

Existing Estimators

SNo	Estimators	Bias	MSE
8	Chanu and Singh(2015) $\bar{y}_{12}^{*}=\bar{y}^{*}\left[\alpha_4\exp\left(\frac{\overline{X}-\bar{x}^{*}}{\overline{X}+\bar{x}^{*}}\right)\right.$ $\left.(1-\alpha_4)\exp\left(\frac{\bar{x}'-\overline{X}}{\bar{x}'+\overline{X}}\right)\right]$	$\overline{Y}(F-\alpha_5 F')$	$\overline{Y}^2\left(V_y^{*}-\frac{V_{yx}^{*2}}{V_x^{*}}\right)$
9	Chanu and Singh (2015) $\bar{y}_{13}^{*}=\bar{y}^{*}\left[\alpha_5\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)\right.$ $\left.(1-\alpha_5)\exp\left(\frac{\bar{x}'-\overline{X}}{\bar{x}'+\overline{X}}\right)\right]$	$\overline{Y}(F''-\alpha_5 F''')$	$\overline{Y}^2(V_y^{*}-\lambda\rho_{yx}^2 C_y^2)$
10	Pal and Singh (2016) $\bar{y}_{14}^{*}=$ $\bar{y}^{*}\left(\frac{\overline{X}}{\bar{x}^{*}}\right)^{\alpha}\exp\left[\frac{\delta(\overline{X}-\bar{x}^{*})}{(\overline{X}+\bar{x}^{*})}\right]$	$\frac{\overline{Y}(\delta+2\alpha)}{8}V_x^{*}R'$	$\overline{Y}^2\left(V_y^{*}-\frac{V_{yx}^{*2}}{V_x^{*}}\right)$
11	Zubir et al.(2018) $\bar{y}_{15}^{*}=\bar{y}^{*}\left[\alpha_6\exp\left(\frac{\overline{X}-\bar{x}^{*}}{\overline{X}+\bar{x}^{*}}\right)\right.$ $\left.+(1-\alpha_6)\exp\left(\frac{\overline{Z}-\bar{z}^{*}}{\overline{Z}+\bar{z}^{*}}\right)\right]$	$\overline{Y}\left(\frac{3}{8}G''-\frac{1}{2}G'''\right)$	$\overline{Y}^2\left(C-\frac{G'^2}{G}\right)$

where,

- $\alpha_1=\frac{\left(V_{yx}^{*}-V_{yx}\right)}{(V_x^{*}-V_x)}$
- $\alpha_2=\frac{\left(V_x^{*}V_{yx}-V_{yx}^{*}V_x\right)}{V_x(V_x^{*}-V_x)}$
- $E=(\alpha_1+1)\,V_x^{*}-2V_{yx}^{*}$
- $E'=C_x^2\,(\alpha_2+1+2\alpha_2)-2C_{yx}$
- $d_1=\frac{\overline{Y}(V_{yx}^{*}-V_{yx})}{\overline{X}(V_x^{*}-V_x)}$
- $d_2=\frac{\overline{Y}V_{yx}}{\overline{X}V_x}$
- $\alpha_3=\frac{1}{2}\left(1+2\frac{V_{yx}^{*}}{V_x^{*}}\right)$
- $\bar{x}'=\frac{N\overline{X}-n\bar{x}}{N-n}$
- $F=\frac{g}{2}V_{yx}^{*}-\frac{g^2}{8}V_x^{*}$
- $F'=\frac{g-1}{2}V_{yx}^{*}+\frac{3+g^2}{8}V_x^{*}$
- $F''=\frac{\lambda g C_{yx}}{2}+\frac{\lambda g^2 C_x^2}{8}$
- $F'''=\frac{\lambda(g-1)C_{yx}}{2}+\frac{\lambda(3+g^2)C_x^2}{8}$
- $g=\frac{n}{N-n}$
- $\alpha_4=\frac{g}{g-1}-\frac{2}{g-1}\frac{V_{yx}^{*}}{V_x^{*}}$
- $\alpha_5=\frac{g-2k_{yx}}{g-1}$
- $k_{yx}=\frac{\rho_{yx}C_y}{C_x}$
- $\delta=-2\alpha$
- $\alpha=\frac{1}{2}$ (Standard Value)
- $R'=\delta+2\alpha-4R+2$
- $R=\frac{V_{yx}^{*}}{V_x^{*}}$
- $\alpha_6=\frac{2(V_z^{*}+V_{yx}^{*}-V_{yz}^{*}-V_{xz}^{*})}{V_z^{*}+4V_x^{*}-2V_{xz}^{*}}$

- $G = \frac{V_z^*}{4} + \frac{V_x^*}{4} - \frac{V_{xz}^*}{2}$
- $G' = \frac{V_z^*}{4} + \frac{V_{yx}^*}{2} - \frac{V_{yz}^*}{2} - \frac{V_{xz}^*}{2}$
- $G'' = \alpha_6 V_x^* + (1-\alpha_6)V_z^*$
- $G''' = \alpha_6 V_{yx}^* + (1-\alpha_6)V_{yz}^*$
- $C = V_y^* + \frac{V_y^*}{4} - V_{yz}^*$

3.4 The Proposed Estimator

$$t^* = \bar{y}^* \left[k_1 \exp\left(\frac{\overline{X} - \bar{x}^*}{\overline{X} + \bar{x}^*} \right) + k_2 \exp\left(\frac{\overline{Z} - \bar{z}^*}{\overline{Z} + \bar{z}^*} \right) \right] \tag{3.4.1}$$

Where constants k_1 and k_2 are chosen such that the Mean Squared Error (MSE) of our estimator is minimum.

3.4.1 Bias and MSE of Suggested Estimator

Bias and MSE of our proposed estimator up to the first order of estimation is given as follows :

$$B(t^*) = \overline{Y} \left[\frac{A}{B} D_2 + \frac{AD + BD_4}{BD_3} D_4 - 1 \right] \tag{3.4.2}$$

$$MSE(t^*) = \overline{Y}^2 \left[k - \frac{A^2}{B} \right] \tag{3.4.3}$$

where,

$$A = \frac{(D_2 D_3 - D D_4)}{D_3}$$
$$B = \frac{(D_1 D_3 - D^2)}{D_3}$$
$$k = 1 - \frac{D_4^2}{D_3}$$
$$D = 1 + V_y^* + \frac{3}{8}(V_x^* + V_z^*) - (V_{yx}^* + V_{yz}^*) + \frac{V_{xz}^*}{4}$$
$$D_1 = 1 + V_y^* + V_x^* - 2V_{yx}^*$$
$$D_2 = 1 + \frac{3}{8}V_x^* - \frac{V_{yx}^*}{2}$$
$$D_3 = 1 + V_y^* + V_z^* - 2V_{yz}^*$$
$$D_4 = 1 + \frac{3}{8}V_z^* - \frac{V_{yz}^*}{2}$$

3.4.2 Derivation of Bias and MSE of Proposed Estimator

We first let the relative error terms as follows :

$$e_0 = \frac{\bar{y}-\overline{Y}}{\overline{Y}} \qquad e_1 = \frac{\bar{x}-\overline{X}}{\overline{X}} \qquad e_2 = \frac{\bar{z}-\overline{Z}}{\overline{Z}}$$
$$e_0^* = \frac{\bar{y}^*-\overline{Y}}{\overline{Y}} \qquad e_1^* = \frac{\bar{x}^*-\overline{X}}{\overline{X}} \qquad e_2^* = \frac{\bar{z}^*-\overline{Z}}{\overline{Z}}$$
$$\text{So,} \quad \bar{y} = \overline{Y}(1+e_0) \quad \bar{x} = \overline{X}(1+e_1) \quad \bar{z} = \overline{Z}(1+e_2)$$
$$\bar{y}^* = \overline{Y}(1+e_0^*) \quad \bar{x}^* = \overline{X}(1+e_1^*) \quad \bar{z}^* = \overline{Z}(1+e_2^*)$$

The expectations of various terms are as follows :

$$\begin{aligned}
E(e_i) &= 0 & E(e_i^*) &= 0 \;;\; i=0,1,2 \\
E(e_0^2) &= E(e_0e_0^*) = \lambda C_y^2 = V_y\,, & E(e_1^2) &= E(e_1e_1^*) = \lambda C_x^2 = V_x \\
E(e_2^2) &= E(e_2e_2^*) = \lambda C_z^2 = V_z\,, & E(e_0^{*2}) &= \lambda C_y^2 + \gamma C_{y(2)}^2 = V_y^* \\
E(e_1^{*2}) &= \lambda C_x^2 + \gamma C_{x(2)}^2 = V_x^*\,, & E(e_2^{*2}) &= \lambda C_z^2 + \gamma C_{z(2)}^2 = V_z^* \\
E(e_0^*e_1^*) &= \lambda C_{yx} + \gamma C_{yx(2)} = V_{yx}^*\,, & E(e_0^*e_2^*) &= \lambda C_{yz} + \gamma C_{yz(2)} = V_{yz}^* \\
E(e_1^*e_2^*) &= \lambda C_{xz} + \gamma C_{xz(2)} = V_{xz}^*\,, & E(e_0^*e_1) &= E(e_0e_1) = \lambda C_{yx} = V_{yx} \\
E(e_0^*e_2) &= E(e_0e_2) = \lambda C_{yz} = V_{yz}\,, & E(e_1^*e_2) &= E(e_1e_2) = \lambda C_{xz} = V_{xz}
\end{aligned}$$

Now rewriting our proposed estimator from equation (3.4.1) as follows :

$$t^* = (1+e_0^*)\overline{Y}\left[k_1 \exp\left\{\frac{\overline{X}-(1+e_1^*)\overline{X}}{\overline{X}+(1+e_1^*)\overline{X}}\right\} + k_2 \exp\left\{\frac{\overline{Z}-(1+e_1^*)\overline{Z}}{\overline{Z}+(1+e_1^*)\overline{Z}}\right\}\right]$$
$$t^* = (1+e_0^*)\overline{Y}\left[k_1 \exp\left\{-e_1^*(2+e_1^*)^{-1}\right\} + k_2 \exp\left\{-e_2^*(2+e_2^*)^{-1}\right\}\right]$$
$$t^* = (1+e_0^*)\overline{Y}\left[k_1 \exp\left\{\frac{-e_1^*}{2}\left(1+\frac{e_1^*}{2}\right)^{-1}\right\} + k_2 \exp\left\{\frac{-e_2^*}{2}\left(1+\frac{e_2^*}{2}\right)^{-1}\right\}\right] \tag{3.4.4}$$

We assume $\left|\frac{e_1^*}{2}\right| < 1$ and $\left|\frac{e_2^*}{2}\right| < 1$ so that $\left(1+\frac{e_1^*}{2}\right)^{-1}$ and $\left(1+\frac{e_2^*}{2}\right)^{-1}$ is expandable. Expanding the right hand side of equation (3.4.4) up to the first order of approximation and after simplification we get,

$$t^* = (1+e_0^*)\overline{Y}\left[k_1\left(1-\frac{1}{2}e_1^*+\frac{3}{8}e_1^{*2}\right) + k_2\left(1-\frac{1}{2}e_2^*+\frac{3}{8}e_2^{*2}\right)\right]$$
$$\mathbf{t^*} = \mathbf{\overline{Y}}\left[\mathbf{k_1}\left(\mathbf{1}+\mathbf{e_0^*}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e_1^*}+\frac{\mathbf{3}}{\mathbf{8}}\mathbf{e_1^{*2}}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e_0^*e_1^*}\right)\right.$$
$$\left.+\mathbf{k_2}\left(\mathbf{1}+\mathbf{e_0^*}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e_2^*}+\frac{\mathbf{3}}{\mathbf{8}}\mathbf{e_2^{*2}}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e_0^*e_2^*}\right)\right] \tag{3.4.5}$$

Subtracting $\overline{Y}$ from both sides we get,

$$(t^*-\overline{Y}) = \overline{Y}\left[k_1\left(1+e_0^*-\frac{1}{2}e_1^*+\frac{3}{8}e_1^{*2}-\frac{1}{2}e_0^*e_1^*\right)\right.$$
$$\left.+k_2\left(1+e_0^*-\frac{1}{2}e_2^*+\frac{3}{8}e_2^{*2}-\frac{1}{2}e_0^*e_2^*\right)-1\right] \tag{3.4.6}$$

Taking expectation on both sides of equation (3.4.6) we get the bias of t^* as follows,

$$B(t^*) = \overline{Y}\left[k_1\left(1+\frac{3}{8}V_x^* - \frac{1}{2}V_{yx}^*\right) + k_2\left(1+\frac{3}{8}V_z^* - \frac{1}{2}V_{yz}^*\right) - 1\right] \tag{3.4.7}$$

Squaring both sides of equation (2.4.5) and solving up to the first order of approximation we get,

$$\begin{aligned}(t^*-\overline{Y})^2 =& \overline{Y}^2\left[k_1^2\left\{1+2e_0^*-e_1^*+(e_0^*-e_1^*)^2\right\}+k_2^2\left\{1+2e_0^*-e_2^*+(e_0^*-e_2^*)^2\right\}+1\right.\\ &-2k_1k_2\left\{1+2e_0^*-\frac{1}{2}e_1^*-\frac{1}{2}e_2^*+e_0^{*2}+\frac{3}{8}\left(e_1^{*2}+e_2^{*2}\right)-e_0^*e_1^*-e_0^*e_2^*+\frac{1}{4}e_1^*e_2^*\right\}\\ &\left.-2k_1\left(1+e_0^*-\frac{1}{2}e_1^*+\frac{3}{8}e_1^{*2}-\frac{1}{2}e_0^*e_1^*\right)-2k_2\left(1+e_0^*-\frac{1}{2}e_2^*+\frac{3}{8}e_2^{*2}-\frac{1}{2}e_0^*e_2^*\right)\right]\end{aligned} \tag{3.4.8}$$

Taking expectation on both sides we obtain the MSE of t^* upto the first order of approximation as follows ,

$$\begin{aligned}M(t^*) =& \overline{Y}^2\left[k_1^2\left(1+V_y^*+V_x^*-2V_{yx}^*\right)+k_2^2\left(1+V_y^*+V_z^*-2V_{yz}^*\right)+1\right.\\ &-2k_1k_2\left\{1+V_y^*+\frac{3}{8}\left(V_x^*+V_z^*\right)-\left(V_{yx}^*+V_{yz}^*\right)+\frac{1}{4}V_{xz}^*\right\}\\ &\left.-2k_1\left(1+\frac{3}{8}V_x^*-\frac{1}{2}V_{yx}^*\right)-2k_2\left(1+\frac{3}{8}V_z^*-\frac{1}{2}V_{yz}^*\right)\right]\\ \mathbf{M(t^*)} =& \mathbf{\overline{Y}^2}\left(\mathbf{k_1^2D_1+k_2^2D_3+1-2k_1k_2D-2k_1D_2-2k_2D_4}\right)\end{aligned} \tag{3.4.9}$$

where,

$$\begin{aligned}D &= 1+V_y^*+\frac{3}{8}(V_x^*+V_z^*)-(V_{yx}^*+V_{yz}^*)+\frac{1}{4}V_{xz}^*\\ D_1 &= 1+V_y^*+V_x^*-2V_{yx}^*\\ D_2 &= 1+\frac{3}{8}V_x^*-\frac{V_{yx}^*}{2}\\ D_3 &= 1+V_y^*+V_z^*-2V_{yz}^*\\ D_4 &= 1+\frac{3}{8}V_z^*-\frac{V_{yz}^*}{2}\end{aligned}$$

Using the method of minimum least square of estimation we get the optimum values of k_1 and k_2 as follows,

$$\frac{\partial M(t^*)}{k_2} = 0 \qquad \text{and} \qquad \frac{\partial^2 M(t^*)}{\partial {k_2}^2} > 0$$

$$\Longrightarrow \quad \overline{Y}^2[2k_2D_3-2k_1D-2D_4]=0$$

$$\Longrightarrow \quad 2k_2D_3 = 2k_1D+2D_4$$

$$\Longrightarrow \quad k_2 = \frac{k_1D+D_4}{D_3} \tag{3.4.10}$$

again,

$$\frac{\partial M(t^*)}{k_1} = 0 \qquad \text{and} \qquad \frac{\partial^2 M(t^*)}{\partial {k_1}^2} > 0$$

$$\implies \overline{Y}^2[2k_1D_1 - 2k_2D - 2D_2] = 0$$

putting the value of k from equation (3.4.10),

$$\implies \overline{Y}^2\left[2k_1D_1 - 2\left(\frac{k_1D + D_4}{D_3}\right)D - 2D_2\right] = 0$$

$$\implies k_1D_1 - \left(\frac{k_1D^2 + D_4D}{D_3}\right) = D_2$$

$$\implies \frac{k_1D_3D_1 - k_1D^2 - D_4D}{D_3} = D_2$$

$$\implies k_1 = \frac{D_2D_3 + D_4D}{D_3D_1 - D^2}$$

$$\implies k_1 = \frac{(D_2D_3 + D_4D)/D_3}{(D_3D_1 - D^2)/D_3} = \frac{A}{B}$$

Hence,

$$k_{1(opt.)} = \frac{A}{B} \qquad \text{and} \qquad K_{2(opt.)} = \frac{AD + BD_4}{BD_3}$$

$$\text{where} \qquad A = \frac{(D_2D_3 + DD_4)}{D_3} \qquad \text{and} \qquad B = \frac{(D_1D_3 - D^2)}{D_3}$$

