

NUMERICAL EXPERIMENTS FOR ANALYSIS OF VIBRATIONS OF TWO DIMENSIONAL PLATES

THESIS

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UTTAR PRADESH, INDIA

2021



Dedicated To

My

Lovely

Parents

DECLARATION

I declare that the work embodied in this Ph.D. thesis entitled as "NUMERICAL EXPERIMENTS FOR ANALYSIS OF VIBRATIONS OF TWO DIMENSIONAL PLATES" is carried out by me under the supervision of Prof. Vipin Saxena, Department of Computer Science, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow, India. The information presented in this Ph.D. thesis has not been submitted for the award of any other degree or diploma. I declare that I have faithfully acknowledged, given credit to and referred to the research workers whenever their works have been cited in the text and the body of the thesis. I further declare that I have not willful lifted up some others work, para, text, data, results, etc. reported in the journals, books, magazines, reports, dissertations, thesis, etc., or available at websites and included them in this thesis and cited my own work.

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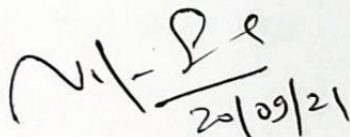
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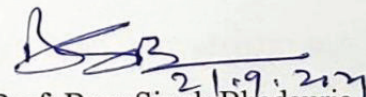
CERTIFICATE

This is to certify that the thesis titled "NUMERICAL EXPERIMENTS FOR ANALYSIS OF VIBRATIONS OF TWO DIMENSIONAL PLATES" submitted by **Ms. Neetu Singh** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other University

The thesis submitted to Babasaheb Bhimrao Ambedkar University, Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations-1999 as amended in 2008/2010/2013* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

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Neetu Singh
Neetu Singh

TABLE OF CONTENTS

| TITLE | PAGE NO. |
|--|--------------|
| DECLARATION | I |
| CERTIFICATE | II |
| ACKNOWLEDGEMENTS | III |
| LIST OF VARIABLES | VII |
| LIST OF FIGURES | IX |
| LIST OF TABLES | X |
| LIST OF PUBLICATIONS | XI |
| ABSTRACT | XVI |
| CHAPTER I INTRODUCTION | 1-25 |
| 1.1 CAUSES OF VIBRATION | 1 |
| 1.2 TYPES OF VIBRATION | 2 |
| 1.3 REQUIREMENTS FOR VIBRATION | 2 |
| 1.4 CLASSICAL PLATE THEORY | 3 |
| 1.5 BOUNDARY CONDITIONS | 3 |
| 1.6 TYPE OF PLATES | 5 |
| 1.7 TRANSVERSE VIBRATION | 8 |
| 1.8 RAYLEIGH METHOD | 14 |
| 1.9 RITZ METHOD | 16 |
| 1.10 RAYLEIGH-RITZ METHOD | 18 |
| 1.11 COMPUTATIONAL TOOLS | 23 |
| 1.12 CONVERGENCE OF RESULTS | 25 |
| CHAPTER II REVIEW OF LITERATURE | 26-54 |
| CHAPTER III NUMERICAL EXPERIMENTS ON QUARTER OF ELLIPTIC PLATE WITH EXPONENT THICKNESS VARIATION ALONG RADIAL DIRECTION | 55-74 |
| 3.1 BACKGROUND | 55 |
| 3.2 MATHEMATICAL FORMULATION | 60 |
| 3.3 NUMERICAL RESULTS AND DISCUSSION | 64 |
| 3.4 MAJOR FINDINGS | 73 |

| | |
|--|----------------|
| CHAPTER IV NUMERICAL EXPERIMENTS ON RECTANGULAR AND SQUARE PLATES WITH TRANSCENDENTAL THICKNESS VARIATION | 75-111 |
| 4.1 BACKGROUND | 75 |
| 4.2 MATHEMATICAL FORMULATION | 93 |
| 4.3 NUMERICAL RESULTS AND DISCUSSION | 95 |
| 4.4 MAJOR FINDINGS | 111 |
| CHAPTER V NUMERICAL EXPERIMENTS ON CIRCULAR PLATE WITH TRANSCENDENTAL THICKNESS VARIATION | 112-120 |
| 5.1 BACKGROUND | 112 |
| 5.2 MATHEMATICAL FORMULATION | 113 |
| 5.3 NUMERICAL RESULTS AND DISCUSSION | 117 |
| 5.4 MAJOR FINDINGS | 120 |
| CHAPTER VI NUMERICAL EXPERIMENTS ON SKEW PLATE WITH LINEAR AND EXPONENTIAL THICKNESS VARIATIONS | 121-135 |
| 6.1 BACKGROUND | 121 |
| 6.2 MATHEMATICAL FORMULATION | 122 |
| 6.3 NUMERICAL RESULTS AND DISCUSSION | 125 |
| 6.4 MAJOR FINDINGS | 135 |
| CHAPTER VII CONCLUDING REMARKS AND FUTURE SCOPE OF WORK | 136-138 |
| REFERENCES | 139-174 |
| APPENDIX REPRINTS OF PUBLISHED RESEARCH PAPERS | |

LIST OF VARIABLES

$W \rightarrow$ Displacement

$M \rightarrow$ Bending moment

$Q \rightarrow$ Shearing force

$R \rightarrow$ Domain of the plate

$R' \rightarrow$ New domain of the plate

$r \rightarrow$ Radius of circular plate

$\rho \rightarrow$ Density of moment

$E \rightarrow$ Young's modulus

$x, y, z \rightarrow$ Coordinate of axis

Q_x and $Q_y \rightarrow$ Shearing force of plate theory or moment resultant

M_x and $M_y \rightarrow$ Moment intensities along x and y axis

$M_{xy} \rightarrow$ Twisting moment intensity form x to y direction

$M_{yx} \rightarrow$ Twisting moment intensity form y to x direction

N_x and $N_y \rightarrow$ Normal force along x and y axis

TT \rightarrow Tangent

N \rightarrow Normal

$h \rightarrow$ Thickness of plate

$\nu \rightarrow$ Poisson's ratio

$C_j \rightarrow$ Arbitrary constant

$\phi_j(x, y), \phi_i(x, y) \rightarrow$ Basis functions

U \rightarrow Potential energy

T \rightarrow Kinetic energy

F \rightarrow Completely-free boundary condition

$S \rightarrow$ Simply-support boundary condition
 $C \rightarrow$ Clamped boundary condition
 $W(x, y) \rightarrow$ Maximum displacement at the time t
 $\omega \rightarrow$ Angular frequency
 $D \rightarrow$ Flexural rigidity
 $n \rightarrow$ Order of approximation
 $h_0 \rightarrow$ Non-dimensional thickness at the origin of plate
 $a, b \rightarrow$ Semi major and semi minor axis
 $X, Y \rightarrow$ Axis of plate
 $\alpha, \beta \rightarrow$ Taper parameters
 $\lambda \rightarrow$ Frequency parameter
 $\langle x_n \rangle \rightarrow$ Sequence of real number
 $m, \mu \rightarrow$ Aspect ratio
 $\theta \rightarrow$ Angle
 $G \rightarrow$ Shear modulus
 $\epsilon_x, \epsilon_y, \epsilon_z, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{zx} \rightarrow$ Strain components
 $\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx} \rightarrow$ Stress components
 $q(x, y) \rightarrow$ External normal force per unit length
 $\xi, \eta \rightarrow$ Non-dimensional variables
 P, q, r and $s \rightarrow$ controlling parameters
 $\gamma \rightarrow$ Density of material of plate

LIST OF FIGURES

| Figure No. | Name of Figure | Pg. No. |
|------------|---|---------|
| Figure 1.1 | Representation of the Boundary Condition | 3 |
| Figure 1.2 | Geometrical Representation of Circular Plate | 6 |
| Figure 1.3 | Geometrical Representation of Rectangular Plate | 6 |
| Figure 1.4 | Geometrical Representation of Elliptic Plate | 7 |
| Figure 1.5 | Geometrical Representation of Skew Plate | 8 |
| Figure 1.6 | Representation of Stresses alongwith Plate Element | 10 |
| Figure 3.1 | Representation of Quarter of Elliptic Plate | 56 |
| Figure 3.2 | Exponential Thickness Variation for Quarter of Elliptic Plate | 63 |
| Figure 3.3 | Representation of First Three Frequencies for CCS, FCC and CCC Plates | 71 |
| Figure 3.4 | Representation of First Three Frequencies for SSS, SSC and FSC Plates | 72 |
| Figure 3.5 | Representation of First Three Frequencies for CSS, CSC and CFC Plates | 72 |
| Figure 4.1 | Representation of Thickness Variation for Rectangular and Square Plates | 95 |
| Figure 4.2 | First Three Mode Shapes for Rectangular Clamped Plate ($\alpha=0.5, \nu=0.3$) | 108 |
| Figure 4.3 | First Three Mode Shapes for Square Clamped Plate ($\alpha=0.5, \nu=0.3$) | 109 |
| Figure 5.1 | Representation of Clamped Circular Plate | 113 |
| Figure 5.2 | Representation of Simply-Supported Circular Plate | 113 |
| Figure 5.3 | Representation of First Three Frequencies for Simply-Supported Circular Plate | 118 |
| Figure 5.4 | Representation of First Three Frequencies for Clamped Circular Plate | 119 |
| Figure 6.1 | Representation of First Three Frequencies ($\mu=1.0$) for Exponential Thickness Variation | 133 |
| Figure 6.2 | Representation of First Three Frequencies ($\mu=1.0$) for Linear Thickness Variation | 133 |
| Figure 6.3 | Representation of First Three Frequencies ($\mu=2.0$) for Exponential Thickness Variation | 134 |
| Figure 6.4 | Representation of First Three Frequencies ($\mu=2.0$) for Linear Thickness Variation | 134 |

LIST OF TABLES

| Table No. | Name of Table | Pg. No. |
|-----------|---|---------|
| Table 1.1 | Convergence of Results | 25 |
| Table 3.1 | Representation of Boundary Conditions on Quarter of Elliptic Plate | 56 |
| Table 3.2 | Non Negative Values of m_i and n_i | 61 |
| Table 3.3 | First Three Frequencies for Quarter of Elliptic Plate ($m=0.25, \nu=0.3$) | 65 |
| Table 3.4 | First Three Frequencies for Quarter of Elliptic Plate ($m=0.5, \nu=0.3$) | 67 |
| Table 3.5 | First Three Frequencies for Quarter of Elliptic Plate ($m=0.75, \nu=0.3$) | 69 |
| Table 3.6 | Comparison of Results for Quarter of Elliptic Plate with CCC ($\alpha=0, m=1$) | 73 |
| Table 3.7 | Convergence of Results for Quarter of Elliptic Plate with CCC ($\alpha=0$) | 73 |
| Table 4.1 | Representation of Boundary Conditions for Rectangular Plate | 76 |
| Table 4.2 | Representation of Boundary Conditions for Square Plate | 84 |
| Table 4.3 | First Three Frequencies for the Rectangular Plate ($\mu=2.0$) | 96 |
| Table 4.4 | First Three Frequencies for the Square Plate ($\mu=1.0$) | 102 |
| Table 4.5 | Convergence of the Results for Rectangular and Square Plates | 110 |
| Table 4.6 | Comparisons of the Results for Rectangular and Square Plates ($\alpha=0.0$) | 111 |
| Table 5.1 | First Three Frequencies for Simply-Supported and Clamped Circular Plate with Transcendental Thickness Variation | 118 |
| Table 5.2 | Convergence of the Results for Circular Plate ($\alpha=0, s=1, 2$) | 119 |
| Table 5.3 | Comparisons of the Results for Uniform Circular Plate | 120 |
| Table 6.1 | Representation of Boundary Conditions with Different Cases | 122 |
| Table 6.2 | First Three Frequencies ($\mu=1.0$) with Exponential Thickness Variation | 126 |
| Table 6.3 | First Three Frequencies ($\mu=1.0$) with Linear Thickness Variation | 127 |
| Table 6.4 | First Three Frequencies ($\mu=2.0$) with Exponential Thickness Variation | 128 |
| Table 6.5 | First Three Frequencies ($\mu=2.0$) with Linear Thickness Variation | 130 |
| Table 6.6 | Convergence of the Results ($\alpha=0.5, \theta=60^\circ, \mu=2.0$) | 131 |
| Table 6.7 | Comparison of Results for Skew Plate ($\alpha=0.0$) | 132 |

LIST OF PUBLICATIONS

A. JOURNALS

1. **Neetu Singh and Vipin Saxena**, “Numerical Experiment on Quarter of an Elliptic Plate with Exponential Thickness Variation”, **International Journal of Mathematics Trends and Technology (IJMTT)**, ISSN (2231-5373), Vol. 50(5), 279-286, Oct 2017.
2. **Neetu Singh and Vipin Saxena**, “Numerical Computing of Frequencies for Rectangular and Square Plates with Transcendental Thickness Variation”, **International Journal of Recent Technology and Engineering (IJRTE, Scopus)**, ISSN (2277-3878), Vol. 8(2), 4017-4030, July 2019.
3. **Neetu Singh and Vipin Saxena**, “Numerical Computation of First Three Frequencies for Circular Plate with Transcendental Thickness Variation”, **International Journal of Recent Technology and Engineering (IJRTE)**, ISSN (2277-3878), Vol. 8(6), 4017-4030, March 2020.
4. **Neetu Singh and Vipin Saxena**, “Numerical Experiments on Skew Plate with Linear and Exponential Thickness Variations”, **Strad Research (Web of Science Group)**, ISSN (0039-2049), Vol. 8(5), 13-27, April 2021.

B. CONFERENCES

B.1 INTERNATIONAL CONFERENCES

5. “Transverse Vibration of a Quarter of a Circular with Exponential Thickness”, Presented in **International Conference on Recent Advances in PDE: Theory Computation and Applications**, held at **Indian Institute of Technology, Bombay**, during **June 8-10th, 2017**.
6. “Numerical Computation of First Few Frequencies for Rectangular Plate with Transcendental Thickness Variation”, Presented in **International Conference**

on **Mathematical Modelling, Applied Analysis and Computation (ICMMAAC)**, held at **Department of Mathematics and Statistics, School of Science, JECRC University, Jaipur, Rajasthan, during July 6-8th, 2018.**

B.2 NATIONAL CONFERENCES

7. “Variational Method for Solution of Vibration of Plate”, Presented in 3rd Lucknow Science Congress and National Conference on “Science for Society: An Interdisciplinary Approach, held at Babasaheb Bhimrao Ambedkar University, Lucknow, during Oct 31-2 Nov, 2015.
8. “A Study of Vibrations of Plates”, Presented in National Conference on Mathematical Techniques in Engineering and Technology (MTET), held at Department of Applied Mathematics, School of Physical Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow, during March 30-31, 2016.
9. “Vibration Problem on Plates”, Presented in 4th Lucknow Science Congress LUSCON-2017, Science Technology and Innovations for Sustainable Development held at Babasaheb Bhimrao Ambedkar University, Lucknow, during March 3-4, 2017.

C. LIST OF WORKSHOP AND SEMINAR ATTENDED

1. Two days National Seminar on “Development of Mathematical Principle with their Advancement in Computer Sciences” Organized by Department of Computer Application of the College Isabelia Thoburn College, Lucknow, 17-18 Nov, 2015.
2. Two Week Workshop on “Modeling Week and Study Group Meeting on Industrial Problems (MWSGMIP-2010)” Organized by National Program on Differential Equation: Theory Computation & Applications (NPDE-TCA) (Sponsored by Department of Science of Technology Government of India),

Institute of Technology & Management (ITM) Universe, Vadodara, 10-19 March, 2016.

3. Two days XXXII IATLIS National Conference on LIS Research: Issues and Challenges, held at Babasaheb Bhimrao Ambedkar University, Lucknow, during Dec 19-20, 2016.
4. One Week Workshop on “Emerging Research Trends in Computer Science, Organized by Department of Computer Science, School for Information Science & Technology, Babasaheb Bhimrao Ambedkar University, Lucknow, 20-24 March, 2017.
5. One Week Workshop and Faculty Development Programme on “Research Methodology” Organized by Department of Computer Science & Engineering, Uttaranchal University, Dehradun, 4-9 Dec, 2017.
6. One Week Short Term Course on “Software Tools for Scientific Research: Mathematica and LATEX” Organized by Department of Mathematics and Mechanical Engineering (Sponsored by TEQIP-Phase III), MNIT Jaipur, 09-13 Jan, 2018.
7. Two Weeks Faculty Development Program on “Ensuring Excellence in Teaching/ Learning/ Research in Higher Educational Institutions Using ICT (ID)” Organized by Department of Information Technology, Babasaheb Bhimrao Ambedkar University, Lucknow, 22 Dec 2018-06 Jan, 2019.
8. One Week National Workshop on “Research Methodology” Organized by Faculty of Education, IGNTU, Amarkantak (MP), 26-30 May, 2020.
9. Online Quiz Competition on “National Level Quiz on Engineering Mathematics” Organized by Department of Mathematics, Mandsaur University, Mandsaur (WSSCWW-CE000170), 2020.

-
10. Online Quiz Competition on “Engineering Mathematics” Organized by Department of Humanities & Mathematics, G. Narayanamma Institute of Technology & Science Shaikpet, Hyderabad, June 2020 (OAASMG-CE00 2153).
 11. Six Days Faculty Development Programme on “Application of Mathematics in Diverse Fields” Organized by Department of Mathematics, Anand Institute of Higher Technology, Kazhipattur, Chennai, 26 June-04 July, 2020.
 12. One Week Online Faculty Development Programme (FDP) on “Soft Computing Techniques and their Applications (SCTA 2020)” Organized by Department of Mathematics of Jaypee Institute of Information Technology, Noida, 13-18 July, 2020.
 13. Three Days Online Short Term Training Programme (STTP) on “MATLAB and MATHEMATICA for Scientific Research” Organized by the Department of Mathematics, Arul Anandar College, Karumathur, Madurai, 27-29 July, 2020.
 14. One Week Online FDP on “Mathematical Modelling & Numerical Technique 2020” Organized by the Department of Mathematics and Humanities, Kaktiya Institute of Technology and Science, Telangana, 27-31 July, 2020.
 15. National Level Faculty Development Programme (FDP) on “MATLAB TOOLS & APPLICATIONS” Organized by the Department of Mathematics, Babu Banarasi Das Northern India Institute of Technology (AKTU College code: 056), Lucknow, 07-08, 10 August, 2020.
 16. One Week Faculty Development Programme on “RESEARCH METHODOLOGY AND TECHNOLOGY LED PARADIGM SHIFT IN TEACHING AND LEARNING PROCESS” Organized by the Internal Quality

Assurance Cell, Research and Consultancy Cell, P.G.DA.V. College (Evening)
in Collaboration with Buniyaad Education Society, Delhi, 4-12 August, 2020.

17. National Level Webinar on “LaTex and its Applications” Organized by the
Department of Computer Science, BCA & IQAC, SBRR Mahajana First Grade
College (Autonomous), Mysuru, Karnataka, 19-08-2020.

ABSTRACT

The present research work is the study on “**NUMERICAL EXPERIMENTS FOR ANALYSIS OF VIBRATIONS OF TWO DIMENSIONAL PLATES**” through Rayleigh-Ritz technique, computer based programming and MATLAB simulation tool. The shapes of plate may be circular, rectangular, square, quarter elliptic with different types of boundary conditions either clamped, simply-supported or completely-free and thickness variations may be linear, exponential and transcendental. The Classical plate theory is induced for all equations for the formulation of the problems. Rayleigh-Ritz technique is an approximate technique and operated by advisable continuous basis function, which considers all types of boundary conditions for a given problem. The presented technique is capable to generate the successive sequence of approximations to establish excellent convergence of results. The whole strategy is implemented through computer programming language like FORTRAN and MATLAB version simulation tool is used for designing the mode shapes, etc. First three natural frequencies have been computed and presented through tables and graphs through MS-Excel, generalized Jacobi method is used for computation of eigenvalues and eigenvectors. In special cases, the computed results are in excellent match with the results available in the literature. The chapter-wise summary is briefly discussed below:

In the Chapter I, causes of vibration, type of vibration, requirements for vibration, Classical plate theory, boundary conditions, types of plate like circular plate, rectangular plate, elliptic plate, skew plate, transverse vibration, Rayleigh method, Ritz method and Rayleigh-Ritz method are briefly discussed. This is an introductory chapter giving information all about the fundamentals used in the present research

work alongwith the concepts of the programming language FORTRAN and MATLAB simulation tool and convergence of results.

Chapter II deals with the information about the important and useful literature available on the beams and plates. In this section, all existing literatures are explained for different types of plates, beams, shells, Classical Plate theory, boundary conditions and different type of methodologies which are used to obtain natural frequencies, mode shapes with continuous basis function, aspect ratio and varying thickness.

In the Chapter III, the actual research work is presented for a quarter of an elliptic plate with exponential thickness by the use of Rayleigh-Ritz method which is very efficient to compute first three natural frequencies with some selected boundary conditions fixed at three edges and taper parameter. Convergence results for quarter of elliptic plate with exponential boundary conditions are computed upto five significant digits and in special cases, the computed results are in good agreement with the existing results in the literature.

Chapter IV deals with the study on the numerical computing of frequencies for rectangular and square plates with transcendental thickness variation. First time, transcendental thickness variation is used on aforesaid plates for getting the natural frequencies in different cases of boundary conditions although it satisfies the boundary conditions of the plate. Further, a well known Rayleigh-Ritz technique is used for computing the natural frequencies and mode shapes which are representing the behaviour of the plate. Computed frequencies are in increasing form due to increase in the taper parameter and results are correct upto five significant digits. In special cases, the computed results are in good match with the existing results available in the literature.

In the Chapter V, again a well known Rayleigh-Ritz method is used for clamped, simply-supported and completely-free for circular plate with transcendental thickness variation. Computed numerical results are in increasing form because taper parameter is increasing and stiffness of the plate is increasing. Results of circular plate are obtained correct upto five significant digits. Convergent table in the case of completely clamped and simply-supported of circular plate is presented and results are compared in special cases with the existing results available in the literature.

Chapter VI deals with the computation of first three frequencies for skew plate with linear and exponential thickness variations. First time exponential thickness variation is used on aforesaid plate for getting the natural frequencies in different cases of boundary conditions although it satisfies the boundary conditions of the plate. Further, a well known Rayleigh-Ritz technique is used for computing the natural frequencies of the plate. Computed frequencies are in increasing form due to increase in the taper parameter and results are correct upto five significant digits. In special cases, the computed results are in good match with the existing results available in the literature.

The last chapter of thesis is related to the conclusions and future scope of the work. In the present work, quarter of an elliptic, circular, rectangular, square, and skew plates have been studied on the linear, transcendental and exponential thickness variations by the use of Rayleigh-Ritz method. The same plates may be studied by the use of Finite Element method and other Polynomial methods. Further, other kinds of plates may be studied for future extension of the present research work.



CHAPTER-I

Introduction

CHAPTER-I

INTRODUCTION

A study of vibrations of thin plates are defined as vibrations in which a solid body is bounded by two parallel lines with flat surfaces and considered as a two dimensional geometry. The study of plates play an important role in the design of chips used in the trains, aircraft, naval structure and at many more places and plates are of the various kinds like skew, rhombus, rectangular, square, triangle, elliptic, circular, etc. A study of vibration of plates is a branch of engineering application and for the solution of the drafted research problems, many numerical methods have been used by the researchers for obtaining the first few frequencies alongwith the mode shapes which are representing the behaviour of the plates at the time of vibrations.

A lot of research work and review papers have already been available in the literature. Many of the papers compute the first few frequencies with different boundary conditions. The problems of vibration of plates have handled by considering the different types of methods like Rayleigh-Ritz, Finite Element, Galerkin methods, etc. with continuous and discontinuous functions. All methods have been developed in analytical in nature with computations of frequencies and mode shapes. Rayleigh-Ritz method is one of the most important methods for obtaining the frequencies and mode shapes for selected plates with different boundary conditions. This method has a fast and efficient convergence of results.

1.1 CAUSES OF VIBRATION

Basic concepts of vibration have mechanical phenomenon of oscillation, an equilibrium position of body, particle and member. Some causes of vibration are listed below:

- ☛ Movement of solid body and vehicles;
- ☛ Change of electric and magnetic;
- ☛ Unequal distribution of force in a rotating body;
- ☛ Friction between two bodies;
- ☛ External energy or forces like earthquakes, blast, wind;
- ☛ Damped vibration;

1.2 TYPES OF VIBRATION

The followings are the different types of vibrations studied by the various researchers across the world:

- ☛ Free Vibration;
- ☛ Damped Vibration;
- ☛ Forced Vibration;
- ☛ Un-damped Vibration;
- ☛ Linear Vibration;
- ☛ Non-Linear Vibration;
- ☛ Deterministic Vibration;
- ☛ Non-Linear Vibration;
- ☛ Transverse Vibration;
- ☛ Longitudinal Vibration;
- ☛ Torsional Vibration;

1.3 REQUIREMENTS FOR VIBRATION

The essential requirements for the vibration of a body or part of body are listed below:

- ☛ There should be in a moving force;
- ☛ The body should have mass;
- ☛ The body should be in equilibrium;

1.4 CLASSICAL PLATE THEORY

In the present work, different shapes are considered only for thin plates based on the assumption of classical plate theory and given by

- ☛ Thickness of the plate is very small as compared with other dimensions.
- ☛ Normal stresses in the direction transverse to the plate are taken to be negligibly small.
- ☛ The middle surface of the plate remains unstretched using deformation.
- ☛ Normal to the undeformed middle surface remains straight and normal to the deformed middle surface.

1.5 BOUNDARY CONDITIONS

The boundary conditions of the plate as represented in figure 1.1 are defined on the basis of the following three categories where ∂R is domain of plate, TT is tangent and N is normal. The different categories of boundary conditions are listed below:

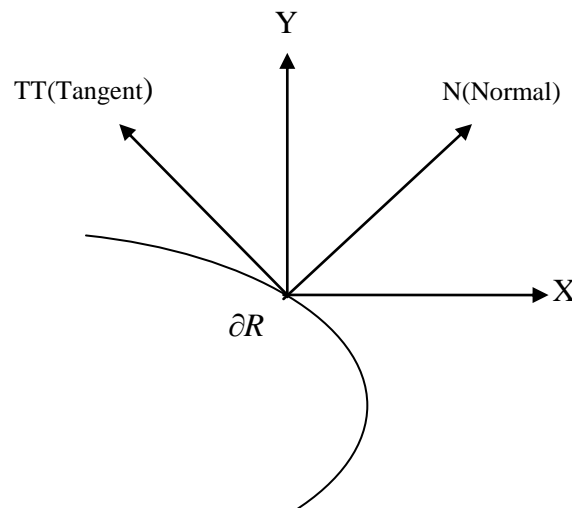


Figure 1.1 Representation of the Boundary Condition

- ☛ **Essential or Geometric Boundary Condition**

This boundary condition is called as Dirichlet boundary condition.

☛ Natural or Force (or dynamic) Boundary Condition

This type of boundary condition is called as Neumann boundary condition.

Any types of problem related to the different types of plates like rectangular, elliptic, circular, skew and triangular, etc. use the followings three basic combinations of boundary conditions:

- ☛ Essential Type
- ☛ Natural Type
- ☛ Mixed Type (one natural and one essential type)

As per Shames and Dym [198], one of the boundary condition is required for vibration on the boundary of plate ∂R as represented in the figure 1.1.

(a) Either $M_N = 0$ or $\frac{\partial w}{\partial N}$ is presented, where N is normal to the boundary of plate.

(b) Either $Q_N + \frac{\partial M_{NTT}}{\partial TT} = 0$ or w is presented, where TT is tangent to the boundary of plate.

On the basis of these aspects, one can get the following boundary conditions.

(i) Clamped Boundary Condition

In this case, displacement and slope are considered as zero. In the mathematical form, clamped boundary conditions on domain of plate ∂R are given by following equations:

$$w = 0 \quad \text{on } \partial R \quad (1.1)$$

$$\frac{\partial w}{\partial N} = 0 \quad \text{on } \partial R \quad (1.2)$$

where, w represents the displacement.

(ii) Simply-Supported Boundary Condition

In this case, displacement and bending moment must be zero. In the mathematical form, the simply-supported boundary conditions on domain of plate ∂R are given following equations:

$$w = 0 \quad \text{on } \partial R \quad (1.3)$$

$$M_N = 0 \quad \text{on } \partial R \quad (1.4)$$

where, M shows the bending moment.

(iii) Completely-Free Boundary Condition

In the free boundary conditions, the bending moment and shear force both must be zero. The mathematical forms of this case on the domain ∂R are given by following equations:

$$M_N = 0 \quad \text{on } \partial R \quad (1.5)$$

$$Q_N + \frac{\partial M_{NTT}}{\partial TT} = 0 \quad \text{on } \partial R \quad (1.6)$$

where Q is the shearing force on the domain of plate, if all the boundary conditions are applied then the type of condition is called as the mixed boundary conditions.

1.6 TYPE OF PLATES

1.6.1 Circular Plate

The circular plate as represented in figure 1.2 is defined as a radius r with origin as $(0,0)$ and can also be defined in terms of polar coordinate system. The solid plate has no internal holes. The outside boundary of plate may be clamped, simply-supported or completely-free. The polar coordinates of a point P in circular plate are given by.

$$x = r \cos \theta, \quad y = r \sin \theta \quad (1.7)$$

$$r^2 = x^2 + y^2 \quad (1.8)$$

$$\theta = \tan^{-1} \frac{x}{y} \quad (1.9)$$

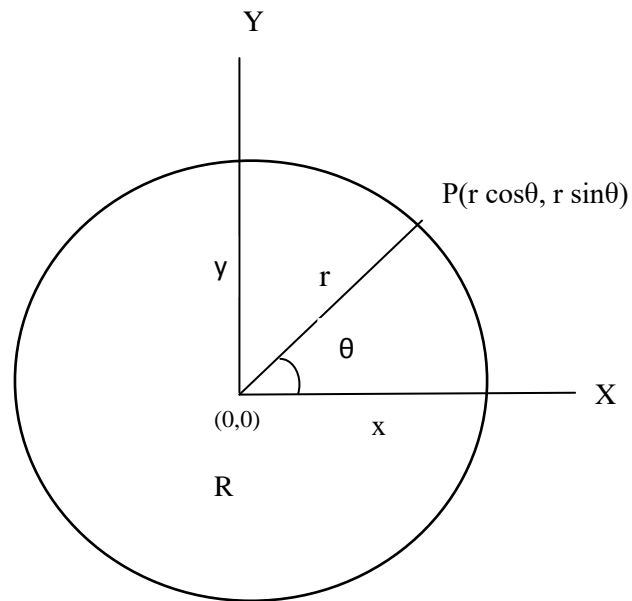


Figure 1.2 Geometrical Representation of Circular Plate

1.6.2 Rectangular Plate

A rectangular plate is defined in two dimensional along X and Y axis by considering length, breadth and outer boundary as clamped, simply-supported and completely-free and it leads to twenty one different combinations of the boundary conditions. When length and breadth are same, then rectangular plate is converted into the square plate. The plate is bounded by $x=0$, $x=a$, $y=0$ and $y=b$ as represented in figure 1.3.

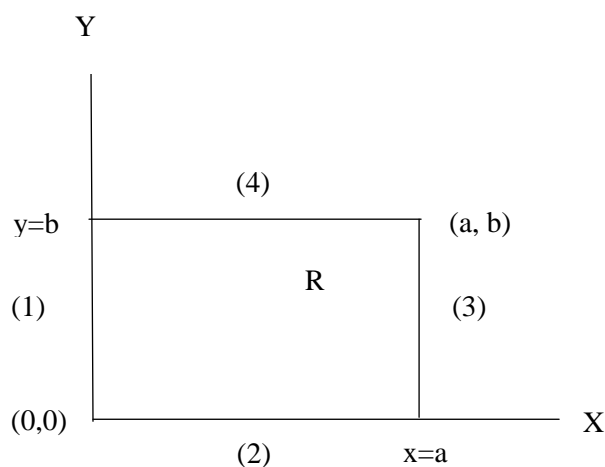


Figure 1.3 Geometrical Representation of Rectangular Plate

1.6.3 Elliptic Plate

The elliptic solid plate can be defined with coordinate (x,y) by the use of following mathematical formula:

$$D = \left\{ (x, y) \mid i.e \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} \quad (1.10)$$

Where a is semi major axes of the plate and b is semi minor axes of the plate. If $a=b$ then elliptic is converted into the circular plate. The geometrical representation of the elliptic plate is shown in figure 1.4.

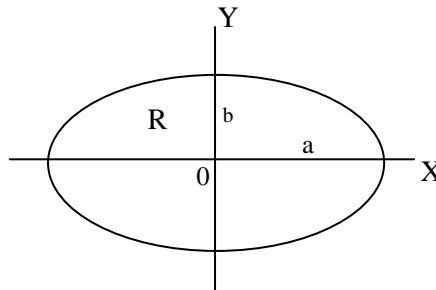


Figure 1.4 Geometrical Representation of Elliptic Plate

$$u = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \quad \text{where } 0 \leq u \leq 1 \quad (1.11)$$

$u = 0$ shows the boundary of the elliptic plate.

1.6.4 Skew Plate

Researchers have done the study of the skew plate with different methods of solution.

In skew plate, the possible boundary conditions are divided into five categories

- ☛ Sides with clamped and completely-free boundary conditions;
- ☛ Sides with clamped and simple-supported boundary conditions;
- ☛ Sides with simply-supported and completely-free boundary conditions;
- ☛ Sides with any of the clamped, simply-supported and completely-free boundary conditions;

- All sides with clamped, simply-supported or completely-free boundary conditions.

The representation of a skew plate on domain R is shown in figure 1.5 and special case of skew domain substituting the angle $\alpha = 90^\circ$ and then domain is mapped into new unit square domain R' and ξ, η are new coordinates as shown in the figure 1.5. The sides (1), (2), (3), and (4) may be taken as clamped, simply-supported or completely-free.

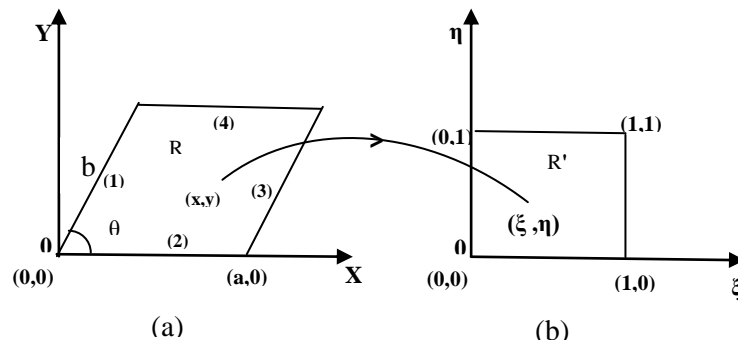


Figure 1.5 Geometrical Representation of Skew Plate

1.7 TRANSVERSE VIBRATION

In the plate theory, the transverse vibration of plate is considered for the thin plates based on assumptions of Classical plate theory in which vibration due to the earthquake, disturbance on the plate, etc. occur.

1.7.1 Transverse Vibration of Plate

In the present work, transverse vibration of plate is considered for the different kinds of thin plates. The vibrations may occur, due to earthquake, disturbance on the plate, etc. In this work, the element of force is considered as zero. Let us first describe basic equations involved to obtain the equation of motion for transverse vibration of plate.

In the present work, cartesian coordinate system is considered and the type of thin plate is considered as isotropic then stress-strain relation are given by.

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \quad (1.12)$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)] \quad (1.13)$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (1.14)$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G}, \varepsilon_{yz} = \frac{\sigma_{yz}}{G} \text{ and } \varepsilon_{zx} = \frac{\sigma_{zx}}{G} \quad (1.15)$$

Where $\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$ are the strain components and $\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ are the stress components at the point (x, y, z) E, ν and G are the young's modulus, Poisson's ratio and shear modulus of the material of plate and there is relation between E and G represented

$$E = 2G(1 + \nu) \quad (1.16)$$

For the thin plate the $\sigma_z = 0$ is taken as zero then

$$\sigma_x = \frac{E}{(1 - \nu^2)} [\varepsilon_x + \nu\varepsilon_y] \quad (1.17)$$

$$\sigma_y = \frac{E}{(1 - \nu^2)} [\varepsilon_y + \nu\varepsilon_x] \quad (1.18)$$

$$\sigma_{xy} = G\varepsilon_{xy}, \sigma_{yz} = G\varepsilon_{yz} \text{ and } \sigma_{zx} = G\varepsilon_{zx} \quad (1.19)$$

Let us consider \bar{u}, \bar{v} and \bar{w} are the displacement at point (x, y, z) on the plate in the x, y, z direction respectively. Then strain components are explained by

$$\varepsilon_x = \frac{\partial \bar{u}}{\partial x}, \varepsilon_y = \frac{\partial \bar{v}}{\partial y} \quad (1.20)$$

$$\varepsilon_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}, \varepsilon_{yz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}, \varepsilon_{zx} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} \quad (1.21)$$

Let u , v , and w show the displacement at the middle surface of the plate at point $(x,y,0)$ then as per classical plate theory, the displacement function \bar{u} , \bar{v} and \bar{w} at point (x,y,z) are defined by

$$\bar{u} = u(x, y) - z \frac{\partial w}{\partial x} \quad (1.22)$$

$$\bar{v} = v(x, y) - z \frac{\partial w}{\partial y} \quad (1.23)$$

$$\bar{w} = w(x, y) \quad (1.24)$$

Let us consider an element on the plate and stresses are shown at mid plate of the plate represented in following figure 1.6.

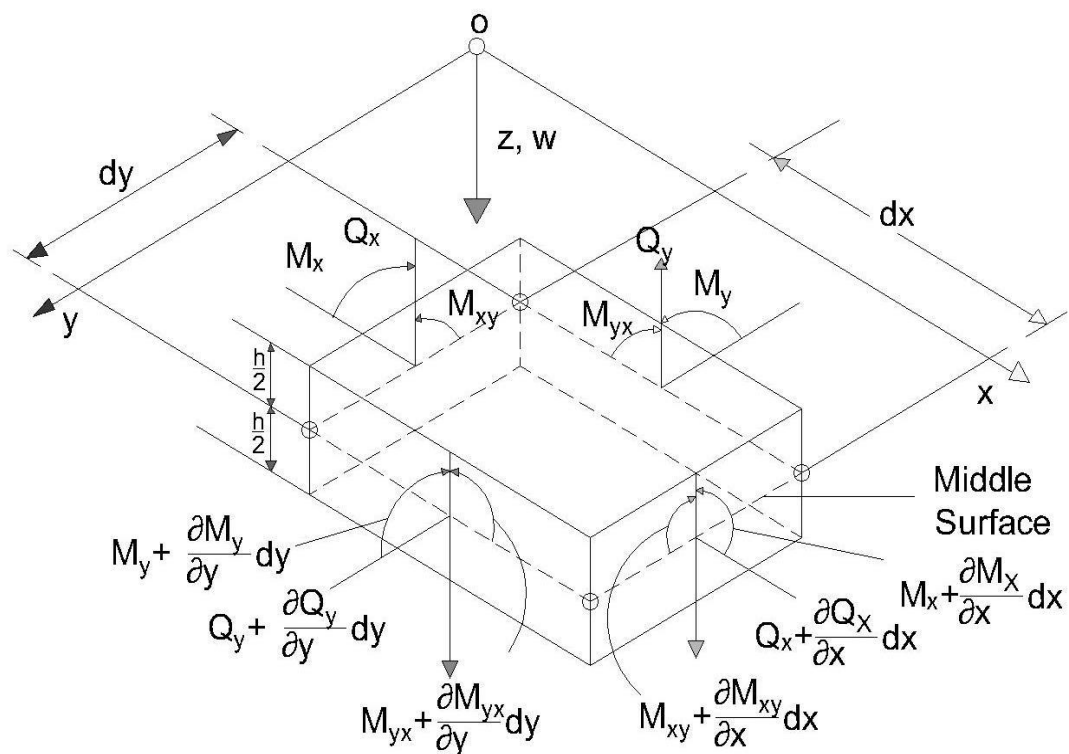


Figure 1.6 Representation of Stresses alongwith Plate Element

Further the stresses may vary according to the thickness of the plate taken as h , then shear force intensities per unit length are explained by following equation.

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz, Q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz \quad (1.25)$$

The bending moment intensities per unit length are given by.

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \quad (1.26)$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz \quad (1.27)$$

The twisting moment intensities per unit length are given by

$$M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \quad (1.28)$$

$$M_{yx} = \int_{-h/2}^{h/2} \sigma_{yx} z dz \quad (1.29)$$

We can easily established the following relation

$$M_{xy} = M_{yx} \quad (1.30)$$

The figure 1.6 shows the shearing forces, bending and twisting moment intensities on the strain perpendicular to x and y axes.

Putting the value of σ_x , σ_y and σ_{xy} from equations (1.17), (1.18), (1.19) into (1.26), (1.27), (1.28) and (1.29) and using strain relations from (1.20) and (1.21) then we get

$$M_x = -D \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] \quad (1.31)$$

$$M_y = -D \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] \quad (1.32)$$

$$M_{xy} = -D(1-\nu) \left[\frac{\partial^2 w}{\partial x \partial y} \right] \quad (1.33)$$

Where

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (1.34)$$

D is known as flexural rigidity of material of plate.

Let us consider equilibrium of plate element as shown in figure 1.6 in the absence of body force and considering equilibrium in x direction then

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad (1.35)$$

Multiply (1.35) by z and integrals over the thickness of plate then one can get

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad (1.36)$$

Similarly, one can get by takes equilibrium equation in y direction then

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad (1.37)$$

Now consider the integration over the thickness, then we get

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \quad (1.38)$$

We can get

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0 \quad (1.39)$$

Where $q(x, y)$ external normal force per unit length.

Now substituting Q_y , Q_z from (1.36) and (1.37) into (1.39) then we can get

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q(x, y) = 0 \quad (1.40)$$

By using (1.31), (1.32) and (1.33), then we get the biharmonic equation of elastic

plate

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{q}{D} = 0 \quad (1.41)$$

or
$$\nabla^4 w = \frac{q}{D} \quad (1.42)$$

Now let us consider parallelepiped cut out of the plate as represented in figure 1.6 and assigning positive and negative internal forces and moments then equation of motion of plate element in transverse direction is given by

$$\rho h dx dy \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q_x}{\partial x} dx dy + \frac{\partial Q_y}{\partial y} dx dy \quad (1.43)$$

Where ρ, h, t are density of moment of plate, thickness of plate and time variable respectively. After dividing $dx dy$ then we get

$$\rho h \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \quad (1.44)$$

After taking moment of all free forces acting on the element about the line through centre of the element in y axis we can get

$$\left[M_x + \frac{\partial M_x}{\partial x} dx \right] dy - M_x dy + \left[M_{yx} + \frac{\partial M_{yx}}{\partial y} dy \right] - M_{yx} dx - \left[Q_x + \frac{\partial Q_x}{\partial x} dx \right] \frac{dy dx}{2} - Q_x \frac{dy dx}{2} = 0 \quad (1.45)$$

Neglecting higher order terms, one can get

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \quad (1.46)$$

Similarly

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y \quad (1.47)$$

Substituting Q_x and Q_y from equation (1.46) and (1.47) into (1.43) then

$$\rho h \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \quad (1.48)$$

After putting the values of M_x , M_y , and M_{xy} from (1.31), (1.32) and (1.33), we get

the differential equation of motion of plate as

$$\nabla^2 [D \nabla^2 w] - (1 - \nu) [D_{yy} w_{xy} - 2D_{xy} w_{xy} + D_{xx} w_{yy}] + \rho h w = 0 \quad (1.49)$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (1.50)$$

1.8 RAYLEIGH METHOD

Rayleigh method is an approximate method which is used for all continuous function.

In this method, maximum kinetic and maximum potential energy must be equal which shows that there is no loss of energy into the system over one cycle of vibration.

Rayleigh method depends on Rayleigh quotient, and applied for computing the natural frequency for different kinds of plates.

Now, kinetic energy T and strain energy U for transverse vibration of plates are given on the following harmonic function for transverse vibration of thin plate, the displacement is given by

$$w(x, y, t) = W(x, y) \cos \omega t \quad (1.51)$$

Where $W(x, y)$ is the maximum displacement at the time t and ω is angular frequency of plate. Kinetic energy (T) of the plate is given by

$$T = \frac{1}{2} \iint_R h\rho \left(\frac{\partial w}{\partial t} \right)^2 dx dy \quad (1.52)$$

The strain energy (U) of the plate is given by

$$U = \frac{1}{2} \iint_R D \left[(\nabla^2 w)^2 + 2(1-\nu) \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \right] dx dy \quad (1.53)$$

Differentiating the equation (1.51), then we get

$$\frac{\partial w}{\partial t} = -\omega W(x, y) \sin \omega t \quad (1.54)$$

$$\left(\frac{\partial w}{\partial t} \right)^2 = \omega^2 W^2(x, y) \sin^2 \omega t \quad (1.55)$$

The maximum value of $\sin^2 \omega t$ is 1 hence the maximum kinetic energy is obtained by using (1.54) and (1.55) as

$$T_{\max} = \frac{1}{2} \omega^2 \iint_R \rho h W^2(x, y) dx dy \quad (1.56)$$

Further putting equations (1.54) and (1.55) into equation (1.53) for getting the maximum strain energy as.

$$U_{\max} = \frac{1}{2} \iint_R \left[D (\nabla^2 W(x, y) \cos \omega t)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \cos^2 \omega t - \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) \cos^2 \omega t \right\} \right] dx dy \quad (1.57)$$

The maximum value of $\cos^2 \omega t$ is 1, then the maximum strain energy given by

$$U_{\max} = \frac{1}{2} \iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) \right\} \right] dx dy \quad (1.58)$$

For getting the Rayleigh quotient, we have to equate maximum kinetic and maximum strain energy of the plate as shown below.

$$T_{\max} = U_{\max} \quad (1.59)$$

$$\frac{1}{2} \omega^2 \iint_R h \rho W^2(x, y) dx dy = \frac{1}{2} \iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) \right\} \right] dx dy \quad (1.60)$$

$$\omega^2 = \frac{\iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) \right\} \right] dx dy}{\iint_R h \rho W^2 dx dy} \quad (1.61)$$

Where $W_{xy} = \frac{\partial^2 W}{\partial x \partial y}$, $W_{xx} = \frac{\partial^2 W}{\partial x^2}$, $W_{yy} = \frac{\partial^2 W}{\partial y^2}$

$$\omega^2 = \frac{\iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \{ W_{xy}^2 - W_{xx} W_{yy} \} \right] dx dy}{\iint_R \rho h W^2 dx dy} \quad (1.62)$$

Where $D = \frac{Eh^3}{12(1-\nu^2)}$ and D is the flexural rigidity

The above method is known as Rayleigh method and equation (1.62) is known as Rayleigh quotient which can be used for computation of natural frequencies for different kinds of plates.

1.9 RITZ METHOD

Ritz method produces excellent approximate solutions using several coordinate functions. It can generate a series of solutions and it is a variational technique. Let us consider the coordinate functions are defined by $\phi_1(x, y)$, $\phi_2(x, y)$, $\phi_3(x, y)$,

$\phi_n(x, y)$ which are known as the basis functions over the constant c_1, c_2, \dots, c_n respectively for getting a series of solution, let us consider the following n terms approximation as

$$W(x, y) = c_1\phi_1(x, y) + c_2\phi_2(x, y) + \dots + c_n\phi_n(x, y) \quad (1.63)$$

$$W(x, y) = \sum_{j=1}^n c_j\phi_j(x, y) \quad (1.64)$$

The above equation gives the approximate solution not as the exact solution over the constant c_j and the basis function $\phi_j(x, y)$ must satisfy the boundary condition. By the use of boundary conditions, differential equation can be solved by the above method. The above technique is known as Ritz method. Let us consider an example of solution of following differential equation by Ritz method.

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = 6x \quad \text{with } y(1) = y(2) = 0 \quad (1.65)$$

The above equation can be written as

$$-\frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) = -6x \quad (1.66)$$

Integrating the above equation, we get

$$I = -\int_1^2 \left(x^2 y \frac{d^2 y}{dx^2} + 2xy \frac{dy}{dx} - 12xy \right) dx \quad (1.67)$$

Now, consider the basis function satisfying the boundary condition $y(1) = y(2) = 0$ as

$$y = c_1\phi_1(x, y) = c_1(x-1)(x-2) \quad (1.68)$$

Where $\phi_j = (x-1)(x-2)$

$$y = c_1(x^2 - 3x + 2) \quad (1.69)$$

Substituting y in (1.67) we get

$$I = -\int_1^2 \left(x^2 c_1^2 \phi_1 \left(\frac{d^2 \phi_1}{dx^2} \right) + 2x c_1^2 \phi_1 \frac{d^2 \phi_1}{dx^2} - 12x c_1 \phi_1 \right) dx \quad (1.70)$$

Maximizing the I with respect to c_1 , we get

$$\int_1^2 \left(x^2 c_1 \phi_1 \left(\frac{d^2 \phi_1}{dx^2} \right) + 2x c_1 \phi_1 \frac{d^2 \phi_1}{dx^2} - 6x \phi_1 \right) dx = 0 \quad (1.71)$$

Substituting ϕ_1 as $(x-1)(x-2)$, we get

$$\int_1^2 \left[(3x^4 - 12x^3 + 15x^2 - 6x) c_1 - 3(x^3 - 3x^2 + 2x) \right] dx = 0 \quad (1.72)$$

Integrating equation (1.72), we get

$$\left[\left(\frac{3}{5} x^5 - 3x^4 + 5x^3 - 3x^2 \right) c_1 - \left(\frac{3}{4} x^4 - 3x^3 + 3x^2 \right) \right]_1^2 = 0 \quad (1.73)$$

From this, we get

$$c_1 = 1.875$$

From (1.68) the value of y_1 is given by

$$y_1 = 1.875(x^2 - 3x + 2)$$

For a complicated differential equation, we can get approximate solution by Ritz method but for simple differential equation, we can also get exact solution by Ritz method.

1.10 RAYLEIGH-RITZ METHOD

Rayleigh-Ritz method is an extension of Rayleigh method. This technique is very useful and important for solving all different types of research problem. In this technique, one assumes a linear combination of the continuous function. Rayleigh-Ritz may be used for obtaining the first few vibration frequencies and mode shapes of plates. The Rayleigh quotient is given by equation (1.62).

After minimizing the Rayleigh quotient, we can determine the natural frequencies and mode shapes of different vibrating plates. The Rayleigh method can be combined with the Ritz method and known as Rayleigh-Ritz method. In this technique, we consider an approximate solution with some unknown constants as given by (1.64) which leads to the homogenous differential equation. The constants c_j 's are arbitrary constants and $\phi_j(x, y)$ are basis functions which satisfy the essential boundary conditions.