Putting the value of k_1 and k_2 in equation (3.4.9) we get the minimum value of MSE of t^* as follows :

$$\mathbf{MSE_{min}(t^*)} = \overline{\mathbf{Y}}^{\mathbf{2}}\left[\mathbf{k} - \frac{\mathbf{A^2}}{\mathbf{B}}\right] \tag{3.4.11}$$

Putting the value of k_1 and k_2 in equation (5.2) we get the Bias of t^* as follows :

$$\mathbf{B(t^*)} = \overline{\mathbf{Y}}\left[\frac{\mathbf{A}}{\mathbf{B}}\mathbf{D_2} + \frac{\mathbf{AD + BD_4}}{\mathbf{BD_3}}\mathbf{D_4} - \mathbf{1}\right] \tag{3.4.12}$$

$$\text{where,} \qquad k = 1 - \frac{D_4^2}{D_3}$$

3.5 Theoretical Efficiency Comparison

The following Table-2 represents the conditions under which proposed class of estimators is more efficient than the existing competing estimators,

Table 3.2: **Efficiency Comparison**

SNo.	**MSE($\mathbf{t}^*$) < MSE($\cdot$)**	**Condition**
1	$MSE(t^*) < MSE(\bar{y}^*)$	$\frac{A^2}{B} > k - V_y^*$
2	$MSE(t^*) < MSE\left(\bar{y}_R^*\right)$	$\frac{A^2}{B} > k - k^{'}$
3	$MSE(t^*) < MSE\left(\bar{y}_P^*\right)$	$\frac{A^2}{B} > k - k^{''}$
4	$MSE(t^*) < MSE\left(\bar{y}_1^*\right)$	$\frac{A^2}{B} > k - k^{'''}$
5	$MSE(t^*) < MSE\left(\bar{y}_2^*\right)$	$\frac{A^2}{B} > k - k^{iv}$
6	$MSE(t^*) < MSE\left(\bar{y}_3^*\right)$	$\frac{A^2}{B} > k + 2V_{yx} - 3V_x - k^{'}$
7	$MSE(t^*) < MSE\left(\bar{y}_4^*\right)$	$\frac{A^2}{B} > k - 2V_{yx} - 3V_x - k^{''}$
8	$MSE(t^*) < MSE\left(\bar{y}_5^*\right)$	$\frac{A^2}{B} > k + m - V_y^*$
9	$MSE(t^*) < MSE\left(\bar{y}_6^*\right)$	$\frac{A^2}{B} > k + m^{'} - V_y^*$
10	$MSE(t^*) < MSE\left(\bar{y}_7^*\right)$	$\frac{A^2}{B} > k - k^{'} + g$
11	$MSE(t^*) < MSE\left(\bar{y}_8^*\right)$	$\frac{A^2}{B} > k - k^{'} + g^{'}$
12	$MSE(t^*) < MSE\left(\bar{y}_9^*\right)$	$\frac{A^2}{B} > k + \lambda C_{yx} - g^{''}$
13	$MSE(t^*) < MSE\left(\bar{y}_{10}^*\right)$	$\frac{A^2}{B} > k - (\lambda C_{yx} + g^{''})$
14	$MSE(t^*) < MSE\left(\bar{y}_{11}^*\right)$	$\frac{A^2}{B} > k - m^{''}$
15	$MSE(t^*) < MSE\left(\bar{y}_{12}^*\right)$	$\frac{A^2}{B} > k - m^{''}$
16	$MSE(t^*) < MSE\left(\bar{y}_{13}^*\right)$	$\frac{A^2}{B} > k - m^{'''}$
17	$MSE(t^*) < MSE\left(\bar{y}_{14}^*\right)$	$\frac{A^2}{B} > k - m^{''}$
18	$MSE(t^*) < MSE\left(\bar{y}_{15}^*\right)$	$\frac{A^2}{B} > (k - C) + \frac{G^{'2}}{G}$

where,

- $k^{'} = V_y^* + V_x^* - 2V_{yx}^*$
- $k^{''} = V_y^* + V_x^* + 2V_{yx}^*$
- $k^{'''} = V_y^* + V_x - 2V_{yx}$
- $k^{iv} = V_y^* + V_x + 2V_{yx}$
- $m = \frac{V_x V_{yx}^* + V_x^* V_{yx}^2 - 2V_{yx} v_{yx}^* V_x}{V_x(V_x^* - V_x)}$
- $m^{'} = \frac{\left(V_{yx}^* - V_{yx}\right)^2}{V_x^* - V_x} - \frac{V_{yx}^2}{V_x}$
- $g = \frac{3}{4}V_x^* - V_{yx}^*$
- $g^{'} = \frac{3}{4}V_x^* - 3V_{yx}^*$
- $g^{''} = \frac{1}{4}\lambda C_x^2 + V_y^*$
- $m^{''} = V_y^* - \frac{V_{yx}^{*2}}{V_x^*}$
- $m^{'''} = V_y^* - \lambda \rho_{yx}^2 C_y^2$

3.6 Computational Study

To prove the theoretical results numerically we have considered 2 datasets for the different values of $h = 2, 4, 6, 8, 16$. The information on the datasets are given in the following table as follows:

Table 3.3: **Parametric Vlues of the Population**

SNo	**Information**	**Data Set 1**	**Data Set 2**
1	Source	Khare and Sinha (2007)	Khare and Sinha (2012)
2	y	Weight of the children in kilograms	Number of literate persons in the village
3	x	Skull circumference of the children in centimeter	Number of workers in the village
4	z	Chest circumference of the children in centimeter	Number of non-workers in the village
5	N	95	105
6	n	30	30
7	w_2	0.25	0.25
8	$\overline{Y}$	19.4968	145.30
9	$\overline{X}$	51.1726	165.26
10	$\overline{Z}$	55.8611	259.08
11	ρ_{yx}	0.3280	0.81
12	$\rho_{yx(2)}$	0.4770	0.78
13	ρ_{yz}	0.8460	0.90
14	$\rho_{yz(2)}$	0.7290	0.87
14	ρ_{xz}	0.2970	0.81
15	$\rho_{xz(2)}$	0.57	0.72
16	C_y	0.1562	0.76
17	$C_{y(2)}$	0.1207	0.68
18	C_x	0.0302	0.68
19	$C_{x(2)}$	0.0247	0.57
20	C_z	0.0586	0.76
21	$C_{z(2)}$	0.0541	0.57

To compute the Percent Relative Efficiency (PRE) for different estimators with respect to Hansan and Horwitz estimators we use the following :

$$\text{PRE} = \frac{V_y^*}{\text{MSE}(\cdot)}$$

Table 3.4: **Data Set 1 : MSE and Bias of the Concerned Estimators**

SNo	**Estimators**	**MSE and Bias**	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
1	$\bar{y}^*$	MSE	0.2577	0.3500	0.4423	0.5346	0.9038
		Bias	-	-	-	-	-
2	$\bar{y}_R^*$	MSE	0.24071	0.3368	0.4330	0.5291	0.9137
		Bias	-0.0002	0.00001	0.0002	0.0004	0.0012
3	$\bar{y}_P^*$	MSE	0.2942	0.3904	0.4866	0.5828	0.9676
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
4	$\bar{y}_1^*$	MSE	0.2388	0.3311	0.4234	0.5157	0.8849
		Bias	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
5	$\bar{y}_2^*$	MSE	0.2923	0.3846	0.4769	0.5692	0.9383
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
6	$\bar{y}_3^*$	MSE	0.2375	0.3337	0.4298	0.5259	0.9105
		Bias	0.0012	0.0012	0.0012	0.0012	0.0012
7	$\bar{y}_4^*$	MSE	0.3445	0.4407	0.5369	0.6331	1.0178
		Bias	0.0018	0.0018	0.0018	0.0018	0.0018

Data Set 1 : MSE and Bias of the Concerned Estimators

SNo	**Esti-mators**	**MSE and Bias**	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
8	$\bar{y}_5^*$	MSE	-2630.62	-877.463	-526.76	-376.40	-175.6852
		Bias	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002
9	$\bar{y}_6^*$	MSE	0.2349	0.3272	0.4195	0.5118	0.8810
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
10	$\bar{y}_7^*$	MSE	0.2467	0.3310	0.4332	0.5265	0.8995
		Bias	-0.0002	-0.00008	-0.00006	0.0001	0.0001
11	$\bar{y}_8^*$	MSE	0.2735	0.3668	0.4600	0.5533	0.9264
		Bias	0.0003	0.0003	0.0002	0.0002	0.0001
12	$\bar{y}_9^*$	MSE	0.2463	0.3386	0.4309	0.5232	0.8923
		Bias	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002
13	$\bar{y}_{10}^*$	MSE	0.2730	0.3653	0.4576	0.5499	0.9191
		Bias	0.0003	0.0003	0.0003	0.0003	0.0003
14	$\bar{y}_{11}^*$	MSE	0.2394	0.3368	0.4320	0.5261	0.8988
		Bias	-0.0008	-0.0005	-0.0004	-0.0003	-0.0001
15	$\bar{y}_{12}^*$	MSE	0.2394	0.3368	0.4320	0.5261	0.8988
		Bias	0.0001	-0.0001	-0.0002	-0.0002	-0.0002
16	$\bar{y}_{13}^*$	MSE	0.2349	0.3272	0.4195	0.5118	0.8810
		Bias	0.0001	-0.0004	-0.0008	-0.0012	-0.0029

Data Set 1 : MSE and Bias of the Concerned Estimators

SNo	Esti-mators	MSE and Bias	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
17	$\bar{y}^*_{14}$	MSE	0.2394	0.3368	0.4320	0.5261	0.8988
		Bias	0	0	0	0	0
18	$\bar{y}^*_{15}$	MSE	0.2170	0.3559	0.4803	0.5993	1.0583
		Bias	0.0005	0.0012	0.0016	0.001840	0.0022
	t*	**MSE**	**0.1665**	**0.2847**	**0.3895**	**0.4892**	**0.8718**
		Bias	**-71.75**	**-34.41**	**-16.88**	**-6.7005**	**10.7563**

Table 3.5: **Data Set 2 : MSE and Bias of the Concerned Estimators**

SNo	Est.	MSE and Bias	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
1	$\bar{y}^*$	MSE	375.96	538.66	701.36	864.07	1514.88
		Bias	-	-	-	-	-
2	$\bar{y}^*_R$	MSE	209.78	422.48	635.19	847.89	1698.70
		Bias	0.4365	1.0019	1.5674	2.1328	4.3946
3	$\bar{y}^*_P$	MSE	1128.14	1469.49	1810.84	2152.19	3517.57
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
4	$\bar{y}^*_1$	MSE	184.78	347.49	510.19	672.892	1323.71
		Bias	0.1537	0.1537	0.1537	0.1537	0.1537

Data Set 2 : MSE and Bias of the Concerned Estimators

SNo	Est.	MSE and Bias	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
5	$\bar{y}_2^*$	MSE	1038.82	1201.53	1364.23	1526.9	2177.75
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
6	$\bar{y}_3^*$	MSE	490.30	703	915.70	1128.41	1979.22
		Bias	4.76	4.54	4.32	4.09	3.21
7	$\bar{y}_4^*$	MSE	2262.7	2604	2945.4	3286.75	4652.14
		Bias	4.67	4.89	5.12	5.34	6.22
8	$\bar{y}_5^*$	MSE	-83698	-31494	-20923	-16300	-9702
		Bias	0.11	0.19	0.27	0.35	0.67
9	$\bar{y}_6^*$	MSE	178.14	331.80	485.45	639.11	1253.73
		Bias	0.0000	0.0000	0.0000	0.0000	0.0000
10	$\bar{y}_7^*$	MSE	219.62	378.74	537.86	696.98	1333.48
		Bias	-0.03	0.151	0.33	0.52	1.26
11	$\bar{y}_8^*$	MSE	678.80	902.24	1125.69	1349.14	2242.92
		Bias	0.54	0.55	0.56	0.57	0.62
12	$\bar{y}_9^*$	MSE	221.4069	384.1107	546.8145	709.5184	1360.3337
		Bias	-0.126	-0.126	-0.126	-0.126	-0.126
13	$\bar{y}_{10}^*$	MSE	648.4267	811.1306	973.8344	1136.5383	1787.3536
		Bias	0.53	0.53	0.53	0.53	0.53

Data Set 2 : MSE and Bias of the Concerned Estimators

SNo	**Est.**	**MSE and Bias**	$h = 2$	$h = 4$	$h = 6$	$h = 8$	$h = 16$
14	$\bar{y}_{11}^{*}$	MSE	196.06	370.46	535.76	696.88	1325.75
		Bias	-0.84	-0.71	-0.63	-0.59	-0.52
15	$\bar{y}_{12}^{*}$	MSE	196.0555	370.4554	535.7632	696.8819	1325.749
		Bias	-0.32	-0.50	-0.62	-0.72	-0.99
16	$\bar{y}_{13}^{*}$	MSE	182.6660	345.3699	508.0737	670.7775	1321.592
		Bias	-0.36	-0.86	-1.36	-1.86	-3.85
17	$\bar{y}_{14}^{*}$	MSE	196.0555	370.4554	535.7632	696.8819	1325.75
		Bias	0	0	0	0	0
18	$\bar{y}_{15}^{*}$	MSE	203.0745	339.7172	518.1972	688.9346	1352
		Bias	1.219106	1.642624	2.084128	2.532933	4.352755
	t*	**MSE**	**177.669**	**320.662**	**459.355**	**595.126**	**1113.70**
		Bias	**23.29**	**69.63**	**84.81**	**91.81**	**98.99**

Table 3.6: **Data Set 1 : PRE of the Concerned Estimators**

SNo	**Estimators**	**h = 2**	**h = 4**	**h = 6**	**h = 8**	**h = 16**
1	$\bar{y}^*$	100	100	100	100	100
2	$\bar{y}^*_R$	107.04784	103.89649	102.14456	101.02926	98.91511
3	$\bar{y}^*_P$	87.58062	89.64394	90.89151	91.72725	93.40849
4	$\bar{y}^*_1$	107.9085	105.7038	104.4604	103.6620	102.1341
5	$\bar{y}^*_2$	88.16358	91.00439	92.74552	93.92194	96.31333
6	$\bar{y}^*_3$	108.47873	104.88513	102.89912	101.63916	99.26005
7	$\bar{y}^*_4$	74.79281	79.41170	82.37557	84.43880	88.79207
8	$\bar{y}^*_5$	-0.0098	-0.0399	-0.0840	-0.1420	-0.5144
9	$\bar{y}^*_6$	109.6871	106.9547	105.4246	104.4463	102.5831
10	$\bar{y}^*_7$	104.4294	102.9341	102.0825	101.5325	100.4730
11	$\bar{y}^*_8$	94.21459	95.41897	96.13497	96.60957	97.55227
12	$\bar{y}^*_9$	104.6315	103.3689	102.6472	102.1802	101.2782
13	$\bar{y}^*_{10}$	94.38354	95.80260	96.64921	97.21162	98.33169
14	$\bar{y}^*_{11}$	107.6362	103.8981	102.3732	101.5994	100.5475
15	$\bar{y}^*_{12}$	107.6362	103.8981	102.3732	101.5994	100.5475
16	$\bar{y}^*_{13}$	109.6871	106.9546	105.4245	104.4463	102.5830
17	$\bar{y}^*_{14}$	107.6362	103.8981	102.3732	101.5994	100.5475
18	$\bar{y}^*_{15}$	118.79701	98.33295	92.07539	89.19150	85.39881
	t*	**154.7405**	**122.9192**	**113.5595**	**109.2788**	**103.6624**

Table 3.7: **Data Set 2 : PRE of the Concerned Estimators**

SNo	Estimators	h = 2	h = 4	h = 6	h = 8	h = 16
1	$\bar{y}^*$	100	100	100	100	100
2	$\bar{y}^*_R$	179.21338	127.49825	110.41847	101.90801	89.17891
3	$\bar{y}^*_P$	33.32513	36.65614	38.73133	40.14826	43.06596
4	$\bar{y}^*_1$	203.4595	155.0164	137.4712	128.4108	114.4423
5	$\bar{y}^*_2$	36.19057	44.83129	51.41095	56.58841	69.56187
6	$\bar{y}^*_3$	76.67882	76.62276	76.59274	76.57404	76.53943
7	$\bar{y}^*_4$	16.61533	20.68544	23.81216	26.28942	32.56310
8	$\bar{y}^*_5$	-0.4492	-1.7103	-3.3521	-5.3010	-15.6146
9	$\bar{y}^*_6$	211.0424	162.3454	144.4756	135.1984	120.8296
10	$\bar{y}^*_7$	171.1873	142.2243	130.3983	123.9721	113.6040
11	$\bar{y}^*_8$	55.38547	59.70219	62.30519	64.04597	67.54066
12	$\bar{y}^*_9$	169.8029	140.2354	128.2634	121.7822	111.3611
13	$\bar{y}^*_{10}$	57.97961	66.40844	72.02077	76.02620	84.75560
14	$\bar{y}^*_{11}$	191.7596	145.4046	130.9092	123.9904	114.2661
15	$\bar{y}^*_{12}$	191.7596	145.4046	130.9092	123.9904	114.2661
16	$\bar{y}^*_{13}$	205.8157	155.9659	138.0436	128.8157	114.6255
17	$\bar{y}^*_{14}$	191.7596	145.4046	130.9092	123.9904	114.2661
18	$\bar{y}^*_{15}$	185.1317	158.5610	135.3467	125.4207	112.0474
	t*	**211.6039**	**167.9834**	**152.6842**	**145.1906**	**136.0226**

3.7 Results and Conclusion

1. Table 3.2 depicts the conditions under which our proposed class is more efficient than the existing competing estimators. Table 3.3 contains the information of two natural populations with which we have verified the results numerically. Tables $3.4, 3.5$ consist of the Bias and MSE of the proposed and considered existing estimators. Tables $3.6, 3.7$ shows the PRE of proposed and existing estimators with respect to Hansan and Horwitz estimator. In Tables $3.6, 3.7$ we can see that values of the estimators which are more efficient than the $\bar{y}^*$ i.e. $PRE > 100$ decrease as the value of h increases and values of the estimators which are less efficient than the $\bar{y}^*$ i.e. $PRE < 100$ increases as the value of h increases.

2. The reason for the poor performance of the some of the estimators i.e. $PRE < 100$ is that most of them are product type of estimators which perform poorly because of the positive correlation in the two data sets between study and the auxiliary variables. We all know that product estimators are used when there is negative correlation between the study variable and auxiliary variables.

3. Since Efficiency is stronger property than the unbiasedness. Hence here we prefer the biased estimator with minimum MSE instead of unbiased estimator with higher variance. From Tables $3.6, 3.7$ we can easily notice that proposed class have the greater PREs from all the considered existing estimators of population mean under simple random sampling scheme.

4. Hence, it has been proven both theoretically and numerically that the proposed estimator is better than the other given competing estimators. Since our computations revolved around a natural population, therefore we can successfully recommed our class of estimators for practical applications.

References

CHANU, W. W. AND SINGH B. K. (2015) : Improved Exponential Ratio cum Exponential Dual to Ratio Estimator of Finite Population Mean in Presence of Non-Response. *J. Stat. Appl. Pro.*, 4(1), 103-111.

COCHRAN, W.G. (1940): The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *The Journal of Agric. Science*, 30, 262-275.

GOODMAN, L.A. (1960) : On the exact variance of products. *J. Amer. Stat. Ass.* , 55, 708-713.

HANSAN, M.H., HURWITZ, W.N. (1946) : The problem of non-response in sampling surveys. *Journal of American Statistical Association*, 41 , 517-529.

KHARE, B. B., SINHA, R, R. (2007) : Estimation of the ratio of the two population means using multi auxiliary characteristics in the presence of non-response. *In Statistical Techniques in Life Testing, Reliability, Sampling Theory and Quality Control.*, 1, 63-171

KHARE, B. B., SINHA, R, R. (2012) : Improved classes of estimators for ratio of two means with with double sampling the non-respondents. *Statistica* , 49(3), 75-83.

KUMAR, S., BHOUGAL, S. (2011) : Estimation of the population mean in presence of non-response. *Communications of the Korean Statistical Society*, 18(4), 537-548.

MUHAMMAD ZUBAIR, ASAD ALI, WAJIHA NASIR AND MUHAMMAD RASHAD (2018) : An Exponential Estimator in presence of Non Response. 16^{th} *International Conference on Statistical Sciences*, 32, 209-222.

PAL, S. K. AND H. P. SINGH, H. P. (2016) : Finite Population Mean Estimation through a Two-Parameter Ratio Estimator Using Auxiliary Information in Presence of Non-Response. *JAMSI*, 12(2), 5-39.

RAO, P.S.R.S. (1986) : Ratio estimation with sub sampling the non-respondents. *Survey Methodology*, 12(2), 217–230.

SINGH, D., CHAUDHARY, F.S. (1986) : Sample Survey Designs. *New Age Int.(P) Ltd.*.

SINGH, H.P., KUMAR, S. (2008) : A regression approach to the estimation of the finite population mean in the presence of non-response. *Aust. N. Z. J. Stat.*, 50(4), 395–408.

Singh, H.P., Kumar, S. (2009) : A general procedure for estimating the population mean in presence of non-response under double sampling using auxiliary information. *SORT*, 33(1), 71-83.

Chapter 4

A REDEFINED CLASS OF RATIO ESTIMATORS USING AUXILIARY INFORMATION FOR THE ESTIMATION OF POPULATION MEAN UNDER SIMPLE RANDOM SAMPLING SCHEME

4.1 Summary

The present paper advocates the improved estimation of population mean through a new class of estimators using the known informations on an auxiliary variable under the Simple Random Sampling Scheme. The existing estimators of population mean have been shown as the members of the proposed class. The Bias and Mean Squared Error (MSE) of the suggested class are derived upto the first order of approximation. The values of Bias and the minimum MSE are obtained by optimizing the chracterizing scalar. The MSE of the suggested estimator has also been compared with the MSEs of the existing estimators for the comparison of these estimators. Some of the new members of the suggested family are also presented in the paper which are more efficient than the existing ones emperically which has been shown using a natural population.

4.2 Introduction

The study variable combined with the auxiliary information provides considerably efficient estimation of the population parameters. Cochran (1940) was the first who used the auxiliary information in development of ratio estimator. Ratio estimator is effective when there is positive correlation between the study and auxiliary variable otherwise in case of negative correlation product estimator would be the wiser choice. The parameters of the auxiliary variable such as the mean, median, variance and the coefficient of variation or the combinations of the parameters of auxiliary variable is used as the components of the concerned estimator. So many researchers have studied on the auxiliary variable and the ratio estimators and tried to obtain the minimum as well as the unbiased estimators. As a result we end up having a lot of estimators. If some are useful for one situation others are useful for other situations. We don't have any universal estimator. Different estimators are useful in different situations. Considering all of them is impossoble for us but we have considered some of them in our literature which are similar to the situation we are studying on.

Upadhyaya and Singh (1991), Kadilar and Cingi (2003), Al-Omari et al.(2009) suggested new improved ratio estimators. Bahl and Tuteja (1991) proposed exponential type of estimators. Jeelani et al.(2013) used the linear combination of coefficient of skewness and quartile deviation for enhanced estimation. Jerajuddin and Kishun (2016) suggested the use of sample size selected from the population to modify the ratio estimator. His estimator did not require the auxiliary information. Singh, Tailor and Kakran (2004) used the power transformation for the improvement. Singh and Tailor(2003), Sisodia and Dwivedi (1981), Yan and Tian (2010), Subramani and Kumarpandiyan (2012), Yadav et al.(2019) took the use of various characteristics of auxiliary information and their combinations to obatain the efficient estimators. Encouraged by these researchers we proposed a generalized class of estimators which has more precision than the existing competing estimators.

We have considered the mean estimation and tried to obtain the efficient estimator under the simple random sampling scheme. Let from the finite population (X, Y) of size N, a bivariate sample $(x_i, y_i); i = 1, 2, \ldots, n$ of size n is taken using SRSWOR scheme. The sample means $\bar{x}$ and $\bar{y}$ are unbiased estimators of population means $\overline{X}$ and $\overline{Y}$ respectively. If we ignore the property of unbiasedness, we would obtain the Mean Square Error much lower than the usual variance. Though we have obtained the bias of our proposed estimator yet we focused on minimizing our Mean Square Error (MSE).

4.3 Existing Estimators

There are a number of estimators which have been developed till date for the elevated estimation of population mean under simple random sampling without replacement (SRSWOR) scheme. Variour estimators from the literature are considered for the review and are presented in the Table-1 along with their Mean Squared Errors and corresponding constants up the approximation of order one.

Table 4.1: **Existing Estimator**

SNo	Estimators	MSE	Constants
1	$t_0=\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$ Sample Mean	$\lambda\overline{Y}^2C_y^2$	
2	$t_1=\bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)$ Cochran(1940)	$\lambda\overline{Y}^2\left(C_y^2+C_x^2-2C_{yx}\right)$	
3	$t_2=\bar{y}\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Bahl and Tuteja(1991)	$\lambda\overline{Y}^2\left(C_y^2+\frac{C_x^2}{4}-\rho C_yC_x\right)$	
4	$t_3=\bar{y}\left(\frac{\overline{X}+C_x}{\bar{x}+C_x}\right)$ Sisodia and Dwivedi (1981)	$\lambda\overline{Y}^2\left(C_y^2+\theta_3^2C_x^2-2\theta_3C_{yx}\right)$	$\theta_3=\frac{\overline{X}}{\overline{X}+C_x}$
5	$t_4=\bar{y}\left(\frac{\overline{X}C_x+\beta_{2(x)}}{\bar{x}C_x+\beta_{2(x)}}\right)$ Upadhyaya and Singh(1999)	$\lambda\overline{Y}^2\left(C_y^2+\theta_4^2C_x^2-2\theta_4C_{yx}\right)$	$\theta_4=\frac{\overline{X}C_x}{\overline{X}C_x+\beta_{2(x)}}$
6	$t_5=\bar{y}\left(\frac{\overline{X}\beta_{2(x)}+C_x}{\bar{x}\beta_{2(x)}+C_x}\right)$ Upadhyaya and Singh(1999)	$\lambda\overline{Y}^2\left(C_y^2+\theta_5^2C_x^2-2\theta_5C_{yx}\right)$	$\theta_5=\frac{\overline{X}\beta_{2(x)}}{\overline{X}\beta_{2(x)}+C_x}$
7	$t_6=\bar{y}\left(\frac{\overline{X}+\rho}{\bar{x}+\rho}\right)$ Singh and Tailor(2003)	$\lambda\overline{Y}^2\left(C_y^2+\theta_6^2C_x^2-2\theta_6C_{yx}\right)$	$\theta_6=\frac{\overline{X}}{\overline{X}+\rho}$
8	$t_7=\bar{y}\left(\frac{\overline{X}+\beta_{2(x)}}{\bar{x}+\beta_{2(x)}}\right)$ Singh et al.(2004)	$\lambda\overline{Y}^2\left(C_y^2+\theta_7^2C_x^2-2\theta_7C_{yx}\right)$	$\theta_7=\frac{\overline{X}}{\overline{X}+\beta_{2(x)}}$
9	$t_8=\bar{y}\left(\frac{\overline{X}+Q_{1(x)}}{\bar{x}+Q_{1(x)}}\right)$ Al-Omari et al.(2009)	$\lambda\overline{Y}^2\left(C_y^2+\theta_8^2C_x^2-2\theta_8C_{yx}\right)$	$\theta_8=\frac{\overline{X}}{\overline{X}+Q_{1(x)}}$

SNo	Estimators	MSE	Constants
10	$t_9 = \bar{y}\left(\frac{\overline{X}+Q_{3(x)}}{\bar{x}+Q_{3(x)}}\right)$ Al-Omari et al.(2009)	$\lambda\overline{Y}^2\left(C_y^2+\theta_9^2C_x^2-2\theta_9C_{yx}\right)$	$\theta_9=\frac{\overline{X}}{\overline{X}+Q_{3(x)}}$
11	$t_{10} = \bar{y}\left(\frac{\overline{X}+\beta_{1(x)}}{\bar{x}+\beta_{1(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{10}^2C_x^2-2\theta_{10}C_{yx}\right)$	$\theta_{10}=\frac{\overline{X}}{\overline{X}+\beta_{1(x)}}$
12	$t_{11} = \bar{y}\left(\frac{\overline{X}\beta_{1(x)}+\beta_{2(x)}}{\bar{x}\beta_{1(x)}+\beta_{2(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{11}^2C_x^2-2\theta_{11}C_{yx}\right)$	$\theta_{11}=\frac{\overline{X}\beta_{1(x)}}{\overline{X}\beta_{1(x)}+\beta_{2(x)}}$
13	$t_{12} = \bar{y}\left(\frac{\overline{X}C_x+\beta_{1(x)}}{\bar{x}C_x+\beta_{1(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{12}^2C_x^2-2\theta_{12}C_{yx}\right)$	$\theta_{12}=\frac{\overline{X}C_x}{\overline{X}C_x+\beta_{1(x)}}$
14	$t_{13} = \bar{y}\left(\frac{\overline{X}+M_x}{\bar{x}+M_x}\right)$ Subramani and Kumarpandiyan(2012a)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{13}^2C_x^2-2\theta_{13}C_{yx}\right)$	$\theta_{13}=\frac{\overline{X}}{\overline{X}+M_x}$
15	$t_{14} = \bar{y}\left(\frac{\overline{X}C_x+M_x}{\bar{x}C_x+M_x}\right)$ Subramani and Kumarpandiyan(2012a)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{14}^2C_x^2-2\theta_{14}C_{yx}\right)$	$\theta_{14}=\frac{\overline{X}C_x}{\overline{X}C_x+M_x}$
16	$t_{15} = \bar{y}\left(\frac{\overline{X}+Q_{r(x)}}{\bar{x}+Q_{r(x)}}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{15}^2C_x^2-2\theta_{15}C_{yx}\right)$	$\theta_{15}=\frac{\overline{X}}{\overline{X}+Q_{r(x)}}$
17	$t_{16} = \bar{y}\left(\frac{\overline{X}+QD}{\bar{x}+QD}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{16}^2C_x^2-2\theta_{16}C_{yx}\right)$	$\theta_{16}=\frac{\overline{X}}{\overline{X}+QD}$
18	$t_{17} = \bar{y}\left(\frac{\overline{X}+Q_{a(x)}}{\bar{x}+Q_{a(x)}}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{17}^2C_x^2-2\theta_{17}C_{yx}\right)$	$\theta_{17}=\frac{\overline{X}}{\overline{X}+Q_{a(x)}}$