Differentiating equation (1.64) as.

$$W_x = \frac{\partial W}{\partial x} = \sum_{j=1}^n c_j \frac{\partial \phi_j}{\partial x} = \sum_{j=1}^n c_j \phi_j^x \quad (1.74)$$

$$W_{xx} = \frac{\partial^2 W}{\partial x^2} = \sum_{j=1}^n c_j \frac{\partial^2 \phi_j}{\partial x^2} = \sum_{j=1}^n c_j \phi_j^{xx} \quad (1.75)$$

$$W_y = \frac{\partial W}{\partial y} = \sum_{j=1}^n c_j \frac{\partial \phi_j}{\partial y} = \sum_{j=1}^n c_j \phi_j^y \quad (1.76)$$

$$W_{yy} = \frac{\partial^2 W}{\partial y^2} = \sum_{j=1}^n c_j \frac{\partial^2 \phi_j}{\partial y^2} = \sum_{j=1}^n c_j \phi_j^{yy} \quad (1.77)$$

$$W_{xy} = \frac{\partial^2 W}{\partial x \partial y} = \sum_{j=1}^n c_j \frac{\partial^2 \phi_j}{\partial x \partial y} = \sum_{j=1}^n c_j \phi_j^{xy} \quad (1.78)$$

Using the equation (1.74), (1.75), (1.76), (1.77) and (1.78) in (1.62) then we get the

Rayleigh quotient as

$$\omega^2 = \frac{\iint_R D \sum_{j=1}^n c_j^2 \left(\frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_j}{\partial y^2} \right)^2 + 2(1-\nu) \left[\left(\sum_{j=1}^n c_j \frac{\partial^2 \phi_j}{\partial x \partial y} \right)^2 - \left(\sum_{j=1}^n c_j^2 \frac{\partial^2 \phi_j}{\partial x^2} \right) \left(\sum_{j=1}^n \frac{\partial^2 \phi_j}{\partial y^2} \right) \right] dx dy}{\iint_R \rho h \left(\sum_{j=1}^n c_j \phi_j \right)^2 dx dy} \quad (1.79)$$

$$\omega^2 = \frac{\iint_R D \left[\sum_{j=1}^n c_j^2 (\phi_j^{xx} + \phi_j^{yy})^2 + 2(1-\nu) \left(\sum_{j=1}^n c_j^2 (\phi_j^{xy})^2 - \sum_{j=1}^n c_j^2 \phi_j^{xx} \phi_j^{yy} \right) \right] dx dy}{\iint_R \rho h \left(\sum_{j=1}^n c_j \phi_j \right)^2 dx dy} \quad (1.80)$$

$$\omega^2 = \left[\frac{\iint_R D \left[\sum_{j=1}^n c_j^2 \left((\phi_j^{xx})^2 + (\phi_j^{yy})^2 \right) + 2\nu \sum_{j=1}^n c_j^2 \phi_j^{xx} \sum_{j=1}^n \phi_j^{yy} + 2(1-\nu) \sum_{j=1}^n c_j^2 (\phi_j^{xy})^2 \right] dx dy}{\iint_R \rho h \left(\sum_{j=1}^n c_j \phi_j \right)^2 dx dy} \right] \quad (1.81)$$

$$\omega^2 = \frac{\iint_R D \left[\sum_{j=1}^n c_j^2 \left((\phi_j^{xx})^2 + (\phi_j^{yy})^2 \right) + 2\nu \sum_{j=1}^n c_j^2 \phi_j^{xx} \phi_j^{yy} + 2(1-\nu) \sum_{j=1}^n c_j^2 (\phi_j^{xy})^2 \right] dx dy}{\iint_R \rho h \left(\sum_{j=1}^n c_j \phi_j \right)^2 dx dy} \quad (1.82)$$

Now introduce some important notation, x and y replace x_1 and x_2 .

$$b_{ij} = \iint_R h \rho \phi_i \phi_j dx_1 dx_2 \quad (1.83)$$

$$a_{ijpq} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_p^2} \frac{\partial^2 \phi_j}{\partial x_q^2} dx_1 dx_2 \quad \left. \begin{array}{l} i, j = 1, 2, 3, \dots, n \\ p, q = 1, 2 \end{array} \right\} \quad (1.84)$$

$$a_{ij11} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \phi_j}{\partial x_1^2} dx_1 dx_2 \quad (1.85)$$

$$a_{ij12} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \phi_j}{\partial x_2^2} dx_1 dx_2 \quad (1.86)$$

$$a_{ij21} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \phi_j}{\partial x_1^2} dx_1 dx_2 \quad (1.87)$$

$$a_{ij22} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \phi_j}{\partial x_2^2} dx_1 dx_2 \quad (1.88)$$

$$c_{ij} = \iint_R D \frac{\partial^2 \phi_i}{\partial x_1 \partial x_2} \frac{\partial^2 \phi_j}{\partial x_1 \partial x_2} dx_1 dx_2 \quad (1.89)$$

ν as the poisson's ratio whose value is 0.3 for isotropic plate and D is the flexural rigidity. Then Rayleigh quotient equation can be written as.

$$\omega^2 = \frac{\sum_{j=1}^n \sum_{i=1}^n c_i c_j^2 \left[\sum_{q=1}^2 \sum_{p=1}^2 a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) \right]}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij}} \quad (1.90)$$

$$\omega^2 = \frac{\sum_{j=1}^n \sum_{i=1}^n c_i c_j^2 \left[\sum_{q=1}^2 (a_{ij1q} + a_{ij2q}) + 2(1-\nu)(c_{ij} - a_{ij12}) \right]}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij}} \quad (1.91)$$

$$\omega^2 = \frac{\sum_{j=1}^n \sum_{i=1}^n c_i c_j^2 \left[(a_{ij11} + a_{ij12} + a_{ij21} + a_{ij22}) + 2(1-\nu)(c_{ij} - a_{ij12}) \right]}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij}} \quad (1.92)$$

Now putting equation (1.85), (1.86), (1.87), (1.88) and (1.89) in equation (1.92) and we get

$$\omega^2 = \frac{\sum_{j=1}^n \sum_{i=1}^n c_i c_j^2 \left[\left(D \iint_R \frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \phi_j}{\partial x_1^2} + \frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \phi_j}{\partial x_2^2} + \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \phi_j}{\partial x_1^2} + \frac{\partial^2 \phi_i}{\partial x_2^2} \frac{\partial^2 \phi_j}{\partial x_2^2} \right) + 2(1-\nu) \left(\frac{\partial^2 \phi_i}{\partial x_1 \partial x_2} \frac{\partial^2 \phi_j}{\partial x_1 \partial x_2} - \frac{\partial^2 \phi_i}{\partial x_1^2} \frac{\partial^2 \phi_j}{\partial x_2^2} \right) \right] dx_1 dx_2}{\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 h \rho \iint_R \phi_i \phi_j dx_1 dx_2} \quad (1.93)$$

Minimizing the value of ω^2 over the arbitrary constants $c_1, c_2, c_3, \dots, c_n$.

$$\frac{\partial \omega^2}{\partial C_i} = 0 \quad (1.94)$$

$$\frac{\left[\left(\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij} \right) \left(\sum_{j=1}^n c_j^2 \left\{ \sum_{q=1}^2 \sum_{p=1}^2 a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) \right\} \right) - \sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 \left\{ \sum_{p=1}^n \sum_{q=1}^n a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) \right\} \sum_{j=1}^n c_j^2 b_{ij} \right]}{\left(\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij} \right)^2} = 0 \quad (1.95)$$

$$\left(\sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij} \right) \left(\sum_{j=1}^n c_j^2 \left\{ \sum_{q=1}^2 \sum_{p=1}^2 a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) \right\} \right) - \left(\omega^2 \sum_{i=1}^n \sum_{j=1}^n c_i c_j^2 b_{ij} \sum_{j=1}^n c_j^2 b_{ij} \right) = 0 \quad (1.96)$$

$$\left(\sum_{j=1}^n c_j^2 \left\{ \sum_{q=1}^2 \sum_{p=1}^2 a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) \right\} \right) - \left(\omega^2 \sum_{j=1}^n c_j^2 b_{ij} \right) = 0 \quad (1.97)$$

$$\sum_{j=1}^n c_j^2 \left[\sum_{q=1}^2 \sum_{p=1}^2 a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12}) - \omega^2 b_{ij} \right] = 0 \quad (1.98)$$

$$\text{Let us consider } \lambda^2 = \omega^2, a_{ij} = \sum_{q=1}^2 \sum_{p=1}^2 (a_{ijpq} + 2(1-\nu)(c_{ij} - a_{ij12})) \quad (1.99)$$

$$\sum_{j=1}^n (a_{ij} - \lambda^2 b_{ij}) c_j = 0 \quad i = 1, 2, \dots, n \quad (1.100)$$

The equation (1.100) is a homogenous system of n equations for the constant c_j . For getting non trival solution, determinant of coefficients must be zero. Hence, we get the following frequency equation for vibrations of plates as given by

$$|a_{ij} - \lambda^2 b_{ij}| = 0 \quad (1.101)$$

From (1.101), we can easily compute the eigenvalues as λ and after putting λ in (1.100), we can get eigenvectors representing modes of vibrating plate. In the present work equation (1.101) can be solved by the generalized Jacobi method for getting the eigenvalues and eigenvectors. The reason for selecting generalized Jacobi method is that when we increase the values of n, then we get the faster convergence of results.

Replacing x_1 as x and x_2 as y , we get the expressions of a_{ij} and b_{ij} from (1.83) to (1.89) and (1.99).

$$a_{ij} = \iint_R \left[\phi_i^{xx} \phi_j^{xx} + \phi_i^{yy} \phi_j^{yy} + \nu (\phi_i^{xx} \phi_j^{yy} + \phi_i^{yy} \phi_j^{xx}) + 2(1-\nu) \phi_i^{xy} \phi_j^{xy} \right] dx dy \quad (1.102)$$

$$b_{ij} = \iint_R h \phi_i \phi_j dx dy \quad (1.103)$$

After solving equations (1.102), (1.103) and substituted in equation (1.101), we can compute first three natural frequencies, mode shapes for given plate with different types of boundary conditions.

1.11 COMPUTATIONAL TOOLS

1.11.1 FORTRAN Language

FORmula TRANslations is called as FORTRAN which is a computer programming language that specially designed for solving Mathematical and scientific problems. FORTRAN is a high level language that was originally developed by IBM in the year 1957. This language provides a user friendly environment for developing software's and solving Mathematical and scientific calculations. It supports a large number of tools and functions that help programmer to design and developed software's like consisting of variables, decision controls, loop controls, functions, etc. There are so many areas in which this language can be used like in the field of numerical analysis and scientific computating, structured programming, modular programming, object-oriented programming, concurrent programming, etc. This software language plays a very important role for computing all numerical results like frequency, mode shape, etc in area of vibration of plate. FORTRAN gives a fast and accurate results in different area, therefore it is widly used by the scientists and engineers.

1.11.2 Microsoft Excel

MS-Excel is an application software program included in the Microsoft Office. This application software program is used to create numeric and calculative documents with the help of spreadsheet program. Spreadsheet is a data sheet which made by multiple rows and columns. First, we enter data in this document and then calculate it with the help of mathematical functions. Excel provides a large number of functions to arrange and manipulate data, like Sum, Average, Count, Max, Min., etc. There are multiple versions of this software is available at present time like that Excel 2003, Excel 2007, Excel 2010, Excel 2013, and Excel 2016 etc.

In the present work, MS-excel version software is used to draw tables and graphs for frequency.

1.11.3 Matlab

Matrix Laboratory is called a MATLAB in short which is a mathematical tool that provides a user-friendly and interactive platform for solving numerical and mathematical problems. There are so many functions that can be performed with the help of MatLab like that matrix manipulations, plotting of data and functions, implementing algorithms, creation of interfaces etc. MatLab supports various tools and commands and mathematical functions. One can solve complicated and difficult numerical problems in a very easily manner. There are so many areas in which MatLab functions are used like dealing with Matrices and Arrays, 2-D and 3-D Plotting and graphics, linear Algebra, non Linear Algebra, Statistics, data Analysis, calculus and Differential Equations, Numerical Calculations, Integration, curve Fitting etc. From this software, one can obtain the results for representing the behaviour of plate in the form of graphs and mode shapes. In present work, MatLab version is also used to draw mode shapes representing the behaviour of plate.

1.12 CONVERGENCE OF RESULTS

A sequence of real numbers i.e $\langle x_n \rangle$ is said to be converge to a real number z , if for any $\varepsilon > 0$ there exists a natural number m such that

$$|x_n - z| < \varepsilon \text{ for all } n \geq m \quad (1.104)$$

$$\text{i.e } z - \varepsilon < x_n < z + \varepsilon \text{ for all } n \geq m \quad (1.105)$$

As per above definition of convergence, we have got the converged results in the form of frequencies for known shapes of plates. The converged results are matching upto five significant digits. After converting the problem into eigenvalue problem, generalized Jacobi method produces faster rate of convergence. The sample results of converged sequences are demonstrated in the following table 1.1.

Table 1.1 Convergence of Results

| n | λ_1 | λ_2 | λ_3 |
|----|-------------|-------------|-------------|
| 3 | 4.9351 | 30.503 | 163.23 |
| 5 | 4.9351 | 29.720 | 74.683 |
| 7 | 4.9351 | 29.720 | 74.683 |
| 9 | 4.9351 | 29.720 | 74.683 |
| 10 | 4.9351 | 29.720 | 74.683 |



CHAPTER-II
Review of Literature

CHAPTER-II

REVIEW OF LITERATURE

The study of transverse vibration of plates plays an important role in the areas of mechanical engineering, aerospace, civil engineering, ocean engineering, and naval architecture etc. The concept of vibration depends on space, time, mass, forces. The study of vibration reduces vibration and produces accurate design of structures. The analysis for vibration problem can be done through mathematical modelling of system. First three frequencies and mode shapes for different type of plates like skew, square, circular, elliptic etc. with several of boundary conditions and thickness of variations may be computed through various numerical methods. A large number of research papers are available in the literature on the vibrations of plates with different types of boundary condition. Transverse vibration of plates is based on Kirchhoff plate theory i.e. Classical plate theory and can be solved by different methodologies like Rayleigh-Ritz, Finite Element, Boundary Integral Equation, Finite Difference methods, etc. Various types of plate problems computed accurate natural frequencies and mode shapes.

The present chapter gives maximum information of existing literature on the plates, shell, beam, etc. based on static and dynamic behavior of frequencies and modes shapes with several boundary conditions and thickness variations. Many authors used different shapes of plates and basis functions for analysis of frequencies and behavior of structures. Some basic boundary conditions include clamped, simply-supported and completely-free which researchers also considered mixed type of boundary conditions. Let us describe brief review of literature available of the plates. The research problems are described as per year-wise development of research work on various shapes of plates.

2.1 ELLIPTICAL AND CIRCULAR PLATES

In 1947, Mclachlan [155] performed vibrations on uniform, homogenous loss-free stretched membrane, water in a lake of uniform depth which is an elliptical ring and elastic elliptical plate with clamped boundary condition. Waller [243] studied vibrations of free elliptical plate for natural frequencies with boundary conditions and aspect ratio. Shibaoka [206] has solved vibration of a uniform thin elliptic plate with clamped boundary condition by using Mathieu function and modified Mathieu function.

In the year 1962, Mcniit [154] has also studied free vibration of elliptic plate with clamped bounded condition. Singh and Tyagi [222] have computed frequencies for symmetric transverse vibration of an elliptic plate by the use of Galerkin's method with clamped edge and variable thickness. The numerical results of the first four frequencies were represented upto five significant digits for taper parameters, and aspect ratio of elliptic and circular plates with uniform and parabolically varying thickness. Singh and Chakraverty [213] have applied Rayleigh-Ritz method by using characteristic orthogonal polynomials for elliptic and circular plates. After applying this technique, authors have computed frequencies and modes shapes with two boundary conditions (simply-supported and completely-free boundary conditions) and taken first four frequencies upto five significant digits and mode shapes have been represented in three dimensional form. Lain et al. [105] have determined natural frequencies and mode shapes using Rayleigh-Ritz method of elliptical and circular plates with classical boundary condition and orthogonal plates function. Singh and Chakraverty [211-212] have also investigated natural frequencies by applying Rayleigh-Ritz method for elliptical, circular and skew plates with characteristic orthogonal polynomial in two dimension and clamped, simply-supported and

completely-free boundary conditions. Hasheminejed et al. [76] computed frequency and model for free in plane vibrations of co-focal annular elliptic plates with a classical and flexible boundary condition.

Scolan [197] has calculated analytic solution of elliptic paraboloid on cylindrical waves. Reddy et al. [191] used Finite Element method and ANSYS for determining first four natural frequency and mode shapes of clamped elliptical plate with aspect ratio of 0.5. Soni [228] has discussed vibration of shell and elastic plates with boundary conditions and variable thickness and also computed numerical values by Rayleigh-Ritz technique. Singh and Chakraverty [209, 210] have also used Rayleigh-Ritz technique for computing first four frequencies and mode shapes of elliptic, circular plates and simply-supported elliptical plates with boundary conditions, quadratically varying thickness, aspect ratio, orthogonal polynomials and two controlling parameters. Hassan and Makary [74] applied Rayleigh-Ritz method for study of transverse vibration of elliptical and circular plates with two boundary conditions first one is half clamped boundary condition and second one rest simply-supported boundary condition and varying thickness. Suganthi and Karthikeyan [233] derived natural frequencies of ring shaped elliptical plate with different types of boundary condition for symmetric and antisymmetric mode. Numerical calculation solved by Finite Element method. Smirnov [225] have used Finite Element method for computed frequencies and modes shapes of elliptic and annular circular plates. Obtain numerical results compared with Rayleigh-Ritz and other analytical methods. Carrington [20] has discussed vibration of flat fixed circular plates for frequencies with Rayleigh-Ritz method and Bessel function. Wah [242] studied a large initial compression of circular plate with two different types of boundary conditions like simply-supported and clamped boundary conditions. This problem depends upon

Poisson Kirchhoff theory. In 1964 Conway et al. [31] studied frequencies of cantilever beams and circular clamped-free, clamped-simply-supported and clamped-clamped with Poisson's ratio as 0.3. Prasad et al. [177] have computed first few natural frequencies by using Rayleigh method for axisymmetric of circular plate and used three boundary conditions i.e. clamped, simply-supported and completely-free with linearly varying thickness.

In year 1978, Laura and Grossi [108] studied the influence of Poisson's ratio on the lower natural frequencies of transverse vibration of a circular plate with linearly varying thickness and elastically restrained against rotation. In year 1980, Celep [23] has applied Classical, Reissner and Mindlin theories for computing numerical results of two circular plates with simply-supported and completely-free boundary conditions and arbitrary thickness. Grossi and Laura [62] have studied transverse vibration of circular plate with boundary conditions and linearly varying thickness. Gupta and Mishra [70] have analysed the numerical results of transverse vibrations of a circular plate by using Hamilton's energy principle. All computed numerical results presented in the form of Bessel function with shear and classical theories. Lessia and Narita [121] have also obtained natural frequencies upto six significant digits for simply-supported circular plate with poisson's ratio ranging from 0 to 0.5.

In year 1981, Gelos et al. [54] applied Galerkin Variational method for transverse vibration of circular plates and computed two lowest frequency coefficients of axisymmetric modes for the several combinations of governing mechanical parameters. Rayleigh-Ritz method which is most important technique was also used by Laura and Grossi [109] for obtaining natural frequencies for polynomial coordinate function. Irie et al. [86] have introduced Ritz method for determining natural frequencies of stability circular plate with boundary conditions, taper parameter and

unidirectional varying thickness. Zhixin [285] has studied circular plate with varying thickness under uniformly distributed load. In the year 1986, Ficcadent and Laura [50] have used Ritz method for obtaining the frequency of transverse vibration of circular plates with uniform and non uniform thickness.

In year 1989, Kim and Dickinson [100] have applied Rayleigh-Ritz method on annular and circular plates for obtaining eigenvalue with orthogonally polynomials and used isotropic or polar orthotropic material of varying thickness. Finite Element analysis has been applied by Weisened [262] for obtaining natural frequencies of circular and annular plates. In year 1993, Yang [276] has also investigated vibration of a circular plate of varying thickness with clamped solid plate. The result of the circular plate has been computed by Perturbation method. Singh and Saxena [220] have also obtained natural frequencies by Rayleigh-Ritz method using three different types of boundary conditions for clamped circular plate with the exponential thickness variations. Singh and Saxena [217-219] have computed frequency of circular plate with clamped, simply-supported and completely-free bounded conditions by using Rayleigh-Ritz technique with linear and double linear variable thickness and obtained natural frequencies upto the five significant. Singh and Saxena [218] have investigated first four frequencies, mode shapes, nodal radii of circular plate with clamped, simply-supported boundary conditions and exponential thickness variation. Frequencies were obtained by Rayleigh-Ritz method through computer programming. In year 1997, Gu and Wang [63] have studied free vibration analysis of circular plates including stepped thickness and used Differential Quadrates Element method for numerical results. Gupta and Ansari [71] have also applied Rayleigh-Ritz method using the linearly varying thickness, rigidity ratio and computing the approximate solution for orthotropic circular plate. Singh and Hassan [216] discussed transverse

vibration of circular plate with uniform thickness and different boundary conditions. Kung and Singh [102] obtained eigenvalue for rectangular plate with damping patches. In the year 1999, Wang [246] discussed classical circular plates of variable thickness and boundary condition using generalized power series solution. Vera et al. [240] applied Finite Element analysis for obtaining the frequencies of circular annular plate with a boundary condition. Gupta and Goyal [69] have studied Eigen Function method for asymmetric linearly tapered circular plate. The approach is based on the Classical Plate theory. Wang [248] has studied fundamental frequency and general solution of circular plate supported by concentric ring in the interior. Wu and Liu [266] have applied Generalized Differential Quadrature technique for solid circular plate with a varying thickness and elastic restraints. Ghorashi and DaneshPazhooh [56] have studied simply-supported circular plates with a variable thickness for study of lower and upper bound. Zaera et al. [282] obtained distribution of the bending moments and membrane forces inside the metallic circular plates. Rdzanek and Rdzanek [187] have studied clamped circular plate for obtaining low frequencies of self power with the time harmonic and axisymmetric. Chen et al. [25] have obtained natural frequencies and boundary mode shape using Meshless approach and radial basic function of circular and rectangular clamped plates. Li and Xiao [126] have discussed bending analysis of thin circular plate with large deflection. Problem was based on linear theory and Incremental Load technique and both were used for the determinate in of numerical results. Differential Quadrature (DQ) method has been applied by Gupta et al. [72] to compute first three natural frequencies and modes shapes with clamped, simply-supported and completely-free boundary conditions and Chebyshev polynomials of nonhomogenous, isotropic circular plates. Rdzanek et al. [188] have obtained low frequency range, radiation efficiency axisymmetric and

asymmetric modes of supported circular plate by the use of Low Frequency Approximated formulas with limited boundary conditions clamped, guided, simply-supported or completely-free boundary conditions. Sun and Luo [234] studied array of dynamic circular loads moving constant and harmonic loads using Fourier transform and generalized Dunamel's integral.

In the year, 2008 Allahverdizadeh et al. [4] applied vonkarman dynamic equation and Semi Analytical approach for computing natural frequency of thin circular functionally graded plates and solved governing equation by using Assumed Time Mode method and Kantorovh Time Averaging technique. Park [171] has also computed high frequency vibration by using cylindrical coordinate, kinetic, potential energy and Hamiton's principle of clamped circular plate with boundary condition. Ebrahimi and Rastgo [46] applied Classical Plate theory on thin circular functionally graded plates with clamped boundary condition and two uniformly distributed actuator layers mode of piezoelectric. Leniowska [122] has studied viscous damping, structural internal damping and fluid loading of circular plate by using Orthogonal Series method and MATLAB computer program. Kumar et al. [101] have studied Finite Element method for composite circular disc with various orthotropic properties. In the year 2010, Yuxin and Masumi [281] have used Governing equation of Thermo-elastic damping for cut of plane vibration of micro-scale circular plate. Thermo-elastic damping was investigated with different environmental temperature, plate dimensions and boundary conditions. Abbas et al. [1] has discussed circular cylindrical shell in high temperature with an open variable thickness. Radzanek et al. [189] used Neumann boundary value problem of circular membrane and computed frequencies. Sinha and Turner [224] also obtained eigenvalues of noncircular cross section plate by Rayleigh-Ritz method with clamped and constraints free edge boundary condition,

aspect ratio with one dimensional. Szemela [235] studied Numerical Approximation techniques for modal acoustic independence efficient of circular plate with boundary of three wall corner region.

In year 2013, Wang et al. [256] has also studied modified couple stress theory of circular micro plate results dependent on size of plates. Rao and Rao [182] carried out study on frequencies of circular plate weakened along an internal concentric circle and elastically restrained edge against translation. Baghani and Fereidoonzed [11] proposed analytical solution of functionally graded circular plate with simply-supported boundary condition. Xing et al. [272] derived the exact eigen solution of circular cylindrical shells with classical boundary conditions by using Variables method. In year 2014, Abbasi et al. [2] attempted semi analytical solution of circular plate by using non-uniform axisymmetric transverse loading resting with Winkler elastic foundation. Qin et al. [180] used Structure-Preserving method for circular thin plates under the impact load of dynamic analysis. Plaut [176] studied axisymmetric problem of circular plates and solved Generalized Reissner theory with linearly elastic, large rotations and strains. Equilibrium equation and Shooting method were used for numerical results of given system. Wang et al. [253] studied Rayleigh-Ritz technique of graded circular, annular and sector plates for obtaining unified solution and general solution. Bahrami and Teimourian [13] analysed the wave power reflection of circular annular nano plates by using Wave Propagation technique for natural frequencies with classical boundary conditions. In the year 1979, Mukhopadhyay [162] used Finite Difference technique for annular sector plates with boundary conditions. Srinivasan and Thruvenkatchari [230] used Integral Equation technique for calculating natural frequencies of isotropic and orthotropic clamped annular sector plates with fixed parameter sector angle and radii ratio. Irie and

Yamada [85] were successfully computed natural frequency by using Transfer Matrix approach of Mindlin annular plate with linearly, parabolically and exponentially varying thickness. Vibration of annular plate was taken as coupled set of first order differential equations by using Transfer matrix of Mindlin annual plate. Dawe and Roufaeil [32] investigated frequencies and mode shape for Mindlin plates by using Rayleigh-Ritz method with boundary conditions. Lee and Ng [110] have obtained natural frequencies, modes and radial nodal with simply-supported, clamped, and completely-free boundary conditions and unequally, exponentially varying thickness for annular disk. Numerical results were determined by using Lagrangian approach. Avalos et al. [9] have obtained frequency by Rayleigh-Ritz approach with exponential parameter and satisfies identically the boundary conditions. Numerical results have approximate solution for the annular plates of stepped thickness. In 1998, Kang and Lissa [91] used the Ritz method for completely-free thick and linearly tapered annular plate for trigonometry function and obtained extensive and accurate frequencies. Rdzanek [185] was successfully computed numerical results with clamped-free, free-clamped and fully clamped boundary conditions for annular plate. Wang [249] has obtained fundamental frequencies of seven cases of axisymmetric or non-axisymmetric or modes of thin annular plates with movable edge. Lee and Schutz [111] have analysed eigenvalues of Timoshenko beam and axisymmetric Mindlin plates with clamped, simply-supported, completely-free and sliding boundary conditions. Rdzanek and Rdzanek [186] used Cauchy Theorem for the residues of the corresponding integrals with supported annular plate with clamped, simple-supported and completely-free boundary conditions. Bahaloo et al. [12] used Finite Element method for transverse vibration and stability of functionally graded rotating annular disk with circumferential crack. Xiao and Feng [267] have used Galerkin method for

the compute algorithm for circular plate. Magnucki et al. [150, 151] presented the symmetrically and axisymmetric flexural vibrations of circular and bending circular plates with mechanical properties and clamped edge and shear effect.

2.2 RECTANGULAR AND SQUARE PLATES

Young [280] has obtained frequency for rectangular plate by using Ritz method with boundary conditions. In year 1971, Cheung and Cheung [29] have computed natural frequency by Finite strip method for rectangular and polygonal plates with clamped, simple-supported and completely-free boundary conditions and behavior of plate as isotropic, orthotropic and distributed concentrated masses with continuous in one direction and thickness plate is constant or a variable. In 1972, Ramachandran and Reddy [184] have adopted approximate solutions for analyzing rectangular plate with cutout by approximate method. Jain and Soni [88] studied rectangular plate with a parabolically varying thickness and boundary conditions with computation of frequencies by the use of Rayleigh-Ritz technique. Dickinson [35] applied Rayleigh's method on the flexural vibration of rectangular plate and demonstrated vibration of plate using simply-supported boundary condition. Dhotarad and Genesan [34] studied thin rectangular plates with one or two dimensional coordinate with clamped and completely-free boundary conditions and satisfy Laplace equation of temperature distribution. All eigenvalues and eigenvectors were obtained by Simultaneous Iteration technique and Finite Element method. All computing results were compared with results of Finite Difference method. Gorman [59] has applied the Superposition method for completely-free rectangular plate and obtained solutions for satisfying the differential equation with boundary conditions and obtained eigenvalues in four digits. Touratier [237] has investigated several problems for bending free undammed vibration and buckling of a three layered (Sandwich and Laminated) symmetric cross-

ply; and rectangular or square plates of simply-supported boundary condition. A numerical result was computed by three Dimensional Elasticity theory and Several Approximated theories. Liew et al. [142] have determined the vibration frequencies for thick rectangular plate with twenty one boundary conditions for all the possible combinations of clamped, simply-supported and completely-free boundary conditions and two different types of method which was applied on Mindlin Plate theory and secondly by the use of Rayleigh-Ritz method. Liew et al. [139] studied an orthogonal polynomial function and three dimensional, small-strain elasticity energy principle for homogenous isotropic, thick rectangular plates with Ritz formulation and boundary constraints. Bardell et al. [16] also studied free vibration of isotropic rectangular plates with boundary condition. Askari et al. [8] successfully applied Galerkin and Rayleigh-Ritz methods. Galerkin method was obtained for unknown coefficient in the velocity potential and Rayleigh-Ritz method has used for obtaining natural frequencies and mode shapes with free edge and submerged in water.

In the year 2000, Filipich and Rosales [51] have obtained precision frequencies and classical natural modes of rectangular thin plate by using Whole Element method with completely-free boundary condition. In another paper [64], author obtained frequencies for rectangular plates with free-side edge in two dimensional coordinate. Zhou and Cheung [291] used Ritz and Rayleigh-Ritz methods for obtaining eigen frequencies for rectangular plate. Cheung and Zhou [28] investigated natural frequencies of rectangular plate by Ritz method. The approach was used to compute the frequencies of the system. Li and Gibeling [124] have obtained mutual radiation resistance of a simply-supported rectangular plate with boundary condition on the radiated sound power. Finite Layer method was also used by Zhou et al. [294] for getting eigen frequencies of thick layered, isotropic and laminated composite

rectangular plates with a simply-supported boundary condition. Wei et al. [260] have determined natural frequencies by using Discrete Singular Convolution of rectangular plate with simply-supported, clamped and transversely supported boundary conditions. Kevorkian and Pascal [92] applied Continuous Element method in place of Finite Element method of vibration analysis for 3-D rectangular plates with boundary condition solution. Fan [49] reported a Model based on Strain Energy method for rectangular plate. Shuyu [208] presented analytic solutions of rectangular thin plate with elasticity and free boundary conditions for two dimensional.

In the year 2002, Wang and Wereley [250] have computed model frequency and displacement mode shapes by using Kantorovich-Krylov Variation method for vibration of rectangular plates with completely-free and clamped boundary conditions. Zhou et al. [293] have applied Ritz technique for thick rectangular plates to set accurate natural frequencies and mode shapes with chebyshev polynomials, boundary function, aspect ratio and poisson's ratio. Results were based on small-strain, and three dimensional elasticity theory. Park et al. [172] have studied two different models i.e. first model was based on wave propagation at the edge and second model was based on Rayleigh-Ritz method for vibration of rectangular plate. Park et al. [173] have also investigated flow induced vibration of visco-elastically supported rectangular plates using Energy Flow Analysis approach. Sakiyama et al. [195] studied characteristic equation for free vibration of rectangular and equivalent rectangular plates, right cantilever triangular plate with non-uniform thickness, uniform thickness, variable thickness various and aspect ratio.

In 2004 Gorman [58] was successfully applied Superposition method for analytical solution of rectangular plate with boundary condition. Singh and Muhammad [214] have obtained natural frequencies, mode shapes and node of non-rectangular plate by

using energy functional and matrix equation for isotropic non-rectangular defined by four curved boundaries. Li [132] has computed modal characteristics and frequencies of rectangular plates by applying Rayleigh-Ritz method with general elastic boundary supports and admissible function. Casimir et al. [21] have investigated dynamic stiffness matrix of Continuous Element method of Kirchhoff of rectangular plate. On the other hand, Gorman [60] demonstrated frequencies and mode shapes for rectangular plate using Superposition method with boundaries and aspect ratios. Hasseini-Hashemi and Arsanyni [75] have obtained eigen function, eigen frequency and mode shape using six distinct cases with four edges and Mindlin Plate theory for thick rectangular plate. Manna [152] has investigated eighteen mode shapes on sides and six internal nodes and free vibration analysis with different thickness ratios, different aspect ratios and different boundary conditions by using high-order triangular Finite Element technique of isotropic on rectangular plate. Bao and Deng [15] have also studied general solution of free vibration for rectangular thin plate on Hamilton systems. Zhong and Zhang [288] have successfully obtained analytical solution of a rectangular thin plate with four edges free boundary condition.

Gupta and Khanna [67] have computed frequency equation by using Rayleigh-Ritz method and Separation of Variables method with a clamped boundary condition and linearly thickness of rectangular plate in both directions. Wu et al. [265] applied Novel Bessel Function method by Bessel functions and computed the natural frequencies for rectangular plate with three boundary conditions i.e. fully simply-supported, fully clamped, two opposed edge simply-supported and two clamped boundary conditions. Eloy et al. [48] have successfully applied Galerkin method and Fourier transforms of rectangular plate for computing flutter modes and frequencies by using clamped and completely-free bounded conditions. Qiao et al. [179] have

studied analytical method of rectangular plates with five different types of boundary conditions and various thicknesses. Du et al. [41] have investigated analytical solution by using Fourier Series method for rectangular plate in two directions with homogeneous boundary condition and periodic function. Lakis et al. [106] have studied Finite Element method and Sander's Shell theory for the natural frequencies of rectangular plate with fluid, bilinear polynomial and exponential functions. Xing and Liu [270] used Rayleigh Quotient Variational Principle to calculate the natural frequency of orthotropic rectangular plates with classical boundary condition. Monterrubio [160] has also performed numerical value on the shallow shells of rectangular plate with classical boundary conditions. In another paper, Li et al. [134] discussed the transverse vibration of rectangular with general elastic boundary conditions. In this paper, exact solutions were obtained by using governing different equation with boundary conditions on the point-wise basis in x-y directions. Zou and Crocker [298] have investigated the sound power radiation of rectangular plate with arbitrary boundary condition. All numerical results were obtained by using Rayleigh Integral technique with six different boundary conditions of given plates. Lim et al. [195] have computed free vibration solution of rectangular Kirchhoff plate by the use of New Elasticity approach. Bardell et al. [16] also studied free vibration of isotropic rectangular plate with boundary condition. Wang et al. [251] were computed mode shapes function of two dimensional plates by using Extended Kantorovics-Krylov method with clamped-free-clamped-free boundary condition (CFCF).

Exact solution for free vibration of thin rectangular plate was studied by Xing and Liu [268]. Zhang et al. [283] discussed sound radiation of baffled rectangular plates with boundary condition, aspect ratio, 2-D and 1-D Fourier and solution computed by using Rayleigh-Ritz technique. Venkatesham et al. [239] applied Finite Element

method and Boundary Element technique for rectangular plenum to obtain natural frequencies and mode shapes. Dozio [40] has also obtained natural frequency and mode shapes by using Ritz method for rectangular plates with classical boundary condition. Yoo [279] calculated critical frequencies with a large ratio and different boundary condition of a rectangular plate. Hunag et al. [80] also obtained natural frequency of rotating Euler beams with the use of power series solution and governing equation solved by Power Series method. Free vibration analysis of rectangular plates with elastically point support edges has studied by Du et al. [43]. Lim [144] applied Elasticity technique for obtaining frequency of rectangular plates. Wang and Xus [254] have studied Discrete Singular Convolution (DSC) method of beams, annular and rectangular plates with completely-free boundary condition. Different Quadrature method was also used by Liu and Xing [146,147] for getting 10 sets of eigenvalue for an isotropic rectangular plate and orthotropic rectangular plate with classical boundary condition. Orthotropic rectangular plate was extension solution of rectangular plates. Dozio [39] has applied Ritz method for natural frequencies modes shapes for a laminated rectangular plate with clamped, completely-free boundary conditions and trigonometric functions. Free vibration of two elasticity coupled rectangular plates with uniform elastic boundary condition studied by Du et al. [42]. Zhu and Wang [296] used Discrete Singular Convolution method for computing numerical solution of thin isotropic and anisotropic rectangular plates with clamped, simply-supported and completely-free boundary conditions. Dynamic behaviour of large deformable rectangular plate is subjected to a moving mass governed by Nonlinear Nonhomogenous which was investigated by Rofooei and Enhaeian [192]. Wefalli and Karoud [261] successfully obtained natural frequencies for isotropic rectangular plates by using Galerkin-Based Finite Element method with an aspect

ratio and number of element. Hosseini-Hashemi et al. [78] calculated the natural frequencies, mode shape of rectangular Mindlin plates with six different types of boundary conditions. Results were generated by Mindlin Plate Theory, Bernoulli's equation and potential function. Gibbs et al. [57] studied theory flutter of a rectangular plate with a fixed leading edge in three dimensional axial flows. Eftekhari and Jafari [47] also obtained natural frequencies of rectangular plate due to an accelerated traveling mass. Chen et al. [27] studied vibration characteristic and power transmission coupled rectangular plates with boundary condition and arbitrary angle. Asadi and Fariborz [7] have applied Differential Quadrature method for computed natural frequencies of rectangular plate with different boundary conditions or mixed boundary condition. Hu et al. [79] studied free vibration and harmonic forced vibration natural frequency.

In the year 2013, Ibearugbulem and Ezeh [82] have investigated CCCC thin-walled rectangular plate and applied Taylor-Mclaurin's Series as shape function on the Ritz method after putting shape function in total potential energy functional and then computed numerical results. Xu and Zhu [274] were successfully able to apply Non-Contact Experimental technique on rectangular aluminium plate with free boundary condition for measuring modal parameter. Khanna [98] has applied Rayleigh-Ritz method for obtaining frequencies of first two modes of nonhomogenous rectangular plate clamped with four edges, thermal gradient, taper constants and aspect ratio. Khorshid and Farhadi [99] have determined natural frequency of laminated composite rectangular plate by using Rayleigh-Ritz method with clamped and simply-supported geometric boundary conditions, aspect ratio and thickness ratio, fiber orientation. Li et al. [129] presented analytic bending solution of free thin rectangular plate with all edges sliding supported. Analytic solution was obtained by using Geometry method.

Papkov [174] has studied Super Position method for quasi-regular infinite system for rectangular orthotropic plate. Ilanko and Monterrubio [84] have proposed Rayleigh-Ritz and Penalty methods and computed the fundamental natural frequencies, mode shapes with an admissible displacement function and boundary condition of beam, shallow shells and rectangular plan. Khanna and Kaur [94] attempted analytic solutions of nonhomogeneous isotropic visco-elastic rectangular plate in one direction with thickness and Poisson's ratio. Khanna and Singh [96] calculated the frequencies and two modes shape of tapered rectangular plate by using Rayleigh-Ritz method with gradient taper constants and aspect ratio in x-y directions. Nikkhoo et al. [168] have studied resonance of rectangular plate with a multiple travelling masses. Breslavsky et al. [19] have determined static deflection and forced large-amplitude vibration of thin rubber rectangular plate with uniformly distribution pressure and neo-hookean law for nonlinear elastic deformation and geometric non linearity study. Zhang et al. [284] have obtained series of solution using Fourier Series method of orthotropic rectangular plate with elastically restrained edges, double Fourier cosine series and four supplementary functions.

In the year 2014, Zhang et al. [287] computed a series solution of orthotropic rectangular plate by using Fourier Series method with boundary conditions, governing different equations and elastically restrained edges in x-y plane. Shi et al. [204] successfully applied Spectro-Geometric and Rayleigh-Ritz technique for orthotropic thin rectangular plates with trigonometric series, arbitrary boundary conditions. Jin et al. [89] have obtained exact solution of graded rectangular plates with general boundary conditions. Wang and Zhou [258] determined eigenvalues of fourth order characteristic of visco-elastic rectangular plate by using Power Series method and obtained exact solution of given plates with two opposite boundary conditions aspect

ratio (λ) and dimension less delay time. Li et al. [130] have applied Symplectic-Superposition method and determined bending solution of rectangular plate with Hamiltonian system. Nefovska and Petronijevic [167] have obtained Dynamic Stiffness matrix by the use of Projection method and obtained natural frequencies with arbitrary boundary conditions of isotropic rectangular plate. Unruh et al. [238] presented numerical study of rectangular plates with sound radiation properties. Banerjee et al. [14] have discussed dynamic stiffness matrix of rectangular plates by using bi-harmonic equation with general case frequency depending on dynamic stiffness matrix and numerical result solved by Dynamic Stiffness method. Shi et al. [205] attempted series solutions of orthotropic rectangular plates for plane vibration with interval line supports and non-uniform elastic boundary conditions. Ghazvini et al. [55] have introduced Orthonormal Polynomial Series Expansion method for transverse vibration of 2-D system and computed natural frequencies and dynamic response of thin rectangular plate with non-uniform thickness. Li et al. [127] also attempted more than three hundred frequencies and mode shapes of rectangular thin plates.

In the year 2016, Shao et al. [199] have computed unified and analytical solution of composite laminated rectangular plates by Method of Reverberation Ray Matrix with general boundary restraints parameters, layer number and orthotropic ratios. Sharma et al. [200] have determined natural frequencies and mode shape of non uniform and homogenous rectangular plate with boundary condition. Sharma et al. [203] have used Rayleigh-Ritz method to compute frequency equation of orthotropic visco-elastic rectangular plate with two term, thermal gradients, aspect ratio, taper constants, thickness variation and two dimensional temperature. Singhatanadgid and Taranajetsada [223] have obtained natural frequency and displacement function of

stepped rectangular plates using Extended Kantorovich method with boundary conditions and governing equation. Liao and Ma [137] have determined vibration analysis of rectangular plate in compressible fluid. Wang et al. [245] found buckling analytic solution of rectangular thin plate by using Lagrangian Multiplier and Symplectic-Superposition methods with combinations of clamped and simply-supported boundary conditions. Li et al. [136] have computed natural frequencies and mode shapes of rectangular thin plate by using Approximate Numerical methods with boundary condition. Wang and Yuan [255] applied Discrete Singular Convolution Taylor Series Expansion Differential Quadrature methods, for high order frequencies analysis, boundary condition and higher number of grid points of beam and rectangular plates with boundaries conditions. Yang and Wang [275] applied Galerkin method and fourth order Runge-Kutta methods for dynamic buckling strength of rectangular plate with elastically restrained edges. Spectral Element method was used by Zhang [286] for natural surface and high-order Shear Deformation theory for constructed FGM plate with Poisson's ratio. Zhou et al. [290] proposed Reverberation Ray Matrix and Golden Section Search of rectangular plate for calculating exact solution, frequencies, mode shapes with classical boundary conditions. Li et al. [128] has developed analytic solution of thick rectangular plates with clamped and simply-supported boundary conditions. Hosseini-Hahemi et al. [77] has successfully computed natural frequencies with six different combinations of boundary conditions by using Reissner-Mindlin Plate theory. Gupta and Khanna [65] have applied Rayleigh-Ritz method to derive frequency equation with some of the important constants i.e aspect ratio, taper constant, and time period deflection.

In year 2018, Xing et al [273] investigated new Separation-of-Variable and Separation of Variable methods for a free vibration of solution of Kirchhoff rectangular thin

plates with homogeneous boundary conditions. Runge-Kutta method was applied by Wang and Zu [257] for getting non-linear dynamic thermo elastic of rectangular FGM plates with longitudinal velocity. Resonance and Jump analysis of a beam subjected to periodic mass transition was studied by Pirmoradian and Karimpour [175]. Geannakakes [53] has computed natural frequencies by Rayleigh-Ritz method and normalized beam of characteristics orthogonal polynomials with displacement function. Xing and Liu [271] found three classical eigenvalues differential equation and free vibration solution of Mindlin plate and rectangular Mindlin plate by using new two Eigen Function theory and Proposed theory with simply-supported and clamped edge, aspect ratio and relative thickness. Duan and Wang [45] have applied Discrete Singular Convolution technique for determining the free vibration analysis solution of multiple-stepped beam. Boscolo and Banerjee [18] used Sophisticated Layer Wise theory for developed Dynamic Stiffness method with high accuracy for laminated composite plates. Li et al. [125] have also obtained natural frequency and mode shapes for rotating hub-functionally graded material beam with chord wise Bending and Stretching (B-S) coupling effect by using Lagrange's equations of the second kind. Su et al. [232] have studied Modified Fourier Series and Rayleigh-Ritz methods. Exact solution of laminated composite plate was calculated by Rayleigh-Ritz method with general case. Effect of horizontal reaction forces on the deflection of short simple supported beam under transverse loading was studied by Li and Lee [135]. Dong et al. [38] have discussed refined shear deformation beam theory of free vibration analysis of composite laminated beams with various boundary conditions by the use of Method of Reverberation Ray Matrix.

Vijayakumar and Ramaiah [241] have successfully obtained flexural frequencies of clamped square plate by using Modified Bolotin's method. First twenty eigenvalues

divide in three or four symmetric group and then authors applied Bolotin, Rayleigh and Rayleigh-Ritz methods to obtain frequencies. Rayleigh method provided much closer eigenvalues from Bolotin method. Rayleigh-Ritz method applied for the computed eigen modes of admissible functions. This method gives more superior results than other methods. Khanna and Sharma [93, 95, 97] successfully applied Rayleigh-Ritz technique for visco-elastic square plate and nonhomogeneous visco-elastic square plate with thermal gradient, thermal effect and thickness various parabolic in one direction. Numerical results (frequencies and mode shapes) calculated by computational technique and software MATLAB program. Prashar and Sofia [178] calculated frequencies equation for the modes of square plate with thickness, Quadratic temperature, two term deflection function and taper constant with Rayleigh-Ritz technique. Sharma et al. [202] analysed vibration frequency of nonhomogeneous square plate with thermal effect. Numerical results obtained by using Rayleigh-Ritz method in one direction with different boundary conditions and temperature.

In year 2017, Sharma et al. [201] computed vibration frequency by Rayleigh-Ritz method of tapered square plate with clamped boundary condition, aspect ratio in x-y linear direction. Aydogdu and Timarci [10] have obtained frequencies for cross-ply laminated square plate with twelve different cases and clamped, simply-supported, and completely-free boundary conditions by using Ritz method and algebraic polynomial. Onyechere et al. [169] used polynomial displacement functions (u, v and w) and shear deformation for obtain fundamental natural frequency of isotropic thick rectangular plate with simply-supports boundary condition (SSSS). Onyeka et al. [170] have computed static analysis of an isotropic and thick rectangular plate with different types of boundary condition. In this work, we obtain exact relationship

actual displacement, stresses, shear deformation by general variation of total displacement, potential energy.

2.3 SKEW PLATE

In the year 1993 Dokainish and Kumar [37] have solved vibration problem using the Small-Deflection theory of thin orthotropic skew plates with linearly varying thickness, aspect ratio, skew angle, taper parameter and clamped boundary condition. Ritz method has also studied by Nair and Durvasula [164] to compute the natural frequencies and modes shapes of skew plates with skew angles, simply-supported, clamped boundary conditions and different combinations of side ratios. Mizusawa et al. [158] also used Rayleigh-Ritz technique to obtain convergence of solution of skew plates for B-spline function. Liew et al. [143] have given the natural frequencies for thick skew plates by using the pb-2 Rayleigh-Ritz method, two dimensional polynomial function and basic functions. Singh and Saxena [221] have also used Rayleigh-Ritz method to obtain first three frequencies and mode shapes of skew plates for different combinations of boundary conditions, aspect ratio and different skew angles and two dimensional thickness variation. Reddy and Palaninathan [190] investigated the free vibration of skew plate by hierarchical Finite Element method for frequencies with boundary condition. Zitnan [297] has also used Rayleigh-Ritz method, Metodo Rayleigh-Ritz methods for skew and rectangular plates. Woo et al. [264] used P-Version Finite Element method for the analysis of natural frequencies and mode shapes of skew plates with skew angle, aspect ratio, thickness-width ratio and different boundary conditions. Legendre polynomials and Mindlin Plate theory were also used for computing numerical results. Zhou et al. [292] have obtained eigenvalues equation of skew thick plates by using Chebyshev-Ritz method with arbitrary boundary condition and variable thickness in 3-D direction.

Malekzadeh and Karami [148] have applied First Order Shear Deformation Theory (FSDT) for skew laminated composite plates with thickness to the length ratio and different types of boundary conditions. Mohazzab and Dozio [161] have also computed eigenvalue using Spectral Collocation method with various skew angles, aspect ratios and boundary conditions of skew plates. Naghsh and Azhari [165] have investigated Element-Free Galerkin (EFG) method for large amplitude free vibration of supported laminated composite skew plates and proposed Lagrange Multiplier method and Orthogonal Transformation technique with support of boundary condition, skew angle, fiber orientation and aspect ratio. Kumar et al. [104] attempted numerical results of composite skew plates for different skew angles with different boundary conditions and boundary characteristics of orthonormal polynomial. Srinivasa et al. [229] have computed natural frequencies for an isotropic skew plates using QUAD Finite Element method. Malekzadeh and Fiouz [149] studied large deformation analysis of orthotropic skew plate with a nonlinear rotationally restrained by using Different Quadratic Method (DQM). Singha and Daripa [215] investigated Finite Element method and Galerkin's method to explain the behavior of symmetrically laminated composite skew plate. Gurses et al. [73] have obtained numerical results by using Discrete Singular Convolution method of laminated skew plates with fibre orientation angle, different geometric parameters and boundary condition. Zhou and Zheng [295] successfully applied Ritz method to derive the governing eigenvalue equation for skew plate and Moving Least Square Ritz method (MLSR) used for vibration of plate with essential boundary condition.

In year 2009, Ceribasi and Altay [22] have used Rayleigh-Ritz method for obtaining frequency equation and studied vibration of visco-elastic parallelogram plate with only one boundary condition, in one direction, aspect ratio and taper parameter. Dey

and singha [33] applied Finite Element method for solved dynamic stability of composite skew plate with different parameter, skew angle and thickness of span ratio. Gupta and Khanna [66] studied thermal effect on vibration of parallelogram plate of linearly thickness in different area space technology, nuclear energy and mechanical science. Numerical results were identified by Rayleigh-Ritz method for clamped parallelogram plate, different value of aspect ratio, thermal gradient constants and skew angle. Lather and Sharma [107] obtained natural frequencies used Rayleigh-Ritz method, passion ratio and clamped, simply-supported and completely-free boundary conditions for skew plate. Liew and Wang [141] have defined Ritz function on three different types of polynomial function with clamped, simply-supported and completely-free edges and defined the equation of internal line supports of skew plate.

2.4 OTHER PLATES

Kuttler and Sigillito [103] have obtained lower-upper bounds for vibrational frequency of clamped plates with variable thickness and linear taper parameters. Bhat [17] has applied Rayleigh-Ritz method for obtaining natural frequencies with orthogonal polynomial function. Orthogonal polynomials were obtained by using Gram-Schmidt process. Monographs produced. Leissa [112-119] are excellent source of information as transverse vibration of different plates and boundary conditions.

In year 1991, Deif [36] has calculated a general technique for interval matrix for set of upper and lower eigenvalue and eigen pairs. Wang [246] investigated Straight Forward method for free vibration of stepped plates for frequencies margins, circumferential modes and radial nodes. Liew [138] used Rayleigh-Ritz method for the analysis of triangular plate with two dimensional polynomial functions. Numerical results were obtained for ten distinct cases including combination of completely-free,

simply-supported or clamped edges. Inman [87] has determined frequencies, damping ratios and mode shapes and proposed specifically referencing MATLAB, MathCAD and Mathematica. Liew and Wang [140] applied Rayleigh-Ritz and Lagrangian Multiplier methods for computing natural frequencies of the triangular plates with mixed boundary condition. Chen et al. [26] used a Perturbation method for obtaining upper and lower bounds of eigenvalues in structured vibration systems with interval parameters.

In year 1995, Gould [61] has discussed Rayleigh-Ritz and Weinstein method for determining the eigenvalue for vibrating string, rod, membrane and plate. Mathias [153] has applied Jacobi method for obtaining eigenvalue and eigenvector of a positive definite matrix and graded matrix. Al-obeid and Cooper [5] have used Rayleigh-Ritz approach to compute frequencies and mode shapes to laminate plates with different boundary conditions and orthotropic ratio stacking sequence. Saliba [196] determined the eigenvalue for a free vibration of triangular thin plate with one simple-support boundary condition. Guo et al. [68] studied vibration analysis of stepped thickness. Karunasena and Kitipornchai [90] have obtained natural frequencies of general triangular Mindlin plates with three different types of condition clamped, simply-supported and completely-free boundary conditions. Computing results by the use of Rayleigh-Ritz method and Gaussi Quartered approach were accurate. Meylan [157] applied Rayleigh-Ritz method for numerical solution of plate liquid system and linear potential problem solving by appropriate green function. Li [131] studied vibration analysis of generally supported beam with linear combination of Fourier series, polynomial function and boundary condition. Sakiyama and Huang [194] successfully obtained eigenvalue by using approximate method for right triangular plates with various boundary conditions and arbitrary

variable thickness. Wei [259] proposed Benchmark Mechanical approaches for obtaining numerical results of mechanical problem. Free vibration analysis of an isotropic right triangular and orthotropic right cantilever triangular plates have been discussed in [81]. Lewis [123] has determined eigenvalues of symmetric and non symmetric matrices by Classical Mathematical technique, Contemporary Optimization theory and Perturbation theory with hyperbolic polynomial Lie algebra. Du et al. [44] have also applied Reliability-Based Design (RBD) method with an uncertain variables characterized by the mixture of probability distributions and interval variables. Shimpi and Patel [207] have used plate theories for free vibration of plate. Gao [52] has applied Random Factor Method (RFM) and Interval Factor Method (IFM) for obtaining natural frequency and mode shape of truss structure with uncertain parameter. Nachum and Altus [163] have applied Functional Perturbation method by using nonhomogenous properties and obtaining natural frequencies and mode shapes. Rao [183] has discussed continuous structural systems like strings, bars, shafts, circular rings, beams, plates and shells, discussed Integral Transform method, Rayleigh, Rayleigh-Ritz and Buter-Bernoulli methods, etc. and obtained eigen function and natural frequencies of continuous system with different boundary conditions.

Lessia [119] has also studied Rayleigh's, Ritz and Rayleigh-Ritz methods for different types of continuous systems (beam, plate, string etc.) with clamped, simply-supported and completely-free boundary conditions and thickness variations. All techniques computed natural frequency, mode shapes and node shapes. Angeli et al. [6] have computed natural frequencies by using Monte Carlo method or numerical optimization with unknown parameters and uncertainty of system.

In the year 2011, Rong et al. [193] have used two methods, Transfer Matrix method

and Finite Element method. Eigenvalue problem was obtained by using Finite Element method and random eigenvalue problem has solved by using Transfer Matrix method. Monterrubio [159] has obtained eigenvalue problem of box-type structures by using Rayleigh-Ritz and Penalty Function methods. Wang and Cheng [244] studied free vibration of model analysis for discrete and continuous system. Ilanko [83] used Penalty method for eigenvalue, emerging challenges and opportunities of continuous system with elastic stiffness or mass. General Ritz method is also applied by Quintana and Nallim [181] for getting free vibration analysis of thick trapezoidal and triangular laminated with boundary condition. Nefovska-Danilovic et al. [163] used the model for predict traffic-induced vibration of two, six and twelve-story building.

In year 2014, Alijani and Amabli [3] have studied linear of shell on panels and reduced other analytical solutions for accuracy. Yin et al. [278] analyzed static bending, buckling numerical and results were obtained by using framework of Isogeometric Analysis (IGA) with effect boundary conditions. Wang et al. [247] used the Perturbation, Random Interval Moment methods, Hybrid Stochastic and Interval approach for natural frequencies and mode shapes with mixed random and interval parameters. Su et al. [231] attempted unified solution of cylindrical, conical shells and annular plates with general boundary conditions for vibration analysis. Ye et al. [277] have studied modified Fourier solution of moderately thick laminated plates with general boundary conditions. Zhong et al. [289] have applied Finite Cosine Integral Transform method for vibration analysis of plates with four edges boundary conditions. Sofi et al. [226] have computed natural frequencies by using Improved Interval analysis with interval parameter and monotonic function. Cho et al. [30] have obtained natural frequencies and mode shapes by using Mode method with

characteristic polynomial and boundary conditions. Xin and Hu [269] studied State Space Approach (SSA) and Discrete Singular Convolution of laminated cylindrical panels with axial direction in three dimensional boundary conditions. Li et al. [133] have included flexural and longitudinal wave for cross coupling effects at the functions with Finite Element method and boundary condition.

Let us also describe some of the important references related to books on vibrational technique, plates etc.

In 1965 Wilkinson [263] has published review book on algebraic eigenvalue by Iterative Triangularisation methods and provided the solution for linear equation and Hermitian matrix by Jacobi method. In year 1974, Szilard [236] has written a review book on dynamics and stability problem of given plates and used classical finite element numerical solution, modern numerical applications and Finite Difference method. Soedel [227] has reviewed a book on the behaviour of vibration, of shell and plates and explained the effects on vibration, influence rotation, shear rotatory inertia and composite layers and determined natural frequencies and mode shapes using Power Series method and strain energy equations of parabolic cylindrical shell.

In year 1985, Sharnes and Dym [198] have reviewed a book for eigenvalue, eigen function, nodal deflection, various boundary value problems and discussed Rayleigh-Ritz and Finite Element methods for beam, torsion, membrane and plates. Meirovitch [156] has reviewed a book for the primary concepts and techniques for vibration analysis. Chakraverty [24] has studied Rayleigh and Rayleigh-Ritz methods, transverse vibration of plate, and Classical Plate theory and implemented the given method to compute eigenvalue, mode shapes for dynamic behaviour of plate with boundary conditions and continuous basis function of circular, rectangular, skew, square annular plates, etc.

In the year 1877 first book was published on Rayleigh method and Classical plate theory.

In the literature, there is large volume of the research papers on the vibration of plates with thickness variations of different types, but limited research papers related to the present work are briefly explained.

CHAPTER-III

Numerical Experiments on Quarter of an Elliptic Plate with Exponent Thickness Variation Along Radial Direction

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CHAPTER-III

NUMERICAL EXPERIMENTS ON QUARTER OF AN ELLIPTIC PLATE WITH EXPONENT THICKNESS VARIATION ALONG RADIAL DIRECTION

In this chapter, numerical experiments on quarter of elliptic plate have been performed with thickness variation in the form of exponent along the radial direction. First few frequencies play crucial role for getting best structural design. The present work is related to consider a plate in the form of quarter of an elliptic and a variable thickness in the form of exponent along the radial direction. A well-known Rayleigh-Ritz method is used for mathematical solution of the problem in the form of eigenvalue problem. The solution of eigenvalue form is further computed through generalized Jacobi method which gives first few frequencies. The aim of this chapter is to compute first three frequencies and computed results are compared with the existing results for the uniform thickness. The new computed results are presented through tables and graphs. Convergence upto five significant digits are also presented.

3.1 BACKGROUND

Variational methods are used for solutions of thin plates and one of the important methods is the Rayleigh-Ritz technique for computing the first few frequencies for the thin plates. Quarter of an elliptical plate with exponential thickness variation along the radial direction is still not studied by the researchers therefore, the present work is an attempt in this direction. The thickness of thin plate is considered in the form of exponent along radial direction. Combinations of boundary conditions are considered as clamped, simply-supported and completely-free which leads to total combinations of twenty seven cases, but eighteen cases of boundary conditions have been studied as some of the cases produce similar kind of results.

3.1.1 Quarter Elliptic Plate

Elliptic plate is defined as a solid body and there is no internal hole. Internal side of elliptic plate is divided into semi major and semi minor axis. If semi major axis is equal to semi minor axis then elliptic plate is converted into circular plate. Quarter of an elliptic plate is one fourth part of elliptic plate, bounded by two perpendicular lines as shown in figure 3.1.

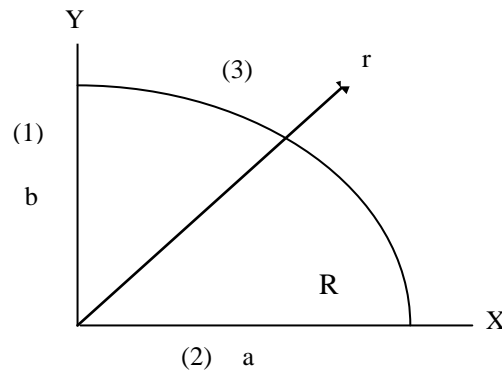


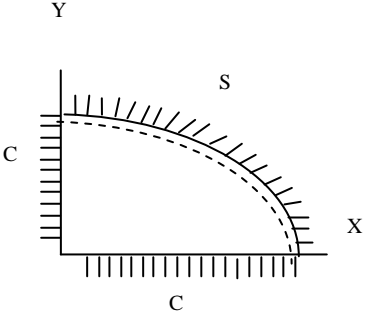
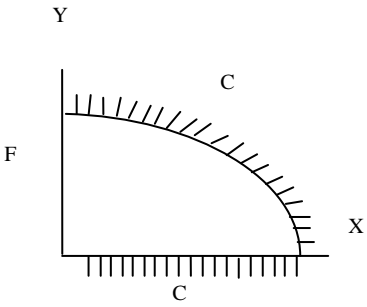
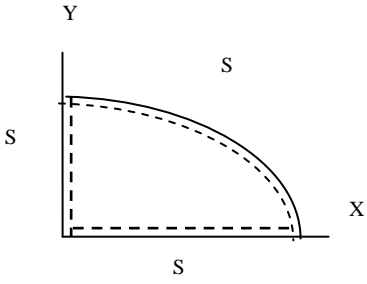
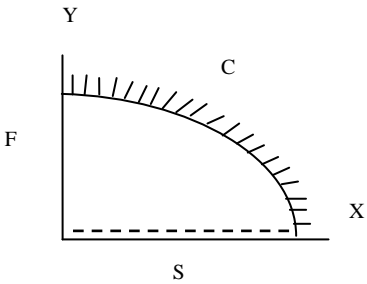
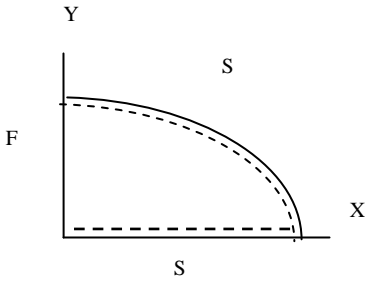
Figure 3.1 Representation of Quarter of Elliptic Plate

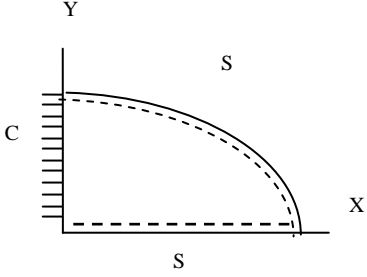
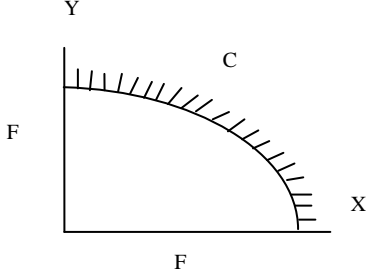
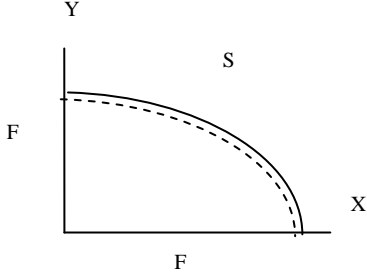
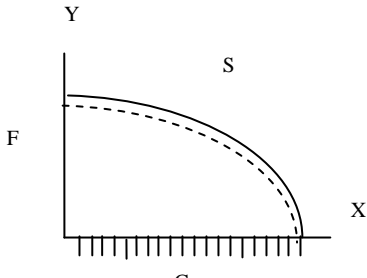
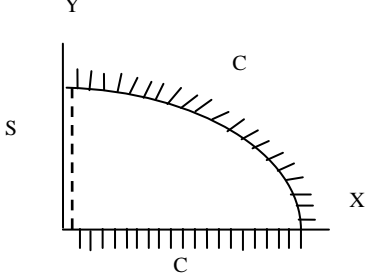
3.1.2 Boundary Conditions

The various combinations of boundary conditions applied on quarter of elliptic plate are shown below in table 3.1, where R is the domain of the plate as shown in figure 3.1. From the figure, the side (1) may be clamped, simply-supported or completely-free. Similar interpretation can be given to other sides represented as (2) and (3).

Table 3.1 Representation of Boundary Conditions on Quarter of Elliptic Plate

| | |
|-----|--|
| CCC | |
|-----|--|

| | |
|------------|--|
| <p>CCS</p> |  |
| <p>FCC</p> |  |
| <p>SSS</p> |  |
| <p>FSC</p> |  |
| <p>FSS</p> |  |

| | |
|------------|--|
| <p>CSS</p> |  |
| <p>FFC</p> |  |
| <p>FFS</p> |  |
| <p>FCS</p> |  |
| <p>SCC</p> |  |

| | |
|------------|--|
| <p>CFS</p> | |
| <p>SSC</p> | |
| <p>SFC</p> | |
| <p>SFS</p> | |
| <p>SCS</p> | |

| | |
|-----|--|
| CFC | |
| CSC | |

3.2 MATHEMATICAL FORMULATION

The equation of elliptic plate can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3.1}$$

$$x^2 + \frac{y^2}{(b/a)^2} = 1 \tag{3.2}$$

Let $m = b/a$ as aspect ratio which controls the length of sides of quarter of elliptic plate and introducing the following non-dimensional variables which reduces the number of variable in the expressions and also reduces the amount of computed numerical data

$$\xi = x/a \quad \eta = y/a \tag{3.3}$$

Therefore, from equation (3.2) can be written as

$$\xi^2 + \frac{\eta^2}{m^2} = 1 \tag{3.4}$$

Let us consider three sides of plate as shown in figure 3.1 and represent as (1), (2), and (3). The three variables p, q, r are taken as 2, 1, 0 which are controlling the boundary conditions, when p=2 then side is clamped, p=1, then side is simply-supported and finally p=0 shows that side is completely-free. The new domain of plate is bounded by equations of three sides taken as $\xi=0$ at side (1), $\eta=0$ at side (2)

and $\xi^2 + \frac{\eta^2}{m^2} = 1$ at side (3). This helps to select the basis function as

$$\phi_i = f(\xi, \eta) \xi^{m_i+p} \eta^{n_i+q}, \quad i = 1, 2, \dots, n \quad (3.6)$$

Where,
$$f(\xi, \eta) = \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^r \quad (3.7)$$

The non negative integers are considered and depicted below for computation purpose for order of matrix as 28.

Table 3.2 Non Negative Values of m_i and n_i

| | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|-------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m_i | 0 | 0 | 1 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 | 0 | 5 | 4 | 3 | 2 | 1 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |
| n_i | 0 | 1 | 0 | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 | 3 | 4 | 5 |

From the Rayleigh-Ritz method as explained in section 1.10, and considering the equation (1.101), the a_{ij} and b_{ij} can be written as for non-dimensional variables ξ, η , as.

$$a_{ij} = \iint_R h^3 \left[\phi_i^{\xi\xi} \phi_j^{\xi\xi} + \phi_i^{\eta\eta} \phi_j^{\eta\eta} + \nu \left(\phi_i^{\xi\xi} \phi_j^{\eta\eta} + \phi_i^{\eta\eta} \phi_j^{\xi\xi} \right) + 2(1-\nu) \phi_i^{\xi\eta} \phi_j^{\xi\eta} \right] d\xi d\eta \quad (3.8)$$

$$b_{ij} = \iint_R h \phi_i \phi_j d\xi d\eta \quad (3.9)$$

Where R is the new domain occupied by the plate and from equation (3.6), we can

get $\phi_i^{\xi\xi}$, $\phi_i^{\eta\eta}$ and $\phi_i^{\xi\eta}$ as given below.

$$\begin{aligned} \phi_i^{\xi\xi} &= (m_i + p)(m_i + p - 1)\xi^{m_i+p-2}\eta^{n_i+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^r + (2(m_i + p) + 1)(-2r) \\ &\xi^{m_i+p}\eta^{n_i+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-1} + 4r(r-1)\xi^{m_i+p+2}\eta^{n_i+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-2} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \phi_i^{\eta\eta} &= (n_i + q)(n_i + q - 1)\xi^{m_i+p}\eta^{n_i+q-2} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^r + (2(n_i + q) + 1)\left(\frac{-2r}{m^2}\right) \\ &\xi^{m_i+p}\eta^{n_i+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-1} + \frac{4r(r-1)}{m^4}\xi^{m_i+p}\eta^{n_i+q+2} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-2} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \phi_i^{\xi\eta} &= (m_i + p)(n_i + q)\xi^{m_i+p-1}\eta^{n_i+q-1} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^r + \left(\frac{-2r}{m^2}\right)(m_i + p)\xi^{m_i+p-1}\eta^{n_i+q+1} \\ &\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-1} + (n_i + q)(-2r)\xi^{m_i+p+1}\eta^{n_i+q-1} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-1} + \frac{4r(r-1)}{m^2} \\ &\xi^{m_i+p+1}\eta^{n_i+q+1} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-2} \end{aligned} \tag{3.12}$$

Similarly, by replacing i into j , we can also get $\phi_j^{\xi\xi}$, $\phi_j^{\eta\eta}$ and $\phi_j^{\xi\eta}$ as

$$\begin{aligned} \phi_j^{\xi\xi} &= (m_j + p)(m_j + p - 1)\xi^{m_j+p-2}\eta^{n_j+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^r + (2(m_j + p) + 1)(-2r) \\ &\xi^{m_j+p}\eta^{n_j+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-1} + 4r(r-1)\xi^{m_j+p+2}\eta^{n_j+q} \left[1 - \left(\xi^2 + \frac{\eta^2}{m^2} \right) \right]^{r-2} \end{aligned} \tag{3.13}$$

$$\begin{aligned} \phi_j^{\eta\eta} &= (n_j + q)(n_j + q - 1)\xi^{m_j+p}\eta^{n_j+q-2}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^r + (2(n_j + q) + 1)\left(\frac{-2r}{m^2}\right) \\ &\xi^{m_j+p}\eta^{n_j+q}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^{r-1} + \frac{4r(r-1)}{m^4}\xi^{m_j+p}\eta^{n_j+q+2}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^{r-2} \end{aligned} \quad (3.14)$$

$$\begin{aligned} \phi_j^{\xi\eta} &= (m_j + p)(n_j + q)\xi^{m_j+p-1}\eta^{n_j+q-1}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^r + \left(\frac{-2r}{m^2}\right)(m_j + p)\xi^{m_j+p-1}\eta^{n_j+q+1} \\ &\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^{r-1} + (n_j + q)(-2r)\xi^{m_j+p+1}\eta^{n_j+q-1}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^{r-1} + \frac{4r(r-1)}{m^2}\xi^{m_j+p+1} \\ &\eta^{n_j+q+1}\left[1 - \left(\xi^2 + \frac{\eta^2}{m^2}\right)\right]^{r-2} \end{aligned} \quad (3.15)$$

Now, considering the thickness of the plate as exponential along the radial direction r and given by

$$h = e^{\alpha r} \quad (3.16)$$

Where α is the taper parameter taken as -0.9 to 0.9. The thickness of the plate is represented in figure 3.2.

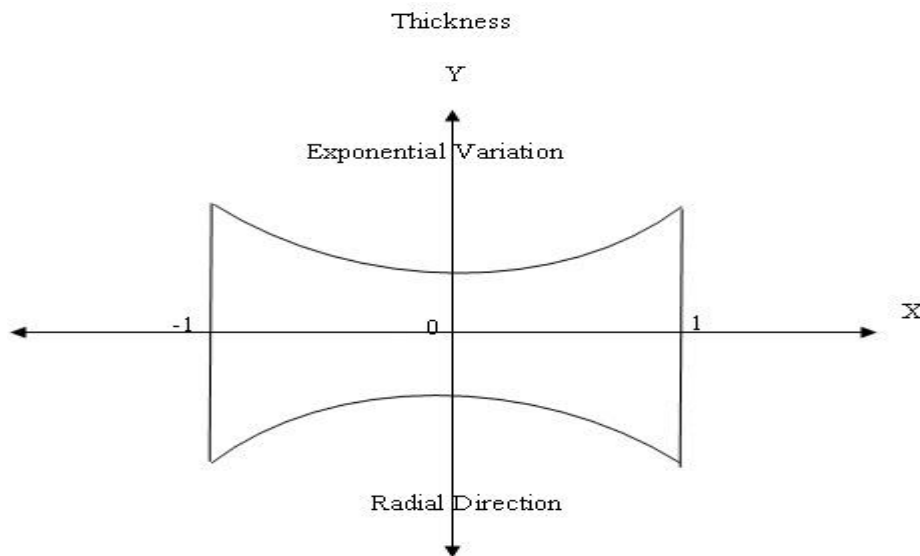


Figure 3.2 Exponential Thickness Variation for Quarter of Elliptic Plate

By putting the values of $h, \phi_i, \phi_i^{\xi\xi}, \phi_i^{\eta\eta}, \phi_i^{\xi\eta}, \phi_j^{\xi\xi}, \phi_j^{\eta\eta}, \phi_j^{\xi\eta}$ in equations (3.8), and (3.9) solving the equation (1.101) by generalized Jacobi method, we can get the first three eigenvalues for quarter of elliptic plate. The computations involve the following types of integrals.

$$\iint_R \xi^k \eta^l r^m (1-r^2)^n d\xi d\eta = \frac{\left(\frac{k+l+m}{2}+1\right)\left(\frac{k+1}{2}\right)\left(\frac{l+1}{2}\right)\overline{(n+1)}}{4\left(\frac{k+l+m}{2}+n+2\right)\left(\frac{k+l}{2}+1\right)}, \quad (3.17)$$

3.3 NUMERICAL RESULTS AND DISCUSSION

On the basis of above formulation of the problem, a Fortran program has been designed for extensive numerical computations and for all the calculations, the following values of the constants and variables have been considered.

1. The value of Poisson's ratio ν is taken as 0.3 for isotropic plate;
2. F, S, C can take value 0, 1, 2 for a completely-free, simply-supported and clamped plates, respectively;
3. The thickness is controlled by the parameter α which is taken as variable from -0.9 to +0.9;
4. The values of m are considered as 0.25, 0.5, and 0.75 for quarter of elliptic plate.

By the use of above parameters, we solved all the integral given in (3.17) and computed a_{ij} and b_{ij} as given in the equation (3.8) & (3.9), respectively. By the use of the generalized Jacobi method, eigenvalue problem can be solved and first few frequencies have been computed for some selected cases as described. Tables 3.3, 3.4 and 3.5 demonstrate the first three frequencies for various boundary conditions for various values of α running from -0.9 to +0.9. From the tables, it is observed that when taper parameter α is increasing from -0.9 to +0.9 then all the frequencies are increasing in the all cases.

Table 3.3 First Three Frequencies for Quarter of Elliptic Plate ($m=0.25$, $U =0.3$)

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -0.1 | 0.0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| FFS | 3.8540 | 4.747 | 5.849 | 7.208 | 8.887 | 9.869 | 10.961 | 13.526 | 16.701 | 20.636 | 25.520 |
| | 25.077 | 28.648 | 32.819 | 37.720 | 43.514 | 46.808 | 50.405 | 58.647 | 68.553 | 80.513 | 95.007 |
| | 86.949 | 99.853 | 114.74 | 131.87 | 151.56 | 162.49 | 174.24 | 200.48 | 231.02 | 266.77 | 308.88 |
| FFC | 24.379 | 30.349 | 37.700 | 46.722 | 57.756 | 64.147 | 71.193 | 87.482 | 107.13 | 130.69 | 158.76 |
| | 47.161 | 55.560 | 65.597 | 77.596 | 91.939 | 100.12 | 109.06 | 129.49 | 153.79 | 182.62 | 216.68 |
| | 114.38 | 132.06 | 152.85 | 177.11 | 205.19 | 220.78 | 237.49 | 274.54 | 317.20 | 366.72 | 424.78 |
| FSS | 99.521 | 112.51 | 126.65 | 141.99 | 158.56 | 167.33 | 176.42 | 195.61 | 216.23 | 238.36 | 262.15 |
| | 129.41 | 146.75 | 166.20 | 188.03 | 212.55 | 225.92 | 240.09 | 271.09 | 306.04 | 345.60 | 390.54 |
| | 220.55 | 251.07 | 285.64 | 325.60 | 372.44 | 398.92 | 427.71 | 493.04 | 570.26 | 661.46 | 769.21 |
| FSC | 141.79 | 162.41 | 185.66 | 211.88 | 241.52 | 257.78 | 275.10 | 313.23 | 356.66 | 406.22 | 462.89 |
| | 175.44 | 200.69 | 229.44 | 262.20 | 299.58 | 320.22 | 342.28 | 391.09 | 446.91 | 510.77 | 583.88 |
| | 269.64 | 310.30 | 355.69 | 406.50 | 464.35 | 496.60 | 531.48 | 610.46 | 704.02 | 815.06 | 946.70 |
| FCS | 174.80 | 190.82 | 207.97 | 226.45 | 246.41 | 256.99 | 267.98 | 291.29 | 316.46 | 343.64 | 373.00 |
| | 195.18 | 217.62 | 242.79 | 270.92 | 302.33 | 319.39 | 337.42 | 376.68 | 420.72 | 470.33 | 526.49 |
| | 280.90 | 319.49 | 364.45 | 417.16 | 479.11 | 514.10 | 552.08 | 638.17 | 739.90 | 860.34 | 1003.2 |
| FSS | 231.22 | 256.32 | 284.20 | 315.39 | 350.46 | 369.64 | 390.03 | 434.85 | 485.73 | 543.64 | 609.69 |
| | 253.73 | 285.50 | 321.53 | 362.38 | 408.69 | 434.11 | 461.19 | 520.77 | 588.44 | 665.47 | 753.37 |
| | 342.91 | 390.98 | 445.52 | 508.58 | 582.42 | 624.11 | 669.40 | 772.08 | 893.37 | 1036.6 | 1206.0 |
| SFS | 16.370 | 18.233 | 20.441 | 23.085 | 26.281 | 28.130 | 30.174 | 34.942 | 40.805 | 48.035 | 56.966 |
| | 40.484 | 45.932 | 52.203 | 59.459 | 67.905 | 72.651 | 77.795 | 89.446 | 103.25 | 119.70 | 139.39 |
| | 121.42 | 138.05 | 156.34 | 177.05 | 201.28 | 215.09 | 230.22 | 264.97 | 306.52 | 355.75 | 413.43 |
| SFC | 37.443 | 44.275 | 52.579 | 62.660 | 74.879 | 81.915 | 89.644 | 107.42 | 128.76 | 154.23 | 184.50 |
| | 64.986 | 75.274 | 87.358 | 101.57 | 118.33 | 127.80 | 138.08 | 161.37 | 188.82 | 221.13 | 259.12 |
| | 149.33 | 171.49 | 196.80 | 225.13 | 256.81 | 274.25 | 293.03 | 335.63 | 386.61 | 447.86 | 520.98 |
| SSS | 115.67 | 131.02 | 148.10 | 167.04 | 188.00 | 199.28 | 211.14 | 236.69 | 264.92 | 296.17 | 330.92 |
| | 149.31 | 170.08 | 193.58 | 220.15 | 250.22 | 266.73 | 284.32 | 323.07 | 367.18 | 417.51 | 475.07 |
| | 249.40 | 288.14 | 334.33 | 388.89 | 452.68 | 488.30 | 526.57 | 611.67 | 709.68 | 823.21 | 955.93 |

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -0.1 | 0.0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| SSC | 160.12 | 183.30 | 209.64 | 239.54 | 273.48 | 292.15 | 312.04 | 355.89 | 405.81 | 462.76 | 527.82 |
| | 197.50 | 226.45 | 259.50 | 297.22 | 340.27 | 364.04 | 389.44 | 445.68 | 510.14 | 584.15 | 669.18 |
| | 301.18 | 343.85 | 393.10 | 451.56 | 521.66 | 561.78 | 605.60 | 705.35 | 822.75 | 959.79 | 1119.1 |
| SCS | 182.77 | 203.03 | 225.26 | 249.70 | 276.57 | 290.99 | 306.12 | 338.65 | 374.50 | 414.09 | 457.96 |
| | 212.18 | 238.98 | 269.23 | 303.41 | 342.06 | 363.24 | 385.77 | 435.27 | 491.39 | 555.17 | 627.91 |
| | 318.26 | 368.61 | 427.26 | 495.39 | 574.53 | 618.83 | 666.69 | 774.47 | 901.07 | 1050.3 | 1227.0 |
| SCC | 240.28 | 269.84 | 302.97 | 340.21 | 382.14 | 405.07 | 429.43 | 482.86 | 543.36 | 612.00 | 690.06 |
| | 272.28 | 308.75 | 350.18 | 397.26 | 450.80 | 480.29 | 511.77 | 581.26 | 660.59 | 751.28 | 855.21 |
| | 369.72 | 424.46 | 489.34 | 566.10 | 656.60 | 707.65 | 762.97 | 887.76 | 1034.1 | 1206.1 | 1408.5 |
| CFS | 19.785 | 21.922 | 24.410 | 27.340 | 30.827 | 32.823 | 35.015 | 40.085 | 46.260 | 53.819 | 63.107 |
| | 47.115 | 53.180 | 60.126 | 68.120 | 77.370 | 82.545 | 88.137 | 100.74 | 115.59 | 133.19 | 154.13 |
| | 78.236 | 138.86 | 164.38 | 193.47 | 221.46 | 235.63 | 250.62 | 284.23 | 324.21 | 372.48 | 431.12 |
| CFC | 42.023 | 49.189 | 57.842 | 68.293 | 80.904 | 88.148 | 96.094 | 114.34 | 136.20 | 162.28 | 193.25 |
| | 72.365 | 83.335 | 96.164 | 111.19 | 128.83 | 138.77 | 149.55 | 173.92 | 202.57 | 236.27 | 275.90 |
| | 141.51 | 163.08 | 189.67 | 222.64 | 263.46 | 287.25 | 313.04 | 361.37 | 413.67 | 476.45 | 552.70 |
| CSS | 118.57 | 134.57 | 152.43 | 172.30 | 194.37 | 206.29 | 218.83 | 245.94 | 276.01 | 309.43 | 346.73 |
| | 154.48 | 176.29 | 201.04 | 229.08 | 260.84 | 278.27 | 296.82 | 337.64 | 384.08 | 437.14 | 498.06 |
| | 256.19 | 292.50 | 335.24 | 386.34 | 447.82 | 483.13 | 521.87 | 610.91 | 717.81 | 845.99 | 999.55 |
| CSC | 163.54 | 187.47 | 214.70 | 245.65 | 280.84 | 300.21 | 320.86 | 366.40 | 418.30 | 477.54 | 545.31 |
| | 203.38 | 233.49 | 267.89 | 307.17 | 352.01 | 376.74 | 403.17 | 461.55 | 528.26 | 604.71 | 692.68 |
| | 283.49 | 335.25 | 398.04 | 467.87 | 539.52 | 580.10 | 624.57 | 726.90 | 850.07 | 998.14 | 1175.8 |
| CCS | 184.82 | 206.01 | 229.38 | 255.18 | 283.66 | 299.00 | 315.12 | 349.90 | 388.44 | 431.28 | 479.09 |
| | 216.75 | 244.91 | 276.71 | 312.59 | 353.13 | 375.37 | 399.06 | 451.30 | 510.98 | 579.47 | 658.42 |
| | 313.30 | 361.00 | 418.20 | 486.67 | 568.43 | 615.04 | 665.90 | 781.97 | 920.16 | 1084.7 | 1280.7 |
| CCC | 242.72 | 273.34 | 307.77 | 346.55 | 390.28 | 414.22 | 439.68 | 495.59 | 559.05 | 631.31 | 713.91 |
| | 277.54 | 315.47 | 358.58 | 407.57 | 463.26 | 493.92 | 526.65 | 599.08 | 682.18 | 778.04 | 889.13 |
| | 379.09 | 435.46 | 501.25 | 579.40 | 672.61 | 725.75 | 783.77 | 916.25 | 1074.0 | 1261.7 | 1485.2 |

Table 3.4 First Three Frequencies for Quarter of Elliptic Plate (m=0.5, $\nu=0.3$)

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -1.0 | 0.0 | 1.0 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| FFS | 2.025 | 2.449 | 2.963 | 3.589 | 4.352 | 4.795 | 5.285 | 6.429 | 7.832 | 9.560 | 11.689 |
| | 13.924 | 16.114 | 18.653 | 21.602 | 25.036 | 26.963 | 29.045 | 33.738 | 39.246 | 45.728 | 53.373 |
| | 38.089 | 43.947 | 50.572 | 58.043 | 66.444 | 71.021 | 75.865 | 86.403 | 98.159 | 111.25 | 125.81 |
| FFC | 7.318 | 9.079 | 11.241 | 13.886 | 17.109 | 18.972 | 21.023 | 25.753 | 31.443 | 38.249 | 46.344 |
| | 21.018 | 24.586 | 28.769 | 33.674 | 39.425 | 42.662 | 46.166 | 54.063 | 63.304 | 74.101 | 86.689 |
| | 51.781 | 59.998 | 69.374 | 80.055 | 92.203 | 98.884 | 106.00 | 121.65 | 139.39 | 159.46 | 182.15 |
| FSS | 25.964 | 29.417 | 33.209 | 37.365 | 41.912 | 44.343 | 46.886 | 52.335 | 58.315 | 64.904 | 72.199 |
| | 44.001 | 49.875 | 56.572 | 64.223 | 72.979 | 77.826 | 83.019 | 94.549 | 107.81 | 123.09 | 140.73 |
| | 102.04 | 114.63 | 128.01 | 142.77 | 159.09 | 167.87 | 177.11 | 197.02 | 219.00 | 243.28 | 270.19 |
| FSC | 37.006 | 42.453 | 48.620 | 55.609 | 63.543 | 67.909 | 72.569 | 82.863 | 94.631 | 108.11 | 123.58 |
| | 57.257 | 65.313 | 74.548 | 85.145 | 97.312 | 104.05 | 111.29 | 127.34 | 145.80 | 166.98 | 191.30 |
| | 119.27 | 137.20 | 157.82 | 177.92 | 199.45 | 211.12 | 223.45 | 250.26 | 280.19 | 313.61 | 350.94 |
| FCS | 44.418 | 48.659 | 53.291 | 58.351 | 63.885 | 66.845 | 69.943 | 76.587 | 83.889 | 91.940 | 100.85 |
| | 59.062 | 66.540 | 75.022 | 84.656 | 95.616 | 101.65 | 108.10 | 122.35 | 138.64 | 157.30 | 178.71 |
| | 116.88 | 135.37 | 156.70 | 180.20 | 199.82 | 210.22 | 221.13 | 244.62 | 270.51 | 299.07 | 330.63 |
| FCC | 58.930 | 65.525 | 72.935 | 81.283 | 90.711 | 95.882 | 101.38 | 113.51 | 127.32 | 143.09 | 161.14 |
| | 75.128 | 85.129 | 96.527 | 109.52 | 124.36 | 132.55 | 141.31 | 160.68 | 182.82 | 208.13 | 237.06 |
| | 135.02 | 155.86 | 180.04 | 208.00 | 240.21 | 257.80 | 272.51 | 303.42 | 337.81 | 376.10 | 418.77 |
| SFS | 9.634 | 10.829 | 12.192 | 13.760 | 15.579 | 16.600 | 17.707 | 20.215 | 23.194 | 26.756 | 31.038 |
| | 28.821 | 32.623 | 36.942 | 41.858 | 47.465 | 50.561 | 53.872 | 61.212 | 69.639 | 79.335 | 90.518 |
| | 57.653 | 65.466 | 74.335 | 84.405 | 95.830 | 102.10 | 108.78 | 123.46 | 140.17 | 159.26 | 181.18 |
| SFC | 15.971 | 18.496 | 21.482 | 25.023 | 29.230 | 31.622 | 34.231 | 40.173 | 47.224 | 55.570 | 65.416 |
| | 38.012 | 43.341 | 49.445 | 56.445 | 64.481 | 68.936 | 73.712 | 84.323 | 96.520 | 110.53 | 126.64 |
| | 69.998 | 79.693 | 90.792 | 103.55 | 118.21 | 126.32 | 135.00 | 154.18 | 176.04 | 200.92 | 229.21 |
| FFF | 35.834 | 40.589 | 45.981 | 52.084 | 58.985 | 62.764 | 66.781 | 75.593 | 85.561 | 96.860 | 109.70 |
| | 58.505 | 66.894 | 76.510 | 87.525 | 100.13 | 107.10 | 114.56 | 131.07 | 149.97 | 171.63 | 196.48 |
| | 118.20 | 133.16 | 149.89 | 168.59 | 189.49 | 200.84 | 212.84 | 239.00 | 268.36 | 301.41 | 338.81 |

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -1.0 | 0.0 | 1.0 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| SSC | 48.208 | 55.091 | 62.978 | 72.007 | 82.338 | 88.047 | 94.153 | 107.66 | 123.11 | 140.77 | 160.97 |
| | 73.385 | 84.135 | 96.467 | 110.59 | 126.77 | 135.71 | 145.27 | 166.40 | 190.52 | 218.02 | 249.38 |
| | 142.57 | 161.10 | 181.92 | 205.28 | 231.51 | 245.80 | 260.93 | 293.94 | 330.99 | 372.66 | 419.60 |
| SCS | 51.433 | 57.682 | 64.697 | 72.571 | 81.407 | 86.222 | 91.324 | 102.46 | 114.98 | 129.09 | 145.02 |
| | 72.671 | 82.970 | 94.745 | 108.19 | 123.56 | 132.04 | 141.11 | 161.15 | 184.06 | 210.27 | 240.28 |
| | 143.87 | 163.71 | 183.67 | 205.92 | 230.74 | 244.20 | 258.42 | 289.31 | 323.80 | 362.41 | 405.78 |
| SCC | 66.841 | 75.575 | 85.487 | 96.738 | 109.50 | 116.53 | 124.01 | 140.48 | 159.19 | 180.47 | 204.66 |
| | 90.158 | 103.16 | 118.05 | 135.06 | 154.48 | 165.19 | 176.63 | 201.88 | 230.64 | 263.38 | 300.66 |
| | 149.62 | 172.32 | 200.21 | 234.42 | 276.16 | 293.78 | 311.35 | 349.60 | 392.42 | 440.35 | 494.05 |
| CFS | 13.669 | 15.083 | 16.686 | 18.515 | 20.616 | 21.787 | 23.049 | 25.886 | 29.219 | 33.162 | 37.859 |
| | 35.422 | 39.902 | 44.967 | 50.701 | 57.203 | 60.777 | 64.589 | 72.999 | 82.598 | 93.578 | 106.16 |
| | 66.174 | 75.215 | 85.385 | 96.853 | 109.83 | 116.96 | 124.55 | 141.20 | 159.97 | 181.09 | 204.89 |
| CFC | 20.726 | 23.526 | 26.811 | 30.680 | 35.247 | 37.832 | 40.644 | 47.027 | 54.571 | 63.475 | 73.960 |
| | 45.363 | 51.426 | 58.335 | 66.213 | 75.205 | 80.170 | 85.477 | 97.218 | 110.64 | 126.01 | 143.59 |
| | 79.259 | 90.053 | 102.30 | 116.21 | 131.89 | 140.45 | 149.55 | 169.62 | 192.52 | 218.66 | 248.46 |
| CSS | 38.662 | 43.931 | 49.917 | 56.708 | 64.404 | 68.627 | 73.121 | 82.994 | 94.183 | 106.88 | 121.32 |
| | 63.724 | 73.023 | 83.684 | 95.899 | 109.88 | 117.61 | 125.88 | 144.19 | 165.13 | 189.09 | 216.53 |
| | 117.70 | 137.01 | 154.63 | 174.27 | 196.28 | 208.28 | 221.00 | 248.87 | 280.41 | 316.30 | 357.39 |
| CSC | 51.497 | 58.961 | 67.517 | 77.317 | 88.534 | 94.734 | 101.36 | 116.04 | 132.81 | 151.99 | 173.89 |
| | 79.168 | 90.896 | 104.34 | 119.75 | 137.38 | 147.11 | 157.52 | 180.50 | 206.71 | 236.54 | 270.47 |
| | 114.86 | 132.22 | 152.14 | 174.98 | 203.05 | 221.13 | 241.12 | 287.41 | 340.94 | 387.40 | 437.88 |
| CCS | 53.978 | 60.868 | 68.634 | 77.383 | 87.236 | 92.618 | 98.329 | 110.82 | 124.90 | 140.79 | 158.77 |
| | 77.661 | 88.951 | 101.87 | 116.66 | 133.57 | 142.91 | 152.89 | 174.96 | 200.19 | 229.08 | 262.22 |
| | 148.65 | 167.50 | 188.39 | 211.69 | 237.70 | 251.84 | 266.81 | 299.48 | 336.32 | 378.03 | 425.45 |
| CCC | 69.808 | 79.265 | 90.020 | 102.24 | 116.14 | 123.79 | 131.94 | 149.90 | 170.32 | 193.53 | 219.92 |
| | 95.690 | 109.76 | 125.88 | 144.31 | 165.36 | 176.97 | 189.37 | 216.74 | 247.90 | 283.37 | 323.80 |
| | 130.15 | 150.16 | 178.75 | 214.12 | 256.84 | 281.34 | 307.98 | 359.36 | 405.40 | 456.63 | 514.49 |

Table 3.5 First Three Frequencies for Quarter of Elliptic Plate ($m=0.75, U =0.3$)

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -0.1 | 0.0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| FFS | 1.4220 | 1.7080 | 2.0510 | 2.4630 | 2.9610 | 3.2480 | 3.5630 | 4.2920 | 5.1800 | 6.2620 | 7.5850 |
| | 10.797 | 12.538 | 14.552 | 16.883 | 19.586 | 21.095 | 22.721 | 26.361 | 30.593 | 35.518 | 41.250 |
| | 30.501 | 34.718 | 39.567 | 45.151 | 51.589 | 55.169 | 59.013 | 67.563 | 77.379 | 88.599 | 101.34 |
| FFC | 4.1070 | 5.0790 | 6.270 | 7.723 | 9.492 | 10.513 | 11.636 | 14.223 | 17.332 | 21.047 | 25.462 |
| | 15.373 | 17.960 | 20.973 | 24.482 | 28.566 | 30.851 | 33.315 | 38.832 | 45.234 | 52.651 | 61.226 |
| | 37.897 | 43.259 | 49.422 | 56.506 | 64.645 | 69.155 | 73.986 | 84.694 | 96.942 | 110.91 | 126.78 |
| FSS | 12.121 | 13.741 | 15.534 | 17.516 | 19.710 | 20.894 | 22.143 | 24.851 | 27.877 | 31.277 | 35.120 |
| | 26.395 | 29.896 | 33.925 | 38.569 | 43.929 | 46.913 | 50.120 | 57.279 | 65.559 | 75.140 | 86.226 |
| | 47.378 | 53.946 | 61.808 | 70.959 | 81.554 | 87.457 | 93.804 | 107.95 | 124.27 | 143.06 | 164.67 |
| FSC | 17.396 | 19.976 | 22.912 | 26.259 | 30.084 | 32.200 | 34.466 | 39.496 | 45.281 | 51.947 | 59.632 |
| | 33.674 | 38.337 | 43.711 | 49.907 | 57.052 | 61.024 | 65.288 | 74.777 | 85.699 | 98.256 | 112.66 |
| | 57.875 | 65.402 | 74.616 | 85.539 | 98.164 | 105.17 | 112.68 | 129.34 | 148.43 | 170.25 | 195.15 |
| FCS | 20.190 | 22.171 | 24.357 | 26.773 | 29.444 | 30.886 | 32.405 | 35.694 | 39.362 | 43.470 | 48.097 |
| | 32.537 | 36.845 | 41.776 | 47.429 | 53.918 | 57.516 | 61.375 | 69.953 | 79.829 | 91.205 | 104.31 |
| | 60.188 | 66.812 | 74.149 | 82.293 | 92.572 | 99.303 | 106.56 | 122.76 | 141.46 | 163.02 | 187.87 |
| FCC | 26.975 | 30.069 | 33.572 | 37.546 | 42.065 | 44.556 | 47.216 | 53.101 | 59.839 | 67.570 | 76.451 |
| | 40.898 | 46.504 | 52.932 | 60.306 | 68.768 | 73.456 | 78.478 | 89.619 | 102.39 | 117.03 | 133.78 |
| | 72.292 | 80.650 | 89.969 | 100.36 | 112.01 | 118.75 | 127.10 | 145.98 | 167.65 | 192.44 | 193.61 |
| SFS | 7.9030 | 8.9330 | 10.088 | 11.389 | 12.863 | 13.674 | 14.542 | 16.468 | 18.693 | 21.279 | 24.309 |
| | 24.743 | 28.069 | 31.874 | 36.223 | 41.187 | 43.923 | 46.843 | 53.276 | 60.573 | 68.827 | 78.135 |
| | 47.153 | 54.210 | 62.341 | 71.706 | 82.474 | 88.439 | 94.821 | 108.90 | 124.84 | 142.73 | 162.68 |
| SFC | 11.897 | 13.693 | 15.773 | 18.188 | 21.002 | 22.581 | 24.288 | 28.131 | 32.629 | 37.891 | 44.042 |
| | 31.566 | 35.968 | 41.010 | 46.779 | 53.369 | 57.004 | 60.885 | 69.442 | 79.165 | 90.192 | 102.67 |
| | 56.800 | 65.274 | 75.000 | 86.144 | 98.883 | 105.90 | 113.39 | 129.86 | 148.41 | 169.14 | 192.10 |
| SSS | 20.192 | 22.842 | 25.875 | 29.341 | 33.299 | 35.483 | 37.816 | 42.974 | 48.867 | 55.610 | 63.341 |
| | 39.244 | 45.014 | 51.651 | 59.279 | 68.038 | 72.890 | 78.084 | 89.593 | 102.76 | 117.80 | 134.97 |
| | 64.538 | 74.741 | 86.544 | 100.18 | 115.92 | 124.67 | 134.06 | 154.94 | 178.96 | 206.61 | 238.41 |

| $\alpha \rightarrow$ | -0.9 | -0.7 | -0.5 | -0.3 | -0.1 | 0.0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| SSC | 26.721 | 30.471 | 34.788 | 39.753 | 45.458 | 48.620 | 52.008 | 59.525 | 68.145 | 78.024 | 89.339 |
| | 48.232 | 55.376 | 63.579 | 72.987 | 83.763 | 89.719 | 96.086 | 110.15 | 126.18 | 144.40 | 165.08 |
| | 76.090 | 87.977 | 101.66 | 117.39 | 135.43 | 145.41 | 156.09 | 179.67 | 206.50 | 236.94 | 271.43 |
| SCS | 26.639 | 30.033 | 33.886 | 38.259 | 43.222 | 45.949 | 48.854 | 55.248 | 62.513 | 70.780 | 80.202 |
| | 45.160 | 51.876 | 59.596 | 68.465 | 78.643 | 84.279 | 90.313 | 103.68 | 118.98 | 136.47 | 156.46 |
| | 72.216 | 82.458 | 95.382 | 110.48 | 127.97 | 137.72 | 148.19 | 171.54 | 198.51 | 229.66 | 265.64 |
| SCC | 34.393 | 39.036 | 44.341 | 50.401 | 57.320 | 61.139 | 65.219 | 74.232 | 84.515 | 96.242 | 109.61 |
| | 55.110 | 63.339 | 72.783 | 83.608 | 96.000 | 102.84 | 110.16 | 126.32 | 144.73 | 165.66 | 189.43 |
| | 83.066 | 95.355 | 110.39 | 128.81 | 148.82 | 159.86 | 171.68 | 197.83 | 227.68 | 261.71 | 300.49 |
| CFS | 12.228 | 13.466 | 14.852 | 16.410 | 18.168 | 19.132 | 20.160 | 22.430 | 25.032 | 28.033 | 31.517 |
| | 29.186 | 33.220 | 37.848 | 43.150 | 49.215 | 52.561 | 56.131 | 63.985 | 72.848 | 82.767 | 93.764 |
| | 52.514 | 60.560 | 69.855 | 80.587 | 92.968 | 99.851 | 107.23 | 123.62 | 142.31 | 163.24 | 185.90 |
| CFC | 16.999 | 19.077 | 21.468 | 24.229 | 27.426 | 29.211 | 31.136 | 35.451 | 40.474 | 46.326 | 53.142 |
| | 36.812 | 42.033 | 48.019 | 54.865 | 62.675 | 66.975 | 71.555 | 81.610 | 92.944 | 105.66 | 119.90 |
| | 62.883 | 72.461 | 83.477 | 96.127 | 110.62 | 118.62 | 127.17 | 145.97 | 167.24 | 191.04 | 217.13 |
| CSS | 23.283 | 26.401 | 29.963 | 34.028 | 38.665 | 41.223 | 43.955 | 49.988 | 56.875 | 64.742 | 73.745 |
| | 44.184 | 50.807 | 58.429 | 67.192 | 77.255 | 82.829 | 88.796 | 102.00 | 117.09 | 134.28 | 153.81 |
| | 70.935 | 82.296 | 95.469 | 110.72 | 128.37 | 138.20 | 148.75 | 172.20 | 199.04 | 229.58 | 264.14 |
| CSC | 30.337 | 34.624 | 39.547 | 45.196 | 51.675 | 55.261 | 59.100 | 67.604 | 77.338 | 88.473 | 101.20 |
| | 53.830 | 61.909 | 71.185 | 81.820 | 93.995 | 100.71 | 107.90 | 123.75 | 141.78 | 162.22 | 185.32 |
| | 76.256 | 86.262 | 97.558 | 112.04 | 131.97 | 143.65 | 156.61 | 186.91 | 224.04 | 261.59 | 299.37 |
| CCS | 29.545 | 33.480 | 37.954 | 43.039 | 48.818 | 51.996 | 55.383 | 62.843 | 71.323 | 80.971 | 91.963 |
| | 50.077 | 57.686 | 66.445 | 76.519 | 88.091 | 94.501 | 101.36 | 116.55 | 133.92 | 153.74 | 176.34 |
| | 78.147 | 90.314 | 104.68 | 121.42 | 140.83 | 151.64 | 163.25 | 189.09 | 218.81 | 252.95 | 292.2 |
| CCC | 37.794 | 43.054 | 49.066 | 55.932 | 63.772 | 68.098 | 72.718 | 82.922 | 94.554 | 107.80 | 122.89 |
| | 60.647 | 69.849 | 80.417 | 92.538 | 106.41 | 114.08 | 122.27 | 140.36 | 160.94 | 184.30 | 210.76 |
| | 88.118 | 100.11 | 113.30 | 128.19 | 152.38 | 167.41 | 183.94 | 216.60 | 249.43 | 286.68 | 328.98 |

The frequencies are increasing due to increase in the stiffness of the plate. These frequencies are also represented through figures 3.3, 3.4 and 3.5. The three cases are represented through different colours. red colour represents CCS, blue colour represents FCC and CCC shows by yellow colour. From the figures, it is observed that the three frequencies are in increasing form. Similar representations also stand for figures 3.4 and 3.5.