Literature Review

SNo	Estimators	MSE	Constants
19	$t_{18} = \bar{y}\left(\frac{\overline{X}+n}{\bar{x}+n}\right)$ Jerajuddin and Kishun(2016)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{18}^2 C_x^2 - 2\theta_{18}C_{yx}\right)$	$\theta_{18} = \frac{\overline{X}}{\overline{X}+n}$
20	$t_{19} = \bar{y}\left(\frac{\overline{X}\beta_{1(x)}+QD}{\bar{x}\beta_{1(x)}+QD}\right)$ Jeelani et al.(2013)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{19}^2 C_x^2 - 2\theta_{19}C_{yx}\right)$	$\theta_{19} = \frac{\overline{X}\beta_{1(x)}}{\overline{X}\beta_{1(x)}+QD}$
21	$t_{20} = \bar{y}\left(\frac{ab\overline{X}+cd}{ab\bar{x}+cd}\right)$ $t_{20(1)} = \bar{y}\left(\frac{\beta_{2(x)}M_x\overline{X}+\rho}{\beta_{2(x)}M_x\bar{x}+\rho}\right)$ $t_{20(2)} = \bar{y}\left(\frac{\beta_{2(x)}M_x\overline{X}+\rho C_x}{\beta_{2(x)}M_x\bar{x}+\rho C_x}\right)$ $t_{20(3)} = \bar{y}\left(\frac{\beta_{1(x)}M_x\overline{X}+\rho}{\beta_{1(x)}M_x\bar{x}+\rho}\right)$ $t_{20(4)} = \bar{y}\left(\frac{\beta_{1(x)}M_x\overline{X}+\rho C_x}{\beta_{1(x)}M_x\bar{x}+\rho C_x}\right)$ $t_{20(5)} = \bar{y}\left(\frac{n\overline{X}+\rho}{n\bar{x}+\rho}\right)$ $t_{20(6)} = \bar{y}\left(\frac{n\overline{X}+C_x}{n\bar{x}+C_x}\right)$ $t_{20(7)} = \bar{y}\left(\frac{n\overline{X}+\rho C_x}{n\bar{x}+\rho C_x}\right)$ $t_{20(8)} = \bar{y}\left(\frac{n\rho\overline{X}+C_x}{n\rho\bar{x}+C_x}\right)$ $t_{20(9)} = \bar{y}\left(\frac{nC_x\overline{X}+\rho}{nC_x\bar{x}+\rho}\right)$ Yadav et al.(2019)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20}^2 C_x^2 - 2\theta_{20}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(1)}^2 C_x^2 - 2\theta_{20(1)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(2)}^2 C_x^2 - 2\theta_{20(2)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(3)}^2 C_x^2 - 2\theta_{20(3)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(4)}^2 C_x^2 - 2\theta_{20(4)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(5)}^2 C_x^2 - 2\theta_{20(5)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(6)}^2 C_x^2 - 2\theta_{20(6)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(7)}^2 C_x^2 - 2\theta_{20(7)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(8)}^2 C_x^2 - 2\theta_{20(8)}C_{yx}\right)$ $\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(9)}^2 C_x^2 - 2\theta_{20(9)}C_{yx}\right)$	$\theta_{20} = \frac{ab\overline{X}}{ab\overline{X}+c.d}$ $\theta_{20(1)} = \frac{\beta_{2(x)}M_x\overline{X}}{\beta_{2(x)}M_x\bar{x}+\rho}$ $\theta_{20(2)} = \frac{\beta_{2(x)}M_x\overline{X}}{\beta_{2(x)}M_x\bar{x}+\rho C_x}$ $\theta_{20(3)} = \frac{\beta_{1(x)}M_x\overline{X}}{\beta_{1(x)}M_x\bar{x}+\rho}$ $\theta_{20(4)} = \frac{\beta_{1(x)}M_x\overline{X}}{\beta_{1(x)}M_x\bar{x}+\rho C_x}$ $\theta_{20(5)} = \frac{n\overline{X}}{n\bar{x}+\rho C_x}$ $\theta_{20(6)} = \frac{n\overline{X}}{n\bar{x}+C_x}$ $\theta_{20(7)} = \frac{n\overline{X}}{n\bar{x}+\rho C_x}$ $\theta_{20(8)} = \frac{n\rho\overline{X}}{n\rho\bar{x}+C_x}$ $\theta_{20(9)} = \frac{nC_x\overline{X}}{nC_x\bar{x}+\rho}$

4.4 Proposed Class of Estimators

Inspired by the literature presented by Yadav et al. (2019), we suggest an improved class of ratio type estimators for the enhanced estimation of population mean of primary variable using information on secondary variable under SRSWOR as follows:

$$t = k\bar{y}\left(\frac{ab\overline{X} + cd}{ab\bar{x} + cd}\right) \tag{4.4.1}$$

where constant kappa k is suitably chosen such that the MSE of the suggested estimator is a least and the constants a, b, c, d are either the constants or the known parameters of the auxiliary variable. Also (a, b, c, d) are free to take those real and parametric values which makes the MSE minimum.

4.4.1 Bias and MSE

To obtain the Bias and MSE we define our expressions approximately as follows :

$$\begin{aligned} e_0 &= \frac{\bar{y} - \overline{Y}}{\overline{Y}} \quad &\text{and}& \quad & e_1 &= \frac{\bar{x} - \overline{X}}{\overline{X}} \\ \text{So,} \quad \bar{y} &= \overline{Y}(1 + e_0) &\text{and}& & \bar{x} &= \overline{X}(1 + e_1) \\ E(e_0) &= E(e_1) = 0 &\text{and}& & E(e_0 e_1) &= \lambda C_{yx} \\ E(e_0^2) &= \lambda C_y^2 &\text{and}& & E(e_1^2) &= \lambda C_x^2 \end{aligned}$$

Now rewriting our proposed estimator from equation (4.4.1) as,

$$\begin{aligned} t &= k\overline{Y}(1 + e_0)\left[\frac{ab\overline{X} + cd}{ab\overline{X}(1 + e_1) + cd}\right] \\ &= k\overline{Y}(1 + e_0)(ab\overline{X} + cd)\left[(ab\overline{X} + cd) + ab\overline{X}e_1\right]^{-1} \\ &= k\overline{Y}(1 + e_0)\left[\frac{ab\overline{X}e_1}{ab\overline{X} + cd}\right]^{-1} \\ t &= k\overline{Y}(1 + e_0)(1 + \theta e_1)^{-1} \end{aligned} \tag{4.4.2}$$

$$\text{Where,} \quad \theta = \frac{ab\overline{X}}{ab\overline{X} + cd}$$

Expanding the equation (4.4.2), then multiplying its terms and retaining the terms upto the first order of approximation we get,

$$\begin{aligned} t &= k\overline{Y}(1 + e_0 - \theta e_1 - \theta e_0 e_1 + \theta^2 e_1^2) \\ (t - \overline{Y}) &= \overline{Y}[k(1 + e_0 - \theta e_1 - \theta e_0 e_1 + \theta^2 e_1^2) - 1] \end{aligned} \tag{4.4.3}$$

Taking expectation on both sides of equation (4.4.3)

$$\begin{aligned} E(t - \overline{Y}) &= \overline{Y}[k(1 - \theta\lambda C_{yx} + \theta^2\lambda C_x^2) - 1] \\ B(t) &= \overline{Y}[k(1 - \theta\lambda C_{yx} + \theta^2\lambda C_x^2) - 1] \end{aligned} \tag{4.4.4}$$

Squaring the equation (4.4.3)

$$
\begin{aligned}
(t-\overline{Y})^2 &= \overline{Y}^2[k^2(1+e_0-\theta e_1-\theta e_0e_1+\theta^2e_1^2)^2+1 \\
&\quad -2k(1+e_0-\theta e_1-\theta e_0e_1+\theta^2+e_1^2)] \\
(t-\overline{Y})^2 &= \overline{Y}^2[k^2(1+e_0^2+\theta^2e_1^2+2e_0-2\theta e_1-2\theta e_0e_1+2\theta^2e_1^2-2\theta e_0e_1)+1 \\
&\quad -2k(1+e_0-\theta e_1-\theta e_0e_1+\theta^2e_1^2)] \\
E[(t-\overline{Y})^2] &= \overline{Y}^2[k^2(1+\lambda C_y^2+\theta^2\lambda C_x^2-4\theta\lambda C_{yx}+2\theta^2\lambda C_x^2)+1 \\
&\quad -2k(1-\theta\lambda C_{yx}+\theta^2\lambda C_x^2)] \\
MSE(t) &= \overline{Y}^2[k^2(1+\lambda C_y^2+3\theta^2\lambda C_x^2-4\theta\lambda C_{yx})+1-2k(1-\theta\lambda C_{yx}+\theta^2\lambda C_x^2)]
\end{aligned} \tag{4.4.5}
$$

For the Optimum value of k we use the Method of least square of estimation as follows :

$$\frac{\partial MSE(t)}{\partial k}=0 \qquad \text{and} \qquad \frac{\partial^2 MSE(t)}{\partial k^2}>0$$

Hence,

$$
\begin{aligned}
&\Longrightarrow \frac{\partial MSE(t)}{\partial k}=0 \\
&\Longrightarrow \overline{Y}^2[2k(1+\lambda C_y^2+3\theta^2\lambda C_x^2-4\theta\lambda C_{yx})-2(1-\theta\lambda C_{yx}+\theta^2\lambda C_x^2)]=0 \\
&\Longrightarrow \mathbf{k_{opt.}}=\frac{(\mathbf{1}-\theta\lambda\mathbf{C_{yx}}+\theta^{\mathbf{2}}\lambda\mathbf{C_x^2})}{(\mathbf{1}+\lambda\mathbf{C_y^2}+\mathbf{3}\theta^{\mathbf{2}}\lambda\mathbf{C_x^2}-\mathbf{4}\theta\lambda\mathbf{C_{yx}})}=\frac{\mathbf{A}}{\mathbf{B}}
\end{aligned}
$$

Putting this value of k in equation (4.4.5) we get

$$MSE(t)=\overline{Y}^2\left[\frac{A^2}{B^2}B+1-2\frac{A}{B}A\right]$$

$$\mathbf{MSE(t)_{min.}}=\overline{\mathbf{Y}}^{\mathbf{2}}\left[\mathbf{1}-\frac{\mathbf{A^2}}{\mathbf{B}}\right] \tag{4.4.6}$$

Putting the value of k in equation (4.4.4) we get the Bias

$$\mathbf{B(t)}=\overline{\mathbf{Y}}\left(\frac{\mathbf{A^2}}{\mathbf{B}}-\mathbf{1}\right) \tag{4.4.7}$$

4.4.2 Existing Members of Proposed Class of Estimators

The proposed class of estimators reduce to some known estimators of population mean $\overline{Y}$ for different values of (a,b,c,d) in equation (4.4.1) which are given in the following Table,

Table 4.2: **Some Existing Members of Proposed Class of estimators t**

SNo.	Estimators	k	a	b	c	d
1	t_1	1	1	1	0	0
2	t_3	1	1	1	C_x	1
3	t_4	1	C_x	1	$\beta_{2(x)}$	1
4	t_5	1	$\beta_{2(x)}$	1	C_x	1
5	t_6	1	1	1	ρ	1
6	t_7	1	1	1	$\beta_{2(x)}$	1
7	t_8	1	1	1	$Q_1(x)$	1
8	t_9	1	1	1	$Q_3(x)$	1
9	t_{10}	1	1	1	$\beta_{1(x)}$	1
10	t_{11}	1	$\beta_{1(x)}$	1	$\beta_{2(x)}$	1
11	t_{12}	1	C_x	1	$\beta_{1(x)}$	1
12	t_{13}	1	1	1	M_x	1
13	t_{14}	1	C_x	1	M_x	1
14	t_{15}	1	1	1	$Q_{r(x)}$	1
15	t_{16}	1	1	1	$Q_{d(x)}$	1
16	t_{17}	1	1	1	$Q_{a(x)}$	1
17	t_{18}	1	$\beta_{1(x)}$	1	QD	1
18	t_{19}	1	1	1	n	1
19	t_{20}	1	a	b	c	d

4.4.3 New Members of Proposed Class of Estimators

We have suggested some new members of the proposed class of estimators which come out to be more efficient than the existing estimators of population mean. These are given as follows,

- $t_{(1)} = k_1\bar{y}\left[\frac{\beta_{2(x)}M_x\overline{X}+\rho}{\beta_{2(x)}M_x\bar{x}+\rho}\right]$
- $t_{(2)} = k_2\bar{y}\left[\frac{\beta_{2(x)}M_x\overline{X}+\rho C_x}{\beta_{2(x)}M_x\bar{x}+\rho C_x}\right]$
- $t_{(3)} = k_3\bar{y}\left[\frac{{}^NC_n\overline{X}+M_x}{{}^NC_n\bar{x}+M_x}\right]$
- $t_{(4)} = k_4\bar{y}\left[\frac{M_x\overline{X}+\rho f}{M_x\bar{x}+\rho f}\right]$
- $t_{(5)} = k_5\bar{y}\left[\frac{TMQ_{3(x)}\overline{X}+\frac{Q_{a(x)}}{QD}}{TMQ_{3(x)}\bar{x}+\frac{Q_{a(x)}}{QD}}\right]$
- $t_{(6)} = k_6\bar{y}\left[\frac{QDQ_{a(x)}\overline{X}+\beta_{2(x)}\beta_{1(x)}}{QDQ_{a(x)}\bar{x}+\beta_{2(x)}\beta_{1(x)}}\right]$
- $t_{(7)} = k_7\bar{y}\left[\frac{M_xS_x\overline{X}+\rho\lambda}{M_xS_x\bar{x}+\rho\lambda}\right]$

4.5 Theoretical Efficiency Comparison

The conditions are derived under which our proposed class of estimators would be more efficient than the existing estimators. First let us consider,

$$MSE(t) = \overline{Y}^2 \left[1 - \frac{A^2}{B}\right]$$
$$MSE(t) = \overline{Y}^2 \left[\frac{B - A^2}{B}\right]$$

Putting the value of A and B and let us take,

$$\begin{aligned}
&B - A^2\\
=&1 + \lambda C_y^2 + 3\theta^2\lambda C_x^2 - 4\theta\lambda C_{yx} - [1 - \theta\lambda C_{yx} + \theta^2\lambda C_x^2]\\
=&B - [1 + \theta^2\lambda^2 C_{yx}^2 + \theta^4\lambda^2 C_x^4 + 2\theta\lambda C_{yx} + 2\theta^2\lambda C_x^2 - 2\theta^3\lambda^2 C_{yx} C_x^2]\\
=&1 + \lambda C_y^2 + 3\theta^2\lambda C_x^2 - 4\theta\lambda C_{yx} - 1 - \theta^2\lambda^2 C_{yx}^2 - \theta^4\lambda^2 C_x^4 - 2\theta\lambda C_{yx}\\
&- 2\theta^2\lambda C_x^2 + 2\theta^3\lambda^2 C_{yx} C_x^2\\
=&\lambda C_y^2 + \theta^2\lambda C_x^2 - 6\theta\lambda C_{yx} - \theta^2\lambda^2 C_{yx}^2 - \theta^4\lambda^2 C_x^4 + 2\theta^3\lambda^2 C_{yx} C_x^2\\
=&\lambda C_y^2 + \theta\lambda C_x^2 \left(\theta - \frac{6\rho C_y C_x}{C_x^2} - \frac{\theta\lambda\rho^2 C_y^2 C_x^2}{C_x^2} - \theta^3\lambda C_x^3 + \frac{2\theta^2\lambda\rho C_y C_x C_x^2}{C_x^2}\right)\\
=&\lambda C_y^2 + \theta\lambda C_x^2 \left(\theta - 6\rho\frac{C_y}{C_x} - \theta\lambda\rho^2 C_y^2 - \theta^3\lambda C_x^3 + 2\theta^2\lambda C_{yx}\right)\\
=&\lambda C_y^2 + \theta\lambda C_x^2 D
\end{aligned}$$

where,

$$\begin{aligned}
D =&\theta[1 - \lambda\rho^2 C_y^2 - \theta^2\lambda C_x^3 + 2\theta\lambda C_{yx}] - 6d\\
d =&\rho\frac{C_y}{C_x}
\end{aligned}$$

Hence,

$$MSE(t) =\overline{Y}^2 \frac{\lambda C_y^2 + \theta\lambda C_x^2 D}{B}$$

1. $\mathbf{MSE(t) < V(t_0)}$ **if**

$$\begin{aligned}
&\Longrightarrow \overline{Y}^2(\lambda C_y^2 + \theta\lambda C_x^2 D) < B\overline{Y}^2\lambda C_y^2\\
&\Longrightarrow C_y^2 + \theta C_x^2 D < BC_y^2\\
&\Longrightarrow C_y^2 - BC_y^2 + \theta C_x^2 D < 0\\
&\Longrightarrow C_y^2(1 - B) + \theta C_x^2 D < 0\\
&\Longrightarrow \theta C_x^2 D < C_y^2(B - 1)\\
&\Longrightarrow \mathbf{C_y^2 > \frac{\theta C_x^2 D}{B - 1}}
\end{aligned}$$

2. $\mathbf{MSE(t) < MSE(t_1)}$ **if**

$$\Longrightarrow \lambda C_y^2 + \theta\lambda C_x^2 D < \lambda B[C_y^2 + C_x^2 - 2C_{yx}]$$
$$\Longrightarrow \lambda C_y^2 + \theta\lambda C_x^2 D < \lambda BC_y^2 + \lambda BC_x^2 - 2\lambda BC_{yx}$$
$$\Longrightarrow \lambda C_y^2 - \lambda BC_y^2 + \theta\lambda C_x^2 D - \lambda BC_x^2 + 2\lambda BC_{yx} < 0$$
$$\Longrightarrow C_y^2(1-B) + C_x^2(\theta D - B) + 2\lambda BC_{yx} < 0$$
$$\Longrightarrow C_y^2(1-B) + C_x^2(\theta D - B + 2Bd) < 0$$
$$\Longrightarrow \mathbf{C_y^2 > \frac{C_x^2(\theta D - B + 2Bd)}{B-1}}$$

3. $\mathbf{MSE(t) < MSE(t_2)}$ **if**

$$\Longrightarrow \lambda C_y^2 + \theta\lambda C_x^2 D < \lambda B\left[C_y^2 + \frac{C_x^2}{4} - C_{yx}\right]$$
$$\Longrightarrow C_y^2(1-B) + \theta C_x^2 D - B\frac{C_x^2}{4} + BC_{yx} < 0$$
$$\Longrightarrow \; - C_y^2(B-1) + C_x^2\left(\theta D - \frac{B}{4} + Bd\right)$$
$$\Longrightarrow \mathbf{C_y^2 > \frac{C_x^2}{(B-1)}\left(\theta D - \frac{B}{4} + Bd\right)}$$

4. $\mathbf{MSE(t) < MSE(t_i)}$; $i = 3, \ldots, 20$ **if**

$$\Longrightarrow \lambda C_y^2 + \theta\lambda C_x^2 D < \lambda B(C_y^2 + \theta_i^2 C_x^2 - \theta_i C_{yx})$$
$$\Longrightarrow C_y^2 + \theta C_x^2 D < BC_y^2 + B\theta_i^2 C_x^2 - 2B\theta_i C_{yx}$$
$$\Longrightarrow C_y^2(1-B) + C_x^2(\theta D - B\theta_i^2 + 2B\theta_i d) < 0$$
$$\Longrightarrow \mathbf{C_y^2 > \frac{C_x^2(\theta D - B\theta_i^2 + 2B\theta_i d)}{B-1}}$$

4.6 Computational Study

To prove the theoretical results numerically we have considered a Natural Population with sample size 5.