To check the validity of computed results, some frequencies are compared in the case of clamped quarter of an elliptic plate. It is observed that the frequencies are matching upto five significant digits as represented through table 3.6.

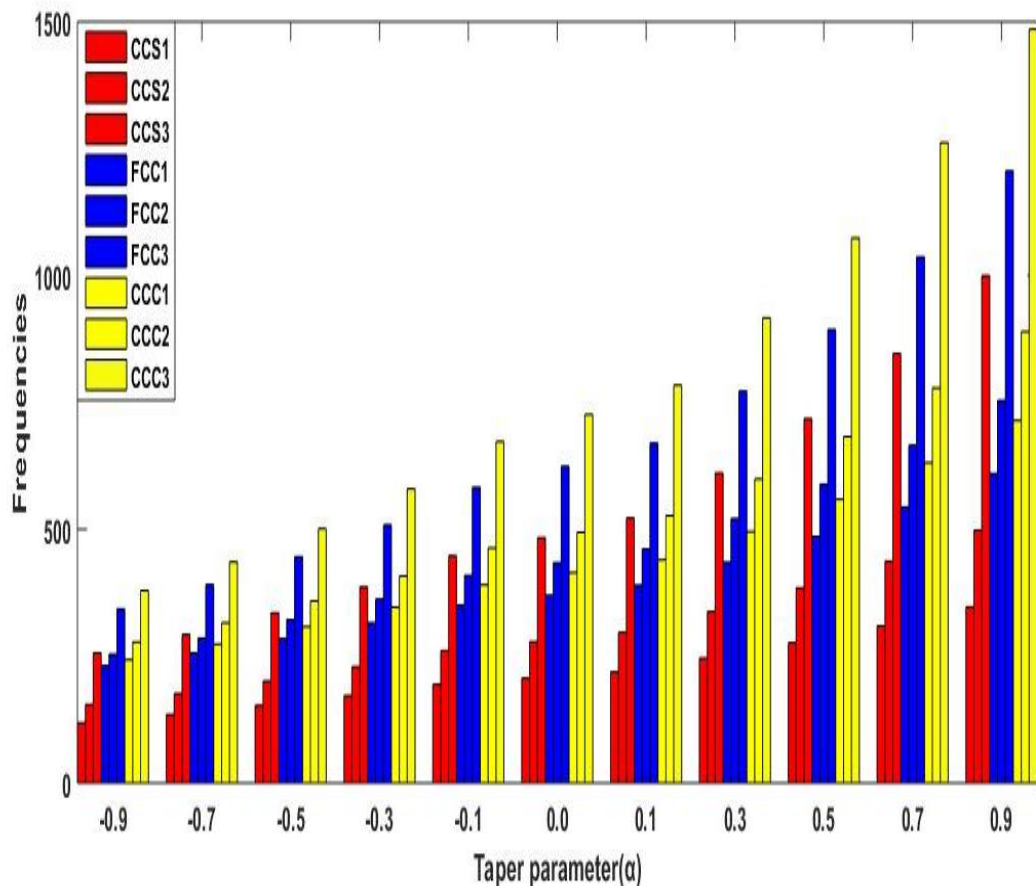


Figure 3.3 Representation of First Three Frequencies for CCS, FCC and CCC Plates

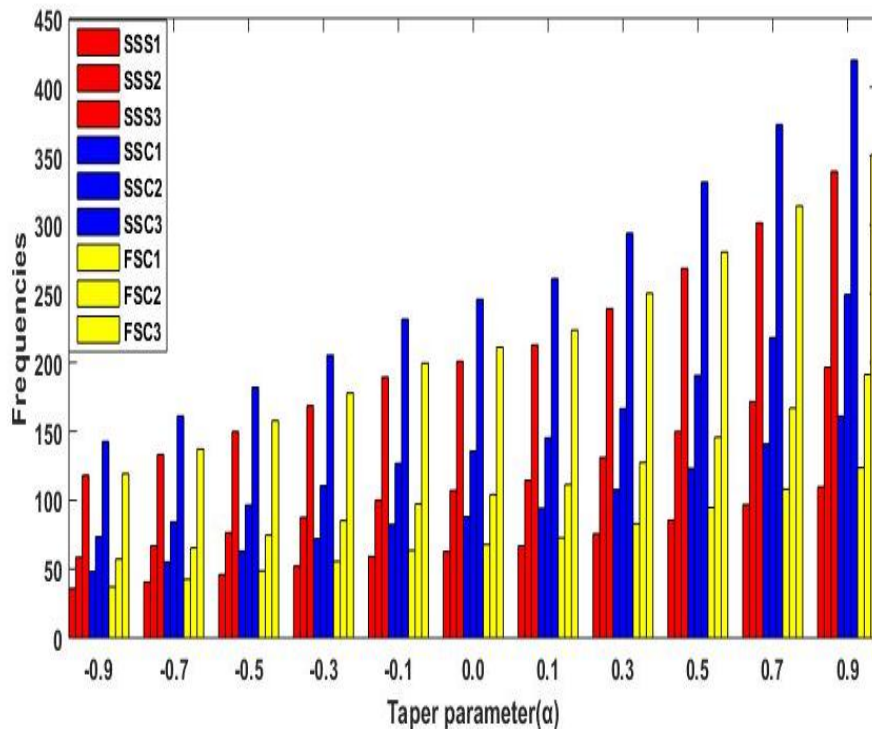


Figure 3.4 Representation of First Three Frequencies for SSS, SSC and FSC Plates

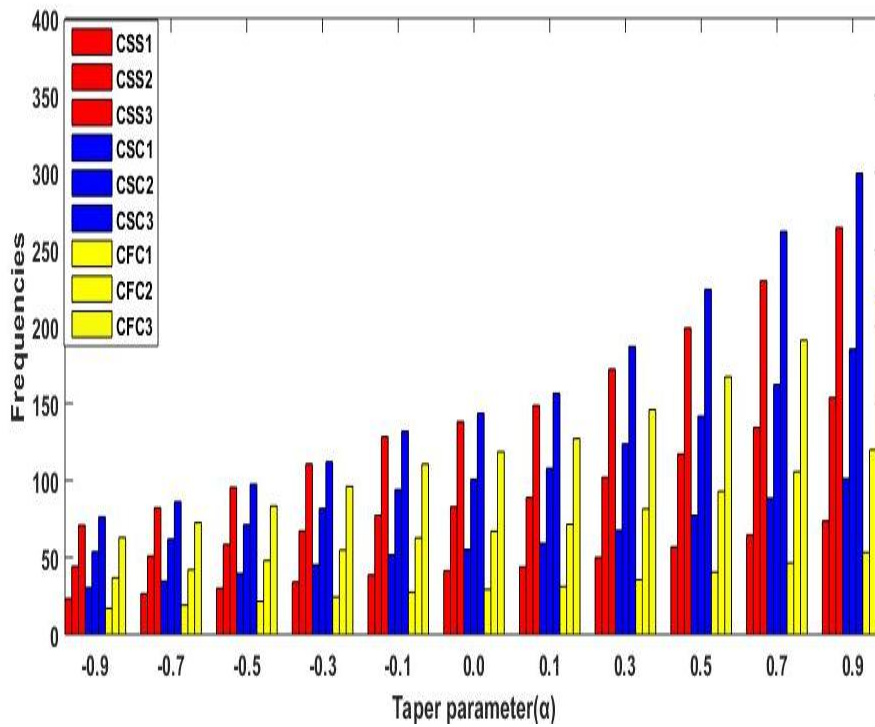


Figure 3.5 Representation of First Three Frequencies for CSS, CSC and CFC Plates

Table 3.6 Comparison of Results for Quarter of Elliptic Plate with CCC

($\alpha = 0, m=1$)

| Ref | First three frequencies | | |
|---------|-------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 |
| Present | 48.788 | 87.787 | 104.872 |
| [219] | 48.788 | 87.787 | 104.872 |
| [29] | 48.820 | 87.860 | 104.970 |
| [162] | 48.200 | 86.890 | 103.020 |
| [230] | 48.700 | 88.130 | 105.060 |
| [100] | 48.786 | 87.779 | 104.890 |

For checking the convergence of results, the first three frequencies are computed by creating a matrix of order 5, 10, 15, 20, 25 and 28 and found that first three frequencies are converging upto five significant as represented in table 3.7.

Table 3.7 Convergence of Results for Quarter of Elliptic Plate with CCC ($\alpha = 0$)

| n | λ_1 | λ_2 | λ_3 |
|----|-------------|-------------|-------------|
| 5 | 49.377 | 91.754 | 119.070 |
| 10 | 48.809 | 87.924 | 105.837 |
| 15 | 48.794 | 87.862 | 104.919 |
| 20 | 48.790 | 87.795 | 104.886 |
| 25 | 48.789 | 87.791 | 104.876 |
| 28 | 48.788 | 87.787 | 104.872 |

3.4 MAJOR FINDINGS

From the above work, it is observed that Rayleigh-Ritz technique is an efficient method used to solve the complex vibrational problems. The first three frequencies have been computed in some selected cases by varying the boundary conditions at the three edges of quarter of elliptic plate. Convergence of result shows the convergence of data upto five significant digits. It is also concluded that as thickness variation is

increasing along the radial direction of quarter of elliptic plate then the frequencies are increasing in all the cases due to increase in the stiffness of the plate. In the case of CCC, the first three frequencies have been compared with the existing results available in the literature and present results are in excellent match with the results available in literature.

CHAPTER-IV

Numerical Experiments on Rectangular and Square Plates with Transcendental Thickness Variation

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CHAPTER-IV

NUMERICAL EXPERIMENTS ON RECTANGULAR AND SQUARE PLATES WITH TRANSCENDENTAL THICKNESS VARIATION

In this chapter, vibrations of rectangular and square plates are demonstrated with various kinds of the combination of boundary conditions and thickness variations. From the study, it is observed that transcendental thickness variation has not been studied by the researchers which satisfy the boundary conditions. Therefore, the present work is an attempt in this direction. Further, Rayleigh-Ritz method has been used for computation of the first three frequencies alongwith design of mode shapes and by varying the boundary conditions at the four sides of the plate. The convergence of the computed results has been presented through table and comparison has also been made with the existing results in the literature.

4.1 BACKGROUND

From the literature it is revealed that exhaustive research work is available on the computation of first few frequencies alongwith representation of mode shapes for rectangular and square plates with various combinations of the boundary conditions applied at the four sides of the plate. The thickness of thin plate is varying as uniformly, linearly, quadratically, etc but it is found from the literature that transcendental thickness variation is not studied by the researchers and scientists which satisfy the various boundary conditions applied on the boundary of the plate. Therefore, the present work is based on computations of first few frequencies for rectangular and square plates with transcendental thickness variation.

4.1.1 Rectangular Plate

Rectangular plate is defined in flat surface and bounded by four fixed edges as shown

in figure 1.3. Clamped, simply-supported and completely-free boundary conditions are applied on the four sides of the plate. The side (1) of rectangular plate may be clamped, simply-supported or completely-free. Similar interpretation is given to other sides represented as (2), (3) and (4).

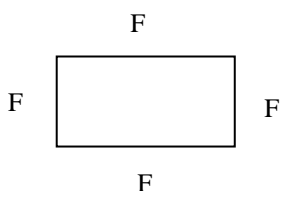
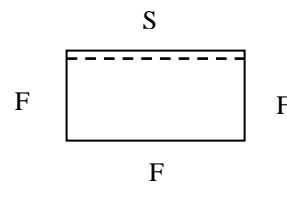
4.1.2 Square Plate

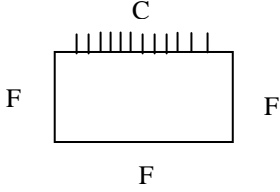
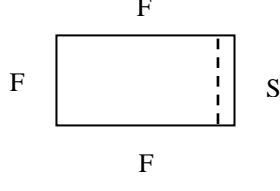
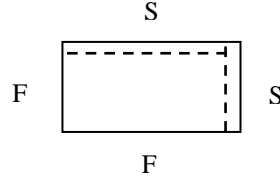
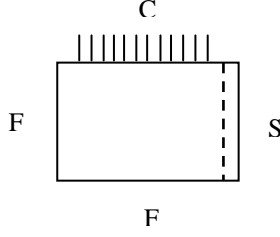
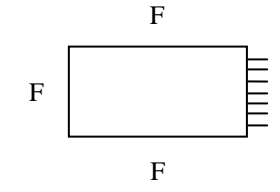
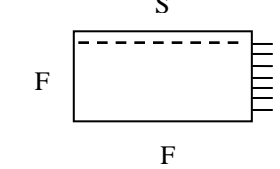
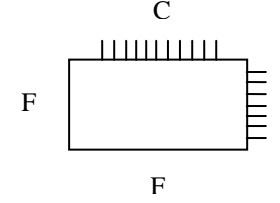
Square plate assembled with four edges, all edge is equal in two dimension domain. When the semi-major axes a is equivalent to semi-minor axes b in figure 1.3, then it is converted into a square plate with side a . further the side of square plate may be clamped, simply-supported or completely-free.

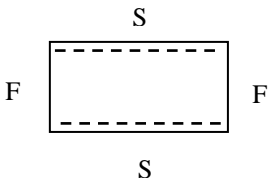
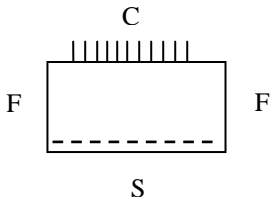
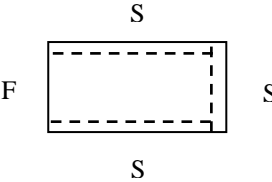
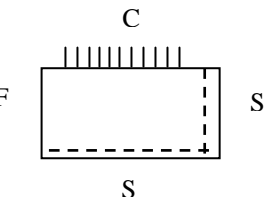
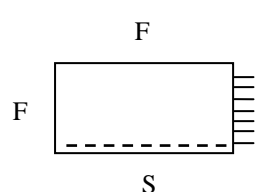
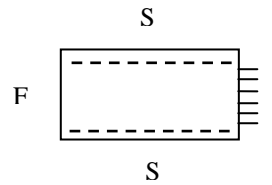
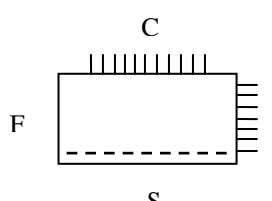
1.3 Boundary Conditions

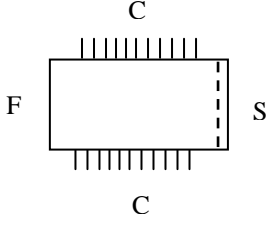
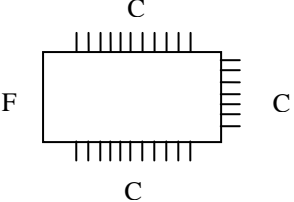
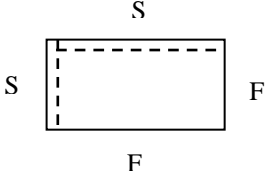
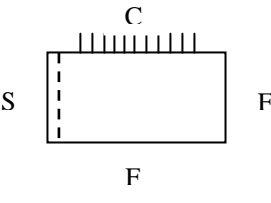
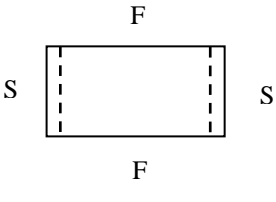
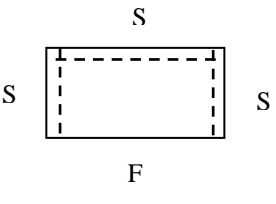
The four side of rectangular and square plates may be clamped, simply-supported or completely-free. This leads to eighty one different combinations of boundary conditions but due to highly computation only fifty two combinations of boundary conditions are represented and considered as shown in tables 4.1 and 4.2 for rectangular and square plates, respectively.

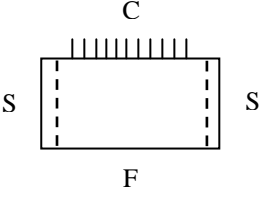
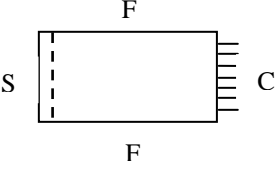
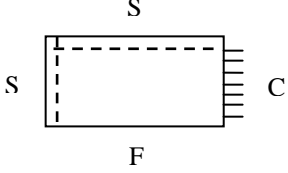
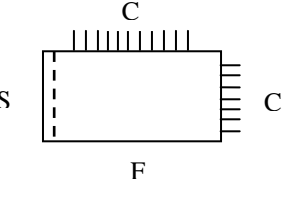
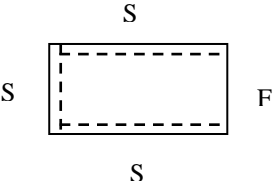
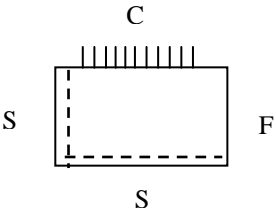
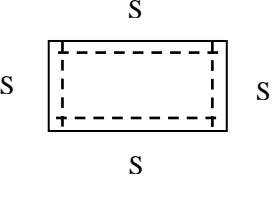
Table 4.1 Representation of Boundary Conditions for Rectangular Plate

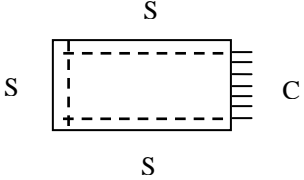
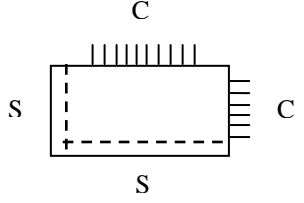
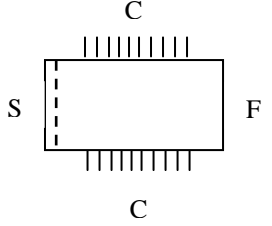
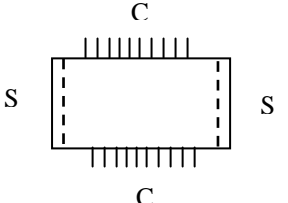
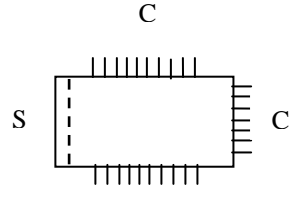
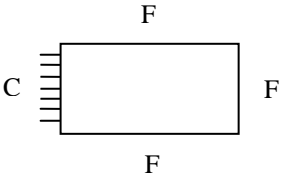
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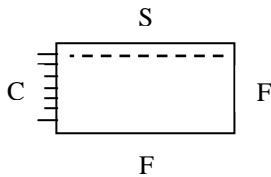
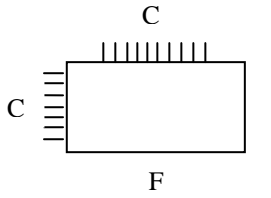
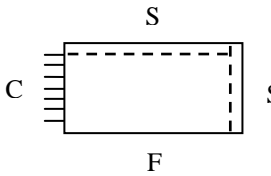
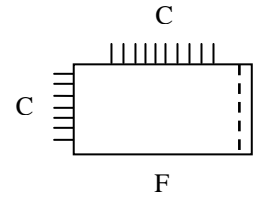
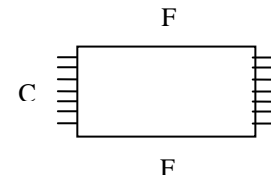
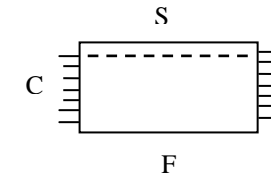
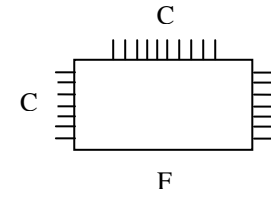
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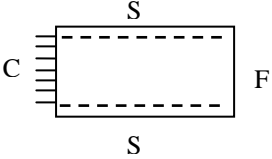
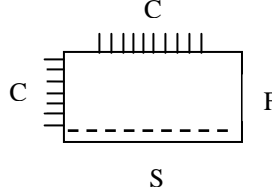
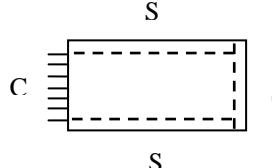
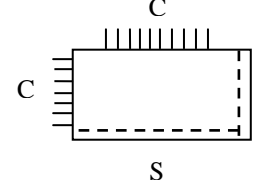
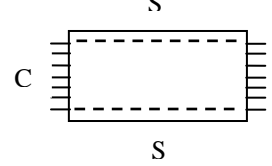
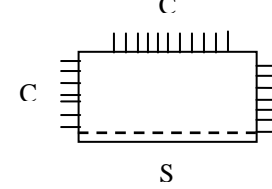
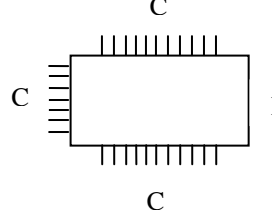
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| <p>FSCS</p> |  |
| <p>FSCC</p> |  |

| | |
|------|--|
| FCFC |  |
| FCSC |  |
| FCCC |  |
| SFFS |  |
| SFFC |  |
| SFSF |  |
| SFSS |  |

| | |
|-------------|--|
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| <p>SFCF</p> |  |
| <p>SFCS</p> |  |
| <p>SFCC</p> |  |
| <p>SSFS</p> |  |
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| <p>SSSS</p> |  |

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| SSCC |  |
| SCFC |  |
| SCSC |  |
| SCCC |  |
| CFFF |  |

| | |
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| CFFC |  |
| CFSS |  |
| CFSC |  |
| CFCF |  |
| CFCS |  |
| CFCC |  |

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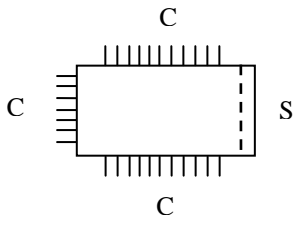
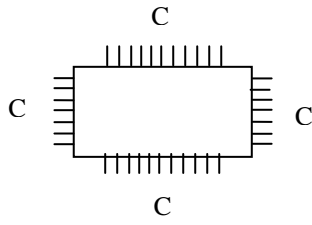
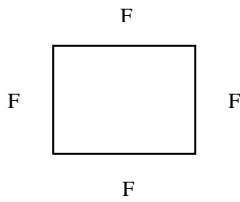
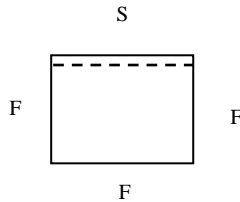
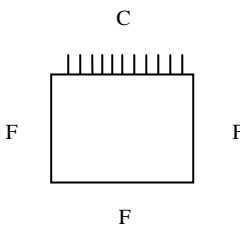
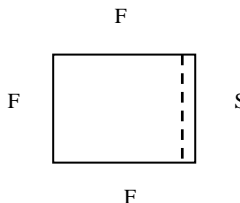
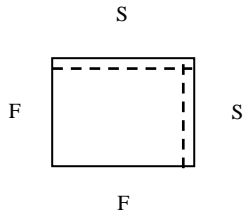
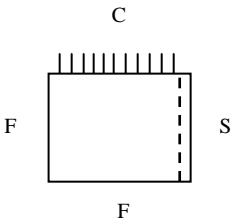
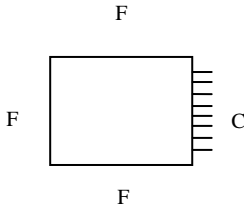
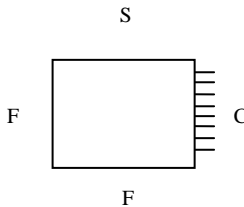
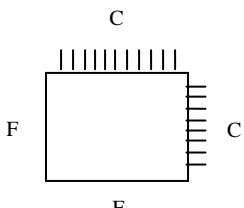
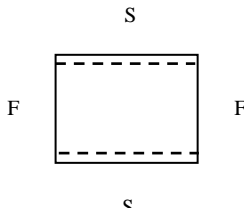
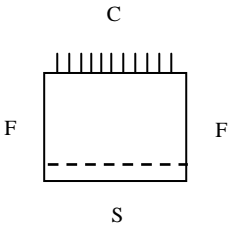
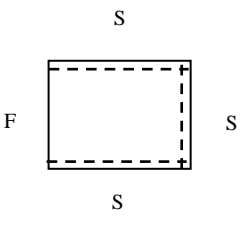
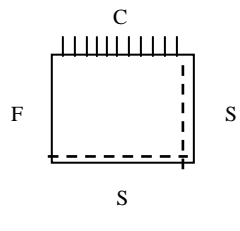
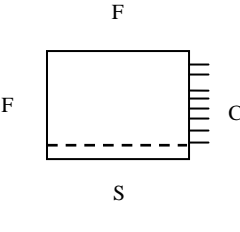
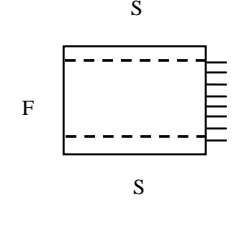
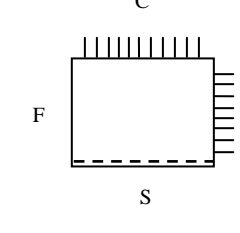
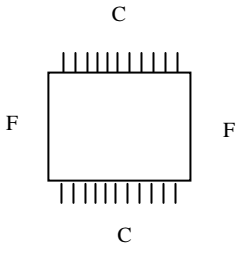
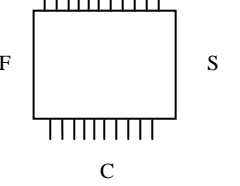
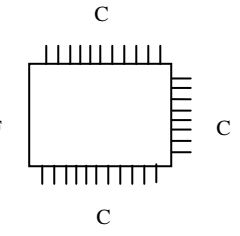
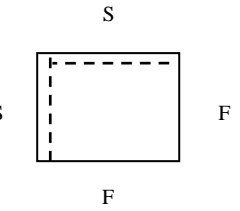
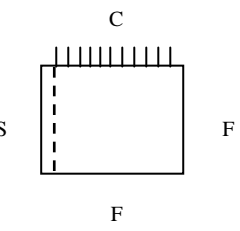
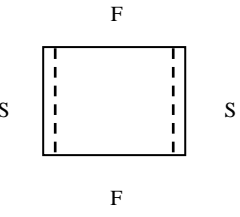
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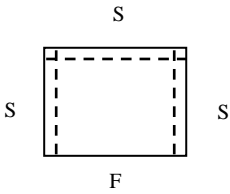
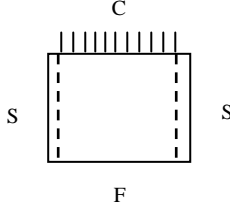
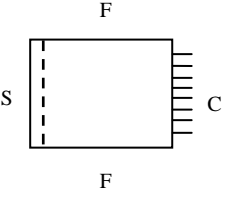
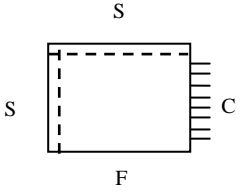
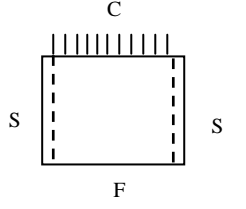
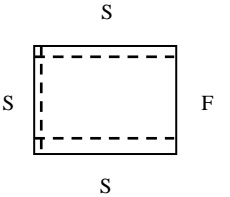
Table 4.2 Representation of Boundary Conditions for Square Plate

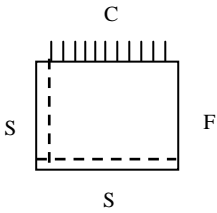
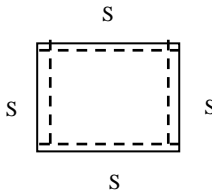
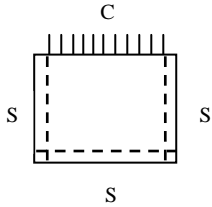
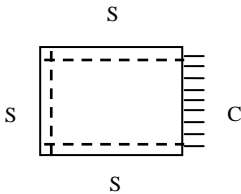
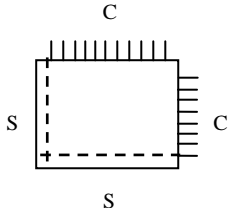
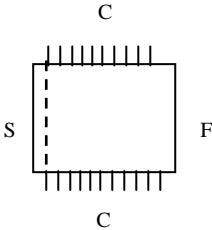
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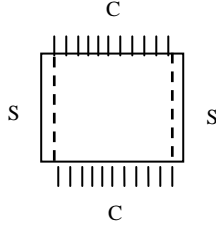
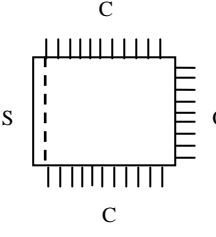
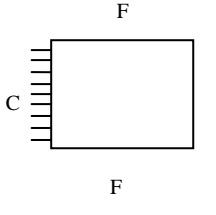
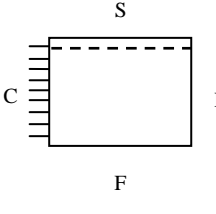
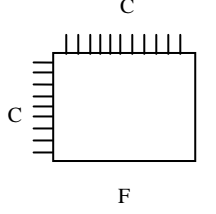
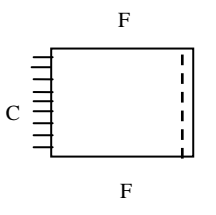
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|-------------|---|
| <p>FFSS</p> |  <p>The diagram shows a rectangular plate with a dashed inner boundary. The top and bottom edges are labeled 'S' (Simply Supported), and the left and right edges are labeled 'F' (Fixed).</p> |
| <p>FFSC</p> |  <p>The diagram shows a rectangular plate with a dashed inner boundary. The top edge is labeled 'C' (Free) and has vertical tick marks. The bottom edge is labeled 'F' (Fixed). The left and right edges are labeled 'F' (Fixed) and 'S' (Simply Supported) respectively.</p> |
| <p>FFCF</p> |  <p>The diagram shows a rectangular plate. The top and bottom edges are labeled 'F' (Fixed). The right edge is labeled 'C' (Free) and has horizontal tick marks. The left edge is labeled 'F' (Fixed).</p> |
| <p>FFCS</p> |  <p>The diagram shows a rectangular plate. The top edge is labeled 'S' (Simply Supported). The bottom edge is labeled 'F' (Fixed). The right edge is labeled 'C' (Free) and has horizontal tick marks. The left edge is labeled 'F' (Fixed).</p> |
| <p>FFCC</p> |  <p>The diagram shows a rectangular plate with a dashed inner boundary. The top edge is labeled 'C' (Clamped) and has vertical tick marks. The bottom edge is labeled 'F' (Fixed). The right edge is labeled 'C' (Clamped) and has horizontal tick marks. The left edge is labeled 'F' (Fixed).</p> |
| <p>FSFS</p> |  <p>The diagram shows a rectangular plate with a dashed inner boundary. The top and bottom edges are labeled 'S' (Simply Supported). The left and right edges are labeled 'F' (Fixed).</p> |

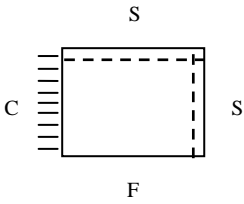
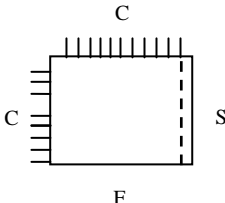
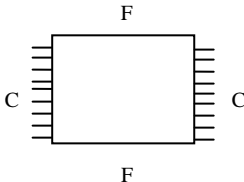
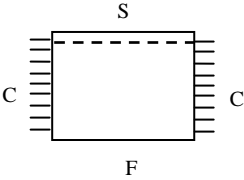
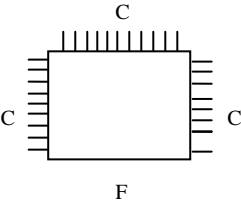
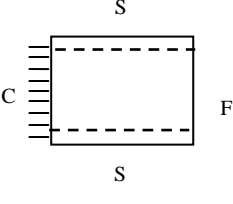
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|-------------|--|
| <p>FSFC</p> |  |
| <p>FSSS</p> |  |
| <p>FSSC</p> |  |
| <p>FSCF</p> |  |
| <p>FSCS</p> |  |
| <p>FSCC</p> |  |

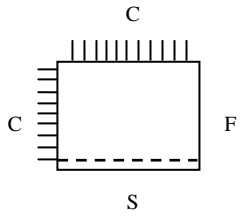
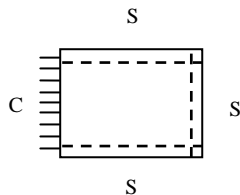
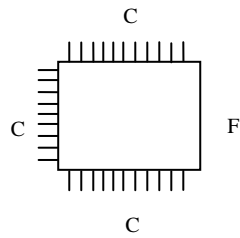
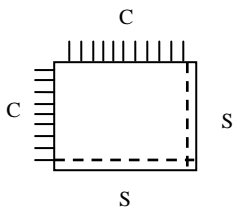
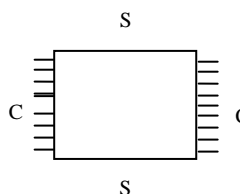
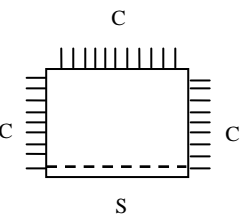
| | |
|-------------|--|
| <p>FCFC</p> |  |
| <p>FCSC</p> |  |
| <p>FCCC</p> |  |
| <p>SFFS</p> |  |
| <p>SFFC</p> |  |
| <p>SFSF</p> |  |

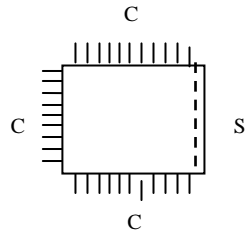
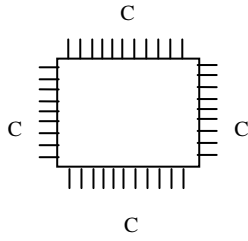
| | |
|-------------|--|
| <p>SFSS</p> |  |
| <p>SFSC</p> |  |
| <p>SFCF</p> |  |
| <p>SFCS</p> |  |
| <p>SFSC</p> |  |
| <p>SSFS</p> |  |

| | |
|------|--|
| SSFC |  |
| SSSS |  |
| SSSC |  |
| SSCS |  |
| SSCC |  |
| SCFC |  |

| | |
|-------------|--|
| <p>SCSC</p> |  |
| <p>SCCC</p> |  |
| <p>CFFF</p> |  |
| <p>CFFS</p> |  |
| <p>CFFC</p> |  |
| <p>CFSF</p> |  |

| | |
|------|--|
| CFSS |  |
| CFSC |  |
| CFCF |  |
| CFCS |  |
| CFCC |  |
| CSFS |  |

| | |
|-------------|--|
| <p>CSFC</p> |  |
| <p>CSSS</p> |  |
| <p>CCFC</p> |  |
| <p>CSSC</p> |  |
| <p>CSCS</p> |  |
| <p>CSCC</p> |  |

| | |
|------|--|
| CCSC |  |
| CCCC |  |

4.2 MATHEMATICAL FORMULATION

Rayleigh-Ritz method is a combination of Rayleigh and Ritz methods can be used for all continuous systems. Considering a rectangular plate as shown in figure 1.3 is bounded in the domain R by $x=0$, $x=a$, $y=0$ and $y=b$ with thickness of the plate is considered as h .

Let us introducing the following non-dimensional variables and parameters.

$$\xi = x/a, \eta = y/b, \mu = a/b, \tag{4.1}$$

The ξ and η are non-dimensional variables and parameter μ is controlling the shape (aspect ratio) of plate when $a = b$ and $\mu = 1$, then it is forming a square plate. Let us consider the following basis function of plate by considering the four sides of the plate as

$$\phi_i(\xi, \eta) = \xi^p \eta^q (1 - \xi)^r (1 - \eta)^s \tag{4.2}$$

Where p , q , r and s parameters are controlling boundary conditions of the plate at sides (1), (2), (3) and (4) respectively. When $p=0$, 1 or 2 then side (1) is completely-

free, simply-supported and clamped. Similar representation is given to q, r and s. The basis function ϕ_i 's are satisfying the essential boundary conditions of the plate.

By solving Rayleigh quotient (1.62) as explained in section 1.10 and considering the equation (1.101). The a_{ij} and b_{ij} can be written as for non-dimensional variables ξ and η

$$a_{ij} = \iint_{R'} h^3 \left[\phi_i^{\xi\xi} \phi_j^{\xi\xi} + \mu^2 \nu \left(\phi_i^{\xi\xi} \phi_j^{\eta\eta} + \phi_j^{\xi\xi} \phi_i^{\eta\eta} \right) + 2\mu^2 (1-\nu) \phi_i^{\xi\eta} \phi_j^{\xi\eta} + \mu^4 \phi_i^{\eta\eta} \phi_j^{\eta\eta} \right] d\xi d\eta \quad (4.3)$$

$$b_{ij} = \iint_{R'} h \phi_i \phi_j d\xi d\eta \quad (4.4)$$

Where R' is the new domain occupied by the plate and from equation (4.2), we get

$\phi_i^{\xi\xi}$, $\phi_i^{\eta\eta}$ and $\phi_i^{\xi\eta}$ given below

$$\phi_i^{\xi\xi} = \eta^p (1-\eta)^s \left[r(r-1) \xi^p (1-\xi)^{r-2} + (-2rp) \xi^{p-1} (1-\xi)^{r-1} + p(p-1) \xi^{p-2} (1-\xi)^r \right] \quad (4.5)$$

$$\phi_i^{\eta\eta} = \xi^p (1-\xi)^r \left[s(s-1) \eta^q (1-\eta)^{s-2} + (-2sq) \eta^{q-1} (1-\eta)^{s-1} + q(q-1) \eta^{q-2} (1-\eta)^s \right] \quad (4.6)$$

$$\phi_i^{\xi\eta} = \left[-s\eta^q (1-\eta)^{s-1} + q\eta^{q-1} (1-\eta)^s \right] \left[-r\xi^p (1-\xi)^{r-1} + p\xi^{p-1} (1-\xi)^r \right] \quad (4.7)$$

Similarly by replacing i and j, we can also get $\phi_j^{\xi\xi}$, $\phi_j^{\eta\eta}$ and $\phi_j^{\xi\eta}$ as

$$\phi_j^{\xi\xi} = \eta^p (1-\eta)^s \left[r(r-1) \xi^p (1-\xi)^{r-2} + (-2rp) \xi^{p-1} (1-\xi)^{r-1} + p(p-1) \xi^{p-2} (1-\xi)^r \right] \quad (4.8)$$

$$\phi_j^{\eta\eta} = \xi^p (1-\xi)^r \left[s(s-1) \eta^q (1-\eta)^{s-2} + (-2sq) \eta^{q-1} (1-\eta)^{s-1} + q(q-1) \eta^{q-2} (1-\eta)^s \right] \quad (4.9)$$

$$\phi_j^{\xi\eta} = \left[-s\eta^q (1-\eta)^{s-1} + q\eta^{q-1} (1-\eta)^s \right] \left[-r\xi^p (1-\xi)^{r-1} + p\xi^{p-1} (1-\xi)^r \right] \quad (4.10)$$

The thickness function $h(\xi, \eta)$ is considered as the thickness variation along a ξ axes as represented in the figure 4.1 and given by for rectangular and square plates

$$h = 1 + \alpha \sin \xi \quad (4.11)$$

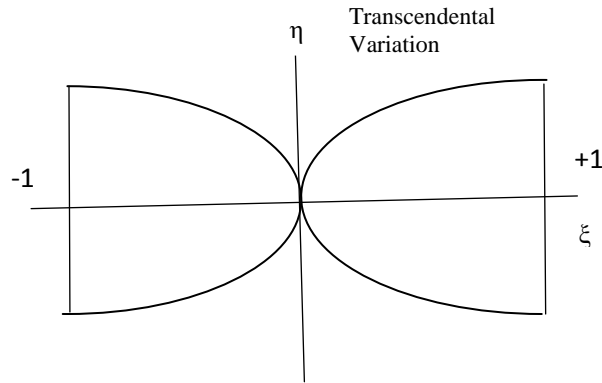


Figure 4.1 Representation of Thickness Variation for Rectangular and Square Plates

By putting the values of $h(\xi, \eta)$, ϕ_i , ϕ_j , $\phi_i^{\xi\xi}$, $\phi_i^{\eta\eta}$, $\phi_i^{\xi\eta}$, $\phi_j^{\xi\xi}$, $\phi_j^{\eta\eta}$, $\phi_j^{\xi\eta}$ in equation (4.3) and (4.4) and solving the equation (1.101) by generalized Jacobi method, we can get the first three frequencies for rectangular and square plates. The computation involve the following integrals.

$$\iint_R \xi^p \eta^q (1-\xi)^r (1-\eta)^s d\xi d\eta = \frac{p!q!r!s!}{(p+r+1)!(q+s+1)!} \quad (4.12)$$

4.3 NUMERICAL RESULTS AND DISCUSSION

For extensive numerical computation on the rectangular and square plates, the following parameters have been selected:

1. The value of Poisson's distribution is considered as $\nu=0.3$ for the isotropic plate;
2. The taper parameter α is varying from 0 to 1 with an interval as 0.2;
3. The numeric values of boundary condition are considered as 2, 1, 0 for the

clamped, simply-supported and completely-free plate, respectively and controlled by the values of p, q, r and s;

4. The frequencies are represented as λ_1 , λ_2 and λ_3 for the first, second and third frequency, respectively;
5. The aspect ratio μ is taken as 1 for square plate and 2.0 for the rectangular plate.

By the use of generalized Jacobi method the first three frequencies and corresponding mode shapes here be computed Table 4.3 represents the first three frequencies for the rectangular plate with different combinations of the boundary conditions. From the table, one can observed that the frequencies are increasing as the value of taper parameter is increasing from 0 to 1. This is because of the stiffness of the plate is increasing.

Table 4.3 First Three Frequencies for the Rectangular Plate ($\mu=2.0$)

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|--------|--------|--------|--------|--------|---------|
| FFFF | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.00000 |
| | 21.467 | 23.580 | 25.731 | 27.900 | 30.076 | 32.254 |
| | 26.585 | 29.168 | 31.781 | 34.390 | 36.980 | 39.547 |
| FFFS | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 13.469 | 14.778 | 16.133 | 17.513 | 18.907 | 20.310 |
| | 34.811 | 38.140 | 41.580 | 45.060 | 48.546 | 52.024 |
| FFFC | 14.002 | 15.339 | 16.738 | 18.170 | 19.620 | 21.082 |
| | 21.460 | 23.516 | 25.665 | 27.864 | 30.089 | 32.332 |
| | 40.767 | 44.653 | 48.681 | 52.762 | 56.853 | 60.933 |
| FFSF | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 13.039 | 15.145 | 17.324 | 19.530 | 21.743 | 23.972 |
| | 42.854 | 46.920 | 51.071 | 55.231 | 59.369 | 63.580 |
| FFSS | 6.6438 | 7.6422 | 8.6819 | 9.7440 | 10.819 | 11.903 |
| | 25.376 | 27.770 | 30.234 | 32.727 | 35.235 | 37.750 |
| | 58.772 | 63.977 | 68.990 | 73.622 | 77.865 | 81.824 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| FFS | 16.140 | 17.480 | 18.870 | 20.288 | 21.726 | 23.177 |
| | 31.487 | 34.331 | 37.255 | 40.213 | 43.186 | 46.165 |
| | 63.453 | 69.342 | 75.357 | 81.388 | 87.379 | 93.300 |
| FFCF | 3.4512 | 4.1566 | 4.9138 | 5.7046 | 6.5185 | 7.3507 |
| | 14.818 | 17.044 | 19.323 | 21.616 | 23.906 | 26.186 |
| | 21.477 | 24.216 | 27.028 | 29.863 | 32.699 | 35.528 |
| FFCS | 8.5156 | 9.7148 | 10.950 | 12.202 | 13.462 | 14.727 |
| | 30.983 | 34.034 | 37.169 | 40.332 | 43.499 | 46.661 |
| | 64.157 | 68.243 | 72.163 | 75.953 | 79.648 | 83.273 |
| FFCC | 17.142 | 18.545 | 19.991 | 21.461 | 22.945 | 24.440 |
| | 36.423 | 39.771 | 43.199 | 46.650 | 50.101 | 53.545 |
| | 73.516 | 80.442 | 87.419 | 94.285 | 100.92 | 107.17 |
| FSFS | 38.945 | 42.238 | 45.029 | 47.512 | 49.832 | 52.061 |
| | 46.751 | 51.477 | 56.905 | 62.648 | 68.487 | 74.318 |
| | 70.748 | 77.472 | 84.465 | 91.566 | 98.697 | 105.82 |
| FSFC | 61.226 | 65.707 | 68.918 | 71.721 | 74.354 | 76.899 |
| | 67.279 | 74.648 | 83.280 | 91.987 | 100.41 | 108.48 |
| | 87.706 | 96.102 | 105.03 | 114.33 | 123.93 | 133.76 |
| FSSS | 41.197 | 43.583 | 45.870 | 48.084 | 50.251 | 52.387 |
| | 59.067 | 64.471 | 70.006 | 75.559 | 81.082 | 86.559 |
| | 94.506 | 103.19 | 112.11 | 121.09 | 130.06 | 138.95 |
| FSSC | 62.922 | 66.161 | 69.060 | 71.766 | 74.360 | 76.886 |
| | 77.408 | 84.479 | 91.757 | 99.015 | 106.17 | 113.20 |
| | 108.94 | 118.96 | 129.31 | 139.77 | 150.23 | 160.64 |
| FSCF | 8.5156 | 9.7148 | 10.950 | 12.202 | 13.462 | 14.727 |
| | 30.983 | 34.034 | 37.169 | 40.352 | 43.499 | 46.661 |
| | 64.157 | 68.243 | 72.162 | 75.953 | 79.648 | 83.273 |
| FSCS | 41.702 | 43.995 | 46.218 | 48.388 | 50.524 | 52.636 |
| | 63.016 | 68.552 | 74.168 | 79.772 | 85.328 | 90.829 |
| | 103.19 | 112.58 | 122.10 | 131.61 | 141.06 | 150.44 |
| FSCC | 63.264 | 66.345 | 69.167 | 71.833 | 74.402 | 76.912 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| | 80.607 | 87.575 | 94.667 | 101.72 | 108.68 | 115.54 |
| | 116.70 | 127.20 | 137.91 | 148.65 | 159.36 | 170.00 |
| FCFC | 89.151 | 94.499 | 98.012 | 101.14 | 104.09 | 106.95 |
| | 93.709 | 104.82 | 116.87 | 127.88 | 137.82 | 147.01 |
| | 110.13 | 120.93 | 133.08 | 146.39 | 160.39 | 174.62 |
| FCSC | 90.400 | 94.564 | 97.990 | 101.10 | 104.05 | 106.90 |
| | 101.65 | 111.00 | 120.57 | 129.90 | 138.92 | 147.64 |
| | 128.31 | 140.12 | 152.54 | 165.25 | 178.03 | 190.75 |
| FCCC | 90.628 | 94.604 | 97.984 | 101.08 | 104.02 | 106.87 |
| | 104.17 | 113.14 | 122.27 | 131.22 | 139.94 | 148.43 |
| | 135.10 | 147.18 | 159.61 | 172.17 | 184.77 | 197.37 |
| SFFS | 6.6438 | 6.9077 | 7.1762 | 7.4435 | 7.7077 | 7.9684 |
| | 25.376 | 27.789 | 30.274 | 32.778 | 35.279 | 37.766 |
| | 58.772 | 64.501 | 70.271 | 75.989 | 81.625 | 87.180 |
| SFFC | 16.140 | 17.878 | 19.7121 | 21.597 | 23.509 | 25.437 |
| | 31.487 | 34.596 | 37.813 | 41.065 | 44.320 | 47.563 |
| | 63.453 | 69.537 | 75.747 | 81.954 | 88.108 | 94.201 |
| SFSF | 9.5125 | 10.403 | 11.275 | 12.123 | 12.946 | 13.747 |
| | 27.522 | 30.057 | 32.728 | 35.463 | 38.227 | 41.004 |
| | 38.531 | 42.144 | 45.787 | 49.406 | 52.983 | 56.514 |
| SFSS | 16.135 | 17.637 | 19.204 | 20.801 | 22.412 | 24.028 |
| | 46.740 | 51.116 | 55.550 | 59.971 | 64.350 | 68.680 |
| | 75.284 | 82.239 | 89.066 | 95.698 | 102.13 | 108.40 |
| SFSC | 22.815 | 24.955 | 27.168 | 29.414 | 31.672 | 33.933 |
| | 50.751 | 55.507 | 60.340 | 65.169 | 69.963 | 74.711 |
| | 98.890 | 108.16 | 117.54 | 126.03 | 134.02 | 141.70 |
| SFCF | 15.082 | 16.839 | 18.632 | 20.423 | 22.199 | 23.955 |
| | 31.310 | 34.154 | 37.110 | 40.108 | 43.116 | 46.122 |
| | 49.095 | 54.000 | 58.979 | 63.934 | 68.829 | 73.656 |
| SFCF | 20.601 | 22.659 | 24.786 | 26.932 | 29.076 | 31.2111 |
| | 56.318 | 61.762 | 67.285 | 72.782 | 78.215 | 83.576 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| | 77.332 | 84.114 | 90.776 | 97.264 | 103.58 | 109.74 |
| SFCC | 26.290 | 28.754 | 31.284 | 33.830 | 36.373 | 38.906 |
| | 59.766 | 65.485 | 71.289 | 77.070 | 82.790 | 88.437 |
| | 101.42 | 110.27 | 118.78 | 126.44 | 134.78 | 142.34 |
| SSFS | 41.197 | 46.429 | 51.717 | 56.886 | 61.877 | 66.683 |
| | 59.067 | 64.841 | 70.885 | 77.107 | 83.457 | 89.898 |
| | 94.506 | 103.63 | 112.97 | 122.31 | 131.59 | 140.80 |
| SSFC | 62.922 | 70.991 | 78.662 | 85.651 | 92.036 | 97.982 |
| | 77.408 | 85.183 | 93.703 | 102.86 | 112.42 | 122.16 |
| | 108.94 | 119.52 | 130.47 | 141.57 | 152.75 | 163.99 |
| SSSS | 49.348 | 53.941 | 58.492 | 62.954 | 67.317 | 71.589 |
| | 78.958 | 86.369 | 93.926 | 101.50 | 109.03 | 116.50 |
| | 128.33 | 140.38 | 152.66 | 164.88 | 176.93 | 188.78 |
| SSSC | 69.327 | 75.733 | 81.918 | 87.853 | 93.562 | 99.082 |
| | 94.586 | 103.52 | 112.70 | 121.96 | 131.21 | 140.40 |
| | 140.33 | 153.52 | 166.97 | 180.42 | 193.75 | 206.93 |
| SSCS | 51.674 | 56.207 | 60.694 | 65.095 | 69.406 | 73.632 |
| | 86.135 | 94.070 | 102.11 | 110.12 | 118.04 | 125.88 |
| | 140.91 | 154.01 | 167.26 | 180.44 | 193.48 | 206.35 |
| SSCC | 71.078 | 77.227 | 83.188 | 88.944 | 94.513 | 99.921 |
| | 100.80 | 110.03 | 119.43 | 128.83 | 138.17 | 147.41 |
| | 151.96 | 166.07 | 180.38 | 194.64 | 208.78 | 222.74 |
| SCFC | 90.400 | 101.65 | 111.07 | 119.04 | 126.27 | 133.06 |
| | 101.65 | 112.33 | 124.97 | 138.58 | 152.25 | 165.57 |
| | 128.31 | 141.00 | 154.36 | 168.16 | 182.37 | 196.97 |
| SCSC | 95.263 | 103.91 | 111.93 | 119.39 | 126.42 | 133.14 |
| | 115.80 | 126.86 | 138.38 | 150.07 | 161.75 | 173.31 |
| | 156.37 | 171.17 | 186.41 | 201.73 | 216.98 | 232.09 |
| SCCC | 96.573 | 104.78 | 112.50 | 119.77 | 126.69 | 133.33 |
| | 121.00 | 132.10 | 143.49 | 154.94 | 166.34 | 177.62 |
| | 167.06 | 182.55 | 198.33 | 214.15 | 229.92 | 245.55 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| CFFF | 3.4512 | 3.4117 | 3.3806 | 3.3551 | 3.3337 | 3.3154 |
| | 14.818 | 15.302 | 15.941 | 16.462 | 16.957 | 17.430 |
| | 21.477 | 22.775 | 24.050 | 25.292 | 26.497 | 27.669 |
| CFFS | 8.5156 | 8.9253 | 9.3326 | 9.7286 | 10.111 | 10.481 |
| | 30.983 | 33.731 | 36.547 | 39.371 | 42.176 | 44.953 |
| | 64.157 | 71.870 | 79.486 | 86.711 | 93.525 | 100.03 |
| CFFC | 17.142 | 18.998 | 20.953 | 22.956 | 24.980 | 27.014 |
| | 36.423 | 39.895 | 43.471 | 47.066 | 50.645 | 54.195 |
| | 73.516 | 80.291 | 87.160 | 94.003 | 100.78 | 107.47 |
| CFSF | 15.082 | 16.094 | 17.092 | 18.065 | 19.009 | 19.926 |
| | 31.310 | 34.214 | 37.243 | 40.317 | 43.401 | 46.479 |
| | 49.310 | 53.275 | 57.476 | 61.630 | 65.715 | 69.726 |
| CFSS | 20.601 | 22.335 | 24.122 | 25.924 | 27.723 | 29.512 |
| | 56.318 | 61.316 | 66.358 | 71.358 | 76.286 | 81.137 |
| | 77.332 | 84.882 | 92.362 | 99.676 | 106.81 | 113.77 |
| CFSC | 26.290 | 28.711 | 31.200 | 33.708 | 36.214 | 38.708 |
| | 59.766 | 65.150 | 70.594 | 76.004 | 81.347 | 86.614 |
| | 101.42 | 111.30 | 120.89 | 130.13 | 139.03 | 147.62 |
| CFCF | 22.076 | 24.067 | 26.100 | 28.125 | 30.124 | 32.091 |
| | 36.018 | 39.307 | 42.693 | 46.097 | 49.487 | 52.852 |
| | 60.897 | 66.476 | 72.118 | 77.709 | 83.210 | 88.616 |
| CFCS | 26.443 | 28.845 | 31.305 | 33.765 | 36.203 | 38.611 |
| | 67.249 | 73.433 | 79.683 | 85.876 | 91.972 | 97.963 |
| | 79.831 | 87.240 | 94.574 | 101.75 | 108.76 | 115.61 |
| CFCC | 31.135 | 33.992 | 36.905 | 39.815 | 42.701 | 45.553 |
| | 70.229 | 76.700 | 83.243 | 89.732 | 96.124 | 102.41 |
| | 103.42 | 113.00 | 122.34 | 131.36 | 140.09 | 148.54 |
| CSFS | 41.702 | 47.167 | 52.775 | 58.340 | 63.787 | 69.091 |
| | 63.016 | 69.330 | 75.863 | 82.502 | 89.201 | 95.936 |
| | 103.19 | 113.10 | 123.21 | 133.32 | 148.37 | 153.32 |
| CSFC | 63.264 | 71.721 | 80.070 | 87.963 | 95.338 | 102.26 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|--------|--------|---------|--------|---------|--------|
| | 80.607 | 88.947 | 97.831 | 107.18 | 116.87 | 126.77 |
| | 116.70 | 128.12 | 139.86 | 151.68 | 163.51 | 175.29 |
| CSSS | 51.674 | 56.759 | 61.833 | 66.825 | 71.715 | 76.504 |
| | 86.135 | 94.287 | 102.57 | 110.83 | 119.02 | 127.11 |
| | 140.91 | 154.05 | 167.30 | 180.46 | 193.45 | 206.29 |
| CSSC | 71.078 | 78.122 | 85.023 | 91.711 | 98.180 | 104.45 |
| | 100.80 | 110.49 | 120.39 | 130.34 | 140.23 | 150.05 |
| | 151.96 | 166.23 | 180.70 | 195.13 | 209.42 | 223.56 |
| CSCS | 54.743 | 59.827 | 64.882 | 69.848 | 74.711 | 79.473 |
| | 94.585 | 103.37 | 112.25 | 121.06 | 129.76 | 138.32 |
| | 154.78 | 169.16 | 183.67 | 198.05 | 212.20 | 226.11 |
| CSCC | 73.396 | 80.222 | 86.911 | 93.410 | 99.720 | 105.86 |
| | 108.22 | 118.33 | 128.57 | 138.78 | 148.88 | 158.85 |
| | 165.03 | 180.39 | 195.92 | 211.34 | 226.56 | 241.56 |
| CCFC | 90.628 | 102.60 | 113.43 | 122.88 | 131.43 | 139.41 |
| | 104.17 | 115.36 | 128.04 | 141.80 | 155.93 | 169.99 |
| | 135.10 | 148.57 | 162.261 | 177.00 | 191.165 | 206.50 |
| CCSC | 96.573 | 106.09 | 115.15 | 123.72 | 131.85 | 139.64 |
| | 121.00 | 132.83 | 145.05 | 157.41 | 169.76 | 182.01 |
| | 167.06 | 182.90 | 199.03 | 215.21 | 231.35 | 247.40 |
| CCCC | 98.313 | 107.40 | 116.12 | 124.43 | 132.40 | 140.06 |
| | 127.32 | 139.30 | 151.52 | 163.76 | 175.92 | 187.95 |
| | 179.09 | 195.83 | 212.83 | 229.76 | 246.54 | 263.11 |

The computations have also been performed on the square plate as a special case of rectangular plate with aspect ratio is considered as $\mu = 1.0$. The results have been computed for the different combinations of the boundary conditions and represented in the following table 4.4 further, it is again observed that the frequencies are represented upto the five significant digits for different combinations of boundary conditions and these are in increasing form as the taper parameter is increasing.

Table 4.4 First Three Frequencies for the Square Plate ($\mu=1.0$)

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| FFFF | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 13.469 | 14.780 | 16.141 | 17.527 | 18.927 | 20.336 |
| | 19.597 | 21.498 | 23.476 | 23.999 | 27.541 | 29.600 |
| FFFS | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 6.6438 | 7.3006 | 8.0110 | 8.7588 | 9.5346 | 10.332 |
| | 14.903 | 16.324 | 17.818 | 19.351 | 20.905 | 22.472 |
| FFFC | 3.4811 | 3.8218 | 4.1951 | 4.5915 | 5.0051 | 5.4316 |
| | 8.5217 | 9.3486 | 10.229 | 11.143 | 12.080 | 13.035 |
| | 21.317 | 23.312 | 25.342 | 27.366 | 29.371 | 31.354 |
| FFSF | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 6.6438 | 7.6422 | 8.6821 | 9.7442 | 10.819 | 11.904 |
| | 14.903 | 16.697 | 18.525 | 20.369 | 22.221 | 24.079 |
| FFSS | 3.3671 | 3.8131 | 4.2870 | 4.7811 | 5.2911 | 5.8140 |
| | 17.317 | 18.909 | 20.531 | 22.165 | 23.804 | 25.446 |
| | 19.293 | 21.188 | 23.149 | 25.145 | 27.163 | 29.194 |
| FFSC | 5.3577 | 5.9060 | 6.4869 | 7.0915 | 7.7146 | 8.3530 |
| | 19.083 | 20.990 | 22.941 | 24.911 | 26.886 | 28.862 |
| | 24.690 | 26.596 | 28.539 | 30.501 | 32.474 | 34.457 |
| FFCF | 3.4811 | 4.1940 | 4.9595 | 5.7590 | 6.5825 | 7.4244 |
| | 8.5217 | 9.7224 | 10.959 | 12.212 | 13.473 | 14.738 |
| | 21.317 | 23.972 | 26.676 | 29.373 | 32.039 | 34.663 |
| FFCS | 5.3577 | 6.1516 | 6.9954 | 7.8717 | 8.7717 | 9.6904 |
| | 19.083 | 20.668 | 22.293 | 23.936 | 25.588 | 27.246 |
| | 24.690 | 27.507 | 30.407 | 33.335 | 36.269 | 39.200 |
| FFCC | 6.9300 | 7.7499 | 8.6182 | 9.5191 | 10.444 | 11.389 |
| | 23.942 | 25.977 | 27.953 | 29.884 | 31.787 | 33.672 |
| | 26.598 | 29.209 | 31.992 | 34.859 | 37.765 | 40.688 |
| FSFS | 9.6314 | 10.539 | 11.475 | 12.424 | 13.80 | 14.342 |
| | 16.135 | 17.701 | 19.386 | 21.145 | 22.953 | 24.793 |
| | 36.727 | 40.252 | 43.900 | 47.556 | 49.880 | 52.111 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| FSFC | 15.211 | 16.592 | 17.920 | 19.198 | 20.444 | 21.673 |
| | 20.613 | 22.641 | 24.898 | 27.299 | 29.787 | 32.323 |
| | 39.773 | 43.583 | 47.537 | 51.556 | 55.601 | 59.649 |
| FSSS | 11.685 | 12.564 | 13.480 | 14.424 | 15.393 | 16.382 |
| | 27.756 | 30.406 | 33.133 | 35.896 | 38.673 | 41.458 |
| | 41.202 | 43.591 | 45.881 | 48.099 | 50.269 | 52.407 |
| FSSC | 16.807 | 17.889 | 18.993 | 20.115 | 21.253 | 22.408 |
| | 31.122 | 34.032 | 37.033 | 40.074 | 43.133 | 46.199 |
| | 51.432 | 54.233 | 56.836 | 59.313 | 61.709 | 64.054 |
| FSCF | 5.3577 | 6.1516 | 6.9954 | 7.8717 | 8.7717 | 9.6904 |
| | 19.083 | 20.668 | 22.293 | 23.936 | 25.588 | 27.246 |
| | 24.690 | 27.507 | 30.407 | 33.335 | 36.269 | 39.200 |
| FSCS | 12.687 | 13.676 | 14.707 | 15.770 | 16.859 | 17.969 |
| | 33.065 | 36.333 | 39.689 | 43.073 | 46.460 | 49.841 |
| | 41.707 | 44.000 | 46.223 | 48.394 | 50.530 | 52.644 |
| FSCC | 17.551 | 18.663 | 19.803 | 20.967 | 22.151 | 23.355 |
| | 36.032 | 39.476 | 43.008 | 46.566 | 50.126 | 53.676 |
| | 51.843 | 54.513 | 57.038 | 59.466 | 61.832 | 64.161 |
| FCFC | 22.200 | 24.081 | 25.693 | 27.159 | 28.562 | 29.941 |
| | 26.478 | 29.188 | 32.362 | 35.762 | 39.256 | 42.781 |
| | 43.630 | 47.817 | 52.205 | 56.700 | 61.251 | 65.830 |
| FCSC | 23.402 | 24.747 | 26.077 | 27.400 | 28.726 | 30.059 |
| | 35.592 | 38.863 | 42.254 | 45.699 | 49.168 | 52.646 |
| | 62.940 | 66.185 | 69.091 | 71.804 | 74.402 | 76.931 |
| FCCC | 23.949 | 25.247 | 26.549 | 27.857 | 29.175 | 30.507 |
| | 40.019 | 43.720 | 47.520 | 51.350 | 55.183 | 59.00 |
| | 63.282 | 66.365 | 69.190 | 71.857 | 74.429 | 76.941 |
| SFFS | 3.3671 | 3.5717 | 3.7912 | 4.0180 | 4.2481 | 4.4794 |
| | 17.317 | 18.992 | 20.707 | 22.417 | 24.105 | 25.766 |
| | 19.293 | 21.117 | 23.028 | 24.997 | 27.004 | 29.037 |
| SFFC | 5.3577 | 5.8404 | 6.3576 | 6.8941 | 7.4421 | 10.530 |

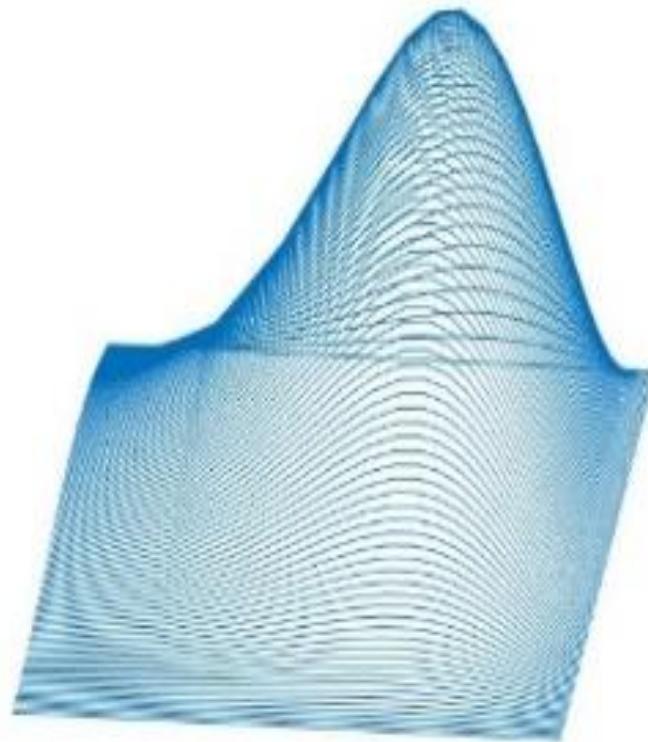
| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| | 19.083 | 20.810 | 22.580 | 24.358 | 26.131 | 35.647 |
| | 24.690 | 27.467 | 30.352 | 33.303 | 36.256 | 52.326 |
| SF _{SF} | 9.6314 | 10.533 | 11.417 | 12.275 | 13.110 | 13.921 |
| | 16.135 | 17.637 | 19.204 | 20.802 | 22.413 | 24.030 |
| | 36.727 | 40.139 | 43.626 | 47.123 | 50.605 | 54.062 |
| SF _{SS} | 11.685 | 12.778 | 13.883 | 14.986 | 16.079 | 17.162 |
| | 27.756 | 30.343 | 32.993 | 35.658 | 38.318 | 40.964 |
| | 41.202 | 45.065 | 48.967 | 52.848 | 56.687 | 60.478 |
| SF _{SC} | 12.687 | 13.876 | 15.087 | 16.301 | 17.512 | 18.715 |
| | 33.065 | 36.150 | 39.285 | 42.420 | 45.535 | 48.624 |
| | 41.707 | 45.618 | 49.570 | 53.505 | 57.399 | 61.247 |
| SF _{CF} | 15.211 | 16.982 | 18.791 | 20.598 | 22.389 | 24.162 |
| | 20.613 | 22.675 | 24.804 | 26.952 | 29.098 | 31.233 |
| | 39.773 | 43.301 | 46.887 | 50.473 | 54.038 | 59.595 |
| SF _{CS} | 16.807 | 18.652 | 20.542 | 22.437 | 24.321 | 26.189 |
| | 31.122 | 33.958 | 36.844 | 39.731 | 42.600 | 45.446 |
| | 51.432 | 56.511 | 61.669 | 66.802 | 71.877 | 76.883 |
| SF _{CC} | 17.551 | 19.437 | 21.370 | 23.311 | 25.242 | 27.157 |
| | 36.032 | 39.259 | 42.523 | 45.775 | 48.999 | 52.190 |
| | 51.843 | 56.956 | 62.148 | 67.317 | 72.427 | 77.469 |
| SS _{FS} | 11.685 | 13.032 | 14.457 | 15.917 | 17.391 | 18.870 |
| | 27.756 | 30.379 | 33.087 | 35.827 | 38.573 | 41.316 |
| | 41.202 | 46.433 | 51.720 | 56.891 | 61.886 | 66.701 |
| SS _{FC} | 16.807 | 18.901 | 21.093 | 23.317 | 25.543 | 27.758 |
| | 31.122 | 34.127 | 37.248 | 40.425 | 43.628 | 46.841 |
| | 51.432 | 58.027 | 64.532 | 70.707 | 76.516 | 82.005 |
| SS _{SS} | 19.739 | 21.588 | 23.474 | 25.368 | 27.256 | 29.134 |
| | 49.349 | 53.942 | 58.493 | 62.956 | 67.321 | 71.595 |
| | 49.349 | 53.974 | 58.658 | 63.328 | 67.955 | 72.531 |
| SS _{SC} | 23.646 | 25.867 | 28.124 | 30.383 | 32.632 | 34.866 |
| | 51.675 | 56.522 | 61.437 | 66.343 | 71.209 | 76.023 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| | 58.646 | 64.092 | 69.427 | 74.609 | 79.642 | 84.543 |
| SSCS | 23.646 | 25.923 | 28.241 | 30.557 | 32.858 | 35.137 |
| | 51.675 | 56.208 | 60.695 | 65.096 | 69.407 | 73.633 |
| | 58.646 | 64.290 | 70.012 | 75.706 | 81.335 | 86.889 |
| SSCC | 27.055 | 29.581 | 32.139 | 34.691 | 37.221 | 39.725 |
| | 60.539 | 65.878 | 71.067 | 76.113 | 81.026 | 85.821 |
| | 60.791 | 66.545 | 72.429 | 78.295 | 84.099 | 89.830 |
| SCFC | 23.402 | 26.431 | 29.549 | 32.655 | 35.708 | 38.696 |
| | 35.592 | 39.109 | 42.813 | 46.636 | 50.541 | 54.503 |
| | 62.940 | 71.005 | 78.671 | 85.658 | 92.048 | 98.012 |
| SCSC | 28.951 | 31.675 | 34.423 | 37.162 | 39.879 | 42.569 |
| | 54.744 | 59.886 | 65.116 | 70.349 | 75.551 | 80.710 |
| | 69.327 | 75.733 | 81.918 | 87.853 | 93.563 | 99.083 |
| SCCC | 31.827 | 34.716 | 37.623 | 40.510 | 43.367 | 46.190 |
| | 63.333 | 69.367 | 75.490 | 81.590 | 87.629 | 93.595 |
| | 71.079 | 77.228 | 83.190 | 88.945 | 94.514 | 99.922 |
| CFFF | 3.4811 | 3.4395 | 3.4065 | 3.3793 | 3.3562 | 3.3363 |
| | 8.5217 | 8.9308 | 9.3375 | 9.7333 | 10.116 | 10.487 |
| | 21.317 | 22.651 | 23.959 | 25.231 | 26.464 | 27.662 |
| CFFS | 5.3577 | 5.5901 | 5.8411 | 6.1002 | 6.3621 | 6.6242 |
| | 19.083 | 21.111 | 23.237 | 25.404 | 27.581 | 29.756 |
| | 24.690 | 26.522 | 28.377 | 30.229 | 32.069 | 33.896 |
| CFFC | 6.9300 | 7.4366 | 7.9802 | 8.5427 | 9.1148 | 9.6920 |
| | 23.942 | 26.242 | 28.468 | 30.600 | 32.658 | 34.661 |
| | 26.598 | 29.182 | 32.007 | 34.976 | 38.016 | 41.088 |
| CFSF | 15.211 | 16.231 | 17.239 | 18.221 | 19.175 | 20.101 |
| | 20.613 | 22.344 | 24.129 | 25.928 | 27.726 | 29.515 |
| | 39.773 | 43.617 | 47.542 | 51.470 | 55.372 | 59.235 |
| CFSS | 16.807 | 18.054 | 19.310 | 20.555 | 21.782 | 22.990 |
| | 31.122 | 34.054 | 37.047 | 40.044 | 43.022 | 45.973 |
| | 51.432 | 55.878 | 60.352 | 64.780 | 69.139 | 73.423 |

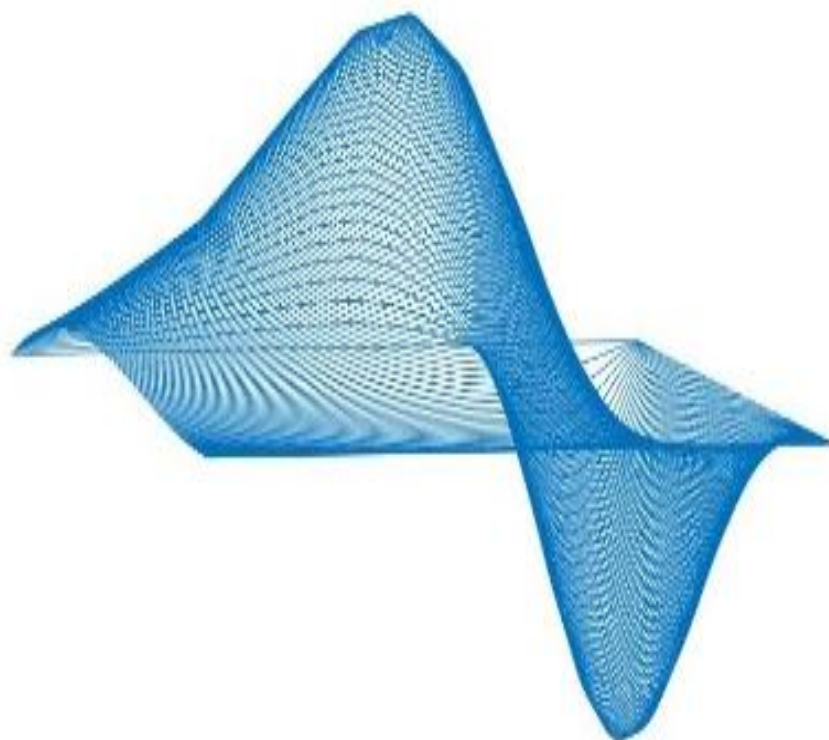
| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|------------|------------|------------|------------|------------|------------|
| CFSC | 17.551 | 18.901 | 20.269 | 21.630 | 22.977 | 24.306 |
| | 36.032 | 39.501 | 43.026 | 46.544 | 50.030 | 53.476 |
| | 51.843 | 56.336 | 60.860 | 65.340 | 69.751 | 74.088 |
| CFCF | 22.200 | 24.204 | 26.250 | 28.288 | 30.300 | 32.281 |
| | 26.478 | 28.880 | 31.339 | 33.796 | 36.231 | 38.636 |
| | 43.630 | 47.656 | 51.742 | 55.814 | 59.845 | 63.826 |
| CFCS | 23.402 | 25.578 | 27.680 | 29.835 | 31.965 | 34.062 |
| | 35.592 | 38.863 | 42.178 | 45.476 | 48.735 | 51.949 |
| | 62.940 | 68.716 | 74.558 | 80.349 | 86.048 | 91.647 |
| CFCC | 23.949 | 26.119 | 28.335 | 30.545 | 32.730 | 34.884 |
| | 40.019 | 43.713 | 47.442 | 51.143 | 54.796 | 58.396 |
| | 63.282 | 69.093 | 74.970 | 80.797 | 86.534 | 92.173 |
| CSFS | 12.687 | 14.112 | 15.622 | 17.169 | 18.731 | 20.296 |
| | 33.065 | 36.009 | 39.029 | 42.066 | 45.092 | 48.099 |
| | 41.707 | 47.175 | 52.787 | 58.360 | 63.821 | 69.145 |
| CSFC | 17.551 | 19.758 | 22.081 | 24.449 | 26.825 | 29.193 |
| | 36.032 | 39.366 | 42.801 | 46.268 | 49.738 | 53.198 |
| | 51.843 | 58.737 | 65.704 | 72.488 | 79.002 | 85.239 |
| CSSS | 23.646 | 25.751 | 27.890 | 30.024 | 32.141 | 34.235 |
| | 51.675 | 56.761 | 61.837 | 66.831 | 71.726 | 76.520 |
| | 58.646 | 63.893 | 69.188 | 74.441 | 79.622 | 84.724 |
| CSSC | 27.055 | 29.564 | 32.108 | 34.646 | 37.161 | 39.648 |
| | 60.539 | 66.125 | 71.658 | 77.144 | 82.555 | 87.884 |
| | 60.791 | 66.676 | 72.609 | 78.419 | 84.087 | 89.617 |
| CSCS | 28.951 | 31.597 | 34.279 | 36.944 | 39.572 | 42.160 |
| | 54.744 | 59.828 | 64.883 | 69.850 | 74.713 | 79.476 |
| | 69.327 | 75.713 | 82.167 | 88.564 | 94.865 | 101.06 |
| CSCC | 31.827 | 34.752 | 37.705 | 40.635 | 43.523 | 46.365 |
| | 63.333 | 69.224 | 75.041 | 80.728 | 86.275 | 91.692 |
| | 71.079 | 77.635 | 84.262 | 90.835 | 97.312 | 103.68 |
| CCFC | 23.949 | 27.145 | 30.478 | 33.839 | 37.176 | 40.468 |

| $\alpha \rightarrow$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|--------|--------|--------|--------|--------|--------|
| | 40.019 | 43.869 | 47.866 | 51.937 | 56.046 | 60.178 |
| | 63.282 | 71.741 | 80.092 | 87.989 | 95.370 | 102.30 |
| CCSC | 31.827 | 34.885 | 37.974 | 41.047 | 44.087 | 47.089 |
| | 63.333 | 69.094 | 74.925 | 80.729 | 86.470 | 92.138 |
| | 71.079 | 78.125 | 85.027 | 91.716 | 98.188 | 104.46 |

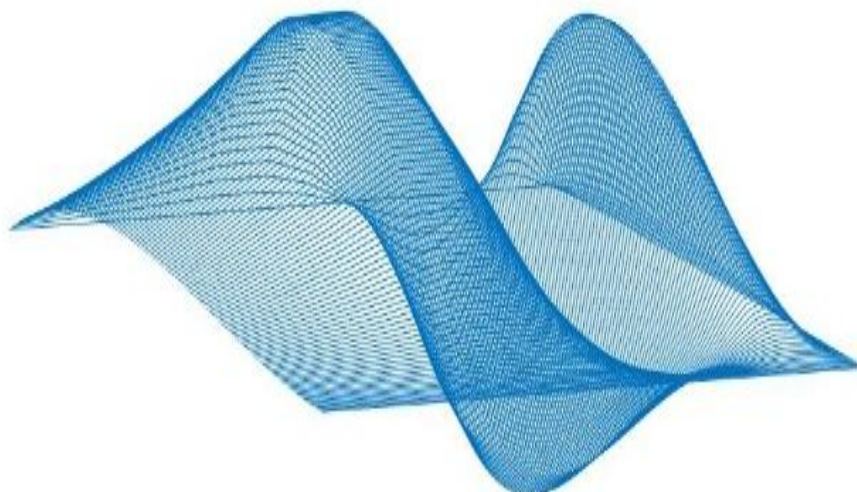
After computing the first three frequencies, the mode shapes have been demonstrated by computing the eigenvectors on the corresponding frequency. These are represented in the following figures 4.2 and 4.3 for the rectangular and square plates, respectively with taper parameter α is considered as 0.5. The presented mode shapes represent the behaviour of the plate during vibration of the plate.



(a) First Mode

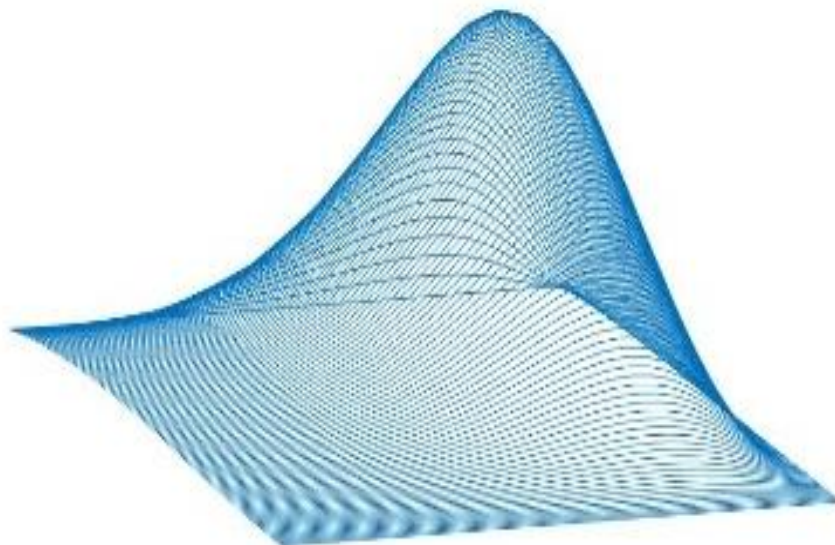


(b) Second Mode

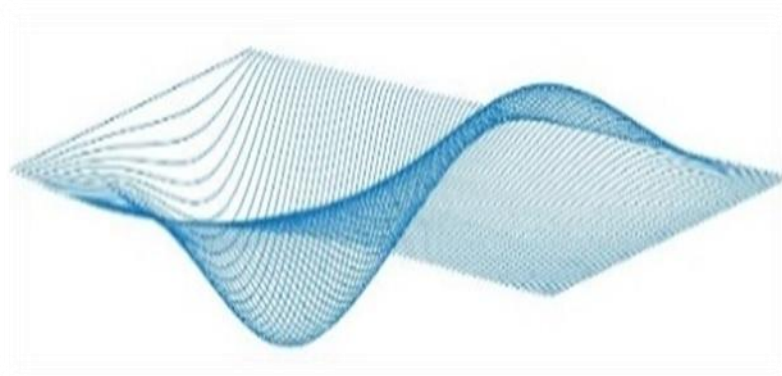


(c) Third Mode

Figure 4.2 First Three Mode Shapes for Rectangular Clamped Plate
($\alpha = 0.5, \nu = 0.3$)



(a) First Mode



(b) Second Mode



(c) Third Mode

Figure 4.3 First Three Mode Shapes for Square Clamped Plate ($\alpha = 0.5, \nu = 0.3$)

The computations in the present work have been performed upto the five significant digits and the results have been computed after the study of the convergence of the results. In the Rayleigh-Ritz method, the convergence is so fast as represented in the following table 4.5. The values of convergence have been obtained for the rectangular and square plates both with taper parameter as $\alpha=0.2$. It is observed from the table that the values are converging upto the five significant digits.

Table 4.5 Convergence of the Results for Rectangular and Square Plates

| n | Rectangular Plate($\mu=a/b=1.0$) | | | Square Plate ($\mu=a/b=1.0$) | | |
|----|------------------------------------|-------------|-------------|--------------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| 1 | 108.19 | - | - | 39.505 | - | - |
| 2 | 107.78 | 140.17 | - | 39.332 | 81.613 | - |
| 3 | 107.78 | 140.17 | 286.36 | 39.332 | 81.534 | 81.613 |
| 4 | 107.59 | 139.40 | 200.94 | 39.328 | 80.989 | 81.534 |
| 5 | 107.59 | 139.40 | 200.94 | 39.328 | 80.989 | 81.265 |
| 6 | 107.42 | 139.40 | 200.23 | 39.328 | 80.989 | 81.265 |
| 7 | 107.42 | 139.35 | 199.04 | 39.328 | 80.365 | 81.265 |
| 8 | 107.42 | 139.35 | 199.04 | 39.328 | 80.365 | 81.083 |
| 9 | 107.42 | 139.35 | 199.04 | 39.328 | 80.239 | 81.083 |
| 10 | 107.42 | 139.35 | 199.04 | 39.328 | 80.239 | 80.266 |
| 11 | 107.42 | 139.33 | 196.31 | 39.328 | 80.214 | 80.266 |
| 12 | 107.42 | 139.33 | 196.31 | 39.328 | 80.214 | 80.266 |
| 13 | 107.41 | 139.33 | 196.19 | 39.317 | 80.211 | 80.266 |
| 14 | 107.41 | 139.33 | 196.19 | 39.317 | 80.211 | 80.263 |
| 15 | 107.41 | 139.33 | 196.19 | 39.316 | 80.211 | 80.263 |
| 16 | 107.41 | 139.32 | 195.99 | 39.316 | 80.199 | 80.263 |
| 17 | 107.41 | 139.32 | 195.99 | 39.316 | 80.199 | 80.258 |
| 18 | 107.41 | 139.30 | 195.98 | 39.316 | 80.186 | 80.258 |
| 19 | 107.41 | 139.30 | 195.98 | 39.316 | 80.186 | 80.243 |
| 20 | 107.41 | 139.30 | 195.98 | 39.316 | 80.181 | 80.243 |
| 21 | 107.41 | 139.30 | 195.98 | 39.316 | 80.181 | 80.225 |
| 22 | 107.41 | 139.30 | 195.88 | 39.316 | 80.180 | 80.225 |
| 23 | 107.41 | 139.30 | 195.88 | 39.316 | 80.180 | 80.225 |
| 24 | 107.40 | 139.30 | 195.84 | 39.315 | 80.180 | 80.225 |
| 25 | 107.40 | 139.30 | 195.84 | 39.315 | 80.180 | 80.225 |
| 26 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |
| 27 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |
| 28 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |

The comparisons of results have also been obtained from the literature for uniform thickness variation for the rectangular and square plates and compiled in the following table 4.6 with different combinations of the boundary conditions. It is observed from the table that computed results in the table are in close agreement with the existing results obtained in the some special cases for rectangular and square plates.

Table 4.6 Comparisons of the Results for Rectangular and Square Plates ($\alpha=0.0$)

| Boundary Condition | Reference | Available Results | | | Present Result | | |
|--------------------------|-----------|-------------------|-------------|-------------|----------------|-------------|-------------|
| | | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| Rectangular Plate | | | | | | | |
| FFFF | [208] | - | 21.789 | - | 0.0 | 21.467 | 26.585 |
| CFCF | - | - | - | - | 22.076 | 36.018 | 60.897 |
| CCFF | [127] | 17.347 | 36.449 | - | 17.142 | 36.423 | 73.516 |
| CSFF | [255] | 15.851 | - | 49.103 | 15.082 | 31.310 | 49.310 |
| SSFF | [255] | - | 27.801 | 38.079 | 9.512 | 27.522 | 38.531 |
| Square Plate | | | | | | | |
| FFFF | [14][248] | - | 13.468 | 19.596 | 0.00 | 13.469 | 19.597 |
| CCCC | [14] | 35.989 | - | - | 35.986 | 73.399 | 73.399 |
| CFCF | [14] | 22.165 | - | 43.589 | 22.200 | 26.478 | 43.630 |
| SFFF | [255] | - | 6.641 | 14.899 | 0.00 | 6.643 | 14.903 |
| SFSF | [255] | 3.470 | 8.498 | 21.281 | 3.481 | 8.521 | 21.317 |
| CCFF | [127] | 9.630 | 16.127 | 36.705 | 9.631 | 16.135 | 36.727 |
| CSFF | [127] | 6.832 | 23.418 | 26.096 | 6.930 | 23.942 | 26.598 |
| SSFF | [127] | - | 20.685 | - | 15.211 | 20.613 | 39.773 |

4.4 MAJOR FINDINGS

The above problem is based upon the study of the transcendental thickness variation applied on the thin rectangular and square plates. It is observed that the Rayleigh-Ritz method is an efficient method for obtaining the results for the first few frequencies for the thin plates. The behaviour of the plate is represented through the computations of the mode shapes. As the taper parameter is controlling the thickness variation which is increasing when the frequencies of the plate are increasing due to increment in the stiffness of the plate. The results have been obtained correct upto five significant digits along with the convergence of the results.

CHAPTER-V

Numerical Experiments on Circular Plate with Transcendental Thickness Variation

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CHAPTER-V

NUMERICAL EXPERIMENTS ON CIRCULAR PLATE WITH TRANSCENDENTAL THICKNESS VARIATION

In the present work, further Rayleigh-Ritz method has been used to compute the first few frequencies of a circular plate. The boundary conditions of circular plate are considered as a clamped and simply-supported. Different types of thickness variations of circular plate have been considered by researchers and a vast numbers of numerical results are available in the literature but none of the researchers consider the transcendental thickness variation which has been considered in the present work. The type of circular plate is considered as isotropic plate and significant numerical computations have been done for finding first three frequencies by varying the order of approximation and also the taper parameter. In special cases, results have been compared for uniform and linearly varying thickness variations and on the basis of these results transcendental thickness variations of circular plate have been studied and computed results are presented in the form of tables and graphs.

5.1 BACKGROUND

The main aim of this paper is to compute first three frequencies for circular plate with two types of boundary conditions for the transcendental thickness variation. Computed results by varying taper parameter have been compared with the existing results in case of uniform and linearly thickness variations. Convergence of computed results has been expressed by taking significant order of approximations.

5.1.1 Circular Plate

The circular plate as represented in the figure 1.2, is assembled a round of disk with sharp boundary. The circular plate has no fold contain in internal area. Distance of

origin to boundary is radius of the plate. Clamped and simply-supported boundary conditions are applied on boundary of circular plate as represented in figure 5.1 and 5.2, respectively. The review of literature on circular plate has already been discussed in the chapter II.

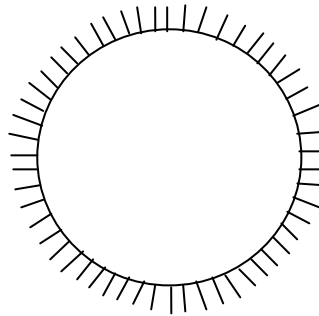


Figure 5.1 Representation of Clamped Circular Plate

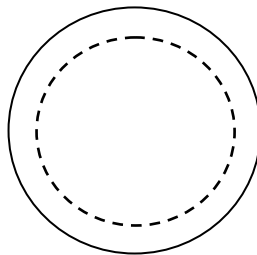


Figure 5.2 Representation of Simply-Supported Circular Plate

5.2 MATHEMATICAL FORMULATION

As represented in figure 1.2, the plate is considered in terms of polar coordinate given by system as

$$x = r \cos \theta \tag{5.1}$$

$$y = r \sin \theta \tag{5.2}$$

where $x^2 + y^2 = r^2$ (5.3)

In terms of polar coordinate system, the maximum kinetic energy is given by

$$T_{\max} = \frac{1}{2} \omega^2 \int_R \gamma h W^2 r dr \quad (5.4)$$

The maximum strain energy is given by

$$U_{\max} = \frac{1}{2} \int_R \frac{E h^3}{12(1-\nu^2)} \left[\left(\frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right)^2 + 2(1-\nu) \frac{d^2 W}{dr^2} \left(\frac{1}{r} \frac{dW}{dr} \right) \right] r dr \quad (5.5)$$

Where $W(x,y)$ is the maximum displacement at point (x,y) given by equation (1.62), ω is angular frequency at the time t , $D = \frac{E h^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate, E is the young modulus, γ is density of material of plate, ν is the poisson's ratio and $h(x,y)$ is the variable thickness of plate at point (x,y) . After equating (5.4) and (5.5) and applying the procedure as explained in section (1.10), the Rayleigh quotient is given by

$$\omega^2 = \frac{E \int_R h^3 \left[\left(\frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right)^2 + 2(1-\nu) \frac{d^2 W}{dr^2} \left(\frac{1}{r} \frac{dW}{dr} \right) \right] r dr}{12(1-\nu^2) \int_R \gamma h W^2 r dr} \quad (5.6)$$

The boundary conditions for clamped and simply-supported circular plate are defined by

$$W = 0, \quad \frac{dW}{dr} = 0 \quad \text{for clamped plate} \quad (5.7)$$

$$\frac{dW}{dr} = 0, \quad \frac{d^2 W}{dr^2} + \left(\frac{\nu}{r} \right) \frac{dW}{dr} = 0 \quad \text{for simply-supported plate} \quad (5.8)$$

To reduce the computations, let us introducing the following non-dimensional variables

$$\rho = r/a, \quad H = h/a = h_0 f(\rho) \quad (5.9)$$

Where h_0 non-dimensional thickness at the origin of the plate i.e centre of the plate while $f(\rho)$ is a function of the non-dimensional radial distance ρ which is controlling the entire thickness variation of the plate. In the present work the transcendental thickness variation is considered and given by

$$f(\rho) = 1 + \sin \alpha \rho \tag{5.10}$$

Where α is a taper parameter which is varying from 0 to 1.0. The equation (5.6) is converted as

$$\lambda^2 = \frac{a^2 \omega^2 \gamma}{D} = \frac{\int_{R'} f^3 \left[\left(W'' + \frac{W'}{\rho} \right)^2 + 2(1-\nu) \frac{W'' W'}{\rho} \right] \rho d\rho}{\int_{R'} f W^2 \rho d\rho} \tag{5.11}$$

Where R' is the new domain occupied by the plate after introducing the non-dimensional variables. For the unit circular disk, ρ is given by

$$\rho \leq 1 \tag{5.12}$$

Now considering the displacement by selecting the basis functions called as trivial functions as

$$W = \sum_{i=1}^n c_i \phi_i(\rho) \tag{5.13}$$

Where c_i 's are the constants and ϕ_i is given by

$$\phi_i = u^{s+i-1} \quad i = 1, 2, \dots, n \tag{5.14}$$

and
$$u = (1 - \rho^2) \tag{5.15}$$

In equation (5.14), parameter s is controlling the boundary condition of the plate, when $s=2$ then plate is called as a simply-supported circular plate and when $s=1$ then the plate clamped circular plate.

Now minimizing the equation (5.11) i.e λ^2 over the c_1, c_2, \dots, c_n . on leads to equation (1.101) i.e.

$$a_{ij} - \lambda^2 b_{ij} = 0 \tag{5.16}$$

Where a_{ij} and b_{ij} are given by

$$a_{ij} = \iint_R f^3 \left(\phi_i'' \phi_j'' + \nu (\phi_j'' \phi_i' + \phi_i'' \phi_j') + \frac{1}{\rho} \phi_i' \phi_j' \right) d\rho \tag{5.17}$$

$$b_{ij} = \iint_R f \phi_i \phi_j \rho d\rho \tag{5.18}$$

These calculations involved the following kinds of derivatives by using (5.14)

$$\phi_i' = -2(s+i-1) \left[\rho (1-\rho^2)^{s+i-2} \right] \tag{5.19}$$

$$\phi_i'' = \left[4(s+i-1)(s+i-2)\rho^2 (1-\rho^2)^{s+i-3} + (-2(s+i-1))(1-\rho^2)^{s+i-2} \right] \tag{5.20}$$

Similarly we can get

$$\phi_j' = -2(s+j-1) \left[\rho (1-\rho^2)^{s+j-2} \right] \tag{5.21}$$

$$\phi_j'' = \left[4(s+j-1)(s+j-2)\rho^2 (1-\rho^2)^{s+j-3} + (-2(s+j-1))(1-\rho^2)^{s+j-2} \right] \tag{5.22}$$

Therefore

$$\begin{aligned} \phi_i'' \phi_j'' = & \left[16(s+i-1)(s+j-1)(s+i-2)(s+j-2)\rho^4 (1-\rho^2)^{2s+i+j-6} + (-8)(s+i-1) \right. \\ & \left. (s+j-1)\rho^2 (1-\rho^2)^{2s+i+j-5} (2s+i+j-4) + 4(s+i-1)(s+j-1)(1-\rho^2)^{2s+i+j-4} \right] \end{aligned} \tag{5.23}$$

$$\phi_i' \phi_j' = \left[4(s+i-1)(s+j-1)\rho^2 (1-\rho^2)^{2s+i+j-4} \right] \tag{5.24}$$

$$\phi_i'' \phi_j' = 4(s+i-1)(s+j-1)\rho(1-\rho^2)^{2s+i+j-5} \left[(1-\rho^2) - 2(s+i-2)\rho^2 \right] \quad (5.25)$$

$$\phi_j'' \phi_i' = 4(s+i-1)(s+j-1)\rho(1-\rho^2)^{2s+i+j-5} \left[(1-\rho^2) - 2(s+i-2)\rho^2 \right] \quad (5.26)$$

By generating the matrices a_{ij} and b_{ij} , which leads to eigenvalue problem for computation of frequencies by generalized Jacobi method. We can obtain the first three frequencies for clamped and simply-supported circular plate by considering the involvement of following types of integrals in the numerical computations

$$\iint_R \sin^k x \cos^l x dx = \frac{\frac{k+1}{2} \frac{l+1}{2}}{2 \frac{k+l+2}{2}} \quad (5.27)$$

5.3 NUMERICAL RESULTS AND DISCUSSION

For computation of the results, different parameters have been considered and discussed below in brief.

1. Taper parameter which is controlling the entire thickness variation of circular plate is considered from 0 to 1 at the interval difference of 0.1;
2. The value of Poisson's ratio is fixed for isotropic plate and taken as 0.3;
3. Approximation n is taken from 1 to 10 for fast convergence of results;
4. The parameter s is controlling the different types of boundary conditions of plate which is taken as 2 for simply-supported and 1 for clamped circular plate;
5. All the involved integral in computation are in closed form and computed by equation (5.27) by the use of Rayleigh-Ritz method, an eigenvalue problem has been solved by generalized Jacobi method. The first three frequencies are represented in table 5.1 for the circular plate on different boundary conditions of plate. Similar results are represented in the figures 5.3 and 5.4 for simply-supported and clamped circular plate, respectively;

Table 5.1 First Three Frequencies for Simply-Supported and Clamped Circular Plate with Transcendental Thickness Variation

| Taper Parameter | Simple-Support Boundary Condition (s=2) | | | Clamped Boundary Condition (s=1) | | |
|-----------------|---|-------------|-------------|----------------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| 0.0 | 4.9351 | 29.720 | 74.156 | 10.215 | 39.771 | 89.104 |
| 0.1 | 5.6337 | 33.924 | 84.643 | 11.677 | 45.414 | 101.72 |
| 0.2 | 6.2635 | 37.705 | 94.061 | 13.020 | 50.517 | 113.08 |
| 0.3 | 6.8385 | 41.140 | 102.60 | 14.268 | 55.180 | 123.42 |
| 0.4 | 7.3672 | 44.277 | 110.40 | 15.436 | 59.462 | 132.87 |
| 0.5 | 7.8547 | 47.147 | 117.51 | 16.532 | 63.403 | 141.53 |
| 0.6 | 8.3048 | 49.773 | 124.01 | 17.562 | 67.030 | 149.45 |
| 0.7 | 8.7202 | 52.173 | 129.93 | 18.531 | 70.365 | 156.70 |
| 0.8 | 9.1030 | 54.363 | 135.32 | 19.442 | 73.425 | 163.31 |
| 0.9 | 9.4550 | 56.357 | 140.22 | 20.298 | 76.228 | 169.33 |
| 1.0 | 9.7777 | 58.170 | 144.65 | 21.102 | 78.790 | 174.79 |

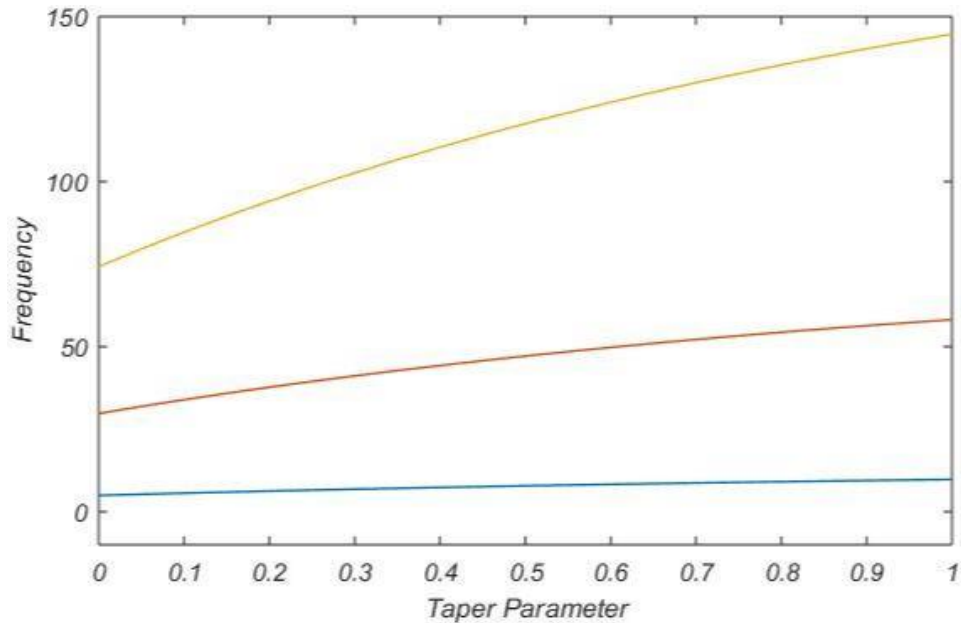


Figure 5.3 Representation of First Three Frequencies for Simply-Supported Circular Plate

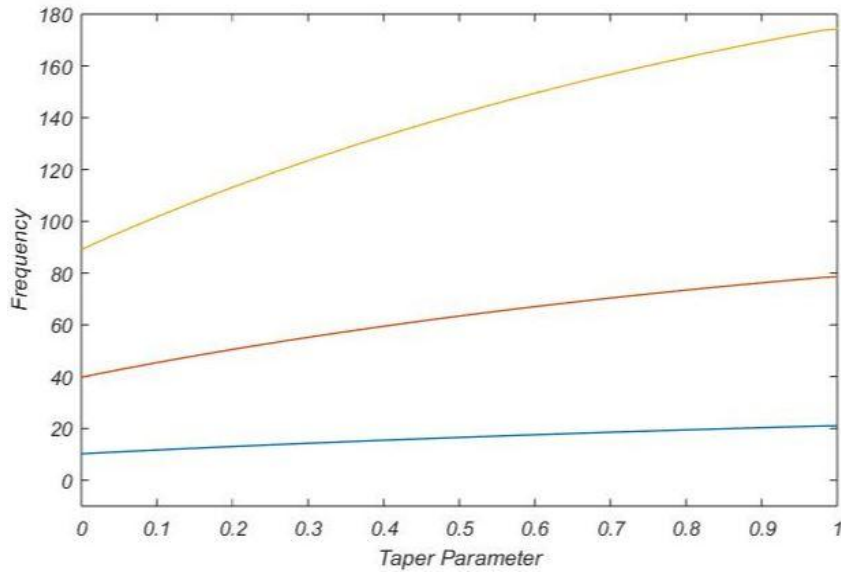


Figure 5.4 Representation of First Three Frequencies for Clamped Circular Plate

In all case, it is observed that due to increase of taper parameter, the frequencies are increasing. This is because of stiffness of circular plate is increasing by increasing the taper parameter. When taper parameter is zero then it leads to results related to the uniform thickness variation of circular plate.