Data Source	: Daroga Singh and F.S. Chaudhary (1986, Page-177)
Data Details	: **Study Variable :**
	: Area under wheat in a region during year 1974
	: **Auxiliary Variable**
	: Cultivated Area under wheat in a region during year 1973

Table 4.3: **Parametric Vlues of the Population**

SNo	Information	Data Set
1	N	34
2	n	5
3	$\overline{Y}$	199.4412
4	$\overline{X}$	208.8824
5	S_y	150.215
6	S_x	150.506
7	C_y	0.7531797
8	C_x	0.7205298
9	M_y	142.5
10	M_x	150
11	ρ	0.9800867
12	C_{yx}	0.5318817
13	$\beta_{1(x)}$	0.8732281
14	$\beta_{2(x)}$	5.912272
15	f	0.1470588
16	λ	0.1705882
17	${}^{N}C_n$	278256
18	$Q_{1(x)}$	94.25
19	$Q_{3(x)}$	275.75
20	$Q_{r(x)}$	160.5
21	$Q_{a(x)}$	166.3333
22	QD	80.25
23	TM	162.25

To compute the Percent Relative Efficiency (PRE) for different estimators with respect to Simple Random Sample Mean we use the following :

$$\text{PRE} = \frac{V(t_0)}{\text{MSE}(\cdot)}$$

Table 4.4: **MSE and PRE of Estimators**

SNo	Estimators	MSE	PRE
1	t_0	3849.248	100
2	t_1	153.8905	2501.29

MSE and PRE of Estimators

SNo	Estimators	MSE	PRE
3	t_2	1120.88	343.413
4	t_3	154.5255	2491.011
5	t_4	165.4474	2326.57
6	t_5	153.9924	2499.635
7	t_6	154.7734	2487.021
8	t_7	161.3104	702.2823
9	t_8	548.1055	2386.237
10	t_9	1312.292	293.3224
11	t_{10}	154.6701	2364.667
12	t_{11}	162.7818	2488.682
13	t_{12}	155.0034	2483.331
14	t_{13}	841.4363	457.4616
15	t_{14}	1117.772	344.368
16	t_{15}	893.9771	430.5757
17	t_{16}	473.1776	813.4891
18	t_{17}	922.6805	417.181
19	t_{18}	535.4868	718.8315
20	t_{19}	159.8507	2408.027
21	$t_{20(1)}$	153.8915	2501.275
22	$t_{20(2)}$	153.8912	2501.279
23	$t_{20(3)}$	153.8967	2501.189
24	$t_{20(4)}$	153.895	2501.217
25	$t_{20(5)}$	154.0555	2498.612
26	$t_{20(6)}$	154.0112	2499.33
27	$t_{20(7)}$	154.0088	2499.369
28	$t_{20(8)}$	154.0137	2499.289
29	$t_{20(9)}$	154.121	2497.549
30	$\mathbf{t}_{(\mathbf{1})}$	152.7766	2519.527
31	$\mathbf{t}_{(\mathbf{2})}$	152.7763	2519.532
32	$\mathbf{t}_{(\mathbf{3})}$	152.7761	2519.535
33	$\mathbf{t}_{(\mathbf{4})}$	152.7765	2519.529
34	$\mathbf{t}_{(\mathbf{5})}$	152.7757	2519.542
35	$\mathbf{t}_{(\mathbf{6})}$	152.776	2519.537
36	$\mathbf{t}_{(\mathbf{7})}$	152.7756	2519.544

4.7 Results and Conclusion

1. Table 4.1 Reviews the Existing literature. Table 4.2 shows the values for which the Existing estimators reduce to the proposed class of estimators. Table 4.3 consists of parametric values of the data with which we verified our results empirically. Table 4.4 shows the MSE and PRE of existing and proposed class of estimators.

2. We study the Bias and MSE of the proposed class up to first order of approximation.Since Efficiency is stronger property than the unbiasedness. Hence here we prefer the biased estimator with minimum MSE instead of unbiased estimator with higher MSE.

3. From table 4.4 we can easily notice that the proposed class of estimators have lesser MSEs thus greater PREs which proves that our class of proposed estimators is efficient enough for the practical purposes.

References

Al-Omari, A.I., Jemain, A.A.,and Ibrahim, K. (2009): New Ratio Estimators of the Mean using Simple Random Sampling and Ranked set Sampling Methods. *Investigacion Operacional*, 30(2), 97-108.

Bahl, S., and Tuteja, R.K. (1991): Ratio and Product Type Exponential Estimators. *Journal of Information and Optimization Sciences*, 12(1), 159-164.

Kadilar, C., and Cingi, H. (2003) : A study on the chain ratio type estimator.*Hacettepe Journal of Mathematics and Statistics*, Vol.32, 105-108.

Cochran, W.G. (1940): The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *The Journal of Agric. Science*, 30, 262-275.

Jeelani, M.I., Maqbool, S., and Mir, S.A. (2013): Modified Ratio Estimators of Population Mean using Linear Combination of Coefficient of Skewness and Quartile Deviation. *International Journal of Model Mathematical Sciences*, 6(3), 174-183.

Jerajuddin, M., and Kishun, J. (2016): Modified Ratio Estimators for Population Mean Using Size of the Sample selected from the Population. *International Journal of Scientific Research in Science, Engineering and Technology*, 2(2), 10-16

Singh, D., Chaudhary, F.S. (1986) : Sample Survey Designs.*New Age Int.(P) Ltd.*, 148, 182, 177.

Singh, H.P., and Tailor, R. (2003): Use of known Correlation Coefficient in Estimating the Finite Population Mean. *Statistics in Transition*, 6(4), 550-560.

Singh, H.P., Tailor, R., and Kakran, M.S. (2004): An improved Estimator of population mean using power transformation. *Journal of Indian Society of Agricultural Statistics*, 58(2), 223-230.

Sisodia, B.V.S. and Dwivedi, V.K (1981) : A modified ratio estimator using coefficient of variation of auxiliary variable.*Journal of the Indian Society of Agricultural Statistics,* 33, 13-18.

Subramani, J. and Kumarapandiyan, G. (2012a): Modified Ratio Estimators for Population Mean using Function of Quartiles of Auxiliary Variable. *Bonfring International Journal of Industrial Engineering and Management Science,* 2(2), pp. 19-23,2012.

Subramani, J. and Kumarapandiyan, G. (2012b): Estimation of Population Mean using known Median and Coefficient of Skewness. *American Journal of Mathematics and Statistics*, 2(5), pp. 101-107.

Upadhyaya, L.N. and Singh, H.P. (1999): Use of Transformed Auxiliary Variable in Estimating the Finite Population Mean. *Biometrical Journal,* 41, pp. 627-636.

Yadav, S.K., Dixit, M.K., Dungana, H.N., Mishra, S.S. (2019): Improved Estimators for Estimating Average Yield Using Auxiliary Variable. *International Journal of Mathematical Engineering and Management Sciences*, Vol.x,No.x,xxx-xxx,20xx

Yan, Z. and Tian, B. (2010): Ratio Method to the Mean Estimation using Coefficient of Skewness of Auxiliary Variable. *ICICA 2010, Part II, CCIS*, 106, pp. 103-110.

Chapter 5

RESTRUCTURED CLASS OF ESTIMATORS UNDER THE SIMPLE RANDOM SAMPLING SCHEME FOR THE ESTIMATION OF POPULATION MEAN USING AN AUXILIARY VARIABLE

5.1 Summary

The present paper provides a remedy for improved estimation of population mean of a study variable, using the information related to an auxiliary variable in the situations under Simple Random Sampling Scheme. We suggest a new class of estimators of population mean and the Bias and MSE of the class are derived upto the first order of approximation. The least value of the MSE for the suggested class of estimators is also obtained for the optimum value of the characterizing scaler. The MSE has also been compared with the considered existing competing estimators both theoretically and empirically. The theoretical conditions for the proposed class to be more efficient than the competing estimators are verified using a natural population.

5.2 Introduction

Auxiliary information has been in practice in sampling theory since the advent of modern sample surveys. Information on auxiliary variable having high correlation with the variable under study is quite useful in improving the sampling design.

Cochran (1940) used the highly positively correlated study and auxiliary variable to propound the ratio estimator. Product estimator requires a high negative correlation between study and auxiliary variable. By reviewing the literature, it is concluded that applying the auxiliary information enhances the efficiencies of the estimators for estimating any parameter under consideration. So it is well established fact that the use of auxiliary variable technique improves the estimation process for target population. It is also noticed that ratio method of estimation is relatively simple and one of the commonly used methods of estimation. Hence we have considered the restructuring of the ratio type estimator in the present study.

Modifications in the usual ratio estimator has been done by various researchers to obtain the MSE as minimum as possible. Bahl and Tuteja (1991) formed exponential type ratio and product estimators. Kadilar and Cingi (2003) studied chain ratio type estimator. Jerajuddin and Kishun (2016) did not use auxiliary variable instead they used size of the sample as supplementary information. Singh, Tailor and Kakran (2004) used power transformation to improve the estimation of population mean. Al-Omari (2009), Jeelani (2013), Singh and Tailor (2003), Sisodia and Dwivedi (1981), Subramani and Kumarpandiyan (2012), Upadhyaya and Singh (1991), Yadav (2019), Yan and Tian (2010) used the functions of auxiliary variable and their combinations to modify the estimator with a greater precision.

the purpose of the current study is also to modify and improve the ratio estimator which would be better than the many of previous derived estimators which are considered in this study. Let the target population is of size N. Y is the study variable and X is the auxiliary variable. A sample of size n has been drawn both for the study and auxiliary variables. The present study would use the information of the variable X combined with the study variable to obtain the more efficient estimators.

5.3 Existing Estimators

A number of modified estimators by various authors have been developed till date for improved estimation of the population mean under various situations under sinple random sampling scheme. The considered existing estimators with there Mean Squared Errors along with their constants are presented in Table-1.

Table 5.1: **Literature Review**

SNo	Estimators	MSE	Constants
1	$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ Sample Mean	$\lambda\overline{Y}^2 C_y^2$	
2	$t_1 = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)$ Cochran(1940)	$\lambda\overline{Y}^2\left(C_y^2 + C_x^2 - 2C_{yx}\right)$	
3	$t_2 = \bar{y}\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Bahl and Tuteja(1991)	$\lambda\overline{Y}^2\left(C_y^2 + \frac{C_x^2}{4} - \rho C_y C_x\right)$	
4	$t_3 = \bar{y}\left(\frac{\overline{X}+C_x}{\bar{x}+C_x}\right)$ Sisodia and Dwivedi (1981)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_3^2 C_x^2 - 2\theta_3 C_{yx}\right)$	$\theta_3 = \frac{\overline{X}}{\overline{X}+C_x}$
5	$t_4 = \bar{y}\left(\frac{\overline{X}C_x+\beta_{2(x)}}{\bar{x}C_x+\beta_{2(x)}}\right)$ Upadhyaya and Singh(1999)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_4^2 C_x^2 - 2\theta_4 C_{yx}\right)$	$\theta_4 = \frac{\overline{X}C_x}{\overline{X}C_x+\beta_{2(x)}}$
6	$t_5 = \bar{y}\left(\frac{\overline{X}\beta_{2(x)}+C_x}{\bar{x}\beta_{2(x)}+C_x}\right)$ Upadhyaya and Singh(1999)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_5^2 C_x^2 - 2\theta_5 C_{yx}\right)$	$\theta_5 = \frac{\overline{X}\beta_{2(x)}}{\overline{X}\beta_{2(x)}+C_x}$
7	$t_6 = \bar{y}\left(\frac{\overline{X}+\rho}{\bar{x}+\rho}\right)$ Singh and Tailor(2003)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_6^2 C_x^2 - 2\theta_6 C_{yx}\right)$	$\theta_6 = \frac{\overline{X}}{\overline{X}+\rho}$
8	$t_7 = \bar{y}\left(\frac{\overline{X}+\beta_{2(x)}}{\bar{x}+\beta_{2(x)}}\right)$ Singh et al.(2004)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_7^2 C_x^2 - 2\theta_7 C_{yx}\right)$	$\theta_7 = \frac{\overline{X}}{\overline{X}+\beta_{2(x)}}$
9	$t_8 = \bar{y}\left(\frac{\overline{X}+Q_{1(x)}}{\bar{x}+Q_{1(x)}}\right)$ Al-Omari et al.(2009)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_8^2 C_x^2 - 2\theta_8 C_{yx}\right)$	$\theta_8 = \frac{\overline{X}}{\overline{X}+Q_{1(x)}}$

SNo	Estimators	MSE	Constants
10	$t_9 = \bar{y}\left(\frac{\overline{X}+Q_{3(x)}}{\bar{x}+Q_{3(x)}}\right)$ Al-Omari et al.(2009)	$\lambda\overline{Y}^2\left(C_y^2+\theta_9^2C_x^2-2\theta_9C_{yx}\right)$	$\theta_9=\frac{\overline{X}}{\overline{X}+Q_{3(x)}}$
11	$t_{10} = \bar{y}\left(\frac{\overline{X}+\beta_{1(x)}}{\bar{x}+\beta_{1(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{10}^2C_x^2-2\theta_{10}C_{yx}\right)$	$\theta_{10}=\frac{\overline{X}}{\overline{X}+\beta_{1(x)}}$
12	$t_{11} = \bar{y}\left(\frac{\overline{X}\beta_{1(x)}+\beta_{2(x)}}{\bar{x}\beta_{1(x)}+\beta_{2(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{11}^2C_x^2-2\theta_{11}C_{yx}\right)$	$\theta_{11}=\frac{\overline{X}\beta_{1(x)}}{\overline{X}\beta_{1(x)}+\beta_{2(x)}}$
13	$t_{12} = \bar{y}\left(\frac{\overline{X}C_x+\beta_{1(x)}}{\bar{x}C_x+\beta_{1(x)}}\right)$ Yan and Tian(2010)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{12}^2C_x^2-2\theta_{12}C_{yx}\right)$	$\theta_{12}=\frac{\overline{X}C_x}{\overline{X}C_x+\beta_{1(x)}}$
14	$t_{13} = \bar{y}\left(\frac{\overline{X}+M_x}{\bar{x}+M_x}\right)$ Subramani and Kumarpandiyan(2012a)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{13}^2C_x^2-2\theta_{13}C_{yx}\right)$	$\theta_{13}=\frac{\overline{X}}{\overline{X}+M_x}$
15	$t_{14} = \bar{y}\left(\frac{\overline{X}C_x+M_x}{\bar{x}C_x+M_x}\right)$ Subramani and Kumarpandiyan(2012a)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{14}^2C_x^2-2\theta_{14}C_{yx}\right)$	$\theta_{14}=\frac{\overline{X}C_x}{\overline{X}C_x+M_x}$
16	$t_{15} = \bar{y}\left(\frac{\overline{X}+Q_{r(x)}}{\bar{x}+Q_{r(x)}}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{15}^2C_x^2-2\theta_{15}C_{yx}\right)$	$\theta_{15}=\frac{\overline{X}}{\overline{X}+Q_{r(x)}}$
17	$t_{16} = \bar{y}\left(\frac{\overline{X}+QD}{\bar{x}+QD}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{16}^2C_x^2-2\theta_{16}C_{yx}\right)$	$\theta_{16}=\frac{\overline{X}}{\overline{X}+QD}$
18	$t_{17} = \bar{y}\left(\frac{\overline{X}+Q_{a(x)}}{\bar{x}+Q_{a(x)}}\right)$ Subramani and Kumarpandiyan(2012b)	$\lambda\overline{Y}^2\left(C_y^2+\theta_{17}^2C_x^2-2\theta_{17}C_{yx}\right)$	$\theta_{17}=\frac{\overline{X}}{\overline{X}+Q_{a(x)}}$