All the computed results have been compared upto the level of five significant digits. These calculated results ensure the convergence of results by varying order of approximate i.e n and is represented below in table 5.2. From the table, it is found that results are matching upto five significant digits. In the case of uniform thickness variation, the computed results are compared with the available results and found that results are matching with the existing results and represented in table 5.3.

Table 5.2 Convergence of the Results for Circular Plate ($\alpha=0, s=2, 1$)

| n | Simply-Supported Circular Plate (s=2) | | | Clamped Circular Plate (s=1) | | |
|---|---------------------------------------|-------------|-------------|------------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| 1 | 5.5856 | - | - | 10.327 | - | - |
| 2 | 4.9405 | 46.706 | - | 10.217 | 43.058 | - |
| 3 | 4.9351 | 30.503 | 163.23 | 10.215 | 39.921 | 109.59 |

| | | | | | | |
|----|--------|--------|--------|--------|--------|--------|
| 4 | 4.9351 | 29.736 | 81.896 | 10.215 | 39.773 | 91.157 |
| 5 | 4.9351 | 29.720 | 74.683 | 10.215 | 39.771 | 89.203 |
| 6 | 4.9351 | 29.720 | 74.171 | 10.215 | 39.771 | 89.106 |
| 7 | 4.9351 | 29.720 | 74.156 | 10.215 | 39.771 | 89.104 |
| 8 | 4.9351 | 29.720 | 74.156 | 10.215 | 39.771 | 89.104 |
| 9 | 4.9351 | 29.720 | 74.156 | 10.215 | 39.771 | 89.104 |
| 10 | 4.9351 | 29.720 | 74.156 | 10.215 | 39.771 | 89.104 |

Table 5.3 Comparisons of the Results for Uniform Circular Plate

| Reference | λ_1 | λ_2 | λ_3 |
|----------------|-------------|-------------|-------------|
| Present Result | 10.215 | 39.771 | 89.104 |
| [23] | 10.217 | 34.940 | - |
| [70] | 13.900 | 48.48 | 102.77 |
| [121] | 10.210 | - | - |
| [262] | 10.215 | 34.877 | - |
| [209] | 10.260 | - | - |
| [276] | 10.216 | - | - |
| [220] | 10.216 | 39.771 | 89.104 |
| [218] | 10.216 | 39.771 | 89.104 |
| [69] | 10.216 | 34.877 | - |

5.4 MAJOR FINDINGS

In this chapter, the natural frequencies have been investigated through well known Rayleigh-Ritz technique with two boundary conditions and one taper parameter on the circular plate. When the taper parameter α is increasing from zero to one at the interval 0.1, then frequencies are also increasing for vibrations of circular plate. The natural frequency has been computed for circular plate upto the five significant digits. Comparison with available results in special cases confirms that accuracy is comparable with the best known result.

CHAPTER-VI

Numerical Experiments on Skew Plate with Linear and Exponential Thickness Variations

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CHAPTER-VI

NUMERICAL EXPERIMENTS ON SKEW PLATE WITH LINEAR AND EXPONENTIAL THICKNESS VARIATIONS

In the present chapter, Rayleigh-Ritz technique has been successfully employed to compute transverse vibration of a skew plate with two boundary conditions and two different thicknesses of variations. Clamped and simply-supported boundary conditions have been used for computation of first three frequencies of two dimensional plate. The entire boundary of the skew plate is either clamped or simply-supported or two sides are clamped or rest two sides are simply-supported. Eigenvalue problem is solved by generalized Jacobi method using aspect ratio, taper parameter, skew angle, exponential and linear thickness of variations in one direction i.e. towards x axis. All the results are represented in the table and frequencies behaviour are represented through graphs. The combination of CCCC, SSSS, CSCS have been obtained with increase in taper parameter. Convergence table is ensured by taking approximation results upto five significant digits and comparison table is also presented for fixed skew angle, taper parameter and aspect ratio of the skew plate. The frequencies are represented in the form of tables and graphs.

6.1 BACKGROUND

6.1.1 SKEW PLATE

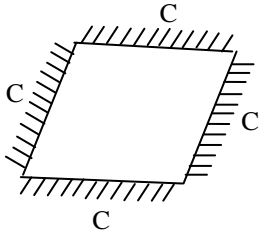
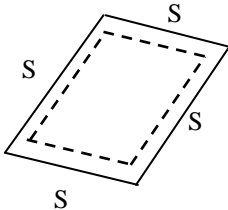
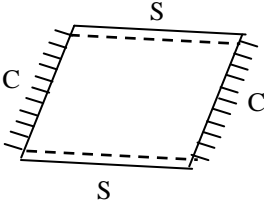
Skew plate is a solid body and bounded by four edges with fixed skew angle in two dimensions in the domain R. let us consider $\theta=90^0$ then skew plate is mapped into a unit square plate as shown in the figure 1.5. The four sides of skew plate are represented by (1), (2), (3), and (4) when we consider CCCC then all sides (1), (2), (3) and (4) are C, C, C and C i.e. clamped side respectively.

6.1.2 BOUNDARY CONDITIONS

All boundary conditions are based on boundary value problem. Boundary conditions are defined in table 6.1 given below as.

Table 6.1 Representation of Boundary Conditions with Different Cases

| case | Edge1 | Edge2 | Edge 3 | Edge4 |
|------|-------|-------|--------|-------|
| 1 | C | C | C | C |
| 2 | S | S | S | S |
| 3 | C | S | C | S |

| | |
|------|--|
| CCCC |  |
| SSSS |  |
| CSCS |  |

6.2 MATHEMATICAL FORMULATION

Rayleigh-Ritz method is based on Rayleigh method and both are used for continuous systems. The Rayleigh technique is used for single function and Rayleigh-Ritz considers linear combination of several functions. The selected function must satisfy the boundary conditions in the domain R. The plate is defined by three parameters

numbers a , b , and θ in the figure 1.5 and $\theta=90^0$ is a special case of skew plate which is transferred into square plate in new domain R' shown in the same figure by using following transformation.

$$x = a\xi + b(\cos \theta)\eta \tag{6.1}$$

$$y = b(\sin \theta)\eta \tag{6.2}$$

Where, ξ and η are generated new coordinates as shown in (1.5), when $\theta = 90^0$ then $\xi = x/a$ and $\eta = y/b$ are treated as non-dimensional variables. Let us consider the thickness h at point (ξ, η) of new domain R' .

In the expansion of $e^{\alpha\xi}$, when two terms are considered, then it is converted into the linear expansion of thickness variation as

$$h = ah_0e^{\alpha\xi}, \quad h = ah_0(1 + \alpha\xi) \tag{6.3}$$

Where, α is taper parameter which is controlling entrie thickness variation and h_0 is the non-dimensional thickness at origin i.e. $(0, 0)$.

Considering the n term approximate as discussed in the chapter 1 and minimizing the Rayleigh quotient over the constants c_j 's ($j=1, 2, 3, \dots$),

We can get equation (1.101) where ϕ_i basis function are considered as

$$\phi_i(\xi, \eta) = \xi^p \eta^q (1-\xi)^r (1-\eta)^s \quad (1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \dots) \tag{6.4}$$

Here a_{ij} and b_{ij} are given by

$$a_{ij} = \frac{1}{\sin^4(\theta)} \iint_{R'} f^3 \left[\phi_i^{\xi\xi} \phi_j^{\xi\xi} - 2\mu \cos(\theta) (\phi_i^{\xi\eta} \phi_j^{\xi\xi} + \phi_i^{\xi\xi} \phi_j^{\xi\eta}) + \mu^2 (\nu \sin^2(\theta) + \cos^2(\theta)) (\phi_i^{\eta\eta} \phi_j^{\xi\xi} + \phi_i^{\xi\xi} \phi_j^{\eta\eta}) \right. \\ \left. + 2\mu^2 (1 + \cos^2(\theta) - \nu \sin^2(\theta)) \phi_i^{\xi\eta} \phi_j^{\xi\eta} - 2\mu^3 \cos(\theta) (\phi_i^{\eta\eta} \phi_j^{\xi\eta} + \phi_i^{\xi\eta} \phi_j^{\eta\eta}) + \mu^4 \phi_i^{\eta\eta} \phi_j^{\eta\eta} \right] d\xi d\eta \tag{6.5}$$

$$a_{ij} = \iint_R f^3 \left[\phi_i^{\xi\xi} \phi_j^{\xi\xi} + A_1 \left(\phi_i^{\xi\eta} \phi_j^{\xi\xi} + \phi_i^{\xi\xi} \phi_j^{\xi\eta} \right) + A_2 \left(\phi_i^{\eta\eta} \phi_j^{\xi\xi} + \phi_i^{\xi\xi} \phi_j^{\eta\eta} \right) + A_3 \phi_i^{\xi\eta} \phi_j^{\xi\eta} \right. \\ \left. + A_4 \left(\phi_i^{\eta\eta} \phi_j^{\xi\eta} + \phi_i^{\xi\eta} \phi_j^{\eta\eta} \right) + A_5 \phi_i^{\eta\eta} \phi_j^{\eta\eta} \right] d\eta d\xi \quad (6.6)$$

Where

$$A_1 = \frac{-2\mu \cos(\theta)}{\sin^4(\theta)} \quad (6.7)$$

$$A_2 = \frac{\mu^2 (\nu \sin^2(\theta) + \cos^2(\theta))}{\sin^4(\theta)} \quad (6.8)$$

$$A_3 = \frac{2\mu^2 (1 + \cos^2(\theta) - \nu \sin^2(\theta))}{\sin^4(\theta)} \quad (6.9)$$

$$A_4 = \frac{-2\mu^3 \cos(\theta)}{\sin^4(\theta)} \quad (6.10)$$

$$A_5 = \frac{\mu^4}{\sin^4(\theta)} \quad (6.11)$$

$$\mu = a/b \quad (6.12)$$

$$b_{ij} = \iint_R f \phi_i \phi_j d\xi d\eta \quad (6.13)$$

The thickness variation along ξ direction is given by

$$f(\xi, \eta) = e^{\alpha\xi} \quad (6.14)$$

When higher terms is expression of $e^{\alpha\xi}$ are neglected then it is converted into linear expression i.e.

$$f(\xi, \eta) = 1 + \alpha\xi \quad (6.15)$$

$$\phi_i^{\xi} = \eta^q (1-\eta)^s \left[p\xi^{p-1} (1-\xi)^r + (-r)\xi^p (1-\xi)^{r-1} \right] \quad (6.16)$$

$$\phi_i^{\xi\xi} = \eta^q (1-\eta)^s \left[p(p-1)\xi^{p-2} (1-\xi)^r + (-2rp)\xi^{p-1} (1-\xi)^{r-1} + r(r-1)\xi^p (1-\xi)^{r-2} \right]$$

$$(6.17)$$

$$\phi_i^\eta = \xi^p (1-\xi)^r \left[q\eta^{q-1}(1-\eta)^s + (-s)\eta^q(1-\eta)^{s-1} \right] \quad (6.18)$$

$$\phi_i^{\eta\eta} = \xi^p (1-\xi)^r \left[s(s-1)\eta^q(1-\eta)^{s-2} + (-2sq)\eta^{q-1}(1-\eta)^{s-1} + q(q-1)\eta^{q-2}(1-\eta)^s \right] \quad (6.19)$$

$$\phi_i^{\xi\eta} = \left[pq\xi^{p-1}\eta^{q-1}(1-\xi)^r(1-\eta)^s + sp\xi^{p-1}\eta^q(1-\xi)^r(1-\eta)^{s-1} + (-rq)\xi^p\eta^{q-1}(1-\xi)^{r-1}(1-\eta)^s + (-rs)\xi^p\eta^q(1-\xi)^{r-1}(1-\eta)^{s-1} \right] \quad (6.20)$$

Similarly by replacing i and j, we can also get $\phi_j^{\xi\xi}$, $\phi_j^{\eta\eta}$ and $\phi_j^{\xi\eta}$ as

$$\phi_j^{\xi\xi} = \eta^q(1-\eta)^s \left[p(p-1)\xi^{p-2}(1-\xi)^r + (-2rp)\xi^{p-1}(1-\xi)^{r-1} + r(r-1)\xi^p(1-\xi)^{r-2} \right] \quad (6.21)$$

$$\phi_j^{\eta\eta} = \xi^p(1-\xi)^r \left[s(s-1)\eta^q(1-\eta)^{s-2} + (-2sq)\eta^{q-1}(1-\eta)^{s-1} + q(q-1)\eta^{q-2}(1-\eta)^s \right] \quad (6.22)$$

$$\phi_j^{\xi\eta} = \left[pq\xi^{p-1}\eta^{q-1}(1-\xi)^r(1-\eta)^s + sp\xi^{p-1}\eta^q(1-\xi)^r(1-\eta)^{s-1} + (-rq)\xi^p\eta^{q-1}(1-\xi)^{r-1}(1-\eta)^s + (-rs)\xi^p\eta^q(1-\xi)^{r-1}(1-\eta)^{s-1} \right] \quad (6.23)$$

The computation of a_{ij} and b_{ij} involve the following integration.

$$\iint_R \xi^p \eta^q (1-\xi)^r (1-\eta)^s d\xi d\eta = \frac{p!q!r!s!}{(p+r+1)!(q+s+1)!} \quad (6.24)$$

The matrixs of a_{ij} and b_{ij} are delved by generalized Jacobi method to obtain the frequencies.

6.3 NUMERICAL RESULTS AND DISCUSSION

Numerical results of various parameters have been discussed and given below;

1. The parameter of aspect ratio $\mu = a/b$ ether 1.0 or 2.0;
2. The approximation of matrix order n is taken 1 to 28 for convergence upto five significant digits of all value of parameter;

3. The value of Poisson's ratio has been fixed as 0.3 for all calculation;
4. Taper parameter α which controls the thickness of skew plate is varying from 0 to 1 with interval 0.1;
5. Each integrals use p, q, r and s which have taken values 2 and 1 for clamped and simply-supported boundary conditions, respectively. When the skew plate is considered CCCC then 1, 2, 3, and 4 sides of the plate are clamped, similarly SSSS means 1, 2, 3, and 4 sides are simply-supported and CSCS means 1, 2, 3, 4 means that sides 1 and 3 are clamped and rest are simply-supported boundary conditions;

The results are obtained for CCCC, SSSS and CSCS with Poisson's ratio as 0.3, $\mu = a/b = 1.0$ which are presented in the tables 6.2 and 6.3. All frequencies are decreasing when taper parameter α is increased from 0 to 1 with fixed skew angle 60° . When $\alpha=0$ is special case of results as uniform thickness variation of given plate. The computing results in table 6.2 are using exponential thickness variation while table 6.3 represents linear thickness variation.

Table 6.2 First Three Frequencies ($\mu=1.0$) with Exponential Thickness Variation

| $\theta=60^\circ$ | α | λ_1 | λ_2 | λ_3 |
|-------------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 46.108 | 81.602 | 105.52 |
| | 0.1 | 42.720 | 76.109 | 99.482 |
| | 0.2 | 41.465 | 74.513 | 97.869 |
| | 0.3 | 40.763 | 73.600 | 96.974 |
| | 0.4 | 40.301 | 72.971 | 96.377 |
| | 0.5 | 39.966 | 72.497 | 95.940 |
| | 0.6 | 39.710 | 72.120 | 95.603 |
| | 0.7 | 39.505 | 71.810 | 95.332 |
| | 0.8 | 39.338 | 71.548 | 95.109 |
| | 0.9 | 39.197 | 71.324 | 94.921 |
| 1.0 | 39.077 | 71.128 | 94.761 | |

| $\theta=60^0$ | α | λ_1 | λ_2 | λ_3 |
|---------------|----------|-------------|-------------|-------------|
| SSSS | 0.0 | 25.162 | 52.659 | 72.713 |
| | 0.1 | 22.597 | 47.122 | 66.461 |
| | 0.2 | 21.463 | 45.342 | 64.751 |
| | 0.3 | 20.814 | 44.422 | 63.869 |
| | 0.4 | 20.387 | 43.838 | 63.308 |
| | 0.5 | 20.082 | 43.425 | 62.910 |
| | 0.6 | 19.852 | 43.112 | 62.609 |
| | 0.7 | 19.671 | 42.865 | 62.371 |
| | 0.8 | 19.524 | 42.664 | 62.177 |
| | 0.9 | 19.402 | 42.495 | 62.014 |
| | 1.0 | 19.300 | 42.352 | 61.876 |
| CSCS | 0.0 | 37.067 | 64.388 | 93.577 |
| | 0.1 | 32.815 | 60.394 | 84.747 |
| | 0.2 | 31.275 | 58.966 | 82.622 |
| | 0.3 | 30.411 | 58.157 | 81.432 |
| | 0.4 | 29.837 | 57.613 | 80.617 |
| | 0.5 | 29.420 | 57.214 | 80.007 |
| | 0.6 | 29.099 | 56.904 | 79.525 |
| | 0.7 | 28.841 | 56.654 | 79.130 |
| | 0.8 | 28.629 | 56.446 | 78.799 |
| | 0.9 | 28.450 | 56.270 | 78.516 |
| | 1.0 | 28.297 | 56.118 | 78.271 |

Table 6.3 First Three Frequencies ($\mu=1.0$) with Linear Thickness Variation

| $\theta=60^0$ | α | λ_1 | λ_2 | λ_3 |
|---------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 46.108 | 81.602 | 105.52 |
| | 0.1 | 40.486 | 66.394 | 94.154 |
| | 0.2 | 36.841 | 59.923 | 88.696 |
| | 0.3 | 34.793 | 57.346 | 85.628 |
| | 0.4 | 33.566 | 55.952 | 83.857 |
| | 0.5 | 32.764 | 55.046 | 82.715 |
| | 0.6 | 32.203 | 54.394 | 81.919 |
| | 0.7 | 31.789 | 53.896 | 81.333 |
| | 0.8 | 31.472 | 53.501 | 80.885 |

| | | | | |
|------|-----|--------|--------|--------|
| | 0.9 | 31.222 | 53.182 | 80.533 |
| | 1.0 | 31.020 | 52.918 | 80.249 |
| SSSS | 0.0 | 30.983 | 73.756 | 91.796 |
| | 0.1 | 26.427 | 63.309 | 82.133 |
| | 0.2 | 22.974 | 63.991 | 80.925 |
| | 0.3 | 20.790 | 67.861 | 82.626 |
| | 0.4 | 19.444 | 70.861 | 86.854 |
| | 0.5 | 18.588 | 72.434 | 92.875 |
| | 0.6 | 18.020 | 73.226 | 99.604 |
| | 0.7 | 17.626 | 73.669 | 106.45 |
| | 0.8 | 17.340 | 73.943 | 113.20 |
| | 0.9 | 17.127 | 74.127 | 119.75 |
| | 1.0 | 16.963 | 74.258 | 126.08 |
| CSCS | 0.0 | 37.067 | 64.388 | 93.577 |
| | 0.1 | 29.676 | 56.662 | 74.941 |
| | 0.2 | 25.531 | 52.582 | 69.553 |
| | 0.3 | 23.198 | 50.403 | 67.054 |
| | 0.4 | 21.717 | 49.013 | 65.527 |
| | 0.5 | 20.690 | 48.026 | 64.461 |
| | 0.6 | 19.932 | 47.277 | 63.663 |
| | 0.7 | 19.347 | 46.683 | 63.037 |
| | 0.8 | 18.881 | 46.198 | 62.534 |
| | 0.9 | 18.500 | 45.792 | 62.119 |
| | 1.0 | 18.183 | 45.447 | 61.773 |

Similarly results of tables 6.4 and 6.5 include results for CCCC, SSSS, CSCS with aspect ratio $\mu = a/b = 2.0$. The behaviour of frequencies are same as in tables 6.2 and 6.3.

Table 6.4 First Three Frequencies ($\mu=2.0$) with Exponential Thickness Variation

| $\theta=60^\circ$ | α | λ_1 | λ_2 | λ_3 |
|-------------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 128.80 | 159.72 | 213.68 |
| | 0.1 | 125.91 | 146.26 | 186.97 |

| $\theta=60^0$ | α | λ_1 | λ_2 | λ_3 |
|---------------|----------|-------------|-------------|-------------|
| | 0.2 | 123.49 | 142.74 | 182.34 |
| | 0.3 | 122.10 | 141.35 | 180.45 |
| | 0.4 | 121.25 | 140.52 | 179.34 |
| | 0.5 | 120.68 | 139.92 | 178.60 |
| | 0.6 | 120.27 | 139.46 | 178.05 |
| | 0.7 | 119.97 | 139.08 | 177.63 |
| | 0.8 | 119.73 | 138.77 | 177.29 |
| | 0.9 | 119.54 | 138.51 | 177.02 |
| | 1.0 | 119.39 | 138.29 | 176.80 |
| SSSS | 0.0 | 64.891 | 126.46 | 227.12 |
| | 0.1 | 60.798 | 106.26 | 209.25 |
| | 0.2 | 56.928 | 96.241 | 242.61 |
| | 0.3 | 54.073 | 90.840 | 264.68 |
| | 0.4 | 51.905 | 87.691 | 266.02 |
| | 0.5 | 50.157 | 85.827 | 265.36 |
| | 0.6 | 48.685 | 84.771 | 264.66 |
| | 0.7 | 47.412 | 84.254 | 264.08 |
| | 0.8 | 46.290 | 84.110 | 263.61 |
| | 0.9 | 45.288 | 84.232 | 263.24 |
| | 1.0 | 44.386 | 84.547 | 262.94 |
| CSCS | 0.0 | 69.927 | 114.45 | 177.00 |
| | 0.1 | 63.939 | 96.270 | 147.70 |
| | 0.2 | 59.854 | 91.204 | 142.06 |
| | 0.3 | 57.533 | 89.020 | 139.55 |
| | 0.4 | 56.089 | 87.667 | 137.96 |
| | 0.5 | 55.111 | 86.686 | 136.82 |
| | 0.6 | 54.406 | 85.920 | 135.95 |
| | 0.7 | 53.874 | 85.298 | 135.25 |
| | 0.8 | 53.458 | 84.780 | 134.67 |
| | 0.9 | 53.124 | 84.342 | 134.19 |
| | 1.0 | 52.849 | 83.966 | 133.78 |

Table 6.5 First Three Frequencies ($\mu=2.0$) with Linear Thickness Variation

| $\theta=60^0$ | α | λ_1 | λ_2 | λ_3 |
|---------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 128.80 | 159.72 | 213.68 |
| | 0.1 | 126.80 | 153.59 | 202.67 |
| | 0.2 | 125.93 | 152.17 | 200.77 |
| | 0.3 | 125.44 | 151.37 | 199.70 |
| | 0.4 | 125.12 | 150.81 | 198.95 |
| | 0.5 | 124.89 | 150.40 | 198.37 |
| | 0.6 | 124.72 | 150.06 | 197.89 |
| | 0.7 | 124.58 | 149.79 | 197.50 |
| | 0.8 | 124.47 | 149.56 | 197.17 |
| | 0.9 | 124.37 | 149.37 | 196.88 |
| | 1.0 | 124.30 | 149.20 | 196.62 |
| SSSS | 0.0 | 63.859 | 96.553 | 147.96 |
| | 0.1 | 61.025 | 88.516 | 133.76 |
| | 0.2 | 59.698 | 86.587 | 131.20 |
| | 0.3 | 58.940 | 85.652 | 130.07 |
| | 0.4 | 58.443 | 85.064 | 129.39 |
| | 0.5 | 58.089 | 84.642 | 128.91 |
| | 0.6 | 57.821 | 84.318 | 128.56 |
| | 0.7 | 57.611 | 84.057 | 128.27 |
| | 0.8 | 57.441 | 83.840 | 128.03 |
| | 0.9 | 57.301 | 83.656 | 127.83 |
| | 1.0 | 57.182 | 83.496 | 127.66 |
| CSCS | 0.0 | 69.927 | 114.45 | 177.00 |
| | 0.1 | 66.469 | 106.40 | 165.02 |
| | 0.2 | 65.170 | 104.50 | 162.79 |
| | 0.3 | 64.455 | 103.44 | 161.52 |
| | 0.4 | 63.989 | 102.70 | 160.62 |
| | 0.5 | 63.656 | 102.15 | 159.93 |
| | 0.6 | 63.402 | 101.71 | 159.37 |
| | 0.7 | 63.202 | 101.34 | 158.89 |

| | | | | |
|--|-----|--------|--------|--------|
| | 0.8 | 63.038 | 101.04 | 158.49 |
| | 0.9 | 62.901 | 100.78 | 158.14 |
| | 1.0 | 62.785 | 100.55 | 157.83 |

Table 6.6 represents the convergence of results upto five significant digits and order of approximation n is taken from 1 to 28 for finding the accuracy of the results. Comparison table 6.7 is also prepared for special case of uniform thickness variation in which $\alpha=0$. It is observed that presented results are in close agreement with the available results.

Table 6.6 Convergence of the Results ($\alpha=0.5, \theta=60^0, \mu=2.0$)

| Boundary Condition | n | λ_1 | λ_2 | λ_3 |
|---------------------------|----------|-------------|-------------|-------------|
| CCCC | 1 | 130.57 | 0 | 0 |
| | 2 | 128.25 | 181.67 | 0 |
| | 3 | 127.94 | 175.52 | 351.58 |
| | 4 | 126.29 | 158.12 | 281.75 |
| | 5 | 122.55 | 153.16 | 271.08 |
| | 6 | 121.74 | 153.03 | 270.51 |
| | 7 | 121.74 | 148.87 | 214.90 |
| | 8 | 121.64 | 143.54 | 197.36 |
| | 9 | 121.63 | 142.36 | 196.35 |
| | 10 | 121.63 | 142.30 | 196.31 |
| | 11 | 121.57 | 142.29 | 190.68 |
| | 12 | 120.97 | 141.69 | 187.15 |
| | 13 | 120.92 | 141.31 | 185.79 |
| | 14 | 120.91 | 141.31 | 185.59 |
| | 15 | 120.91 | 141.31 | 185.59 |
| | 16 | 120.91 | 141.26 | 185.51 |
| | 17 | 120.91 | 140.23 | 181.66 |
| | 18 | 120.84 | 140.08 | 180.08 |
| | 19 | 120.84 | 140.08 | 180.05 |
| | 20 | 120.84 | 140.07 | 180.05 |
| | 21 | 120.84 | 140.07 | 180.05 |
| | 22 | 120.84 | 140.07 | 180.04 |

| | | | | |
|--|----|--------|--------|--------|
| | 23 | 120.79 | 139.96 | 178.73 |
| | 24 | 120.69 | 139.96 | 178.60 |
| | 25 | 120.68 | 139.92 | 178.60 |
| | 26 | 120.68 | 139.92 | 178.60 |
| | 27 | 120.68 | 139.92 | 178.60 |
| | 28 | 120.68 | 139.92 | 178.60 |

Table 6.7 Comparison of Results for Skew Plate ($\alpha=0.0$)

| Case | Reference | λ_1 | λ_2 | λ_3 |
|---------------------------------|-----------|-------------|-------------|-------------|
| $\theta=60^0, \mu=1.0$ CCCC, | Present | 46.108 | 81.602 | 105.52 |
| | [141] | 46.490 | 69.930 | - |
| | [221] | 46.166 | 81.602 | 105.52 |
| $\theta=60^0, \mu=1.0$ SSSS | Present | 25.162 | 52.659 | 72.713 |
| | [141] | 25.310 | 53.320 | - |
| | [221] | 25.314 | 52.660 | 72.714 |
| | [165] | - | 52.389 | 80.776 |
| $\theta=60^0, \mu=1.0$ CSCS | Present | 37.067 | 64.388 | 93.577 |
| | [221] | 37.193 | 64.389 | 93.577 |
| $\theta=60^0, \mu=2.0$ CCCC | Present | 128.80 | 159.72 | 213.68 |
| | [212] | 128.74 | 159.72 | 213.38 |
| | [221] | 128.90 | 159.73 | 215.29 |
| | [295] | 128.42 | 159.96 | - |
| $\theta=60^0, \mu=2.0$ SSSS | Present | 63.859 | 96.553 | 147.96 |
| | [212] | 64.069 | 96.558 | 153.76 |
| | [221] | 64.069 | 96.553 | 153.76 |
| | [295] | 61.778 | 104.96 | 147.30 |
| $\theta=60^0, \mu=2.0$ CSCS | Present | 69.927 | 114.45 | 177.00 |
| | [212] | 70.165 | 114.46 | 179.91 |
| | [221] | 70.166 | 114.46 | 176.91 |
| | [295] | - | 115.35 | 187.94 |

The behaviour of the frequencies is also represented in the form of graphs as shown in the figures 6.1, 6.2, 6.3 and 6.4. The similar behaviour is also seen in the graphs.

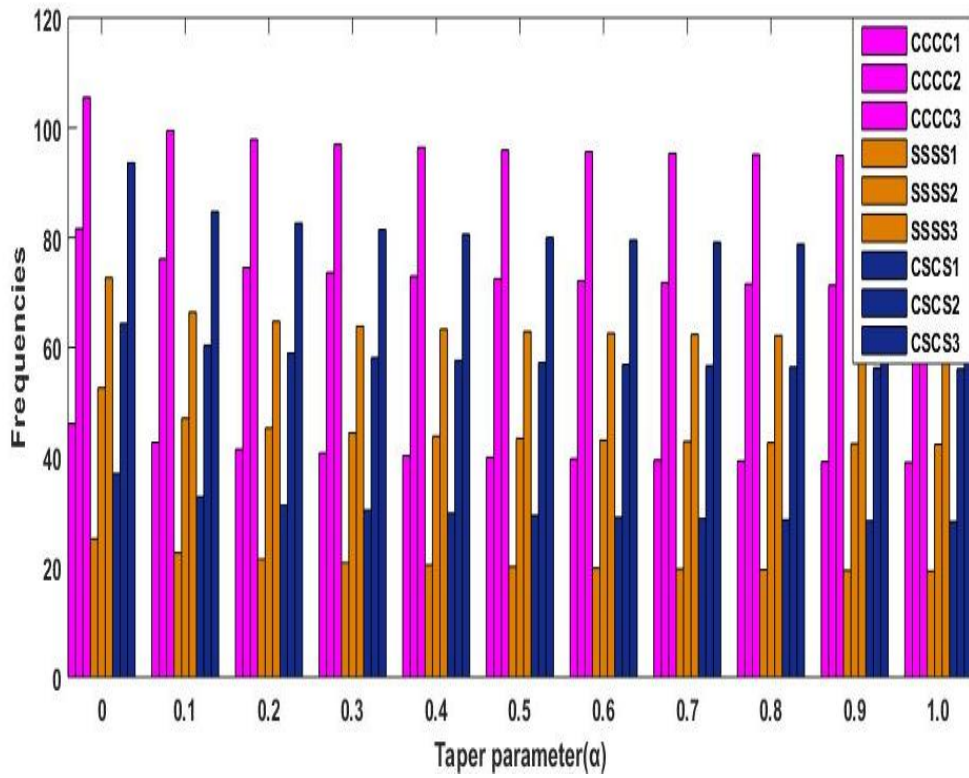


Figure 6.1 Representation of First Three Frequencies ($\mu=1.0$) for Exponential Thickness Variation

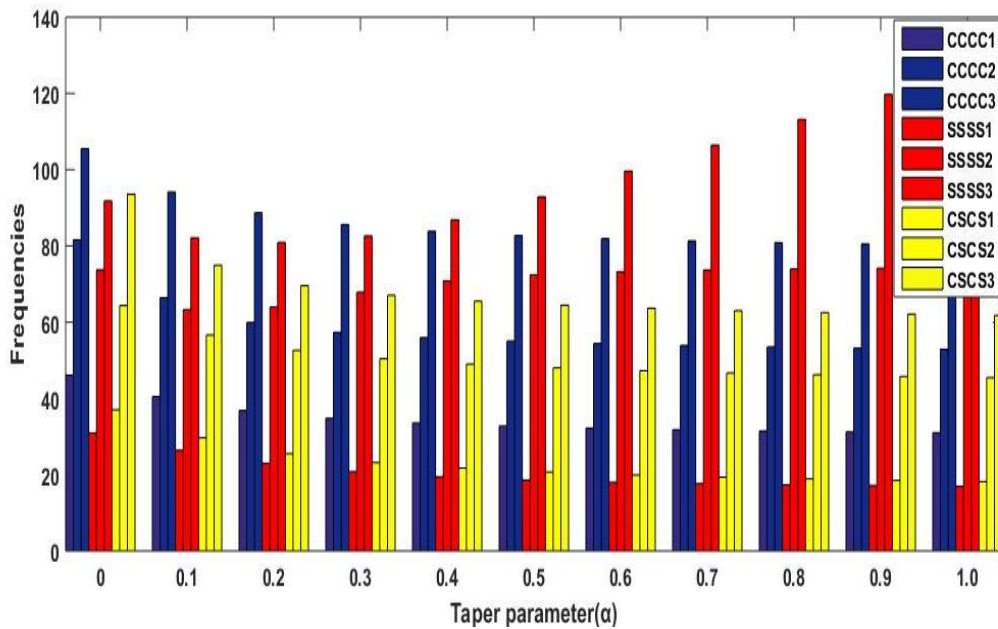


Figure 6.2 Representation of First Three Frequencies ($\mu=1.0$) for Linear Thickness Variation

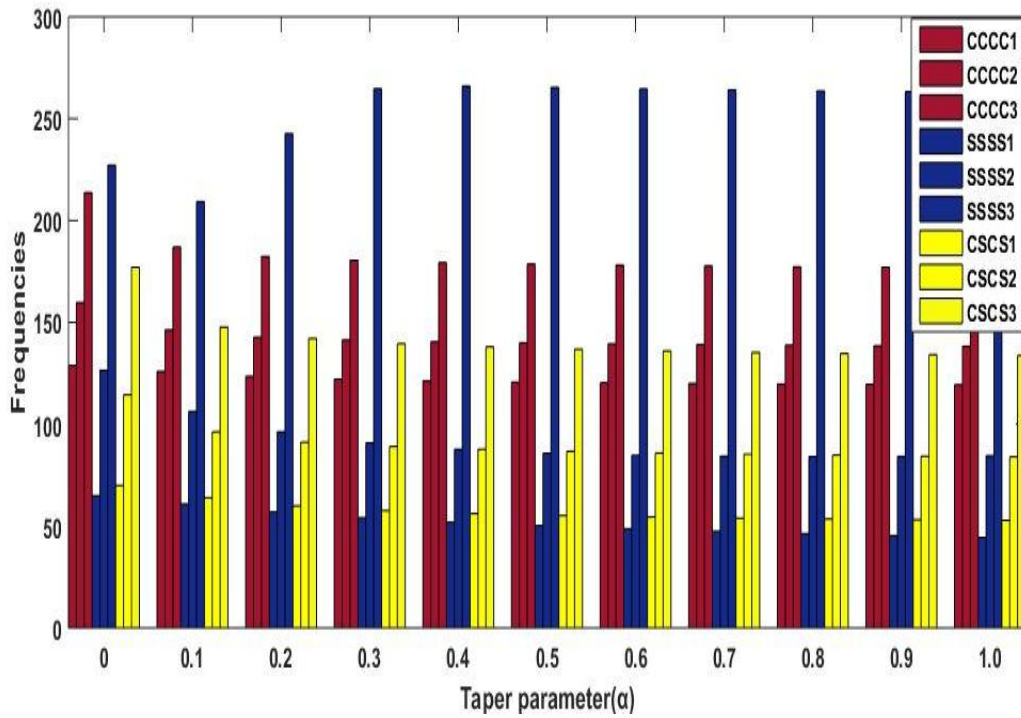


Figure 6.3 Representation of First Three Frequencies ($\mu=2.0$) for Exponential Thickness Variation

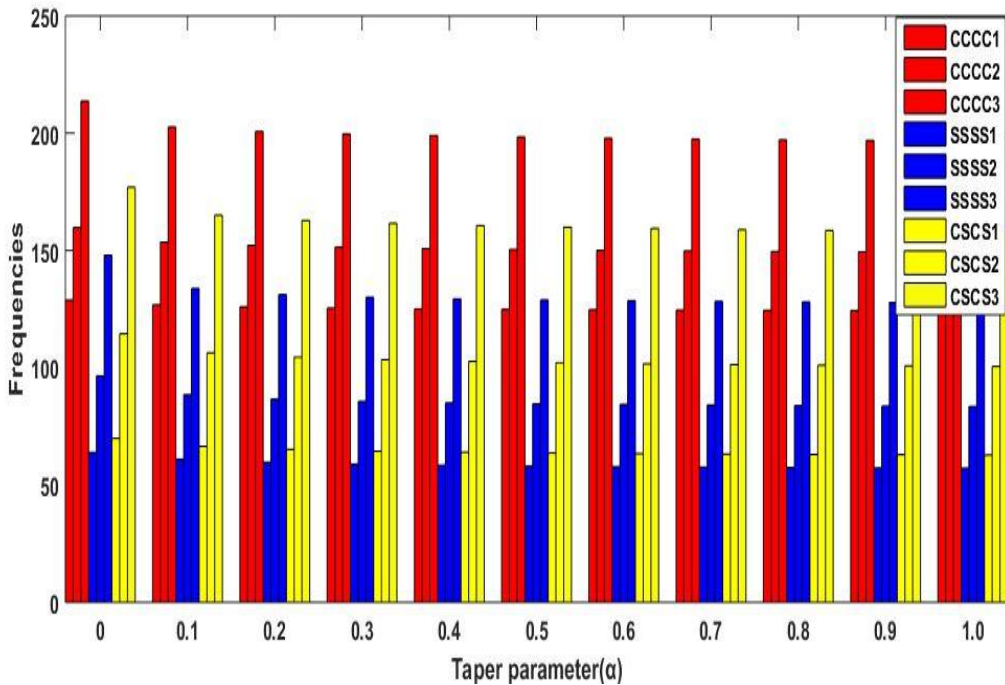


Figure 6.4 Representation of First Three Frequencies ($\mu=2.0$) for Linear Thickness Variation

6.4 MAJOR FINDINGS

In this work, it is found that the first three natural frequencies of skew plate by using Rayleigh-Ritz technique and skew angle 60^0 have been accurately computed for different combinations of boundary conditions and thickness variation of given system handled by single program. When the taper parameter for skew plate is increasing then the behaviour of first three frequencies are in the decreasing form. Convergence result represents that the presented results are in excellent convergence form. In special case of uniform thickness, the results are in close agreement with the existing results. The presented work can be extended for other combinations of boundary conditions and other categories of thickness variations.



CHAPTER-VII

*Concluding Remarks and
Future Scope of Work*

CHAPTER-VII

CONCLUDING REMARKS AND FUTURE SCOPE OF WORK

From the presented work, it is concluded that the Rayleigh-Ritz method is a very useful technique to solve the complex research problems for various shapes of the plate. It produces the excellent convergence of the results as it has been compared from the available results for the quarter of circular, rectangular, square, elliptic and skew plates in the special cases. The generalized Jacobi method gives accurate results for computations of the frequencies and mode shapes. It is also observed that when the stiffness of the plate is increasing, the frequencies are increasing as it is expected and shown that the observed values are correct. MATLAB tool is also useful for depicting the mode shapes in three dimensional forms which are showing the behaviour of the plate at the time of vibrations. Further, the major observations from the computed results for various types of the plates are summarized below:

- ☛ From the literature and presented work, it is observed that Rayleigh-Ritz method is very useful and gives accurate results for the complex types of the research problems. When the researchers are comparing the results with the other methods, then convergence of the results computed by this method is much faster in comparison of the other numerical methods;
- ☛ The presented work is for the isotropic plate while the same work can be extended for the anisotropic plate;
- ☛ The quarter circular, elliptical, rectangular, square and skew plates are considered in the presented work while it can be extended for the other shapes of the plates;

- ☛ In the present work, generalized Jacobi method is used for the faster convergence of the results and it is found that the computed results in special cases are in good agreement with the existing results available in the literature;
- ☛ In some cases accurate mode shapes have been presented for the rectangular and the square plates which represents the behaviour of the plate at the time of vibrations;
- ☛ For representing the behaviour of the plate, MATLAB tool is used for by inserting the eigenvectors which are computed from first three frequencies.
- ☛ For different types of the plates, first three frequencies have been converged upto the five significant digits which have been computed by taking all the parameters in the double precision although the precision can be further increased as desired by the researchers;
- ☛ For the computation purpose, FORTRAN programming has been done for finding the first three frequencies and mode shapes for the different types of the plate;

Further, the presented work may be extended in the following manner:

- ☛ It can be extended by considering the selected plates with central circular and rectangular cut outs;
- ☛ More complex plates i.e. semi-circular plate, octagonal, hybrid triangular plate, and hexagonal plate etc. with different thickness variation may be considered;
- ☛ More general boundary conditions like in plane forces, elastically restrained edges, point supports etc. may be used;

- Other methods like the Finite Element method and Boundary Integral Equation method which are again an efficient computer based techniques for getting accurate results, may be used;
- More general type of thickness variations may be used but keeping in mind the integration involved must be in closed form.



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REFERENCES

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A graphic of a scroll with a black outline and a light gray fill. The scroll is unrolled at the top and bottom, with the unrolled portions shown as rounded rectangles. The word "APPENDIX" is written in a bold, black, serif font in the center of the scroll.

APPENDIX

Numerical Experiments on Quarter of an Elliptic Plate with Exponential Thickness Variation

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Abstract— A study of transverse vibrations of plates plays an important role in the design of naval architecture, engineering design, aircraft design, etc. Due to wide variety of its application, first few frequencies play crucial role for getting best structural design. The present work is related to consider a plate in the form of quarter of an elliptic and a variable thickness in the form of exponent is considered. A well-known Rayleigh Ritz method is used for mathematical solution of the problem in the form of eigenvalue problem. The solution of eigenvalue form is further computed through generalized Jacobin method which gives first few frequencies. The aim of this paper is to compute first three frequencies and computed results are compared with the existing results for the uniform thickness. The new computed results are represented through tables and graphs. Convergence up-to five significant digits are also presented.

Keywords — Rayleigh-Ritz Method, quarter elliptic Plate, eigen value, Jacobi Method.

I. INTRODUCTION

Variational methods are used for solutions of thin plates and one of the important methods is the Rayleigh-Ritz technique for computing the first few frequencies for the thin plates. During the survey, it is found that a study of an exponential thickness variation is done by some researchers on different shapes of thin plates. Let us describe source of the important reference related to present work. The frequencies of vibration of flat circular plates fixed at the circumference have discussed by Carrington [1]. Authors [2-4] have also analysed frequencies of vibration problems for different types of plate like, elliptical plate, circular plate, etc. by Rayleigh-Ritz technique. Shibaolay [5] has briefly discussed transverse vibration of an elliptic plate with clamped boundary condition. Wah [6] has demonstrated vibrations of circular plate with different boundary conditions of the plate as clamped or simply supported. Free vibration of a clamped elliptic plate was discussed by Micnitt [7]. Wilkison [8] has analysed frequencies of exponentially varying thickness. Many of researchers discussed

frequencies, mode shapes and nodal radii for various shapes of the plates. Vibration of a circular plate with variable thickness is also explained by Leissa [9]. Cheung and Cheung [10] have discussed the flexural vibration of rectangular and polygonal plates. Soni [11] has defined axisymmetric vibration of a circular plate which is taken as clamped or simply supported plate in the field. Leissa [12] is the best source of vibration of plate with frequencies, mode shapes by Rayleigh-Ritz method. Mukhopadhyay [13] has investigated the case of semi analytic solution for free vibration of annular sector plate. Srinivasan and Threuvekatchari [14] used integral equation technique solve for free vibration of annular sector plate. Kim and Dickinson [15] have obtained free transverse vibration of annular and circular, thin, sectorial plates subject to certain complicating effect. Singh and Chakraverty [16] have also demonstrated transverse vibration of circular and elliptic plates with quadratically varying thickness. Singh and Saxena [17] have defined quarter of circular plate with variable thickness by Rayleigh-Ritz method. Hassan and Makery [18] have found the first four frequencies of elliptic plate using the boundary conditions like clamped and simple-supported. Leissa [19] has analysed Mathematics of eigenvalue through Mathematical programming for Pseudospectral method and found the eigenvalues of axisymmetric Mindlin plate and Timoshenko beams by Lee and Schutz [20]. The entire coverage on the research done on thin plates is available in monograph of Lesissa [21] which is excellent source of information on vibration of a specific plate i.e. circular plates with variable thickness. Gupta et al. [22] have introduced vibration of analysis for non-homogeneous circular plate of nonlinear thickness variation by differential quadrature method. Rayleigh-Ritz method is defined the free vibration analysis of super elliptical plates with constant and variables thickness by Ceribasi and Altany [23]. Lakshmi et al. [24] have discussed vibration analysis for the elliptical with clamped plate.

On the basis of above, it is observed that quarter of an elliptical plate with exponential thickness variation is still not studied by the researchers,

therefore, the present work is in this direction and Rayleigh-Ritz method is used for computation of the first three frequencies for vibration of quarter of elliptic plate. The thickness of thin plate is considered as exponential. Combinations of boundary conditions are considered as clamped, simply-supported and completely free which leads to total combinations of twenty seven cases, some of the important cases are reported here along with frequencies and mode shapes are also depicted through programming language.

II. MATERIAL AND METHODS

Classical Plate Theory

According to Kirchhoff plate theory, (classical plate theory) is based on some important points, which are given below.

- 1) Thickness of plate is considered as small in comparison of other dimensions.
- 2) Rotatory of inertia is considered as negligible.
- 3) Normal to the undeformed and deformed middle surface remains straight and unstretched due to effect of light,
- 4) Plate is taken as negligibly small of normal stresses in the transform direction.

Boundary Conditions

The plate is shown below in figure 1, where R is the domain of the plate, N is normal to the boundary of plate as shown in the figure and the boundary conditions are defined below in brief.

1) **Clamped Boundary Condition** when the boundary of plate considers as clamped then there is no deflection in the plate represented as W and N is normal to the plate then on the curve surface of the plate, the following boundary condition of the plate is applicable.

$$W = 0 \tag{1}$$

and $\frac{\partial W}{\partial N} = 0$ on the curved surface (C) (2)

2) **Supported Boundary Condition** when the curved boundary of plate is simply-supported then there will be a little deflection represent as W. Let M represents bending moment with respect to the normal of the plate then the following boundary condition of applicable.

$$W = 0 \tag{3}$$

and $\frac{\partial M}{\partial N} = 0$ on the curved surface of plate (C) (4)

3) **Free Boundary Condition** when the boundary of plate is considered as completely-free then there will be deflection (W), bending moment (M) and shear force (Q) with respect to normal of the plate then the following boundary condition is applicable.

$$M_N = 0 \tag{5}$$

$Q_N + \frac{\partial M_{NT}}{\partial T} = 0$ on the curved surface of plate (C) (6)

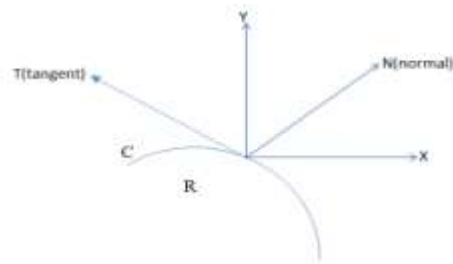


Fig 1 Portion of path C with normal and tangential

On the basis of the above different kind of thin plates are consider by the various authors namely circular, elliptic, square, rectangular, skew, rhombus, etc. From the literature, it is observed that some authors have also considered plate with small whole inside the plate for observing the behaviour of plate. During vibration of thin plate stiffness of the plate is considered by varying the thickness as linear, quadratic, exponential, etc.

Rayleigh-Ritz Method

In the Rayleigh-Ritz method, we equate the maximum kinetic with the maximum strain energy to obtain the Rayleigh quotient which is given by the following equation.

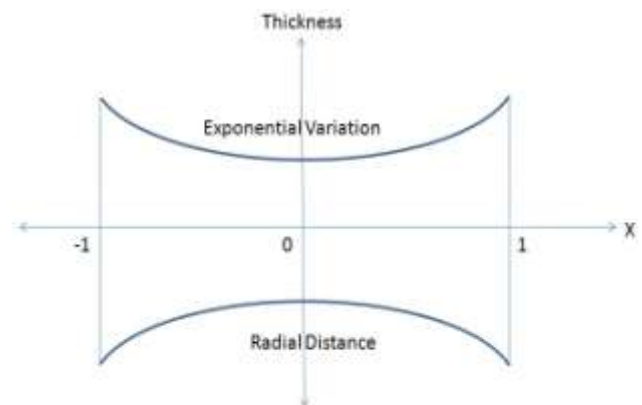


Fig 2 Exponential thickness variation for quarter of an elliptic plate.

$$\omega^2 = \frac{E}{12\rho(1-\nu^2)} \left[\frac{\iint_R h^3 \left[(\nabla^2 W)^2 + 2(1-\nu) \{ (W^{x'y'})^2 - W^{x'x'} W^{y'y'} \} \right] dx' dy'}{\iint_R h W^2 dx' dy'} \right] \tag{7}$$

where E, ρ, ν and h are Young's modulus, density, Poisson ratio and variable thickness respectively. R is domain of the plate. Let us consider non-dimensional variables x and y which are given below by following equation.

$$x' = x/a, y' = y/b, m = b/a \tag{8}$$

where m is the aspect ratio. On the basis of these non-dimensional parameters the deflection of plate is given by equation (9).

$$W(x, y) = \sum_{j=1}^N C_j \phi_j(x, y), \tag{9}$$

Where ϕ_j are taken to satisfy the boundary conditions of given problem and C_j ($j \in \Delta_N = \{1, 2, \dots, N\}$) are and putting the value of $W(x, y)$ in equation (7) an eigenvalue problem is given below;

$$\sum_{j=1}^N (a_{ij} - \lambda^2 b_{ij}) C_j = 0, \quad i \in \Delta_N, \tag{10}$$

where

$$a_{ij} = \iint_R H^3 \left[\phi_i^{xx} \phi_j^{xx} + \phi_i^{yy} \phi_j^{yy} + \nu \left(\phi_i^{xx} \phi_j^{yy} + \phi_i^{yy} \phi_j^{xx} \right) + 2(1-\nu) \phi_i^{xy} \phi_j^{xy} \right] dx dy, \tag{11}$$

$$b_{ij} = \iint_R H \phi_i \phi_j dx dy, \tag{12}$$

$$\lambda^2 = \left[12 \rho a^4 (1-\nu^2) \omega^2 \right] / (E h_0^2), \tag{13}$$

where h_0 is thickness of given elliptic plate. Let us consider a basis function ϕ_i which satisfies the boundary conditions

$$\phi_i(x, y) = f(x, y) x^{m_i+p} y^{n_i+q}, \tag{14}$$

$$f(x, y) = \left[1 - \left(x^2 + \frac{y^2}{m^2} \right) \right]^r \tag{15}$$

Since ϕ_i satisfies the boundary conditions, therefore $f(x, y)$ also satisfies the given boundary conditions. The parameter r is controlling type of boundary conditions i.e. $r=0, 1, 2$ shows that the plate is completely-free, simply-supported and clamped, respectively. The variables m_i+p and n_i+q are non-negative integers given by following table.

The thickness of the plate is given by following equation and represented in.

$$H = e^{\alpha r} \tag{16}$$

By putting the values of f and H in equations (10), (11) and (12) and solving the expressions of a_{ij} and b_{ij} which contains all the integrals in closed form given by following formula.

TABLE I
Non negative value of m_i and n_i

| m_i | n_i |
|-------|-------|
| 0 | 0 |
| 0 | 1 |
| 1 | 0 |
| 2 | 0 |
| 1 | 1 |
| 0 | 2 |
| 3 | 0 |
| 2 | 1 |
| 1 | 2 |
| 0 | 3 |
| 4 | 0 |
| 3 | 1 |
| 2 | 2 |
| 1 | 3 |
| 0 | 4 |
| 5 | 0 |
| 4 | 1 |
| 3 | 2 |
| 2 | 3 |
| 1 | 4 |
| 0 | 5 |
| 6 | 0 |
| 5 | 1 |
| 4 | 2 |
| 3 | 3 |
| 2 | 4 |
| 1 | 5 |
| 0 | 6 |

$$\iint_R x^k y^l r^m (1-r^2)^n dx dy = \frac{G\left(\frac{k+l+m}{2}+1\right) G\left(\frac{k+1}{2}\right) G\left(\frac{l+1}{2}\right) G(n+1)}{4G\left(\frac{k+l+m}{2}+n+2\right) G\left(\frac{k+l}{2}+1\right)}, \tag{17}$$

where G stands for the gamma function.

III. COMPUTATION OF NUMERICAL RESULTS

On the basis of above formulation of the problem, a program has been defined for extensive numerical computations and for all the calculation, the following values of the constants and variables have been considered.

- 1) The value of Poisson's ratio ν is taken as 0.3 for isotropic plate.
- 2) F, S, C can take value 0, 1, 2 for a completely free, simply-supported and clamped plate, respectively.
- 3) The thickness is controlled by the parameter α which is taken as variable from -1 to +1.
- 4) The values of m are considered as 0.25, 0.5, and 0.75 for quarter of elliptic plate.

By the use of above parameters, we solved all the integral and computed a_{ij} and b_{ij} as given in the equation (10) & (11) respectively. By the use of the generalized Jacobi method, eigenvalue problem can

be solved and first few frequencies have been computed in some selected cases as described below in brief. Tables 2, 3 and 4 demonstrate the first three frequencies for various boundary conditions for various value of α running from -1 to +1. From the

tables, it is observed that when taper parameter α is increasing from -1 to +1 then all the frequencies are increasing in the all cases.

TABLE II
First three frequencies for quarter of on elliptic plates (m=0.25, ν =0.3, N=28)

| α | -1.0 | -0.8 | -0.6 | -0.4 | -0.2 | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|----------|----------|
| CCC | 228.690 | 257.590 | 290.052 | 326.586 | 367.758 | 414.227 | 466.766 | 526.308 | 593.991 | 671.211 | 759.679 |
| | 260.340 | 295.900 | 336.340 | 382.300 | 434.524 | 493.921 | 561.646 | 639.179 | 728.375 | 831.508 | 951.331 |
| | 301.352 | 344.989 | 394.730 | 451.425 | 516.361 | 591.307 | 678.463 | 780.416 | 900.154 | 1041.152 | 1207.506 |
| CCS | 174.981 | 195.158 | 217.410 | 241.968 | 269.075 | 299.004 | 332.075 | 368.675 | 409.287 | 454.517 | 505.131 |
| | 203.911 | 230.406 | 260.337 | 294.113 | 332.242 | 375.373 | 424.330 | 480.136 | 544.035 | 617.530 | 702.423 |
| | 242.908 | 276.970 | 315.676 | 359.998 | 411.185 | 470.702 | 540.224 | 621.673 | 717.288 | 829.714 | 962.103 |
| FCC | 219.564 | 243.459 | 269.882 | 299.040 | 332.404 | 369.642 | 411.741 | 459.478 | 513.746 | 575.577 | 646.156 |
| | 239.307 | 269.121 | 302.957 | 341.322 | 384.813 | 434.118 | 490.040 | 553.524 | 625.705 | 707.960 | 801.971 |
| | 270.995 | 308.197 | 350.821 | 399.366 | 454.463 | 517.005 | 588.227 | 669.746 | 763.574 | 872.155 | 998.426 |
| SSS | 108.613 | 123.143 | 139.340 | 157.333 | 177.262 | 199.287 | 223.603 | 250.542 | 280.144 | 313.081 | 349.788 |
| | 139.851 | 159.377 | 181.474 | 206.461 | 234.724 | 266.739 | 303.078 | 344.410 | 391.518 | 445.322 | 506.925 |
| | 178.969 | 204.895 | 234.786 | 269.416 | 309.655 | 356.412 | 410.624 | 473.275 | 545.469 | 628.552 | 724.291 |
| SSC | 149.604 | 171.345 | 196.055 | 224.117 | 255.976 | 292.152 | 333.264 | 380.040 | 433.347 | 494.202 | 563.787 |
| | 184.417 | 211.498 | 242.431 | 277.743 | 318.041 | 364.042 | 416.615 | 476.811 | 545.868 | 625.190 | 716.332 |
| | 226.420 | 260.229 | 299.076 | 343.609 | 394.711 | 453.603 | 521.783 | 600.874 | 692.512 | 798.358 | 920.280 |
| FSC | 132.576 | 151.797 | 173.689 | 198.376 | 226.251 | 257.789 | 293.559 | 334.238 | 380.616 | 433.604 | 494.238 |
| | 163.990 | 187.666 | 214.603 | 245.283 | 280.272 | 320.220 | 365.871 | 418.066 | 477.767 | 546.090 | 624.343 |
| | 201.189 | 230.469 | 264.216 | 303.078 | 347.078 | 398.757 | 457.038 | 523.589 | 599.825 | 687.577 | 789.068 |
| FSS | 93.448 | 105.874 | 119.439 | 134.176 | 150.126 | 167.331 | 185.848 | 205.742 | 227.100 | 250.039 | 274.727 |
| | 121.466 | 137.833 | 156.196 | 176.804 | 199.937 | 225.92 | 255.133 | 288.038 | 325.203 | 367.342 | 415.339 |
| | 155.532 | 177.447 | 202.540 | 231.18 | 263.799 | 300.956 | 343.384 | 392.013 | 447.970 | 512.586 | 587.444 |
| CSS | 111.226 | 126.351 | 143.260 | 162.106 | 183.052 | 206.290 | 232.043 | 260.589 | 292.278 | 327.563 | 367.028 |
| | 144.572 | 165.047 | 188.282 | 214.626 | 244.472 | 278.275 | 316.588 | 360.107 | 409.716 | 466.532 | 531.946 |
| | 185.716 | 213.082 | 244.271 | 279.675 | 319.909 | 365.913 | 418.974 | 480.695 | 552.977 | 638.034 | 738.469 |
| FFC | 21.833 | 27.208 | 33.835 | 41.982 | 51.964 | 64.147 | 78.951 | 96.853 | 118.385 | 144.127 | 172.699 |
| | 43.488 | 51.173 | 60.354 | 71.329 | 84.449 | 100.125 | 118.834 | 141.123 | 167.604 | 198.955 | 235.904 |
| | 69.709 | 80.964 | 94.138 | 109.598 | 127.782 | 149.198 | 174.421 | 204.083 | 238.865 | 279.517 | 326.894 |
| FFS | 3.473 | 4.277 | 5.269 | 6.493 | 8.004 | 9.869 | 12.176 | 15.029 | 18.563 | 22.946 | 28.388 |
| | 23.483 | 26.795 | 30.651 | 35.169 | 40.493 | 46.808 | 54.339 | 63.370 | 74.249 | 87.409 | 103.384 |
| | 81.175 | 93.166 | 107.035 | 123.01 | 141.373 | 162.494 | 186.873 | 215.16 | 248.181 | 286.961 | 332.735 |
| FCS | 167.145 | 182.685 | 199.244 | 217.04 | 236.243 | 256.993 | 279.417 | 303.642 | 329.798 | 358.042 | 388.571 |
| | 184.92 | 206.076 | 229.857 | 256.472 | 286.200 | 319.394 | 356.495 | 398.058 | 444.776 | 497.525 | 557.402 |
| | 215.041 | 243.496 | 275.77 | 312.281 | 353.608 | 400.519 | 454.001 | 515.293 | 585.927 | 667.777 | 763.110 |
| SCC | 226.712 | 254.647 | 285.931 | 321.052 | 360.556 | 405.073 | 455.328 | 512.170 | 576.594 | 649.773 | 733.090 |
| | 255.726 | 289.938 | 328.808 | 372.970 | 423.172 | 480.296 | 545.375 | 619.609 | 704.410 | 801.464 | 912.829 |
| | 294.311 | 336.506 | 384.842 | 440.312 | 504.124 | 577.66 | 662.494 | 760.461 | 873.786 | 1005.222 | 1158.18 |
| CFS | 18.825 | 20.814 | 23.116 | 25.813 | 29.006 | 32.823 | 37.427 | 43.018 | 49.847 | 58.223 | 68.528 |
| | 44.370 | 50.046 | 56.534 | 63.980 | 72.573 | 82.545 | 94.188 | 107.863 | 124.017 | 143.201 | 166.091 |
| | 78.236 | 88.546 | 100.298 | 113.770 | 129.276 | 147.167 | 167.852 | 191.829 | 219.738 | 252.317 | 290.133 |
| SFC | 34.497 | 40.693 | 48.225 | 57.377 | 68.479 | 81.915 | 98.127 | 117.615 | 140.942 | 168.729 | 201.696 |
| | 60.424 | 69.925 | 81.072 | 94.177 | 109.61 | 127.802 | 149.253 | 174.54 | 204.325 | 239.367 | 280.526 |
| | 94.661 | 108.553 | 124.613 | 143.232 | 164.855 | 189.085 | 219.217 | 253.302 | 293.189 | 340.023 | 395.074 |
| SFS | 15.543 | 17.264 | 19.289 | 21.701 | 24.606 | 28.130 | 32.436 | 37.721 | 44.230 | 52.264 | 62.196 |
| | 38.025 | 43.114 | 48.955 | 55.696 | 63.518 | 72.651 | 83.374 | 96.052 | 111.113 | 129.102 | 150.684 |
| | 69.239 | 78.603 | 89.323 | 101.620 | 115.768 | 132.115 | 151.116 | 173.348 | 199.501 | 230.382 | 266.909 |
| SCS | 173.327 | 192.670 | 213.887 | 237.195 | 262.817 | 290.996 | 322.000 | 356.142 | 393.803 | 435.456 | 481.713 |
| | 199.945 | 225.178 | 253.647 | 285.801 | 322.144 | 363.245 | 409.755 | 462.445 | 522.25 | 590.329 | 668.126 |
| | 236.595 | 270.076 | 308.864 | 353.709 | 405.439 | 465.013 | 533.601 | 612.671 | 704.077 | 810.136 | 933.707 |
| CFC | 38.914 | 45.437 | 53.311 | 62.821 | 74.304 | 88.148 | 104.805 | 124.787 | 148.675 | 177.112 | 210.808 |
| | 67.485 | 77.637 | 89.497 | 103.381 | 119.661 | 138.777 | 161.244 | 187.667 | 218.745 | 255.291 | 298.247 |
| | 104.169 | 119.128 | 136.34 | 156.184 | 179.136 | 205.755 | 236.665 | 272.589 | 314.419 | 363.319 | 420.825 |
| CSC | 152.697 | 175.124 | 200.652 | 229.685 | 262.689 | 300.213 | 342.897 | 391.504 | 446.937 | 510.279 | 582.826 |
| | 189.772 | 217.938 | 250.122 | 286.882 | 328.851 | 376.748 | 431.399 | 493.793 | 565.171 | 647.127 | 741.694 |
| | 233.997 | 269.15 | 309.539 | 355.783 | 408.502 | 468.535 | 537.206 | 616.463 | 708.849 | 817.419 | 945.712 |

TABLE III
First three frequencies for quarter of on elliptic plates (m=0.5, ν =0.3, N=28)

| α | -1.0 | -0.8 | -0.6 | -0.4 | -0.2 | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| CCC | 65.517 | 74.385 | 84.470 | 95.938 | 108.975 | 123.796 | 140.642 | 159.789 | 181.554 | 206.303 | 234.457 |
| | 89.340 | 102.489 | 117.551 | 134.787 | 154.487 | 176.974 | 202.612 | 231.815 | 265.662 | 302.925 | 346.105 |
| | 114.991 | 136.888 | 161.241 | 185.846 | 214.037 | 246.289 | 283.142 | 325.247 | 373.431 | 428.767 | 492.633 |
| CCS | 50.831 | 57.30 | 64.635 | 72.879 | 82.163 | 92.618 | 104.390 | 117.652 | 132.608 | 149.796 | 168.646 |
| | 72.567 | 83.115 | 95.196 | 109.024 | 124.837 | 142.910 | 163.558 | 187.154 | 214.145 | 229.080 | 280.629 |
| | 102.127 | 117.961 | 136.252 | 157.354 | 181.667 | 209.650 | 241.855 | 278.971 | 321.874 | 345.831 | 429.733 |
| FCC | 55.906 | 62.133 | 69.121 | 76.983 | 85.852 | 95.882 | 107.258 | 120.196 | 134.950 | 151.814 | 171.127 |
| | 70.596 | 79.965 | 90.641 | 102.813 | 116.700 | 132.557 | 150.674 | 171.382 | 195.055 | 222.117 | 253.047 |
| | 94.041 | 107.562 | 123.073 | 140.881 | 161.332 | 184.816 | 211.767 | 242.671 | 278.082 | 318.647 | 365.140 |
| SSS | 33.674 | 38.136 | 43.201 | 48.939 | 55.429 | 62.764 | 71.052 | 80.422 | 91.032 | 103.074 | 116.786 |
| | 54.723 | 62.556 | 71.538 | 81.831 | 93.617 | 107.107 | 122.541 | 140.206 | 160.437 | 183.631 | 210.253 |
| | 83.143 | 95.655 | 110.071 | 126.662 | 145.741 | 167.694 | 192.995 | 222.218 | 256.018 | 295.077 | 339.958 |
| SSC | 45.105 | 51.532 | 58.900 | 67.340 | 76.999 | 88.047 | 100.682 | 115.129 | 131.649 | 150.537 | 172.132 |
| | 68.541 | 78.575 | 90.091 | 103.293 | 118.415 | 135.715 | 155.488 | 178.065 | 203.821 | 233.189 | 266.670 |
| | 99.432 | 114.380 | 131.529 | 151.195 | 173.733 | 199.530 | 229.002 | 262.627 | 300.992 | 344.839 | 395.055 |
| FSC | 34.525 | 39.645 | 45.441 | 52.004 | 59.449 | 67.909 | 77.546 | 88.549 | 101.142 | 115.582 | 132.161 |
| | 53.620 | 61.148 | 69.772 | 79.664 | 91.018 | 104.058 | 119.041 | 136.255 | 156.028 | 178.728 | 204.762 |
| | 78.817 | 90.155 | 103.173 | 118.128 | 135.321 | 155.096 | 177.837 | 203.970 | 233.957 | 268.314 | 307.617 |
| CFC | 19.482 | 22.071 | 25.102 | 28.666 | 32.868 | 37.832 | 43.702 | 50.642 | 58.840 | 68.505 | 79.87 |
| | 42.614 | 48.296 | 54.768 | 62.144 | 70.560 | 80.170 | 91.151 | 103.706 | 118.069 | 134.506 | 153.325 |
| | 68.355 | 77.958 | 88.974 | 101.592 | 116.020 | 132.481 | 151.196 | 172.389 | 196.304 | 223.240 | 253.611 |
| SCS | 48.57 | 54.467 | 61.088 | 68.521 | 76.861 | 86.222 | 96.731 | 108.539 | 121.826 | 136.811 | 153.762 |
| | 68.019 | 77.648 | 88.661 | 101.248 | 115.626 | 132.047 | 150.801 | 172.229 | 196.727 | 224.765 | 256.900 |
| | 95.589 | 110.151 | 126.938 | 146.297 | 168.64 | 194.454 | 224.312 | 258.881 | 298.938 | 345.384 | 399.241 |
| SFS | 9.091 | 10.213 | 11.488 | 12.948 | 14.635 | 16.600 | 18.908 | 21.640 | 24.840 | 28.797 | 33.502 |
| | 27.092 | 30.661 | 34.713 | 39.320 | 44.569 | 50.561 | 57.417 | 65.279 | 74.316 | 84.726 | 96.744 |
| | 50.659 | 57.440 | 65.186 | 74.019 | 84.073 | 95.489 | 108.425 | 123.059 | 139.602 | 158.315 | 179.512 |
| FCS | 42.432 | 46.492 | 50.924 | 55.765 | 61.056 | 66.884 | 73.187 | 80.150 | 87.815 | 96.281 | 105.673 |
| | 55.659 | 62.684 | 70.647 | 79.685 | 89.958 | 101.654 | 114.99 | 130.223 | 147.653 | 167.637 | 190.602 |
| | 77.176 | 88.148 | 100.781 | 115.332 | 132.096 | 151.409 | 173.662 | 199.318 | 228.919 | 263.158 | 302.814 |
| FSS | 24.358 | 27.649 | 31.269 | 35.240 | 39.587 | 44.343 | 49.548 | 55.254 | 61.528 | 68.456 | 76.150 |
| | 41.337 | 46.842 | 53.113 | 60.27 | 68.452 | 77.826 | 88.583 | 100.947 | 115.182 | 131.597 | 150.56 |
| | 64.288 | 73.351 | 83.788 | 95.831 | 109.742 | 125.815 | 144.384 | 165.833 | 190.609 | 219.234 | 252.307 |
| FFC | 6.565 | 8.153 | 10.105 | 12.498 | 15.419 | 18.972 | 23.277 | 28.468 | 34.696 | 42.124 | 50.930 |
| | 19.434 | 22.731 | 26.595 | 31.124 | 36.435 | 42.662 | 49.959 | 58.502 | 68.493 | 80.155 | 93.734 |
| | 41.771 | 47.86 | 54.869 | 62.937 | 72.229 | 82.934 | 95.27 | 109.480 | 125.832 | 144.611 | 166.117 |
| FFS | 1.841 | 2.226 | 2.693 | 3.261 | 3.952 | 4.795 | 5.828 | 7.094 | 8.651 | 10.569 | 12.935 |
| | 12.943 | 14.979 | 17.336 | 20.072 | 23.253 | 26.963 | 31.298 | 36.381 | 42.354 | 49.392 | 57.701 |
| | 32.589 | 37.161 | 42.403 | 48.429 | 55.371 | 63.381 | 72.633 | 83.318 | 95.642 | 109.83 | 126.106 |
| SCC | 62.87 | 71.07 | 80.375 | 90.934 | 102.921 | 116.53 | 131.985 | 149.542 | 169.495 | 192.181 | 217.983 |
| | 84.284 | 96.44 | 110.362 | 126.276 | 144.453 | 165.197 | 188.848 | 215.795 | 246.483 | 281.421 | 321.207 |
| | 113.969 | 131.269 | 151.145 | 173.957 | 200.116 | 230.084 | 264.398 | 303.687 | 348.694 | 400.302 | 459.577 |
| SFC | 14.856 | 17.181 | 19.926 | 23.176 | 27.063 | 31.622 | 37.074 | 43.549 | 51.223 | 60.292 | 70.970 |
| | 35.606 | 40.586 | 46.288 | 52.824 | 60.324 | 68.936 | 78.833 | 90.209 | 103.285 | 118.311 | 135.566 |
| | 64.688 | 73.765 | 84.106 | 95.810 | 109.001 | 123.881 | 140.686 | 159.666 | 181.092 | 205.297 | 232.737 |
| CSS | 36.272 | 41.212 | 46.829 | 53.206 | 60.436 | 68.627 | 77.904 | 88.413 | 100.330 | 113.866 | 129.280 |
| | 59.532 | 68.214 | 78.172 | 89.584 | 102.655 | 117.616 | 134.735 | 154.316 | 176.715 | 202.35 | 231.722 |
| | 89.935 | 103.658 | 119.48 | 137.729 | 158.749 | 182.892 | 210.519 | 242.021 | 277.878 | 318.713 | 365.309 |
| CFS | 13.023 | 14.355 | 15.859 | 18.00 | 19.528 | 21.787 | 24.412 | 27.484 | 31.106 | 35.406 | 40.542 |
| | 33.379 | 37.594 | 42.357 | 47.745 | 53.849 | 60.777 | 68.657 | 77.639 | 87.902 | 99.653 | 113.133 |
| | 53.950 | 61.161 | 69.407 | 78.824 | 89.558 | 101.774 | 115.657 | 131.42 | 149.311 | 169.635 | 192.782 |
| CSC | 48.132 | 55.101 | 63.093 | 72.251 | 82.736 | 94.734 | 108.458 | 124.148 | 142.083 | 162.579 | 185.994 |
| | 73.884 | 84.830 | 97.391 | 111.789 | 128.271 | 147.117 | 168.638 | 193.184 | 221.145 | 252.961 | 289.142 |
| | 100.872 | 116.272 | 135.136 | 158.239 | 186.412 | 215.471 | 247.373 | 283.623 | 324.68 | 371.114 | 423.779 |

TABLE IV
First three frequencies for quarter of on elliptic plates (m=0.75, ν =0.3, N=28)

| α | -1.0 | -0.8 | -0.6 | -0.4 | -0.2 | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------|--------|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| CCC | 35.417 | 40.336 | 45.960 | 52.385 | 59.723 | 68.098 | 77.653 | 88.548 | 100.965 | 115.108 | 131.209 |
| | 56.506 | 65.088 | 74.951 | 86.271 | 99.244 | 114.085 | 131.029 | 150.328 | 172.259 | 197.122 | 225.272 |
| | 66.865 | 79.818 | 95.653 | 114.965 | 135.788 | 153.362 | 172.960 | 194.980 | 219.794 | 247.866 | 279.773 |
| CCS | 27.760 | 31.449 | 35.645 | 40.415 | 45.836 | 51.996 | 58.934 | 66.947 | 75.991 | 86.287 | 98.028 |
| | 46.658 | 53.747 | 61.912 | 71.307 | 82.106 | 94.501 | 108.705 | 124.954 | 143.510 | 164.673 | 188.798 |
| | 70.052 | 79.737 | 90.199 | 101.773 | 114.677 | 129.106 | 145.283 | 163.489 | 184.085 | 207.530 | 234.394 |
| FCC | 25.564 | 28.474 | 31.765 | 35.495 | 39.732 | 44.556 | 50.060 | 56.355 | 63.571 | 71.856 | 81.380 |
| | 38.368 | 43.605 | 49.608 | 56.492 | 64.392 | 73.556 | 83.858 | 95.788 | 109.465 | 125.129 | 143.050 |
| | 59.071 | 67.891 | 78.034 | 89.692 | 103.076 | 117.975 | 131.851 | 147.130 | 164.188 | 183.252 | 204.583 |
| SSS | 18.996 | 21.472 | 24.308 | 27.550 | 31.255 | 35.483 | 40.310 | 45.822 | 52.124 | 59.342 | 67.628 |
| | 36.649 | 42.028 | 48.217 | 55.333 | 63.508 | 72.890 | 83.644 | 95.956 | 110.034 | 126.108 | 144.431 |
| | 56.749 | 63.923 | 71.972 | 81.001 | 91.126 | 102.476 | 115.208 | 129.518 | 145.645 | 163.887 | 184.606 |
| SSC | 25.036 | 28.529 | 32.554 | 37.184 | 42.507 | 48.620 | 55.638 | 63.688 | 72.916 | 83.490 | 95.595 |
| | 45.016 | 51.680 | 59.336 | 68.123 | 78.194 | 89.719 | 102.888 | 117.907 | 135.002 | 154.420 | 176.436 |
| | 67.798 | 76.586 | 86.473 | 97.599 | 110.115 | 124.189 | 140.014 | 157.811 | 177.831 | 200.408 | 225.903 |
| FSC | 16.225 | 18.644 | 21.396 | 24.530 | 28.108 | 32.200 | 36.893 | 42.287 | 48.496 | 55.653 | 63.906 |
| | 31.578 | 35.924 | 40.929 | 46.699 | 53.352 | 61.024 | 69.865 | 80.047 | 91.760 | 105.215 | 120.646 |
| | 52.563 | 60.162 | 68.374 | 77.046 | 86.600 | 97.256 | 109.168 | 122.498 | 137.418 | 154.126 | 172.849 |
| CSC | 28.407 | 32.406 | 37.000 | 42.275 | 48.325 | 55.261 | 63.208 | 72.307 | 82.719 | 94.624 | 108.227 |
| | 50.193 | 57.729 | 66.387 | 76.322 | 87.703 | 100.719 | 115.574 | 132.487 | 151.690 | 173.426 | 197.956 |
| | 67.419 | 76.986 | 88.776 | 103.084 | 116.986 | 132.112 | 149.118 | 168.275 | 189.895 | 214.357 | 242.131 |
| CSS | 21.873 | 24.790 | 28.123 | 31.928 | 36.270 | 41.223 | 46.872 | 53.317 | 60.677 | 69.091 | 78.728 |
| | 41.206 | 47.379 | 54.484 | 62.658 | 72.051 | 82.829 | 95.180 | 109.304 | 125.417 | 143.745 | 164.532 |
| | 60.348 | 68.119 | 76.829 | 86.596 | 97.551 | 109.849 | 123.680 | 139.277 | 156.938 | 177.044 | 200.080 |
| CFC | 16.063 | 18.002 | 20.230 | 22.799 | 25.769 | 29.211 | 33.212 | 37.867 | 43.289 | 49.605 | 56.958 |
| | 34.459 | 39.333 | 44.925 | 51.328 | 58.643 | 66.975 | 76.429 | 87.110 | 99.125 | 112.590 | 127.646 |
| | 43.695 | 48.979 | 54.954 | 61.720 | 69.400 | 78.137 | 88.108 | 99.527 | 112.648 | 127.765 | 145.206 |
| CFS | 11.659 | 12.830 | 14.139 | 15.608 | 17.262 | 19.132 | 21.257 | 23.686 | 26.478 | 29.708 | 33.471 |
| | 27.368 | 31.133 | 35.455 | 40.409 | 46.082 | 52.561 | 59.936 | 68.287 | 77.674 | 88.129 | 99.678 |
| | 35.898 | 40.000 | 44.602 | 49.770 | 55.587 | 62.152 | 69.595 | 78.086 | 87.856 | 99.213 | 112.529 |
| SCC | 32.293 | 36.637 | 41.600 | 47.270 | 53.746 | 61.139 | 69.577 | 79.204 | 90.186 | 102.707 | 116.978 |
| | 51.405 | 59.082 | 67.899 | 78.012 | 89.596 | 102.847 | 117.981 | 135.234 | 154.868 | 177.175 | 202.479 |
| | 77.135 | 88.830 | 101.468 | 114.509 | 129.027 | 145.290 | 163.527 | 183.990 | 206.967 | 232.794 | 261.867 |
| SCS | 25.097 | 28.282 | 31.899 | 36.004 | 40.662 | 45.949 | 51.949 | 58.764 | 66.512 | 75.336 | 85.405 |
| | 42.139 | 48.401 | 55.602 | 63.877 | 73.379 | 84.279 | 96.771 | 111.076 | 127.440 | 146.141 | 167.493 |
| | 65.588 | 75.617 | 85.603 | 96.324 | 108.263 | 122.263 | 136.567 | 153.342 | 172.200 | 193.463 | 217.538 |
| SFC | 11.090 | 12.763 | 14.695 | 16.935 | 19.541 | 22.581 | 26.134 | 30.291 | 35.157 | 40.847 | 47.492 |
| | 29.581 | 33.692 | 38.404 | 43.798 | 49.965 | 57.004 | 65.025 | 74.149 | 84.507 | 96.241 | 109.511 |
| | 37.604 | 42.496 | 48.066 | 54.412 | 61.646 | 69.896 | 79.308 | 90.051 | 102.316 | 116.324 | 132/329 |
| SFS | 7.430 | 8.404 | 9.493 | 10.718 | 12.102 | 13.635 | 15.319 | 17.153 | 19.136 | 21.269 | 23.552 |
| | 23.241 | 26.350 | 29.908 | 33.977 | 38.404 | 43.798 | 49.965 | 56.811 | 64.575 | 73.343 | 83.217 |
| | 30.137 | 33.864 | 38.080 | 42.854 | 48.066 | 54.412 | 61.383 | 69.331 | 78.413 | 88.830 | 100.835 |
| FCS | 19.271 | 21.156 | 23.237 | 25.535 | 28.075 | 30.886 | 34.005 | 37.477 | 41.356 | 45.713 | 50.633 |
| | 30.589 | 34.618 | 39.227 | 44.505 | 50.561 | 57.516 | 65.514 | 74.717 | 85.316 | 97.528 | 111.607 |
| | 49.252 | 56.601 | 65.081 | 74.866 | 86.056 | 96.066 | 106.568 | 118.195 | 131.086 | 145.395 | 161.316 |
| FSS | 11.371 | 12.911 | 14.615 | 16.500 | 18.585 | 20.894 | 23.460 | 26.321 | 29.526 | 33.138 | 37.235 |
| | 24.818 | 28.085 | 31.840 | 36.165 | 41.153 | 46.913 | 53.569 | 61.268 | 70.175 | 80.481 | 92.405 |
| | 43.453 | 49.276 | 55.331 | 61.193 | 69.193 | 77.252 | 86.183 | 96.086 | 107.071 | 119.279 | 132.885 |
| FFC | 3.690 | 4.568 | 5.544 | 6.961 | 8.565 | 10.513 | 12.870 | 15.707 | 19.108 | 23.161 | 27.963 |
| | 14.221 | 16.617 | 19.409 | 22.661 | 26.447 | 30.851 | 35.970 | 41.915 | 48.807 | 56.785 | 65.996 |
| | 23.203 | 26.940 | 31.212 | 36.085 | 41.635 | 47.948 | 55.122 | 63.269 | 72.515 | 83.001 | |
| FFS | 1.297 | 1.558 | 1.871 | 2.247 | 2.700 | 3.248 | 3.910 | 4.714 | 5.694 | 6.890 | 8.354 |
| | 10.017 | 11.636 | 13.508 | 15.675 | 18.185 | 21.095 | 24.473 | 28.397 | 32.962 | 38.275 | 44.460 |
| | 17.094 | 19.770 | 22.804 | 26.232 | 30.097 | 34.444 | 39.323 | 44.793 | 50.915 | 57.610 | 65.417 |

This is because of stiffness of the plate is increasing. These frequencies are also represented through figures 3, 4 and 5. The three cases are represented through different colours. Red colour represents CCC, blue colour represents CCS and FCC shows yellow colour. From the

figures, it is observed that the three frequencies are in increasing form. Similar representations stand for figures 4 and 5 also. To check the validity of computed results, source frequencies are compared in the case of clamped quarter of an elliptic plate. It is observed that the

frequencies are matching up-to five significant digits.

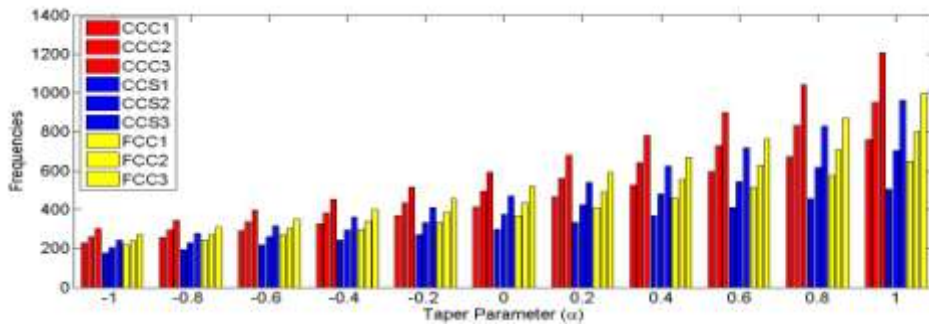


Figure 3 variation of frequency with CCC, CCS, FCC

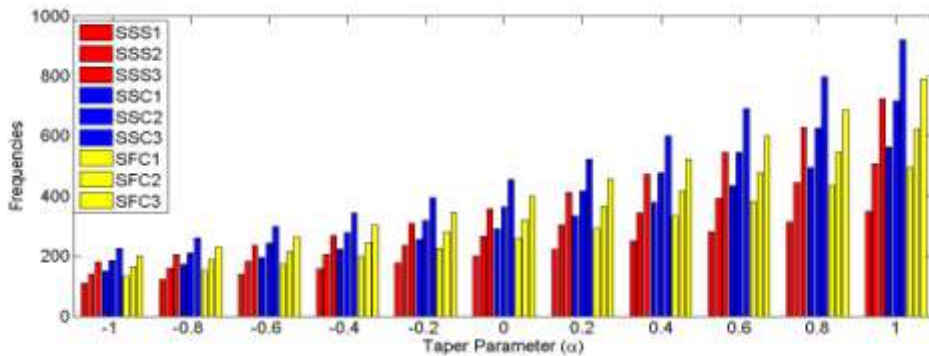


Figure 4 variation of frequency with SSS, SSC, SFC

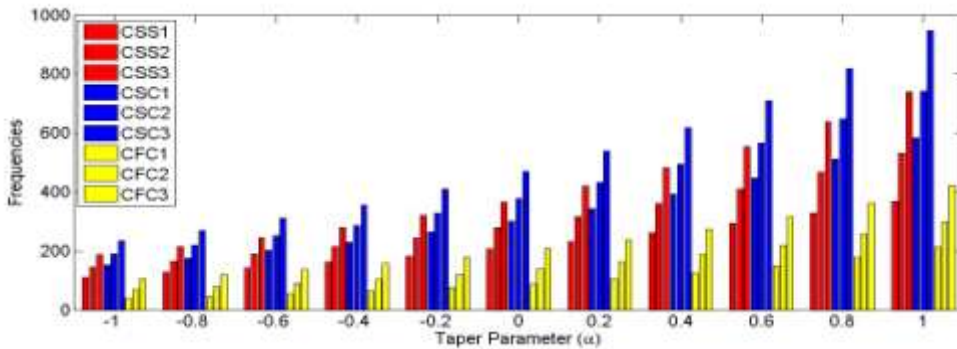


Figure 5 variation of frequency with CSS, CSC, CFC

Table V
Comparison of result for a quarter of circular plate with CCC ($\alpha=0$) $m=1$

| Ref | First three frequencies | | |
|---------|-------------------------|--------|---------|
| | I | II | III |
| Present | 48.788 | 87.787 | 104.872 |
| [17] | 48.788 | 87.787 | 104.872 |
| [10] | 48.820 | 87.860 | 104.970 |
| [13] | 48.200 | 86.890 | 103.020 |
| [14] | 48.700 | 88.130 | 105.060 |
| [15] | 48.786 | 87.779 | 104.890 |

For checking the convergence of results the first three frequencies are computed by creating a matrix of order 5, 10, 15, 20, 25 and 28 and found the first three frequencies are converging up-to five significant

Table VI
Convergence of result for CCC plate with ($\alpha=0$)

| N | λ_1 | λ_2 | λ_3 |
|----|-------------|-------------|-------------|
| 5 | 49.377 | 91.754 | 119.070 |
| 10 | 48.809 | 87.924 | 105.837 |
| 15 | 48.794 | 87.862 | 104.919 |
| 20 | 48.790 | 87.795 | 104.886 |
| 25 | 48.789 | 87.791 | 104.876 |
| 28 | 48.788 | 87.787 | 104.872 |

IV. CONCLUDING REMARKS

From the above work, it is observed that Rayleigh-Ritz technique is an efficient method used to solve the complex vibrational problems. The first three frequencies have been computed in some selected cases by varying the boundary conditions at

the three edges of quarter of elliptic plate. Convergence of result shows the convergence of data up-to five significant digits. It is also concluded that as thickness variation is increasing then the frequencies are increasing in all the cases due to increase in the stiffness of the plate. In the case of CCC, the first three frequencies have been compared with the existing result available in the literature and up-to five significant digits; it is observed that the frequencies are matching up-to four significant digit.

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Numerical Computing of Frequencies for Rectangular and Square Plates with Transcendental Thickness Variation

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Abstract: *The frequencies of the thin plate play an important role for the design of the engineering machines, naval structure, etc. Tremendous amount of the literature on the study of vibrations of rectangular and square plate are available with various kinds of the thickness variations. From the study it is observed that transcendental thickness variation has not been studied by the researchers which satisfy the boundary conditions. Therefore, the present work is the attempt in this direction. A well-known Rayleigh-Ritz method has been used for computation of the first three frequencies alongwith mode shapes and by varying the boundary conditions at the side of the plate. The convergence of the computed results has been presented through table and comparison has also been made with the existing result.*

Index Terms: *Rayleigh-Ritz Method, Frequencies, Mode Shape, Convergence and Comparison, Plates.*

I. INTRODUCTION

The study of vibration of plates has significant role in the engineering design. Exhaustive literature is available on the computation of first few frequencies alongwith representation of mode shapes for rectangular and square plates with various combinations of the boundary conditions applied at the side of the plate. The thickness of thin plate is varying, uniformly, linearly, quadratically, etc but it is revealed from the exhaustive literature that transcendental thickness variation is not studied by the researchers and the scientists which satisfy the various boundary conditions applied on the boundary of the plate.

Let us briefly explain related literature available on the study of rectangular and square plates. The monographs of Leissa [1-10] are an excellent source of information on the study of the various kinds of the plates with varying different types of thickness variation. Extensive numerical results are available in these monographs and very helpful for the researchers and scientists for comparing the results. One of the researchers Laura did tremendous research in the field of vibration of plates. Laura and Grossi [11] have obtained transverse vibration of rectangular plates with edges of the plates are elastically restrained against translation and rotation of the plate. There is lots of work on the thin rectangular plate with different boundary conditions. Shuyu [12] has studied the flexural vibration of thin rectangular plates when all the sides

of the plate are completely free and performed the experiments to compute the resonance frequencies with excellent agreement between the measured results and the existing one in the literature. Rayleigh-Ritz method, finite element method, superposition method, wavelet method and many more have been applied by scientists and researchers for computation of first frequencies for various shapes of thin plate. Superposition technique has been applied by Gormen [13] for free vibration analysis of rectangular plate, with and without elastic super normal to the boundaries convergence with rapidly convergence of the results alongwith the comparisons of the results. In another paper [14], author computed the eigen-values for the elastically supported plates with various aspect ratios and dimension less elastic support coefficients. Wu et al. [15] used the method based on Bessel's function for obtaining the exact value for the free vibration of rectangular thin plates with three boundary conditions like fully simply-supported, fully clamped, two opposite sides are simply supported and other two sides are clamped. The method provides simple, direct, and highly accurate solutions for these kinds of plates. In the year 2009, Xing and Liu [16] have applied two-eigen function theory by the use of amplitude deflection and the generalized curvature for obtaining the results for free vibration solutions of rectangular Mindlin plate. Dozio [17] applied the Ritz method by the use of trigonometric function and obtained accurate in plate modal properties of the rectangular plate with arbitrary non-uniform elastic edge restraints. Dozio observed that there is a support of free in-plate vibration of plates having triangularly and parabolically varying elastic edge supports. By varying the stiffness of plate, natural frequencies and modal shapes are also computed.

In the year 2011, Liu and Xing [18] were successfully obtained exact solutions with 10 sets of distinct eigen solutions for the free in-plane vibration of isotropic rectangular plates with simply-support boundary conditions at two opposite edges with computation of classical boundary conditions at the two other edges. The exact solutions are validated and compared with differential quadrature method. In the year 2013, Zhang [19] has studied model and analysis of FGM rectangular plates based on the physical natural surface and high order shear deformation theory in the rectangular plate. Material properties are used with Poisson's ratio depending on the change of temperature and position of the plate and considered as constant. The presented approach has many merits engineering application.

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Hamiltonian system based benchmark bending solutions for the rectangular thin plates with supported corner point have studied by Li et al. [20] and obtained the comprehensive numerical results with future validation of various, approximate numerical method.

Further, Zhang et al. [21] have obtained orthotropic rectangular plate by the use of two dimensional improved Fourier series method, extensive numerical computations are represented through the reliability and effectiveness of obtained solutions. Banerjee et al. [22] have also studied rectangular plate and solved the bi-harmonic equation to obtain the solution in terms of force displacement relationship of the freely vibrating plate. Wittrick-Williams algorithm is used for ensuing dynamic stiffness matrix to provide the solutions of the problem. Dynamic stiffness method is also used by Danilovic and Petrojjevic [23] for getting the eigenvalues of rectangular plate by converting it as isotropic. The results have been compared with the superposition method used by the Gorman's. The obtained results are high precision accuracy and low memory requirement in comparison of finite element method. In the year 2015, Li et al. [24] reported a benchmark bending solutions of rectangular thin plates with the point-supported at the corner and efficient results were obtained by the author.

Wang and Yuan [25] have applied Discrete Singular Convolution (DSC) and Taylor series expansion method for free vibration analysis of beam and rectangular plates with free boundary conditions and it is observed that the Taylor series expansion method gives accurate results in general and performance proposed by discrete singular decomposition results are excellent and independent of boundary conditions. Recently in the year 2017, Zhou et al. [26] also obtained exact solutions for the free in-plane vibration of rectangular plates with arbitrary boundary conditions, exact natural frequencies and mode shapes can be easily obtained by authors. When two opposite edges of plate have either type of simply-supported boundary conditions and resting plate edges have the arbitrary boundary conditions. Xing et al. [27] have also studied overall assessment of closed form solution methods for vibrations of the rectangular plates. The presented results are giving high accuracy and reliable for different type of rectangular plate.

On the basis of above literature, it is observed that the thickness variation in the closed form of transcendental function has not been studied by the researchers for the various shapes of the plates for which the boundary condition must be satisfied. Hence, the present work is attempt in this direction to perform the exhaustive numerical computations for the rectangular thin plate with transcendental thickness variation. The obtained results have been compared for the different combinations of the boundary conditions with the existing one.

II. BACKGROUND

Let us briefly explain some of the fundamentals used for computation of the frequencies and mode shapes for the rectangular and square plate with thickness variation.

A. Thin Plate

A thin plate may be defined from the theory of beam which was developed by Kirchoff in the year 1888 by dividing the mid-surface plan which is used to represent three dimensional plate in two dimensional form with following assumptions:

- 1) Straight line normal to mid surface remain straight after deformation;
- 2) Straight line normal to mid surface remain normal to mid surface after deformation.
- 3) The thickness of the plate never changes during deformation.

B. Boundary Condition

The boundary condition is divided into the two classes namely essential or geometric boundary condition and natural or dynamic (or force) boundary condition. Essential (or geometric) boundary conditions are also known as Dirichlet boundary and natural (or dynamic) boundary conditions are also known as Neumann boundary conditions.

Any problem related to the plate may basically use three possible combinations of boundary conditions, one is all essential type, second is all natural type and third is mixed type (some natural and essential type).

In the present work, the three kinds of the boundary conditions have been used one is clamped, second is simply-supported and third is completely-free which are defined below and representation of the boundary condition is depicted in figure [1] on the domain of plate as represented as R.

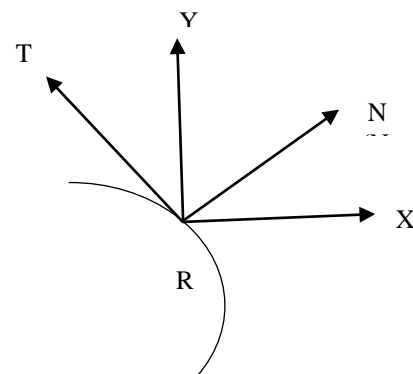


Fig. 1 Representation of the boundary condition

a. Clamped Boundary Condition

In clamped boundary conditions, displacement and slope both are must be zero. In the Mathematical term, it is represented as:

$$\omega = 0 \tag{1}$$

and $\frac{\partial \omega}{\partial \xi} = 0 \tag{2}$

where ω represents the displacement and clamped boundary conditions are the type of essential boundary conditions.

b. Simply-Supported Boundary Condition

In this case, displacement and bending moment must be zero and these boundary conditions written in mathematical term are given below:

$$\omega = 0 \tag{3}$$

$$M_{\xi} = 0 \tag{4}$$

Where M shows the bending moment and these boundary conditions are essential and natural both.

c. Completely-Free Boundary Condition

In this case, it is called as free boundary conditions in which, the bending moment and shear force must be zero. In Mathematical form, these may be represented as:

$$M_{\xi} = 0 \tag{5}$$

$$Q_{\xi} + \frac{\partial M_{\xi\eta}}{\partial \eta} = 0 \tag{6}$$

This type of boundary condition is of the category of natural boundary condition.

III. NUMARICAL FORMULATION

Rayleigh's-Ritz method is the extension of Rayleigh's method and this method is used for all continuous systems. Let us consider a plate which is defined in the domain R and bounded by x=0, x=a, y=0 and y=b as shown below in figure [2] with h is thickness of the plate. Displacement ω at point (x,y) at time t is given below:

$$w(x, y) = W(x, y) \sin \omega t \tag{7}$$

Where w(x,y) is the maximum displacement at (x,y).

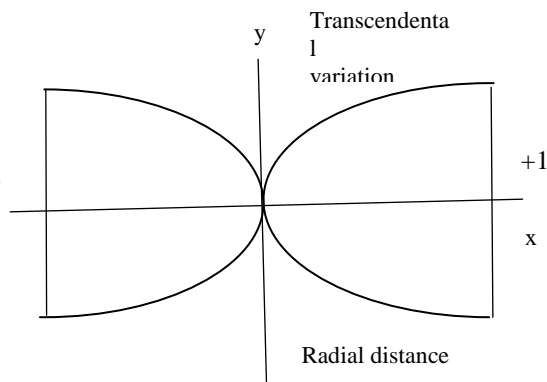
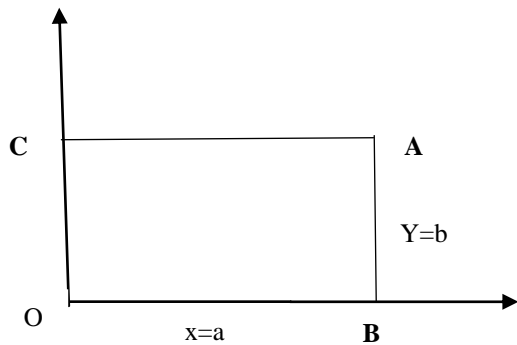


Fig.2 Representation of the thin plate with thickness variation

Now maximum kinetic energy and maximum strain energy are given by

$$T_{\max} = \frac{1}{2} \omega^2 \iint_R \rho h W^2 dx dy \tag{8}$$

$$U_{\max} = \frac{1}{2} \iint_R D [(\nabla^2 W)^2 + 2(1-\nu)\{W_{xy}^2 - W_{xx}W_{yy}\}] dx dy \tag{9}$$

After solving the maximum kinetic energy and strain energy, one can get the Rayleigh quotient given by

$$\omega^2 = \frac{\iint_R D [(\nabla^2 W)^2 + 2(1-\nu)\{W_{xy}^2 - W_{xx}W_{yy}\}] dx dy}{\iint_R h \rho W^2 dx dy} \tag{10}$$

where, $D = Eh^3 / (12(1-\nu^2))$, $\tag{11}$

D is the flexural rigidity and E is young modulus, ρ as mass per unit volume and ν as the passion ratio (0.3) fixed. Let us introduce the following non-dimensional variables and parameters

$$\xi = x/a, \eta = y/b, \mu = a/b, \tag{12}$$

Let us consider the N-term approximation with displacement W

$$W = \sum_{j=1}^N C_j \phi_j(\xi, \eta), \tag{13}$$

where C_j are constants and $\phi_j(\xi, \eta)$ are the basis function which satisfying the essential boundary conditions of the plate. By putting equation (13) in (10) and minimizing ω^2 with constants C_1, C_2, \dots, C_N , then one can get the following:

$$\sum_{j=1}^N (a_{ij} - a^4 \omega^2 p b_{ij}) c_j = 0 \tag{14}$$

The above equation can be rewritten in the form

$$\sum_{j=1}^N (a_{ij} - \lambda^2 b_{ij}) c_j = 0 \quad i = 1, 2, \dots, N \tag{15}$$

Where $\lambda^2 = a^4 \omega^2 p$ and $p = \frac{12\rho(1-\nu^2)}{E}$ then final value of

$$\lambda^2 = \frac{12a^4 \rho (1-\nu^2) \omega^2}{E h_0} \tag{16}$$

$$a_{ij} = \iint_R f^3 \left[\phi_i^{\xi\xi} \phi_j^{\xi\xi} + \mu^2 \nu (\phi_i^{\xi\xi} \phi_j^{\nu\nu} + \phi_j^{\xi\xi} \phi_i^{\nu\nu}) + 2\mu^2 (1-\nu) \phi_i^{\xi\nu} \phi_j^{\xi\nu} + \mu^4 \phi_i^{\nu\nu} \phi_j^{\nu\nu} \right] d\xi d\eta \tag{17}$$

$$b_{ij} = \iint_R f \phi_i \phi_j d\xi d\eta \tag{18}$$

The f is considered as the thickness variation and given by

$$f = 1 + \alpha \sin \xi \tag{19}$$

where, R is the new domain in non-dimensional form bounded by $\xi = 0, \xi = 1, \eta = 0$ and $\eta = 1$, parameter λ represents the frequency parameter.

After solving the above equations for λ^2 with constants values of $C_1,$



C_2, \dots, C_N , with the basis function ϕ_i satisfying the essential boundary conditions, we get an Eigen value problem. The basis ϕ_i is taken of the form of

$$\phi_i(\xi, \eta) = \xi^p \eta^q (1-\xi)^r (1-\eta)^s \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \dots\} \tag{20}$$

Where $p=0, 1$ or 2 , depending on side $\xi = 0$ is free, simply-supported and clamped. Similar representation is given to q, r and s . The parameters, $\eta = 0, \xi = 1, \eta = 1$ are controlling the boundary conditions at other sides. The basis function $\phi_i(\xi, \eta)$ satisfies the all the essential boundary conditions. The complicated integrals involved in the equations are given by.

$$\iint_R \xi^p \eta^q (1-\xi)^r (1-\eta)^s d\xi d\eta = \frac{p!q!r!s!}{(p+r+1)!(q+s+1)!} \tag{21}$$

The Eigen-value problem can be solved by the generalized Jacobi method.

IV. NUMARICAL DISCUSSION

For extensive numerical computation on the rectangular and square plates, the following parameters have been selected:

- 1) The value of Poisson distribution is considered as $\nu=0.3$ for the isotropic plate;
- 2) The taper parameter α is varying from 0 to 1 with an interval as 0.2;
- 3) The numeric values of boundary condition are considered as 2,1,0 for the clamped, simply-supported and completely-free plate, respectively and controlled by the values of p, q, r and s ;
- 4) The frequencies are represented as λ_1, λ_2 and λ_3 for the first, second and third frequency, respectively;
- 5) The aspect ratio a/b is taken as 1 for square plate and 2.0 for the rectangular plate.