SNo	Estimators	MSE	Constants
19	$t_{18} = \bar{y}\left(\frac{\overline{X}+n}{\bar{x}+n}\right)$ Jerajuddin and Kishun(2016)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{18}^2 C_x^2 - 2\theta_{18}C_{yx}\right)$	$\theta_{18} = \frac{\overline{X}}{\overline{X}+n}$
20	$t_{19} = \bar{y}\left(\frac{\overline{X}\beta_{1(x)}+QD}{\bar{x}\beta_{1(x)}+QD}\right)$ Jeelani et al.(2013)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{19}^2 C_x^2 - 2\theta_{19}C_{yx}\right)$	$\theta_{19} = \frac{\overline{X}\beta_{1(x)}}{\overline{X}\beta_{1(x)}+QD}$
21	$t_{20} = \bar{y}\left(\frac{ab\overline{X}+cd}{ab\bar{x}+cd}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20}^2 C_x^2 - 2\theta_{20}C_{yx}\right)$	$\theta_{20} = \frac{ab\overline{X}}{ab\overline{X}+c.d}$
	$t_{20(1)} = \bar{y}\left(\frac{\beta_{2(x)}M_x\overline{X}+\rho}{\beta_{2(x)}M_x\bar{x}+\rho}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(1)}^2 C_x^2 - 2\theta_{20(1)}C_{yx}\right)$	$\theta_{20(1)} = \frac{\beta_{2(x)}M_x\overline{X}}{\beta_{2(x)}M_x\bar{x}+\rho}$
	$t_{20(2)} = \bar{y}\left(\frac{\beta_{2(x)}M_x\overline{X}+\rho C_x}{\beta_{2(x)}M_x\bar{x}+\rho C_x}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(2)}^2 C_x^2 - 2\theta_{20(2)}C_{yx}\right)$	$\theta_{20(2)} = \frac{\beta_{2(x)}M_x\overline{X}}{\beta_{2(x)}M_x\bar{x}+\rho C_x}$
	$t_{20(3)} = \bar{y}\left(\frac{\beta_{1(x)}M_x\overline{X}+\rho}{\beta_{1(x)}M_x\bar{x}+\rho}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(3)}^2 C_x^2 - 2\theta_{20(3)}C_{yx}\right)$	$\theta_{20(3)} = \frac{\beta_{1(x)}M_x\overline{X}}{\beta_{1(x)}M_x\bar{x}+\rho}$
	$t_{20(4)} = \bar{y}\left(\frac{\beta_{1(x)}M_x\overline{X}+\rho C_x}{\beta_{1(x)}M_x\bar{x}+\rho C_x}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(4)}^2 C_x^2 - 2\theta_{20(4)}C_{yx}\right)$	$\theta_{20(4)} = \frac{\beta_{1(x)}M_x\overline{X}}{\beta_{1(x)}M_x\bar{x}+\rho C_x}$
	$t_{20(5)} = \bar{y}\left(\frac{n\overline{X}+\rho}{n\bar{x}+\rho}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(5)}^2 C_x^2 - 2\theta_{20(5)}C_{yx}\right)$	$\theta_{20(5)} = \frac{n\overline{X}}{n\bar{x}+\rho C_x}$
	$t_{20(6)} = \bar{y}\left(\frac{n\overline{X}+C_x}{n\bar{x}+C_x}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(6)}^2 C_x^2 - 2\theta_{20(6)}C_{yx}\right)$	$\theta_{20(6)} = \frac{n\overline{X}}{n\bar{x}+C_x}$
	$t_{20(7)} = \bar{y}\left(\frac{n\overline{X}+\rho C_x}{n\bar{x}+\rho C_x}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(7)}^2 C_x^2 - 2\theta_{20(7)}C_{yx}\right)$	$\theta_{20(7)} = \frac{n\overline{X}}{n\bar{x}+\rho C_x}$
	$t_{20(8)} = \bar{y}\left(\frac{n\rho\overline{X}+C_x}{n\rho\bar{x}+C_x}\right)$	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(8)}^2 C_x^2 - 2\theta_{20(8)}C_{yx}\right)$	$\theta_{20(8)} = \frac{n\rho\overline{X}}{n\rho\bar{x}+C_x}$
	$t_{20(9)} = \bar{y}\left(\frac{nC_x\overline{X}+\rho}{nC_x\bar{x}+\rho}\right)$ Yadav et al.(2019)	$\lambda\overline{Y}^2\left(C_y^2 + \theta_{20(9)}^2 C_x^2 - 2\theta_{20(9)}C_{yx}\right)$	$\theta_{20(9)} = \frac{nC_x\overline{X}}{nC_x\bar{x}+\rho}$

5.4 Proposed Class of Estimators

Inspired by the literature of improved estimators and adopting the Yadav et al. (2019) estimator, we suggest an improved class of ratio type estimators for the estimation of population mean using auxiliary information as follows:

$$\mathbf{t} = \alpha\mathbf{\bar{y}} + (\mathbf{1}-\alpha)\mathbf{\bar{y}}\left[\frac{\mathbf{ab\overline{X}+cd}}{\mathbf{ab\bar{x}+cd}}\right] \tag{5.4.1}$$

where α is a characterizing constatnt and a, b, c, d are either constatnts or the known parameters of the auxiliary variable. The constant α is chosen such that the Mean Squared Error(MSE) of the suggested estimator is minimum. The (a, b, c, d) may also take those real and parametric values which makes the MSE of the proposed estimator a least possible.

5.4.1 Bias and MSE

To obtain the Bias and MSE of the suggested estimator, we define the following approximations as:

$$\begin{aligned}
e_0 &= \frac{\bar{y}-\overline{Y}}{\overline{Y}} && \text{and} \quad e_1 = \frac{\bar{x}-\overline{X}}{\overline{X}} \\
\text{So,} \quad \bar{y} &= \overline{Y}(1+e_0) && \text{and} \quad \bar{x} = \overline{X}(1+e_1) \\
E(e_0) &= E(e_1) = 0 && \text{and} \quad E(e_0e_1) = \lambda C_{yx} \\
E(e_0^2) &= \lambda C_y^2 && \text{and} \quad E(e_1^2) = \lambda C_x^2
\end{aligned}$$

Now rewriting our proposed estimator from equation (5.4.1) as,

$$t = \alpha\overline{Y}(1+e_0) + (1-\alpha)\overline{Y}(1+e_0)\left[\frac{ab\overline{X}+cd}{ab\overline{X}(1+e_1)+cd}\right]$$

$$t = \alpha\overline{Y}(1+e_0) + (1-\alpha)\overline{Y}(1+e_0)(1+\theta e_1)^{-1} \tag{5.4.2}$$

$$\text{Where,} \quad \theta = \frac{ab\overline{X}}{ab\overline{X}+cd}$$

Expanding the equation (5.4.2), then multiplying its terms and retaining the terms upto the first order of approximation we get,

$$t = \alpha\overline{Y}[1+e_0-1-e_0+\theta e_1+\theta e_0e_1-\theta^2e_1^2] + \overline{Y}(1+e_0-\theta e_1-\theta e_0e_1+\theta^2e_1^2)$$

$$(t-\overline{Y}) = \overline{Y}[\alpha(\theta e_1+\theta e_0e_1-\theta^2e_1^2) + (e_0-\theta e_1-\theta e_0e_1+\theta^2e_1^2)] \tag{5.4.3}$$

Taking expectation on both sides of equation (5.4.3)

$$E(t-\overline{Y}) = \overline{Y}[\alpha(\theta\lambda C_{yx}-\theta^2\lambda C_x^2) + (\theta^2\lambda C_x^2-\theta\lambda C_{yx})]$$

$$B(t) = \overline{Y}[\alpha(\theta\lambda C_{yx}-\theta^2\lambda C_x^2) + (\theta^2\lambda C_x^2-\theta\lambda C_{yx})] \tag{5.4.4}$$

Squaring the equation (5.4.3), retaining the terms up to the approximation of order one and putting values of various expectations, we get the Mean Squared Error of the proposed class of estimators as ,

$$E(t-\overline{Y})^2 = \overline{Y}^2E[\alpha^2\theta^2e_1^2+e_0^2+\theta^2e_1^2-2\theta e_0e_1+2\alpha(\theta e_0e_1-\theta^2e_1^2)]$$

$$MSE(t) = \overline{Y}^2[\alpha^2\theta^2\lambda C_x^2+(\lambda C_y^2+\theta^2\lambda C_x^2-2\theta\lambda C_{yx})+2\alpha(\theta\lambda C_{yx}-\theta^2\lambda C_x^2)] \tag{5.4.5}$$

Differentiating it with respect to α and puuting it equal to zero we would get the optimum value of α,

$$\frac{\partial MSE(t)}{\partial \alpha} = 0$$
$$\implies \lambda \overline{Y}^2[2\alpha(\theta^2 C_x^2) + 2(\theta C_{yx} - \theta^2 C_x^2)] = 0$$
$$\implies \alpha\theta^2 C_x^2 = \theta^2 C_x^2$$
$$\implies \alpha = \frac{\theta^2 C_x^2 - \theta C_{yx}}{\theta^2 C_x^2} \tag{5.4.6}$$

$$\implies \alpha = \frac{\theta^2 C_x^2}{\theta^2 C_x^2} - \frac{\theta C_{yx}}{\theta^2 C_x^2}$$
$$\implies \alpha = 1 - \frac{\rho C_y C_x}{\theta C_x^2}$$
$$\implies \alpha = 1 - \frac{\rho C_y}{\theta C_x}$$
$$\implies \alpha = \mathbf{1} - \frac{\mathbf{d}}{\theta} \tag{5.4.7}$$

and, $$\frac{\partial^2 MSE(t)}{\partial \alpha^2} = \lambda \overline{Y}^2(2\theta^2 C_x^2) > 0 \quad (minimum)$$

Putting this value of α in equation (5.4.5) we would get the result,

$$MSE(t)_{min} = \lambda \overline{Y}^2 \left[\left(1 - \frac{d}{\theta}\right)^2 \theta^2 C_x^2 + (C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx}) + 2\left(1 - \frac{d}{\theta}\right)(\theta C_{yx} - \theta^2 C_x^2) \right]$$
$$= \lambda \overline{Y} \left[\left(\frac{\theta^2 C_x^2 - \theta C_{yx}}{\theta^2 C_x^2} \right)^2 \theta^2 C_x^2 + (C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx}) - 2\left(\frac{\theta^2 C_x^2 - \theta C_{yx}}{\theta^2 C_x^2} \right)(\theta^2 C_x^2 - \theta C_{yx}) \right]$$
$$= \lambda \overline{Y}^2 \left[\frac{(\theta^2 C_x^2 - \theta C_{yx})^2}{\theta^2 C_x^2} + (C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx}) - 2\frac{(\theta^2 C_x^2 - \theta C_{yx})^2}{\theta^2 C_x^2} \right]$$
$$= \lambda \overline{Y}^2 \left[(C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx}) - \frac{(\theta^2 C_x^2 - \theta C_{yx})^2}{\theta^2 C_x^2} \right]$$
$$= \lambda \overline{Y}^2 \left[(C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx}) - \left(\frac{\theta^4 C_x^4 + \theta^2 C_{yx}^2 - 2\theta^3 C_x^2 C_{yx}}{\theta^2 C_x^2} \right) \right]$$
$$= \lambda \overline{Y}^2 [C_y^2 + \theta^2 C_x^2 - 2\theta C_{yx} - \theta^2 C_x^2 - \rho^2 C_y^2 + 2\theta C_{yx}]$$
$$= \lambda \overline{Y}^2 (C_y^2 - \rho^2 C_y^2)$$
$$= \lambda \overline{Y}^2 \left(C_y^2 - \frac{C_{yx}^2 C_y^2}{C_y^2 C_x^2} \right)$$
$$\mathbf{MSE(t)_{min}} = \lambda \overline{\mathbf{Y}}^{\mathbf{2}} \left(\mathbf{C_y^2} - \frac{\mathbf{C_{yx}^2}}{\mathbf{C_x^2}} \right) \tag{5.4.8}$$

For this MSE

$$\mathbf{B(t) = 0} \tag{5.4.9}$$

5.5 Efficiency Comparison

The theoretical conditions are derived under which our proposed estimator is more efficient than the exsiting many competing estimators. These are as follows :

1. $\mathbf{MSE(t)_{min} < V(t_0)}$ **if**

$$\lambda\overline{Y}^2\left(C_y^2 - \frac{C_{yx}^2}{C_x^2}\right) < \lambda\overline{Y}^2 C_y^2$$

$$C_y^2 - \frac{C_{yx}^2}{C_x^2} < C_y^2$$

$$C_y^2 - C_y^2 < \frac{C_{yx}^2}{C_x^2}$$

$$\mathbf{\frac{C_{yx}^2}{C_x^2} > 0}$$

2. $\mathbf{MSE(t)_{min} < MSE(t_1)}$ **if**

$$\lambda\overline{Y}^2\left(C_y^2 - \frac{C_{yx}^2}{C_x^2}\right) < \lambda\overline{Y}^2\left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right)$$

$$\left(C_y^2 - \frac{C_{yx}^2}{C_x^2}\right) < \left(C_y^2 + \frac{C_x^2}{4} - C_{yx}\right)$$

$$C_y^2 - C_y^2 + C_{yx} < \frac{C_{yx}^2}{C_x^2} + \frac{C_x^2}{4}$$

$$C_{yx} < \frac{C_{yx}^2}{C_x^2} + \frac{C_x^2}{4}$$

$$C_{yx} - \frac{C_{yx}^2}{C_x^2} < \frac{C_x^2}{4}$$

$$C_{yx}\left(1 - \frac{C_{yx}}{C_x^2}\right) < \frac{C_x^2}{4}$$

$$\mathbf{C_x^2 > 4C_{yx}\left(1 - \frac{C_{yx}}{C_x^2}\right)}$$

3. $\mathbf{MSE(t)_{min} < MSE(t_2)}$ **if**

$$\lambda\overline{Y}^2\left(C_y^2 - \frac{C_{yx}^2}{C_x^2}\right) < \lambda\overline{Y}^2(C_y^2 + C_x^2 - 2C_{yx})$$

$$2C_{yx} < C_x^2 + \frac{C_{yx}^2}{C_x^2}$$

$$2C_{yx} - \frac{C_{yx}^2}{C_x^2} < C_x^2$$

$$\mathbf{C_x^2 > C_{yx}\left(2 - \frac{C_{yx}}{C_x^2}\right)}$$

4. $\mathbf{MSE(t)_{min}} < \mathbf{MSE(t_i)} \qquad ; i = 3, \ldots, 20 \qquad$ **if**

$$\lambda \overline{Y}^2 \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) < \lambda \overline{Y}^2 (C_y^2 + \theta_i^2 C_x^2 - 2\theta_i C_{yx})$$

$$-\frac{C_{yx}^2}{C_x^2} < \theta_i^2 C_x^2 - 2\theta_i C_{yx}$$

$$2\theta_i C_{yx} - \frac{C_{yx}^2}{C_x^2} < \theta_i^2 C_x^2$$

$$\mathbf{C_x^2} > \frac{\mathbf{C_{yx}}}{\boldsymbol{\theta_i^2}} \left(\mathbf{2}\boldsymbol{\theta_i} - \frac{\mathbf{C_{yx}^2}}{\mathbf{C_x^2}} \right)$$

5.6 Computational Study

To prove the theoretical results numerically we have considered a Natural Population with sample size 5.

Data Source : Daroga Singh and F.S. Chaudhary (1986, Page-177)

Data Details : **Study Variable :**

: Area under wheat in a region during year 1974

: **Auxiliary Variable**

: Cultivated Area under wheat in a region during year 1973

Table 5.2: **Parametric Vlues of the Population**

SNo	**Information**	**Data Set**
1	N	34
2	n	5
3	$\overline{Y}$	199.4412
4	$\overline{X}$	208.8824
5	S_y	150.215
6	S_x	150.506
7	C_y	0.7531797
8	C_x	0.7205298
9	M_y	142.5
10	M_x	150

Parametric Vlues of the Population

SNo	Information	Data Set
11	ρ	0.9800867
12	C_{yx}	0.5318817
13	$\beta_{1(x)}$	0.8732281
14	$\beta_{2(x)}$	5.912272
15	f	0.1470588
16	λ	0.1705882
17	${}^{N}C_{n}$	278256
18	$Q_{1(x)}$	94.25
19	$Q_{3(x)}$	275.75
20	$Q_{r(x)}$	160.5
21	$Q_{a(x)}$	166.3333
22	QD	80.25
23	TM	162.25

To compute the Percent Relative Efficiency (PRE) for different estimators with respect to Hansan and Horwitz estimators we use the following :

$$\text{PRE} = \frac{V(t_0)}{\text{MSE}(\cdot)}$$

Table 5.3: **MSE and PRE of Estimators**

SNo	Estimators	MSE	PRE
1	t_0	3849.248	100
2	t_1	153.8905	2501.29
3	t_2	1120.88	343.413
4	t_3	154.5255	2491.011
5	t_4	165.4474	2326.57

MSE and PRE of Estimators

SNo	Estimators	MSE	PRE
6	t_5	153.9924	2499.635
7	t_6	154.7734	2487.021
8	t_7	161.3104	702.2823
9	t_8	548.1055	2386.237
10	t_9	1312.292	293.3224
11	t_{10}	154.6701	2364.667
12	t_{11}	162.7818	2488.682
13	t_{12}	155.0034	2483.331
14	t_{13}	841.4363	457.4616
15	t_{14}	1117.772	344.368
16	t_{15}	893.9771	430.5757
17	t_{16}	473.1776	813.4891
18	t_{17}	922.6805	417.181
19	t_{18}	535.4868	718.8315
20	t_{19}	159.8507	2408.027
21	$t_{20(1)}$	153.8915	2501.275
22	$t_{20(2)}$	153.8912	2501.279
23	$t_{20(3)}$	153.8967	2501.189
24	$t_{20(4)}$	153.895	2501.217
25	$t_{20(5)}$	154.0555	2498.612
26	$t_{20(6)}$	154.0112	2499.33
27	$t_{20(7)}$	154.0088	2499.369
28	$t_{20(8)}$	154.0137	2499.289
29	$t_{20(9)}$	154.121	2497.549
30	$\mathbf{t_{(min)}}$	**151.7764**	**2536.131**

5.7 Results and Conclusion

1. Table 5.1 Reviews the Existing literature.Conditions are derived for which our proposed class estimators is better than the existing estimators. Table 5.2 consists of parametric values of the data with which we verified our results empirically. Table 5.3 shows the MSE and PRE of existing and proposed class of estimators.

2. We study the MSE of the proposed class up to the first order of approximation. For the optimum value of α which makes the MSE minimum our proposed class of estimators is unbiased.

3. We have also suggested some members of the proposed class which come out be more efficient than the existing competing estimators.

4. From Tables 5.3 we can easily notice that proposed class have the greater PREs from all the considered existing estimators of population mean under simple random sampling scheme.

5. Hence, it has been proven both theoretically and numerically that the proposed estimator is better than the other given competing estimators.

6. Since our computations revolved around a natural population, therefore we can successfully recommend our class of estimators for practical applications.

References

Al-Omari, A.I., Jemain, A.A.,and Ibrahim, K. (2009): New Ratio Estimators of the Mean using Simple Random Sampling and Ranked set Sampling Methods. *Investigacion Operacional*, 30(2), 97-108.

Bahl, S., and Tuteja, R.K. (1991): Ratio and Product Type Exponential Estimators. *Journal of Information and Optimization Sciences*, 12(1), 159-164.

Kadilar, C., and Cingi, H. (2003) : A study on the chain ratio type estimator.*Hacettepe Journal of Mathematics and Statistics*, Vol.32, 105-108.

Cochran, W.G. (1940): The Estimation of the Yields of the Cereal Experiments by Sampling for the Ratio of Grain to Total Produce. *The Journal of Agric. Science*, 30, 262-275.

Jeelani, M.I., Maqbool, S., and Mir, S.A. (2013): Modified Ratio Estimators of Population Mean using Linear Combination of Coefficient of Skewness and Quartile Deviation. *International Journal of Model Mathematical Sciences*, 6(3), 174-183.

Jerajuddin, M., and Kishun, J. (2016): Modified Ratio Estimators for Population Mean Using Size of the Sample selected from the Population. *International Journal of Scientific Research in Science, Engineering and Technology*, 2(2), 10-16

Singh, D., Chaudhary, F.S. (1986) : Sample Survey Designs.*New Age Int.(P) Ltd.*, 148, 182, 177.

Singh, H.P., and Tailor, R. (2003): Use of known Correlation Coefficient in Estimating the Finite Population Mean. *Statistics in Transition*, 6(4), 550-560.

Singh, H.P., Tailor, R., and Kakran, M.S. (2004): An improved Estimator of population mean using power transformation. *Journal of Indian Society of Agricultural Statistics*, 58(2), 223-230.

Sisodia, B.V.S. and Dwivedi, V.K (1981) : A modified ratio estimator using coefficient of variation of auxiliary variable.*Journal of the Indian Society of Agricultural Statistics,* 33, 13-18.

Subramani, J. and Kumarapandiyan, G. (2012a): Modified Ratio Estimators for Population Mean using Function of Quartiles of Auxiliary Variable. *Bonfring International Journal of Industrial Engineering and Management Science,* 2(2), pp. 19-23,2012.

SUBRAMANI, J. AND KUMARAPANDIYAN, G. (2012b): Estimation of Population Mean using known Median and Coefficient of Skewness. *American Journal of Mathematics and Statistics*, 2(5), pp. 101-107.

UPADHYAYA, L.N. AND SINGH, H.P. (1999): Use of Transformed Auxiliary Variable in Estimating the Finite Population Mean. *Biometrical Journal,* 41, pp. 627-636.

YADAV, S.K., DIXIT, M.K., DUNGANA, H.N., MISHRA, S.S. (2019): Improved Estimators for Estimating Average Yield Using Auxiliary Variable. *International Journal of Mathematical Engineering and Management Sciences*, Vol.x,No.x,xxx-xxx,20xx

YAN, Z. AND TIAN, B. (2010): Ratio Method to the Mean Estimation using Coefficient of Skewness of Auxiliary Variable. *ICICA 2010, Part II, CCIS*, 106, pp. 103-110.

Appendix A

R Calculations

Chapter 2

```
# Secondry data
N=c(34,34,34,34)
n=c(3,3,5,5)
Ncn=choose(N,n)
f=n/N
l=(1-f)/n
# study variable population means
Y=c(856.4118,856.4118,
856.4118,856.4118)
# AuxiliaryVariable population means
X=c(208.8824,199.4412,
208.8824,199.4412)
# Medians of PopulationStudyVariable
M=c(767.5,767.5,767.5,767.5)
# variances of study variable
vy=c(163356.4086,163356.4086,
91690.3713,91690.3713)
# variances of auxiliary variable
vx=c(6884.4455,6857.8555,
3864.1726,3849.248)
# coefficients of correlation
r=c(0.4491,0.4453,0.4491,0.4453)
r2=rˆ 2
cy2=vy/((Y∗ Y)∗ l)
cy=sqrt(cy2)
cx2=vx/(X∗ X∗ l)
cx=sqrt(cx2)
sx2=vx/l
c=r∗ (cy/cx)
# Proposed Estimator
do1=(M)/(M-r)
d1=(M+r)/(M-r)
do2=(Ncn)/(Ncn-M)
d2=(Ncn+M)/(Ncn-M)
do3=(M)/(M-(f∗ r))
d2=(M+(f∗ r))/(M-(f∗ r))
do4=(M)/(M-cx)
d4=(M+cx)/(M-cx)
do5=sx2/(sx2-M)
d5=(sx2-M)/(sx2+M)
t¡-function(d,do){
A1¡-1-(d∗ l∗ cx2∗ (c-do))
B1¡-1+(l∗ cy2)-(d∗ l∗ cx2∗ ((4∗ c)-d-(2∗
do)))
B¡-(((A1/B1)-1)∗ Y)+((A1/B1)∗ Y∗
((do-c)∗ d∗ l∗ cx2))
M1¡-(Y∗ Y)∗ (1-((A1∗ A1)/B1))
print(paste(”Bias=”,B))
print(paste(”MSE=”, M1))}
t(d1,do1)
t(d2,do2)
t(d2,do3)
t(d4,do4)
t(d5,do5)
# Review of Litrature...............
# MSE and Biasedness................
options(scipen = 100)
# usual estimator
M0¡-l∗ Yˆ 2∗ cy2
# Regression estimator (Watson 1937)
# Ratio Estimator(Cochran 1940)
M2¡-l∗ Yˆ 2∗ (cy2+cx2-(2∗ r∗ cy∗ cx))
```

```
# Product estimator( Goodman 1960)..
M4¡-l∗ (Yˆ 2)∗ (cy2+cx2+(2∗ r∗ cy∗ cx))
B4¡-(l∗ r∗ cy∗ cx)/(Xˆ 2∗ Y)
# Srivastav(1967)...................
M5¡-function(m){
M6¡-l∗ Yˆ 2∗ (cy2+(mˆ 2∗ cx2)-(2∗ m∗
r∗ cy∗ cx))
print(M6)
}
M5(c)
B5¡-function(b){
B5¡-((l∗ b∗ cx2)/2)∗ (b-(2∗ c)+1)
print(B5)
}
B5(c)

# Chakrabarty(1979).................
M6¡-M5(c)
B6¡-l∗ Y∗ ((c/2)∗ cx2-(c∗ r∗ cy∗ cx))
# Walsh(1970) and Reddy(1974).......
M7¡-M5
B7¡-function(e){
B7¡-l∗ cx2∗ (e-(c/(Yˆ 2∗ Xˆ 2)))
print(B7)
}
B7(c)
# Sahai (1979)......................
BM8¡-function(g){
v=(1-g)/(1+g)
h=cx/cy
B8¡-h∗ l∗ v∗ Yˆ 2∗ cy2∗ (r-((h∗ (1-
v))/2))
M8¡-l∗ Yˆ 2∗ cy2∗ (1+hˆ 2+(2∗ h∗ r∗ v))
print(B8)
print(M8)
}
BM8(c)
# Ray et al (1979)..................
M9¡-M5
M10¡-l∗ Yˆ 2∗ (cy2+(cˆ 2∗ cx2)+(2∗ r∗
c∗ cy∗ cx))
B9¡- (-c∗ Y∗ r∗ cy∗ cx)
B10¡-c∗ Y∗ (c-1)∗ cx2
# Ray and Sahai (1980)..............
B11¡-l∗ Y∗ ((((c(c-1))/2)∗ cx2)-(c∗ r∗
cy∗ cx))
M11¡-M5
# Sisodia and Dwivedi(1981).........
R12¡-X/(X+cx)
BM12¡-function(p){
B¡-l∗ Y∗ (pˆ 2∗ cx2-(p∗ r∗ cy∗ cx))
M¡-l∗ Yˆ 2∗ (cy2+(pˆ 2∗ cx2)-(2∗ p∗ r∗
cy∗ cx))
print(B)
print(M)
}
BM12(R12)
# Srivastava(1983)..................
B13¡-l∗ Y∗ (r∗ cy∗ cx+(f/(1-f))∗ cx2)
M13¡-l∗ (Yˆ 2)∗ (cy2+cx2+(2∗ r∗ cy∗
cx))
# Bahl and Tuteja(1991).............
M15¡-l∗ Yˆ 2∗ (cy2+(cx2/4)-(r∗ cy∗ cx))
B15¡-(l/8)∗ Y∗ ((3∗ cx2)-(4∗ r∗ cy∗ cx))
# Cadilar and Cingi(2003)...........
M16¡-l∗ Yˆ 2∗ (cy2+(4∗ cx2)-(4∗ r∗ cy∗
cx))
B16¡-l∗ Y∗ ((3∗ cx2)-(2∗ r∗ cy∗ cx))
# Singh and Tailor(2003)............
R15¡-X/(X+r)
BM12(R15)
# Pandey et al. (2011)..............
BM12(c)

# Singh et al. (2014)...............
M15
B18¡-((l∗ Y)/8)∗ ((1-f)∗ cx2-(4∗ r∗ cy∗
cx))
# Yadav and Mishra (2015)...........
a1¡-(r∗ cy)/(4∗ cx)
M19¡-l∗ Yˆ 2∗ (cy2+((a1ˆ 2∗ cx2)/4)-
(a1∗ r∗ cy∗ cx))
B19¡-Y∗ ((((4∗ a1)-1-(7∗ f)+(4∗ fˆ
2))/8)∗ cx2-((((2∗ a1)-1)/(2∗ (1-f)))∗ r∗
cy∗ cx))
# Singh and Pal (2015)..............
M20¡-l∗ Yˆ 2∗ (cy2+((9∗ cx2)/4)-(3∗ r∗
cy∗ cx))
B20¡-(3/8)∗ l∗ Y∗ ((5∗ cx2)-(4∗ r∗ cy∗
cx))
# Jerajuddin and Kishun (2016).....
R21¡-X/(X+n)
BM12(R21)
# Vishwakarma et al.(2016)......
c1=1-(2∗ c)
M22¡-l∗ Yˆ 2∗ (cy2+((cx2∗ (1-c1)ˆ
```

```
2)/4)-(1-c1)* r* cy* cx)
B22¡-l* Y* cx2* (((3/8)-(c/2))-c1* ((3/8)-(c/2)))
# Kadilar(2016)..................
d1=(cx-(2* r* cy))/(2* cx)
B23¡-l* Y* (((((d1ˆ 2)/2)+(3/8))* cx2)-((d1+0.5)* r* cy* cx))
# Singh , Pal and Solanki(2016)....
c2=cˆ 2
do6=(cx2)/(cx2-r)
d6=(cx2+r)/(cx2-r)
do7=c2/(c2-cx)
d7=(c2+cx)/(c2-cx)
do8=(r2)/(r2-(2* f))
d8=(r2+(2* f))/(r2-(2* f))
do9=(cy)/(cy-cx)
d9=(cy+cx)/(cy-cx)
t24¡-function(d1,do1){
B¡-l* Y* cx2* (d1* (do1-c))
M¡-l* (Y* Y)* (cy2+(d1* cx2* (d1-(2* c))))
print(paste("Bias=",B))
print(paste("MSE=" ,M))
}
t24(d6,do6)
t24(d7,do7)
t24(d8,do8)
t24(d9,do9)
# Soponviwatkul and Lawson(2017)...
z1¡-c/R12
B25¡-l* Y* (((((z1+1)* z1)/2)* R12* cx2)-(z1* R12* r* cy* cx))
M25¡-l* Yˆ 2* (cy2+(z1ˆ 2* R12ˆ 2* cx2)-(2* z1* R12* r* cy* cx))
z2¡-c/R15
B26¡-l* Y* (((((z2+1)* z2)/2)* R15* cx2)-(z2* R15* r* cy* cx))
M26¡-l* Yˆ 2* (cy2+(z1ˆ 2* R15ˆ 2* cx2)-(2* z1* R15* r* cy* cx))
# PRE...............................................
PRE¡-function(mse){
 PRE¡-(M0/mse)* 100
 print(PRE)
}
```

Chapter 3

```
# Data Set 1.......................
# calculation......................
options(scipen = 100)
h=c(2,4,6,8,16)
f=n/N
l=(1-f)/n
w1=w2* ((h-1)/n)
cyx=ryx* cy* cx
cyz=ryz* cy* cz
cxz=rxz* cx* cz
cyx2=ryx2* cy2* cx2
cyz2=ryz2* cy2* cz2
cxz2=rxz2* cx2* cz2
vy2=(l* cyˆ 2)+(w1* cy2ˆ 2)
vx2=(l* cxˆ 2)+(w1* cx2ˆ 2)
vz2=(l* czˆ 2)+(w1* cz2ˆ 2)
vyx2=(l* cyx)+(w1* cyx2ˆ 2)
vyz2=(l* cyz)+(w1* cyz2ˆ 2)
vxz2=(l* cxz)+(w1* cxz2ˆ 2)
vx=l* cxˆ 2
vyx=l* cyx
# # Proposed estimator..............
d=1+vy2+((3/8)* vx2)+((3/8)* vz2)-vyx2-vyz2+(vxz2/4)
d1=1+vy2+vx2-(2* vyx2)
d2=1+((3/8)* vx2)-(vyx2/2)
d3=1+vy2+vz2-(2* vyz2)
d4=1+((3/8)* vz2)-(vyz2/2)
k=1-(d4ˆ 2/d3)
A=((d2* d3)-(d* d4))/d3
B=((d1* d3)-(dˆ 2))/d3
k1=A/B
k2=((k1* d)+d4)/d3
Bt=y* ((k1* d2)+(k2* d4)-1)
t=yˆ 2* (k-(Aˆ 2/B))
t
# Other Existing Estimators.........
# # Hansan and Horwitz..............
M=yˆ 2* vy2
# Ratio and product................
Mr=yˆ 2* (vy2+vx2-(2* vyx2))
Br=y* (vx2-vyx2)
```

```
Mp=yˆ 2∗ (vy2+vx2+(2∗ vyx2))
Bp=vyx2/(xˆ 2∗ y)
# Rao(1986).........................
M1=yˆ 2∗ (vy2+vx-(2∗ vyx))
B1=y∗ (vx-vyx)
M2=yˆ 2∗ (vy2+vx+(2∗ vyx))
B2=vyx/(xˆ 2∗ y)
# Singh and Kumar(2008).............
a1=(vyx2-vyx)/(vx2-vx)
a2=((vx2∗ vyx)-(vyx2∗ vx))/(vx∗ (vx2-
vx))
B3=y∗ (l∗ ((3∗ cxˆ 2)+cyx)-vyx2)
B4=y∗ (l∗ (cxˆ 2+cyx)+vyx2)
B5=y∗ ((l/2)∗ cxˆ 2∗ (a1+a2)∗
(a1+a2+1)+(w1/2)∗ cx2ˆ 2∗ a1∗
(a1+1)-(l∗ a2∗ cyx)-(a1∗ vyx2))
M3=yˆ 2∗ (vy2+vx2+(3∗ vx)-(2∗
(vyx2+vyx)))
M4=yˆ 2∗ (vy2+vx2+(3∗ vx)+(2∗
(vyx2+vyx)))
M5=yˆ 2∗ (vy2-(((vx∗ vyx2)+(vx2∗
vyxˆ 2)-(2∗ vyx∗ vyx2∗ vx))/(vx∗ (vx2-
vx))))
M6=yˆ 2∗ (vy2-((vyx2-vyx)ˆ 2/(vx2-
vx))-((vyxˆ 2)/(vx)))
# Singh et al. (2009)............
B7=(y/8)∗ ((3∗ vx2)-(4∗ vyx2))
B8=y∗ ((vyx2/2)-(vx2/8))
B9=l∗ y∗ (((3/8)∗ cxˆ 2)-((1/2)∗ cyx))
B10=l∗ y∗ (((1/2)∗ cyx)-((1/8)∗ cxˆ 2))
M7=yˆ 2∗ (vy2+((1/4)∗ vx2)-vyx2)
M8=yˆ 2∗ (vy2+((1/4)∗ vx2)+vyx2)
M9=yˆ 2∗ ((((1/4)∗ cxˆ 2-cyx)∗ l)+vy2)
M10=yˆ 2∗ ((((1/4)∗ cxˆ 2+cyx)∗
l)+vy2)
# Kumar and Bhougal (2011)..........
a=1/2∗ (1+(2∗ (vyx2/vx2)))
B11=(1/4)∗ y∗ (1+2∗ (1-(2∗ a)))∗ vyx2
M11=yˆ 2∗ (vy2-(vyx2ˆ 2/vx2))
# Chanu and Singh (2015)............
x1=((N∗ x)-(n∗ x))/(N-n)
g=n/(N-n)
a3=(g/(g-1))-((2/(g-1))∗ (vyx2/vx2))
kyx=ryx∗ (cy/cx)
a4=(g-(2∗ kyx))/(g-1)
B12=y∗ (((g/2)∗ vyx2)-((gˆ 2/8)∗ vx2)-
a3∗ ((((g-1)/2)∗ vyx2)+(((3+gˆ 2)/8)∗
vx2)))
M12=yˆ 2∗ (vy2-(vyx2ˆ 2/vx2))
B13=y∗ (((g/2)∗ vyx2)+((gˆ 2/8)∗ vx2)-
a4∗ ((((g-1)/2)∗ vyx2)+(((3+gˆ 2)/8)∗
vx2)))
M13=yˆ 2∗ (vy2-(l∗ ryxˆ 2∗ cyˆ 2))
# Pal and Singh (2016).............
R=vyx2/vx2
a5=-1
a6=2∗ (R-1)
a7=(a6+1)/2
B14=(y∗ (a5+1)/8)∗ vx2∗ (a5+1-(4∗
R)+2)
M14=yˆ 2∗ (vy2-(vyx2ˆ 2/vx2))
# Zubair et al. (2018).............
a8=(2∗ (vz2+vyx2-vyz2-vxz2))/(vz2+(4∗
vx2)-(2∗ vxz2))
a9=vy2+(vy2/4)-vyz2
A2=(vz2/4)+(vyx2/2)-(vyz2/2)-(vxz2/2)
B2=(vz2/4)+(vx2/4)-(vxz2/2)
B15=y∗ ((3/8)∗ (a8∗ vx2)+(1-a8)∗ vz2-
((1/2)∗ ((a8∗ vyx2+((1-a8)∗ vyz2)))))
M15=yˆ 2∗ (a9-(A2ˆ 2/B2))

# PRE...............................
PRE¡-function(m){
pre=(M/m)∗ 100
print(pre)
}
PRE(t)
```

Chapter 4

```
# Secondry data(From Daroga and
Singh)
# Calculation of parameters
Y¡-c(50,149,284,381,278,111,
634,278,112,355,99,498,111,6,339,80,
105,27,515,249,85,221,133,144,103,
175,335,219,62,79,60,100,141,263)
X¡-c(70,163,320,440,250,125,558,
254,101,359,109,481,125,5,427,78,75,
45,564,238,92,247,134,131,129,190,
```

```
363,235,73,62,71,137,196,255)
# mean
y¡-mean(Y)
x¡-mean(X)
# standard deviation
sy¡-sqrt(var(Y))
sx¡-sqrt(var(X))
# coeffcient of variation
cy¡-sy/y
cx¡-sx/x
# median
mx¡-median(X)
my¡-median(Y)
# correlation coefficient
r¡-cor(X,Y)
cyx¡-r* cy* cx
library(moments)
# coefficient of skewness b1(beta 1)
be1¡-skewness(X)ˆ 2
# coefficient of kurtosis b2( beta 2)
be2¡-kurtosis(X)+3
N=34
n=5
f=n/N
l=(1-f)/n
Ncn=choose(N,n)
summary(X)
# quartile deviation
qd=(q3-q1)/2
# interquartile range
qr=q3-q1
# quartile average
qa=(q1+mx+q3)/3
# tri mean
tm¡-(q1+2* mx+q3)/4
# # Review of literature
# Sample mean
m0¡-l* yˆ 2* cyˆ 2
# Cochran (1940)
m1¡-l* yˆ 2* (cyˆ 2+cxˆ 2-(2* cyx))
# Bahl and Tuteja(1991)
m2¡-l* yˆ 2* (cyˆ 2+(cxˆ 2/4)-cyx)
# Function of MSE calculation
m¡-function(c){
mse=l* yˆ 2* (cyˆ 2+(cˆ 2* cxˆ 2)-(2* c* cyx))
print(mse)
}
# Sisodia and Dwivedi(1981)
c3¡-x/(x+cx)
m(c3)
# # [1] 154.5255
# Upadhyaya and Singh (1999)........
c4¡-(x* cx)/((x* cx)+be2)
c5¡-(x* be2)/((x* be2)+cx)
m(c4)
m(c5)
# Singh and Tailor(2003)......
c6¡-x/(x+r)
m(c6)
# singh et al.(2004)........
c7¡-x/(x+be2)
m(c7)
# Al-Omari et al. (2009).
c8¡-x/(x+q1)
m(c8)
# # Al-Omari et al. (2009)
c9¡-x/(x+q3)
m(c9)
# Yan and Tian(2010)
c10¡-x/(x+be1)
c11¡-(x* be1)/((x* be1)+be2)
c12¡-(x* cx)/((x* cx)+be1)
m(c10)
m(c11)
m(c12)
# Subramani and Kumarpandiyan(2012a)
c13¡-x/(x+mx)
c14¡-(x* cx)/((x* cx)+mx)
m(c13)
m(c14)
# Subramani and Kumarpandiyan(2012b)
c15¡-x/(x+qr)
c16¡-x/(x+qd)
c17¡-x/(x+qa)
m(c15)
m(c16)
m(c17)
# Jeelani et al.(2013).............
c18¡-(x* be1)/((x* be1)+qd)
m(c18)
# Jerajuddin and Kishun(2016)....
c19¡-x/(x+n)
m(c19)
# Yadav et al. (2019).....
c20_1¡-(be2* mx* x)/((be2* mx* x)+(r))
```

```
c20_2¡-(be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗ cx))
c20_3¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r))
c20_4¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r∗ cx))
c20_5¡-(n∗ x)/((n∗ x)+(r))
c20_6¡-(n∗ x)/((n∗ x)+(cx))
c20_7¡-(n∗ x)/((n∗ x)+(cx∗ r))
c20_8¡-(n∗ r∗ x)/((n∗ r∗ x)+(cx))
c20_9¡-(n∗ cx∗ x)/((n∗ cx∗ x)+(r))
m(c20_1)
m(c20_2)
m(c20_3)
m(c20_4)
m(c20_5)
m(c20_6)
m(c20_7)
m(c20_8)
m(c20_9)
# Proposed Estimators.............
mse¡-function(th){
A=1-(th∗ l∗ r∗ cy∗ cx)+(thˆ 2∗ l∗ cxˆ 2)
B=1+(l∗ cyˆ 2)+(thˆ 2∗ l∗ cxˆ 2)-(4∗ th∗ l∗ r∗ cy∗ cx)+(2∗ thˆ 2∗ l∗ cxˆ 2)
M=yˆ 2∗ (1-(Aˆ 2/B))
print(M)
}
th1=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r)))
mse(th1)
th2=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗ cx)))
mse(th2)
th3=(Ncn∗ x)/((Ncn∗ x)+(mx))
mse(th3)
th4=(mx∗ x)/((mx∗ x)+(f∗ r))
mse(th4)
th5=(tm∗ q3∗ x)/((tm∗ q3∗ x)+(qa/qd))
mse(th5)
th6=(qd∗ qa∗ x)/((qd∗ qa∗ x)+(be2∗ be1))
mse(th6)
th7=(mx∗ sx∗ x)/((mx∗ sx∗ x)+(r∗ l))
mse(th7)
PRE¡-function(me){
PRE¡-(m0/me)∗ 100
print(PRE)
}
```

Chapter 5

```
# Secondry data(From Daroga and Singh)
# Calculation of parameters
Y¡-c(50,149,284,381,278,111,
634,278,112,355,99,498,111,6,339,80,
105,27,515,249,85,221,133,144,103,
175,335,219,62,79,60,100,141,263)
X¡-c(70,163,320,440,250,125,558,
254,101,359,109,481,125,5,427,78,75,
45,564,238,92,247,134,131,129,190,
363,235,73,62,71,137,196,255)
# mean
y¡-mean(Y)
x¡-mean(X)
# standard deviation
sy¡-sqrt(var(Y))
sx¡-sqrt(var(X))
# coeffcient of variation
cy¡-sy/y
cx¡-sx/x
# median
mx¡-median(X)
my¡-median(Y)
# correlation coefficient
r¡-cor(X,Y)
cyx¡-r∗ cy∗ cx
library(moments)
# coefficient of skewness b1(beta 1)
be1¡-skewness(X)ˆ 2
# coefficient of kurtosis b2( beta 2)
be2¡-kurtosis(X)+3
N=34
n=5
f=n/N
l=(1-f)/n
Ncn=choose(N,n)
summary(X)
# quartile deviation
qd=(q3-q1)/2
# interquartile range
```

```
qr=q3-q1
# quartile average
qa=(q1+mx+q3)/3
# tri mean
tm¡-(q1+2∗ mx+q3)/4
# # Review of literature
# Sample mean
m0¡-l∗ yˆ 2∗ cyˆ 2
# Cochran (1940)
m1¡-l∗ yˆ 2∗ (cyˆ 2+cxˆ 2-(2∗ cyx))
# Bahl and Tuteja(1991)
m2¡-l∗ yˆ 2∗ (cyˆ 2+(cxˆ 2/4)-cyx)
# Function of MSE calculation
m¡-function(c){
mse=l∗ yˆ 2∗ (cyˆ 2+(cˆ 2∗ cxˆ 2)-(2∗
c∗ cyx))
print(mse)
}
# Sisodia and Dwivedi(1981)
c3¡-x/(x+cx)
m(c3)
# # [1] 154.5255
# Upadhyaya and Singh (1999)........
c4¡-(x∗ cx)/((x∗ cx)+be2)
c5¡-(x∗ be2)/((x∗ be2)+cx)
m(c4)
m(c5)
# Singh and Tailor(2003)......
c6¡-x/(x+r)
m(c6)
# singh et al.(2004)........
c7¡-x/(x+be2)
m(c7)
# Al-Omari et al. (2009).
c8¡-x/(x+q1)
m(c8)
# # Al-Omari et al. (2009)
c9¡-x/(x+q3)
m(c9)
# Yan and Tian(2010)
c10¡-x/(x+be1)
c11¡-(x∗ be1)/((x∗ be1)+be2)
c12¡-(x∗ cx)/((x∗ cx)+be1)
m(c10)
m(c11)
m(c12)
# Subramani and Kumarpandiyan(2012a)
c13¡-x/(x+mx)
c14¡-(x∗ cx)/((x∗ cx)+mx)
m(c13)
m(c14)
# Subramani and Kumarpandiyan(2012b)
c15¡-x/(x+qr)
c16¡-x/(x+qd)
c17¡-x/(x+qa)
m(c15)
m(c16)
m(c17)
# Jeelani et al.(2013).............
c18¡-(x∗ be1)/((x∗ be1)+qd)
m(c18)
# Jerajuddin and Kishun(2016)....
c19¡-x/(x+n)
m(c19)
# Yadav et al. (2019).....
c20_1¡-(be2∗ mx∗ x)/((be2∗ mx∗ x)+(r))
c20_2¡-(be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗
cx))
c20_3¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r))
c20_4¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r∗
cx))
c20_5¡-(n∗ x)/((n∗ x)+(r))
c20_6¡-(n∗ x)/((n∗ x)+(cx))
c20_7¡-(n∗ x)/((n∗ x)+(cx∗ r))
c20_8¡-(n∗ r∗ x)/((n∗ r∗ x)+(cx))
c20_9¡-(n∗ cx∗ x)/((n∗ cx∗ x)+(r))
m(c20_1)
m(c20_2)
m(c20_3)
m(c20_4)
m(c20_5)
m(c20_6)
m(c20_7)
m(c20_8)
m(c20_9)
# proposed estimator...............
th1=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r)))
m(th1)
th2=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗
cx)))
m(th2)
th3=(Ncn∗ x)/((Ncn∗ x)+(mx))
m(th3)
th4=(mx∗ x)/((mx∗ x)+(f∗ r))
m(th4)
th5=(tm∗ q3∗ x)/((tm∗ q3∗ x)+(qa/qd))
```

```
m(th5)
th6=(qd* qa* x)/((qd* qa* x)+(be2*
be1))
m(th6)
th7=(mx* sx* x)/((mx* sx* x)+(r* l))
m(th7)
l* y^ 2* (cy^ 2-(cyx^ 2/cx^ 2))
PRE¡-function(me){
PRE¡-(m0/me)* 100
print(PRE)
}
```