The problem has been solved by the generalized Jacobi method which is used for computation of the first three frequencies and corresponding mode shapes. Table [1] represents the first three frequencies for the rectangular plate with different combinations of the boundary conditions. From the table, one can observed that the frequencies are increasing as the value of taper parameter is increasing from 0 to 1. This is because of the stiffness of the plate is increasing.

Table 1 First three frequencies for the rectangular plate ($a/b=2.0$)

| α | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|------------------|---------|--------|--------|--------|--------|--------|
| FFFF λ_1 | 0.00000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| λ_2 | 21.467 | 23.580 | 25.731 | 27.900 | 30.076 | 32.254 |
| λ_3 | 26.585 | 29.168 | 31.781 | 34.390 | 36.980 | 39.547 |
| FFFS λ_1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| λ_2 | 13.469 | 14.778 | 16.133 | 17.513 | 18.907 | 20.310 |
| λ_3 | 34.811 | 38.140 | 41.580 | 45.060 | 48.546 | 52.024 |
| FFFC λ_1 | 14.002 | 15.339 | 16.738 | 18.170 | 19.620 | 21.082 |
| λ_2 | 21.460 | 23.516 | 25.665 | 27.864 | 30.089 | 32.332 |
| λ_3 | 40.767 | 44.653 | 48.681 | 52.762 | 56.853 | 60.933 |
| FFSF λ_1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| λ_2 | 13.039 | 15.145 | 17.324 | 19.530 | 21.743 | 23.972 |
| λ_3 | 42.854 | 46.920 | 51.071 | 55.231 | 59.369 | 63.580 |
| FFSS λ_1 | 6.6438 | 7.6422 | 8.6819 | 9.7440 | 10.819 | 11.903 |
| λ_2 | 25.376 | 27.770 | 30.234 | 32.727 | 35.235 | 37.750 |
| λ_3 | 58.772 | 63.977 | 68.990 | 73.622 | 77.865 | 81.824 |
| FFSC λ_1 | 16.140 | 17.480 | 18.870 | 20.288 | 21.726 | 23.177 |
| λ_2 | 31.487 | 34.331 | 37.255 | 40.213 | 43.186 | 46.165 |
| λ_3 | 63.453 | 69.342 | 75.357 | 81.388 | 87.379 | 93.300 |
| FFCF λ_1 | 3.4512 | 4.1566 | 4.9138 | 5.7046 | 6.5185 | 7.3507 |
| λ_2 | 14.818 | 17.044 | 19.323 | 21.616 | 23.906 | 26.186 |
| λ_3 | 21.477 | 24.216 | 27.028 | 29.863 | 32.699 | 35.528 |
| FFCS λ_1 | 8.5156 | 9.7148 | 10.950 | 12.202 | 13.462 | 14.727 |
| λ_2 | 30.983 | 34.034 | 37.169 | 40.332 | 43.499 | 46.661 |
| λ_3 | 64.157 | 68.243 | 72.163 | 75.953 | 79.648 | 83.273 |



| | | | | | | | |
|------|-------------|--------|--------|---------|--------|--------|--------|
| FFCC | λ_1 | 17.142 | 18.545 | 19.991 | 21.461 | 22.945 | 24.440 |
| | λ_2 | 36.423 | 39.771 | 43.199 | 46.650 | 50.101 | 53.545 |
| | λ_3 | 73.516 | 80.442 | 87.419 | 94.285 | 100.92 | 107.17 |
| FSFS | λ_1 | 38.945 | 42.238 | 45.029 | 74.512 | 49.832 | 52.061 |
| | λ_2 | 46.751 | 51.477 | 56.905 | 62.648 | 68.487 | 74.318 |
| | λ_3 | 70.748 | 77.472 | 84.465 | 91.566 | 98.697 | 105.82 |
| FSFC | λ_1 | 61.226 | 65.707 | 68.918 | 71.721 | 74.354 | 76.899 |
| | λ_2 | 67.279 | 74.648 | 83.280 | 91.987 | 100.41 | 108.48 |
| | λ_3 | 87.706 | 96.102 | 105.03 | 114.33 | 123.93 | 133.76 |
| FSSS | λ_1 | 41.197 | 43.583 | 45.870 | 48.084 | 50.251 | 52.387 |
| | λ_2 | 59.067 | 64.471 | 70.006 | 75.559 | 81.082 | 86.559 |
| | λ_3 | 94.506 | 103.19 | 112.11 | 121.09 | 130.06 | 138.95 |
| FSSC | λ_1 | 62.922 | 66.161 | 69.060 | 71.766 | 74.360 | 76.886 |
| | λ_2 | 77.408 | 84.479 | 91.757 | 99.015 | 106.17 | 113.20 |
| | λ_3 | 108.94 | 118.96 | 129.31 | 139.77 | 150.23 | 160.64 |
| FSCF | λ_1 | 8.5156 | 9.7148 | 10.950 | 12.202 | 13.462 | 14.727 |
| | λ_2 | 30.983 | 34.034 | 37.169 | 40.352 | 43.499 | 46.661 |
| | λ_3 | 64.157 | 68.243 | 72.162 | 75.953 | 79.648 | 83.273 |
| FSCS | λ_1 | 41.702 | 43.995 | 46.218 | 48.388 | 50.524 | 52.636 |
| | λ_2 | 63.016 | 68.552 | 74.168 | 79.772 | 85.328 | 90.829 |
| | λ_3 | 103.19 | 112.58 | 122.10 | 131.61 | 141.06 | 150.44 |
| FSCC | λ_1 | 63.264 | 66.345 | 69.167 | 71.833 | 74.402 | 76.912 |
| | λ_2 | 80.607 | 87.575 | 94.667 | 101.72 | 108.68 | 115.54 |
| | λ_3 | 116.70 | 127.20 | 137.91 | 148.65 | 159.36 | 170.00 |
| FCFC | λ_1 | 89.151 | 94.499 | 98.012 | 101.14 | 104.09 | 106.95 |
| | λ_2 | 93.709 | 104.82 | 116.87 | 127.88 | 137.82 | 147.01 |
| | λ_3 | 110.13 | 120.93 | 133.08 | 146.39 | 160.39 | 174.62 |
| FCSC | λ_1 | 90.400 | 94.564 | 97.990 | 101.10 | 104.05 | 106.90 |
| | λ_2 | 101.65 | 111.00 | 120.57 | 129.90 | 138.92 | 147.64 |
| | λ_3 | 128.31 | 140.12 | 152.54 | 165.25 | 178.03 | 190.75 |
| FCCC | λ_1 | 90.628 | 94.604 | 97.984 | 101.08 | 104.02 | 106.87 |
| | λ_2 | 104.17 | 113.14 | 122.27 | 131.22 | 139.94 | 148.43 |
| | λ_3 | 135.10 | 147.18 | 159.61 | 172.17 | 184.77 | 197.37 |
| SFFS | λ_1 | 6.6438 | 6.9077 | 7.1762 | 7.4435 | 7.7077 | 7.9684 |
| | λ_2 | 25.376 | 27.789 | 30.274 | 32.778 | 35.279 | 37.766 |
| | λ_3 | 58.772 | 64.501 | 70.271 | 75.989 | 81.625 | 87.180 |
| SFFC | λ_1 | 16.140 | 17.878 | 19.7121 | 21.597 | 23.509 | 25.437 |
| | λ_2 | 31.487 | 34.596 | 37.813 | 41.065 | 44.320 | 47.563 |
| | λ_3 | 63.453 | 69.537 | 75.747 | 81.954 | 88.108 | 94.201 |
| SFSF | λ_1 | 9.5125 | 10.403 | 11.275 | 12.123 | 12.946 | 13.747 |
| | λ_2 | 27.522 | 30.057 | 32.728 | 35.463 | 38.227 | 41.004 |
| | λ_3 | 38.531 | 42.144 | 45.787 | 49.406 | 52.983 | 56.514 |
| SFSS | λ_1 | 16.135 | 17.637 | 19.204 | 20.801 | 22.412 | 24.028 |
| | λ_2 | 46.740 | 51.116 | 55.550 | 59.971 | 64.350 | 68.680 |
| | λ_3 | 75.284 | 82.239 | 89.066 | 95.698 | 102.13 | 108.40 |

Numerical Computing of Frequencies for Rectangular and Square Plates with Transcendental Thickness Variation

| | | | | | | | |
|------|-------------|--------|--------|--------|--------|--------|---------|
| SFSC | λ_1 | 22.815 | 24.955 | 27.168 | 29.414 | 31.672 | 33.933 |
| | λ_2 | 50.751 | 55.507 | 60.340 | 65.169 | 69.963 | 74.711 |
| | λ_3 | 98.890 | 108.16 | 117.54 | 126.03 | 134.02 | 141.70 |
| SFCF | λ_1 | 15.082 | 16.839 | 18.632 | 20.423 | 22.199 | 23.955 |
| | λ_2 | 31.310 | 34.154 | 37.110 | 40.108 | 43.116 | 46.122 |
| | λ_3 | 49.095 | 54.000 | 58.979 | 63.934 | 68.829 | 73.656 |
| SFCS | λ_1 | 20.601 | 22.659 | 24.786 | 26.932 | 29.076 | 31.2111 |
| | λ_2 | 56.318 | 61.762 | 67.285 | 72.782 | 78.215 | 83.576 |
| | λ_3 | 77.332 | 84.114 | 90.776 | 97.264 | 103.58 | 109.74 |
| SFCC | λ_1 | 26.290 | 28.754 | 31.284 | 33.830 | 36.373 | 38.906 |
| | λ_2 | 59.766 | 65.485 | 71.289 | 77.070 | 82.790 | 88.437 |
| | λ_3 | 101.42 | 110.27 | 118.78 | 126.44 | 134.78 | 142.34 |
| SSFS | λ_1 | 41.197 | 46.429 | 51.717 | 56.886 | 61.877 | 66.683 |
| | λ_2 | 59.067 | 64.841 | 70.885 | 77.107 | 83.457 | 89.898 |
| | λ_3 | 94.506 | 103.63 | 112.97 | 122.31 | 131.59 | 140.80 |
| SSFC | λ_1 | 62.922 | 70.991 | 78.662 | 85.651 | 92.036 | 97.982 |
| | λ_2 | 77.408 | 85.183 | 93.703 | 102.86 | 112.42 | 122.16 |
| | λ_3 | 108.94 | 119.52 | 130.47 | 141.57 | 152.75 | 163.99 |
| SSSS | λ_1 | 49.348 | 53.941 | 58.492 | 62.954 | 67.317 | 71.589 |
| | λ_2 | 78.958 | 86.369 | 93.926 | 101.50 | 109.03 | 116.50 |
| | λ_3 | 128.33 | 140.38 | 152.66 | 164.88 | 176.93 | 188.78 |
| SSSC | λ_1 | 69.327 | 75.733 | 81.918 | 87.853 | 93.562 | 99.082 |
| | λ_2 | 94.586 | 103.52 | 112.70 | 121.96 | 131.21 | 140.40 |
| | λ_3 | 140.33 | 153.52 | 166.97 | 180.42 | 193.75 | 206.93 |
| SSCS | λ_1 | 51.674 | 56.207 | 60.694 | 65.095 | 69.406 | 73.632 |
| | λ_2 | 86.135 | 94.070 | 102.11 | 110.12 | 118.04 | 125.88 |
| | λ_3 | 140.91 | 154.01 | 167.26 | 180.44 | 193.48 | 206.35 |
| SSCC | λ_1 | 71.078 | 77.227 | 83.188 | 88.944 | 94.513 | 99.921 |
| | λ_2 | 100.80 | 110.03 | 119.43 | 128.83 | 138.17 | 147.41 |
| | λ_3 | 151.96 | 166.07 | 180.38 | 194.64 | 208.78 | 222.74 |
| SCFC | λ_1 | 90.400 | 101.65 | 111.07 | 119.04 | 126.27 | 133.06 |
| | λ_2 | 101.65 | 112.33 | 124.97 | 138.58 | 152.25 | 165.57 |
| | λ_3 | 128.31 | 141.00 | 154.36 | 168.16 | 182.37 | 196.97 |
| SCSC | λ_1 | 95.263 | 103.91 | 111.93 | 119.39 | 126.42 | 133.14 |
| | λ_2 | 115.80 | 126.86 | 138.38 | 150.07 | 161.75 | 173.31 |
| | λ_3 | 156.37 | 171.17 | 186.41 | 201.73 | 216.98 | 232.09 |
| SCCC | λ_1 | 96.573 | 104.78 | 112.50 | 119.77 | 126.69 | 133.33 |
| | λ_2 | 121.00 | 132.10 | 143.49 | 154.94 | 166.34 | 177.62 |
| | λ_3 | 167.06 | 182.55 | 198.33 | 214.15 | 229.92 | 245.55 |
| CFFF | λ_1 | 3.4512 | 3.4117 | 3.3806 | 3.3551 | 3.3337 | 3.3154 |
| | λ_2 | 14.818 | 15.302 | 15.941 | 16.462 | 16.957 | 17.430 |
| | λ_3 | 21.477 | 22.775 | 24.050 | 25.292 | 26.497 | 27.669 |
| CFFS | λ_1 | 8.5156 | 8.9253 | 9.3326 | 9.7286 | 10.111 | 10.481 |
| | λ_2 | 30.983 | 33.731 | 36.547 | 39.371 | 42.176 | 44.953 |
| | λ_3 | 64.157 | 71.870 | 79.486 | 86.711 | 93.525 | 100.03 |
| CFFC | λ_1 | 17.142 | 18.998 | 20.953 | 22.956 | 24.980 | 27.014 |
| | λ_2 | 36.423 | 39.895 | 43.471 | 47.066 | 50.645 | 54.195 |
| | λ_3 | 73.516 | 80.291 | 87.160 | 94.003 | 100.78 | 107.47 |

| | | | | | | |
|------------------|--------|--------|---------|--------|---------|--------|
| CFSF λ_1 | 15.082 | 16.094 | 17.092 | 18.065 | 19.009 | 19.926 |
| λ_2 | 31.310 | 34.214 | 37.243 | 40.317 | 43.401 | 46.479 |
| λ_3 | 49.310 | 53.275 | 57.476 | 61.630 | 65.715 | 69.726 |
| CFSS λ_1 | 20.601 | 22.335 | 24.122 | 25.924 | 27.723 | 29.512 |
| λ_2 | 56.318 | 61.316 | 66.358 | 71.358 | 76.286 | 81.137 |
| λ_3 | 77.332 | 84.882 | 92.362 | 99.676 | 106.81 | 113.77 |
| CFSC λ_1 | 26.290 | 28.711 | 31.200 | 33.708 | 36.214 | 38.708 |
| λ_2 | 59.766 | 65.150 | 70.594 | 76.004 | 81.347 | 86.614 |
| λ_3 | 101.42 | 111.30 | 120.89 | 130.13 | 139.03 | 147.62 |
| CFCF λ_1 | 22.076 | 24.067 | 26.100 | 28.125 | 30.124 | 32.091 |
| λ_2 | 36.018 | 39.307 | 42.693 | 46.097 | 49.487 | 52.852 |
| λ_3 | 60.897 | 66.476 | 72.118 | 77.709 | 83.210 | 88.616 |
| CFCS λ_1 | 26.443 | 28.845 | 31.305 | 33.765 | 36.203 | 38.611 |
| λ_2 | 67.249 | 73.433 | 79.683 | 85.876 | 91.972 | 97.963 |
| λ_3 | 79.831 | 87.240 | 94.574 | 101.75 | 108.76 | 115.61 |
| CFCC λ_1 | 31.135 | 33.992 | 36.905 | 39.815 | 42.701 | 45.553 |
| λ_2 | 70.229 | 76.700 | 83.243 | 89.732 | 96.124 | 102.41 |
| λ_3 | 103.42 | 113.00 | 122.34 | 131.36 | 140.09 | 148.54 |
| CSFS λ_1 | 41.702 | 47.167 | 52.775 | 58.340 | 63.787 | 69.091 |
| λ_2 | 63.016 | 69.330 | 75.863 | 82.502 | 89.201 | 95.936 |
| λ_3 | 103.19 | 113.10 | 123.21 | 133.32 | 143.37 | 153.32 |
| CSFC λ_1 | 63.264 | 71.721 | 80.070 | 87.963 | 95.338 | 102.26 |
| λ_2 | 80.607 | 88.947 | 97.831 | 107.18 | 116.87 | 126.77 |
| λ_3 | 116.70 | 128.12 | 139.86 | 151.68 | 163.51 | 175.29 |
| CSSS λ_1 | 51.674 | 56.759 | 61.833 | 66.825 | 71.715 | 76.504 |
| λ_2 | 86.135 | 94.287 | 102.57 | 110.83 | 119.02 | 127.11 |
| λ_3 | 140.91 | 154.05 | 167.30 | 180.46 | 193.45 | 206.29 |
| CSSC λ_1 | 71.078 | 78.122 | 85.023 | 91.711 | 98.180 | 104.45 |
| λ_2 | 100.80 | 110.49 | 120.39 | 130.34 | 140.23 | 150.05 |
| λ_3 | 151.96 | 166.23 | 180.70 | 195.13 | 209.42 | 223.56 |
| CSCS λ_1 | 54.743 | 59.827 | 64.882 | 69.848 | 74.711 | 79.473 |
| λ_2 | 94.585 | 103.37 | 112.25 | 121.06 | 129.76 | 138.32 |
| λ_3 | 154.78 | 169.16 | 183.67 | 198.05 | 212.20 | 226.11 |
| CSCC λ_1 | 73.396 | 80.222 | 86.911 | 93.410 | 99.720 | 105.86 |
| λ_2 | 108.22 | 118.33 | 128.57 | 138.78 | 148.88 | 158.85 |
| λ_3 | 165.03 | 180.39 | 195.92 | 211.34 | 226.56 | 241.56 |
| CCFC λ_1 | 90.628 | 102.60 | 113.43 | 122.88 | 131.43 | 139.41 |
| λ_2 | 104.17 | 115.36 | 128.04 | 141.80 | 155.93 | 169.99 |
| λ_3 | 135.10 | 148.57 | 162.261 | 177.00 | 191.165 | 206.50 |
| CCSC λ_1 | 96.573 | 106.09 | 115.15 | 123.72 | 131.85 | 139.64 |
| λ_2 | 121.00 | 132.83 | 145.05 | 157.41 | 169.76 | 182.01 |
| λ_3 | 167.06 | 182.90 | 199.03 | 215.21 | 231.35 | 247.40 |
| CCCC λ_1 | 98.313 | 107.40 | 116.12 | 124.43 | 132.40 | 140.06 |
| λ_2 | 127.32 | 139.30 | 151.52 | 163.76 | 175.92 | 187.95 |
| λ_3 | 179.09 | 195.83 | 212.83 | 229.76 | 246.54 | 263.11 |

The computations have also been performed on the square plate as a special case of rectangular plate with aspect ratio is considered as $a/b=1.0$. The results have been computed for the different combinations of the boundary conditions and represented in the following table [2]. From the table, it is

again observed that the frequencies are represented upto the five significant digits for different combinations of boundary conditions and these



are in increasing form as the taper parameter is increasing.

Table 2 First few frequencies for the square plate (a/b=1.0)

| α | | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------|-------------|--------|--------|--------|--------|--------|--------|
| FFFF | λ_1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | λ_2 | 13.469 | 14.780 | 16.141 | 17.527 | 18.927 | 20.336 |
| | λ_3 | 19.597 | 21.498 | 23.476 | 23.999 | 27.541 | 29.600 |
| FFFS | λ_1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | λ_2 | 6.6438 | 7.3006 | 8.0110 | 8.7588 | 9.5346 | 10.332 |
| | λ_3 | 14.903 | 16.324 | 17.818 | 19.351 | 20.905 | 22.472 |
| FFFC | λ_1 | 3.4811 | 3.8218 | 4.1951 | 4.5915 | 5.0051 | 5.4316 |
| | λ_2 | 8.5217 | 9.3486 | 10.229 | 11.143 | 12.080 | 13.035 |
| | λ_3 | 21.317 | 23.312 | 25.342 | 27.366 | 29.371 | 31.354 |
| FFSF | λ_1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | λ_2 | 6.6438 | 7.6422 | 8.6821 | 9.7442 | 10.819 | 11.904 |
| | λ_3 | 14.903 | 16.697 | 18.525 | 20.369 | 22.221 | 24.079 |
| FFSS | λ_1 | 3.3671 | 3.8131 | 4.2870 | 4.7811 | 5.2911 | 5.8140 |
| | λ_2 | 17.317 | 18.909 | 20.531 | 22.165 | 23.804 | 25.446 |
| | λ_3 | 19.293 | 21.188 | 23.149 | 25.145 | 27.163 | 29.194 |
| FFSC | λ_1 | 5.3577 | 5.9060 | 6.4869 | 7.0915 | 7.7146 | 8.3530 |
| | λ_2 | 19.083 | 20.990 | 22.941 | 24.911 | 26.886 | 28.862 |
| | λ_3 | 24.690 | 26.596 | 28.539 | 30.501 | 32.474 | 34.457 |
| FFCF | λ_1 | 3.4811 | 4.1940 | 4.9595 | 5.7590 | 6.5825 | 7.4244 |
| | λ_2 | 8.5217 | 9.7224 | 10.959 | 12.212 | 13.473 | 14.738 |
| | λ_3 | 21.317 | 23.972 | 26.676 | 29.373 | 32.039 | 34.663 |
| FFCS | λ_1 | 5.3577 | 6.1516 | 6.9954 | 7.8717 | 8.7717 | 9.6904 |
| | λ_2 | 19.083 | 20.668 | 22.293 | 23.936 | 25.588 | 27.246 |
| | λ_3 | 24.690 | 27.507 | 30.407 | 33.335 | 36.269 | 39.200 |
| FFCC | λ_1 | 6.9300 | 7.7499 | 8.6182 | 9.5191 | 10.444 | 11.389 |
| | λ_2 | 23.942 | 25.977 | 27.953 | 29.884 | 31.787 | 33.672 |
| | λ_3 | 26.598 | 29.209 | 31.992 | 34.859 | 37.765 | 40.688 |
| FSFS | λ_1 | 9.6314 | 10.539 | 11.475 | 12.424 | 13.80 | 14.342 |
| | λ_2 | 16.135 | 17.701 | 19.386 | 21.145 | 22.953 | 24.793 |
| | λ_3 | 36.727 | 40.252 | 43.900 | 47.556 | 49.880 | 52.111 |
| FSFC | λ_1 | 15.211 | 16.592 | 17.920 | 19.198 | 20.444 | 21.673 |
| | λ_2 | 20.613 | 22.641 | 24.898 | 27.299 | 29.787 | 32.323 |
| | λ_3 | 39.773 | 43.583 | 47.537 | 51.556 | 55.601 | 59.649 |
| FSSS | λ_1 | 11.685 | 12.564 | 13.480 | 14.424 | 15.393 | 16.382 |
| | λ_2 | 27.756 | 30.406 | 33.133 | 35.896 | 38.673 | 41.458 |
| | λ_3 | 41.202 | 43.591 | 45.881 | 48.099 | 50.269 | 52.407 |
| FSSC | λ_1 | 16.807 | 17.889 | 18.993 | 20.115 | 21.253 | 22.408 |
| | λ_2 | 31.122 | 34.032 | 37.033 | 40.074 | 43.133 | 46.199 |
| | λ_3 | 51.432 | 54.233 | 56.836 | 59.313 | 61.709 | 64.054 |
| FSCF | λ_1 | 5.3577 | 6.1516 | 6.9954 | 7.8717 | 8.7717 | 9.6904 |
| | λ_2 | 19.083 | 20.668 | 22.293 | 23.936 | 25.588 | 27.246 |
| | λ_3 | 24.690 | 27.507 | 30.407 | 33.335 | 36.269 | 39.200 |



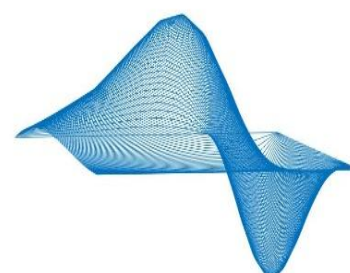
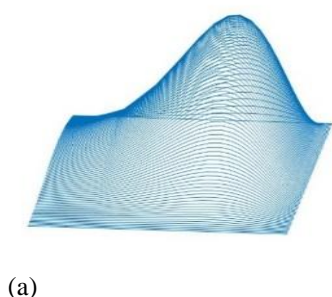
| | | | | | | |
|------------------|--------|--------|--------|--------|--------|--------|
| FSCS λ_1 | 12.687 | 13.676 | 14.707 | 15.770 | 16.859 | 17.969 |
| λ_2 | 33.065 | 36.333 | 39.689 | 43.073 | 46.460 | 49.841 |
| λ_3 | 41.707 | 44.000 | 46.223 | 48.394 | 50.530 | 52.644 |
| FSCC λ_1 | 17.551 | 18.663 | 19.803 | 20.967 | 22.151 | 23.355 |
| λ_2 | 36.032 | 39.476 | 43.008 | 46.566 | 50.126 | 53.676 |
| λ_3 | 51.843 | 54.513 | 57.038 | 59.466 | 61.832 | 64.161 |
| FCFC λ_1 | 22.200 | 24.081 | 25.693 | 27.159 | 28.562 | 29.941 |
| λ_2 | 26.478 | 29.188 | 32.362 | 35.762 | 39.256 | 42.781 |
| λ_3 | 43.630 | 47.817 | 52.205 | 56.700 | 61.251 | 65.830 |
| FCSC λ_1 | 23.402 | 24.747 | 26.077 | 27.400 | 28.726 | 30.059 |
| λ_2 | 35.592 | 38.863 | 42.254 | 45.699 | 49.168 | 52.646 |
| λ_3 | 62.940 | 66.185 | 69.091 | 71.804 | 74.402 | 76.931 |
| FCCC λ_1 | 23.949 | 25.247 | 26.549 | 27.857 | 29.175 | 30.507 |
| λ_2 | 40.019 | 43.720 | 47.520 | 51.350 | 55.183 | 59.00 |
| λ_3 | 63.282 | 66.365 | 69.190 | 71.857 | 74.429 | 76.941 |
| SFFS λ_1 | 3.3671 | 3.5717 | 3.7912 | 4.0180 | 4.2481 | 4.4794 |
| λ_2 | 17.317 | 18.992 | 20.707 | 22.417 | 24.105 | 25.766 |
| λ_3 | 19.293 | 21.117 | 23.028 | 24.997 | 27.004 | 29.037 |
| SFFC λ_1 | 5.3577 | 5.8404 | 6.3576 | 6.8941 | 7.4421 | 10.530 |
| λ_2 | 19.083 | 20.810 | 22.580 | 24.358 | 26.131 | 35.647 |
| λ_3 | 24.690 | 27.467 | 30.352 | 33.303 | 36.256 | 52.326 |
| SFSF λ_1 | 9.6314 | 10.533 | 11.417 | 12.275 | 13.110 | 13.921 |
| λ_2 | 16.135 | 17.637 | 19.204 | 20.802 | 22.413 | 24.030 |
| λ_3 | 36.727 | 40.139 | 43.626 | 47.123 | 50.605 | 54.062 |
| SFSS λ_1 | 11.685 | 12.778 | 13.883 | 14.986 | 16.079 | 17.162 |
| λ_2 | 27.756 | 30.343 | 32.993 | 35.658 | 38.318 | 40.964 |
| λ_3 | 41.202 | 45.065 | 48.967 | 52.848 | 56.687 | 60.478 |
| SFSC λ_1 | 12.687 | 13.876 | 15.087 | 16.301 | 17.512 | 18.715 |
| λ_2 | 33.065 | 36.150 | 39.285 | 42.420 | 45.535 | 48.624 |
| λ_3 | 41.707 | 45.618 | 49.570 | 53.505 | 57.399 | 61.247 |
| SFCF λ_1 | 15.211 | 16.982 | 18.791 | 20.598 | 22.389 | 24.162 |
| λ_2 | 20.613 | 22.675 | 24.804 | 26.952 | 29.098 | 31.233 |
| λ_3 | 39.773 | 43.301 | 46.887 | 50.473 | 54.038 | 59.595 |
| SFCS λ_1 | 16.807 | 18.652 | 20.542 | 22.437 | 24.321 | 26.189 |
| λ_2 | 31.122 | 33.958 | 36.844 | 39.731 | 42.600 | 45.446 |
| λ_3 | 51.432 | 56.511 | 61.669 | 66.802 | 71.877 | 76.883 |
| SFCC λ_1 | 17.551 | 19.437 | 21.370 | 23.311 | 25.242 | 27.157 |
| λ_2 | 36.032 | 39.259 | 42.523 | 45.775 | 48.999 | 52.190 |
| λ_3 | 51.843 | 56.956 | 62.148 | 67.317 | 72.427 | 77.469 |
| SSFS λ_1 | 11.685 | 13.032 | 14.457 | 15.917 | 17.391 | 18.870 |
| λ_2 | 27.756 | 30.379 | 33.087 | 35.827 | 38.573 | 41.316 |
| λ_3 | 41.202 | 46.433 | 51.720 | 56.891 | 61.886 | 66.701 |
| SSFC λ_1 | 16.807 | 18.901 | 21.093 | 23.317 | 25.543 | 27.758 |
| λ_2 | 31.122 | 34.127 | 37.248 | 40.425 | 43.628 | 46.841 |
| λ_3 | 51.432 | 58.027 | 64.532 | 70.707 | 76.516 | 82.005 |

Numerical Computing of Frequencies for Rectangular and Square Plates with Transcendental Thickness Variation

| | | | | | | |
|------------------|--------|--------|--------|--------|--------|--------|
| SSSS λ_1 | 19.739 | 21.588 | 23.474 | 25.368 | 27.256 | 29.134 |
| λ_2 | 49.349 | 53.942 | 58.493 | 62.956 | 67.321 | 71.595 |
| λ_3 | 49.349 | 53.974 | 58.658 | 63.328 | 67.955 | 72.531 |
| SSSC λ_1 | 23.646 | 25.867 | 28.124 | 30.383 | 32.632 | 34.866 |
| λ_2 | 51.675 | 56.522 | 61.437 | 66.343 | 71.209 | 76.023 |
| λ_3 | 58.646 | 64.092 | 69.427 | 74.609 | 79.642 | 84.543 |
| SSCS λ_1 | 23.646 | 25.923 | 28.241 | 30.557 | 32.858 | 35.137 |
| λ_2 | 51.675 | 56.208 | 60.695 | 65.096 | 69.407 | 73.633 |
| λ_3 | 58.646 | 64.290 | 70.012 | 75.706 | 81.335 | 86.889 |
| SSCC λ_1 | 27.055 | 29.581 | 32.139 | 34.691 | 37.221 | 39.725 |
| λ_2 | 60.539 | 65.878 | 71.067 | 76.113 | 81.026 | 85.821 |
| λ_3 | 60.791 | 66.545 | 72.429 | 78.295 | 84.099 | 89.830 |
| SCFC λ_1 | 23.402 | 26.431 | 29.549 | 32.655 | 35.708 | 38.696 |
| λ_2 | 35.592 | 39.109 | 42.813 | 46.636 | 50.541 | 54.503 |
| λ_3 | 62.940 | 71.005 | 78.671 | 85.658 | 92.048 | 98.012 |
| SCSC λ_1 | 28.951 | 31.675 | 34.423 | 37.162 | 39.879 | 42.569 |
| λ_2 | 54.744 | 59.886 | 65.116 | 70.349 | 75.551 | 80.710 |
| λ_3 | 69.327 | 75.733 | 81.918 | 87.853 | 93.563 | 99.083 |
| SCCC λ_1 | 31.827 | 34.716 | 37.623 | 40.510 | 43.367 | 46.190 |
| λ_2 | 63.333 | 69.367 | 75.490 | 81.590 | 87.629 | 93.595 |
| λ_3 | 71.079 | 77.228 | 83.190 | 88.945 | 94.514 | 99.922 |
| CFFF λ_1 | 3.4811 | 3.4395 | 3.4065 | 3.3793 | 3.3562 | 3.3363 |
| λ_2 | 8.5217 | 8.9308 | 9.3375 | 9.7333 | 10.116 | 10.487 |
| λ_3 | 21.317 | 22.651 | 23.959 | 25.231 | 26.464 | 27.662 |
| CFFS λ_1 | 5.3577 | 5.5901 | 5.8411 | 6.1002 | 6.3621 | 6.6242 |
| λ_2 | 19.083 | 21.111 | 23.237 | 25.404 | 27.581 | 29.756 |
| λ_3 | 24.690 | 26.522 | 28.377 | 30.229 | 32.069 | 33.896 |
| CFFC λ_1 | 6.9300 | 7.4366 | 7.9802 | 8.5427 | 9.1148 | 9.6920 |
| λ_2 | 23.942 | 26.242 | 28.468 | 30.600 | 32.658 | 34.661 |
| λ_3 | 26.598 | 29.182 | 32.007 | 34.976 | 38.016 | 41.088 |
| CFSF λ_1 | 15.211 | 16.231 | 17.239 | 18.221 | 19.175 | 20.101 |
| λ_2 | 20.613 | 22.344 | 24.129 | 25.928 | 27.726 | 29.515 |
| λ_3 | 39.773 | 43.617 | 47.542 | 51.470 | 55.372 | 59.235 |
| CFSS λ_1 | 16.807 | 18.054 | 19.310 | 20.555 | 21.782 | 22.990 |
| λ_2 | 31.122 | 34.054 | 37.047 | 40.044 | 43.022 | 45.973 |
| λ_3 | 51.432 | 55.878 | 60.352 | 64.780 | 69.139 | 73.423 |
| CFSC λ_1 | 17.551 | 18.901 | 20.269 | 21.630 | 22.977 | 24.306 |
| λ_2 | 36.032 | 39.501 | 43.026 | 46.544 | 50.030 | 53.476 |
| λ_3 | 51.843 | 56.336 | 60.860 | 65.340 | 69.751 | 74.088 |
| CFCF λ_1 | 22.200 | 24.204 | 26.250 | 28.288 | 30.300 | 32.281 |
| λ_2 | 26.478 | 28.880 | 31.339 | 33.796 | 36.231 | 38.636 |
| λ_3 | 43.630 | 47.656 | 51.742 | 55.814 | 59.845 | 63.826 |
| CFCS λ_1 | 23.402 | 25.578 | 27.680 | 29.835 | 31.965 | 34.062 |
| λ_2 | 35.592 | 38.863 | 42.178 | 45.476 | 48.735 | 51.949 |
| λ_3 | 62.940 | 68.716 | 74.558 | 80.349 | 86.048 | 91.647 |

| | | | | | | |
|------------------|--------|--------|--------|--------|--------|--------|
| CFCC λ_1 | 23.949 | 26.119 | 28.335 | 30.545 | 32.730 | 34.884 |
| λ_2 | 40.019 | 43.713 | 47.442 | 51.143 | 54.796 | 58.396 |
| λ_3 | 63.282 | 69.093 | 74.970 | 80.797 | 86.534 | 92.173 |
| CSFS λ_1 | 12.687 | 14.112 | 15.622 | 17.169 | 18.731 | 20.296 |
| λ_2 | 33.065 | 36.009 | 39.029 | 42.066 | 45.092 | 48.099 |
| λ_3 | 41.707 | 47.175 | 52.787 | 58.360 | 63.821 | 69.145 |
| CSFC λ_1 | 17.551 | 19.758 | 22.081 | 24.449 | 26.825 | 29.193 |
| λ_2 | 36.032 | 39.366 | 42.801 | 46.268 | 49.738 | 53.198 |
| λ_3 | 51.843 | 58.737 | 65.704 | 72.488 | 79.002 | 85.239 |
| CSSS λ_1 | 23.646 | 25.751 | 27.890 | 30.024 | 32.141 | 34.235 |
| λ_2 | 51.675 | 56.761 | 61.837 | 66.831 | 71.726 | 76.520 |
| λ_3 | 58.646 | 63.893 | 69.188 | 74.441 | 79.622 | 84.724 |
| CSSC λ_1 | 27.055 | 29.564 | 32.108 | 34.646 | 37.161 | 39.648 |
| λ_2 | 60.539 | 66.125 | 71.658 | 77.144 | 82.555 | 87.884 |
| λ_3 | 60.791 | 66.676 | 72.609 | 78.419 | 84.087 | 89.617 |
| CSCS λ_1 | 28.951 | 31.597 | 34.279 | 36.944 | 39.572 | 42.160 |
| λ_2 | 54.744 | 59.828 | 64.883 | 69.850 | 74.713 | 79.476 |
| λ_3 | 69.327 | 75.713 | 82.167 | 88.564 | 94.865 | 101.06 |
| CSCC λ_1 | 31.827 | 34.752 | 37.705 | 40.635 | 43.523 | 46.365 |
| λ_2 | 63.333 | 69.224 | 75.041 | 80.728 | 86.275 | 91.692 |
| λ_3 | 71.079 | 77.635 | 84.262 | 90.835 | 97.312 | 103.68 |
| CCFC λ_1 | 23.949 | 27.145 | 30.478 | 33.839 | 37.176 | 40.468 |
| λ_2 | 40.019 | 43.869 | 47.866 | 51.937 | 56.046 | 60.178 |
| λ_3 | 63.282 | 71.741 | 80.092 | 87.989 | 95.370 | 102.30 |
| CCSC λ_1 | 31.827 | 34.885 | 37.974 | 41.047 | 44.087 | 47.089 |
| λ_2 | 63.333 | 69.094 | 74.925 | 80.729 | 86.470 | 92.138 |
| λ_3 | 71.079 | 78.125 | 85.027 | 91.716 | 98.188 | 104.46 |
| CCCC λ_1 | 35.986 | 39.313 | 42.656 | 45.964 | 49.220 | 52.423 |
| λ_2 | 73.399 | 80.180 | 86.914 | 93.414 | 99.725 | 105.86 |
| λ_3 | 73.399 | 80.225 | 87.035 | 93.839 | 100.55 | 107.15 |

After computing the first three frequencies, the mode shapes have been demonstrated by computing the eigenvectors on the corresponding frequency. These are represented in the following figures [3] and [4] for the rectangular and square plate, respectively with taper parameter is considered as 0.5. The mode shapes represent the behaviour of the plate during vibrat



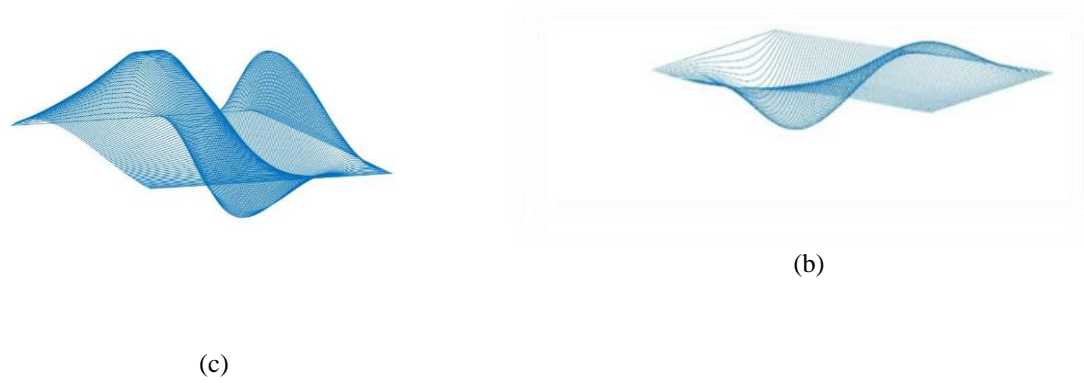


Fig. 3 First three mode shapes for rectangular clamped plate ($\alpha=0.5, \nu=0.3$)
 (a) first frequency (b) second frequency (c) third frequency

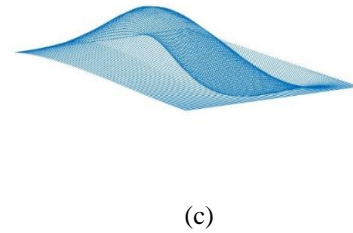
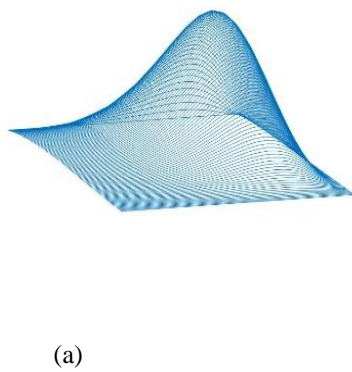


Fig. 4 First three mode shapes for square clamped plate ($\alpha=0.5, \nu=0.3$)
 (a) first frequency (b) second frequency (c) third frequency
 The computations are upto the five significant digits and computed after the study of the convergence of the result. In the Rayleigh-Ritz method, the convergence is so fast and represented in the following table [3]. The values of convergence have been obtained for the rectangular and square plate both with taper parameter as $\alpha=0.2$. It is observed from the table that the values are converging upto the five significant digits.

Table 3 Convergence of the results for rectangular and square plate

| N | Rectangular Plate ($\mu=a/b=1.0$) | | | Square plate ($\mu=a/b=1.0$) | | |
|----|-------------------------------------|-------------|-------------|--------------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| 1 | 108.19 | - | - | 39.505 | - | - |
| 2 | 107.78 | 140.17 | - | 39.332 | 81.613 | - |
| 3 | 107.78 | 140.17 | 286.36 | 39.332 | 81.534 | 81.613 |
| 4 | 107.59 | 139.40 | 200.94 | 39.328 | 80.989 | 81.534 |
| 5 | 107.59 | 139.40 | 200.94 | 39.328 | 80.989 | 81.265 |
| 6 | 107.42 | 139.40 | 200.23 | 39.328 | 80.989 | 81.265 |
| 7 | 107.42 | 139.35 | 199.04 | 39.328 | 80.365 | 81.265 |
| 8 | 107.42 | 139.35 | 199.04 | 39.328 | 80.365 | 81.083 |
| 9 | 107.42 | 139.35 | 199.04 | 39.328 | 80.239 | 81.083 |
| 10 | 107.42 | 139.35 | 199.04 | 39.328 | 80.239 | 80.266 |
| 11 | 107.42 | 139.33 | 196.31 | 39.328 | 80.214 | 80.266 |
| 12 | 107.42 | 139.33 | 196.31 | 39.328 | 80.214 | 80.266 |
| 13 | 107.41 | 139.33 | 196.19 | 39.317 | 80.211 | 80.266 |
| 14 | 107.41 | 139.33 | 196.19 | 39.317 | 80.211 | 80.263 |
| 15 | 107.41 | 139.33 | 196.19 | 39.316 | 80.211 | 80.263 |

| | | | | | | |
|----|--------|--------|--------|--------|--------|--------|
| 16 | 107.41 | 139.32 | 195.99 | 39.316 | 80.199 | 80.263 |
| 17 | 107.41 | 139.32 | 195.99 | 39.316 | 80.199 | 80.258 |
| 18 | 107.41 | 139.30 | 195.98 | 39.316 | 80.186 | 80.258 |
| 19 | 107.41 | 139.30 | 195.98 | 39.316 | 80.186 | 80.243 |
| 20 | 107.41 | 139.30 | 195.98 | 39.316 | 80.181 | 80.243 |
| 21 | 107.41 | 139.30 | 195.98 | 39.316 | 80.181 | 80.225 |
| 22 | 107.41 | 139.30 | 195.88 | 39.316 | 80.180 | 80.225 |
| 23 | 107.41 | 139.30 | 195.88 | 39.316 | 80.180 | 80.225 |
| 24 | 107.40 | 139.30 | 195.84 | 39.315 | 80.180 | 80.225 |
| 25 | 107.40 | 139.30 | 195.84 | 39.315 | 80.180 | 80.225 |
| 26 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |
| 27 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |
| 28 | 107.40 | 139.30 | 195.83 | 39.313 | 80.180 | 80.225 |

The comparisons of results have also obtained from the literature [22, 24, 27] for uniform thickness variation for the rectangular and square plate and represented in the following

Table [4] with different combinations of the boundary conditions. It is observed from the table that the available results are in excellent agreement.

Table 4 Comparisons of the Results for Rectangular and Square Plate ($\alpha=0.0$)

| Boundary Condition | Reference | Available Results | | | Present Result | | |
|--------------------|-----------|-------------------|-------------|-------------|----------------|-------------|-------------|
| | | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| Rectangular Plate | | | | | | | |
| FFFF | [12] | - | 21.789 | - | 0.0 | 21.467 | 26.585 |
| CFCF | - | - | - | - | 22.076 | 36.018 | 60.897 |
| CCFF | [24] | 17.347 | 36.449 | - | 17.142 | 36.423 | 73.516 |
| CSFF | [27] | 15.851 | - | 49.103 | 15.082 | 31.310 | 49.310 |
| SSFF | [27] | - | 27.801 | 38.079 | 9.512 | 27.522 | 38.531 |
| Square Plate | | | | | | | |
| FFFF | [22][27] | - | 13.468 | 19.596 | 0.00 | 13.469 | 19.597 |
| CCCC | [22] | 35.989 | - | - | 35.986 | 73.399 | 73.399 |
| CFCF | [22] | 22.165 | - | 43.589 | 22.200 | 26.478 | 43.630 |
| SFFF | [27] | - | 6.641 | 14.899 | 0.00 | 6.643 | 14.903 |
| SFSF | [27] | 3.470 | 8.498 | 21.281 | 3.481 | 8.521 | 21.317 |
| CCFF | [24] | 9.630 | 16.127 | 36.705 | 9.631 | 16.135 | 36.727 |
| CSFF | [24] | 6.832 | 23.418 | 26.096 | 6.930 | 23.942 | 26.598 |
| SSFF | [24] | - | 20.685 | - | 15.211 | 20.613 | 39.773 |
| | | 3.208 | 17.510 | - | 3.367 | 17.317 | 19.293 |

V. CONCLUSION

The above problem is based upon the study of the transcendental thickness variation applied on the thin plate. It is observed that the Rayleigh-Ritz method is an efficient method for obtaining the results for the first few frequencies for the thin plate. The behaviour of the plate is represented through the computations of the mode shapes. As the taper parameter controlling the thickness variation is increasing, the frequencies of the plate are increasing due to increment in the stiffness of the plate. The results have been obtained for the rectangular and square plates correct upto the five significant digits along with the convergence.

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Numerical Computation of First Three Frequencies for Circular Plate with Transcendental Thickness

Neetu Singh, Vipin Saxena

Abstract: In the present work, a very important approach Rayleigh-Ritz method has been used to compute the first few frequencies of a circular plate. The boundary conditions of circular plate are considered as a clamped and simply-supported. Different types of thickness variation of circular plate have been considered by researchers and a vast numbers of numerical results are available in the literature but none of the researchers consider the transcendental thickness variation which has been considered in the present work. The type of circular plate is considered as isotropic plate and significant numerical computations have been done for finding first three frequencies by varying the order of approximation and also the taper parameter. In special cases, results have been compared for uniform, linearly varying and transcendental thickness variations of circular plate and computed result are presented in the form of tables and graphs.

Keywords: Rayleigh-Ritz, Natural Frequencies, Transcendent Thickness, Circular Plate, Taper Parameter, Boundary Conditions.

I. NOMENCLATURE

x, y = Co-ordinate of a point on the plate
 t = Time
 w = Displacement
 ω = Angular frequency
 W = Defined by $W(x, y, t) = W(x, y) \cos \omega t$
 R = Domain occupied of the plate
 E = Young modulus of the material
 ρ = Density of the plate
 ν = Poisson ratio of the plate
 h = Thickness of circular plate
 x, y = Non-dimensional co-ordinates
 $\phi_j(x, y)$ = Basis function
 N = Order of approximation
 C_j = Constants
 a_{ij}, b_{ij} = Matrices ($N \times N$)
 h_0 = Thickness at a origin
 λ = Frequency

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α = Taper parameter

$f(x, y)$ = Function for thickness variation.

II. INTRODUCTION

The study of vibration of plates plays an important role in the design of aircraft, naval structure and also for the engineering design. The various research papers and books are available on the transverse vibrations of different types of plates, thickness variations and boundary conditions. Let us describe some of the important references related to the present work. Conway et.al [1] has obtained transverse vibration resonant frequencies of clamped tapered circular plated with taper parameter and boundary conditions. Leissa [2-8] is admirable source of information on survey vibration of different types of plates. This is one of the oldest references related of the circular plate. Celep [9] has computed and compared numerical results for use of free vibration of the circular plate by the Classical, Reissner and Mindlin theories for free vibration of two circular plates.

In the year 1980, Grossi and Laura [10] have done exhalative study for transverse vibrations of a circular plate with boundary conditions and linearly varying thickness. Gupta and Mishra [11] used Bessel function in mathematical modeling to obtain the numerical results for circular plate. All numerical results have been compared between two theories classical theories and shear theories. Leissa and Narita [12] have evaluated the natural frequencies of simply-supported circular plate with boundary conditions. Geloset.al. [13] have studied vibrations of circular plate with variable thicknesses which have been obtained by varying the thickness variation. Zhixin [14] has studied large deflection mathematics analysis of clamped circular plate with boundary conditions and in constant thickness and also consider the distribution of loads. Irie et.al. [15] have obtained stability and vibration of circular plate with uniformly varying thickness.

In the year 1986, Ficcaclenti and Laura [16] have investigated natural frequencies of circular plate of non-uniform variable thickness with rational conditions. Kim and Dickinson [17] examined the free transverse vibration of annular and circular thin sectional plates with boundary conditions. Mode shapes and frequencies have been obtained for circular and annular elastic plates computing the natural frequencies by Weisened [18]. Singh and Chakraverty [19] have used the Rayleigh-Ritz mechanism for figuring the frequencies and mode shapes with orthogonal polynomials for elliptic and circular plates.



In the year 1992 vibration of circular plate with stepped on non-homogenous elastic foundation were discussed by Wang [20]. Further Singh and Chakraverty [21] have discussed the first few natural frequencies and mode shapes of elliptic and circular plates with aspect, poisson ratio and different boundary conditions. All frequencies have been calculated by Rayleigh-Ritz method and compared with existing result in special cases. Yang [22] has investigated the vibration of circular plate with boundary condition and varying thickness and computed frequencies and mode shapes. Singh and Saxena [23-25] have used exponential, double linear thickness variation in the Rayleigh-Ritz technique for computation of frequencies, mode shapes and nodal radii with clamped and simply-supported boundary conditions. Roy [26] has computed eigen system by the use of Jacobi method. This method has investigated frequencies of different types of plates. Singh and Hassan [27] have determined numerical results by Rayleigh-Ritz approach with uniform thickness and boundary conditions. Gupta and Goyal [28] have used eigen function method classical plate theory for obtaining the numerical results of linearly tapered circular plate.

Huge quantities of research papers are available on vibration of circular plate with various types of thickness variation and different types of boundary conditions. The type of method may be numerical or approximate or analytical for different types of plates. Researchers take the thickness variation as linearly thickness, quadratic thickness, exponential thickness etc. Some important are references [29-31] and author computed natural frequencies, mode shapes for circular plate. Monterrubio [32] has solved eigen value problem and Rayleigh-Ritz approach for computation of positive and negative penalty parameter for simple condition system. Askari et. al. [33] studied hydro-elastic vibration of circular plate immersed in a liquid filled container with a free surface. Xing et. al. [34] have computed exact solution of circular cylindrical shells with classical boundary and obtained the frequencies based on Donnell-Mushtari shell theory.

In the year 2013, Baghani and Fereidoonzed [35] used the limit analysis theorem, variation iteration method on a circular plate with the simply- supported boundary condition. Wang et.al. [36] have also determined the fundamental frequency and mode shape of circular micro plates and numerical result determined by Kantorovich and shooting methods of a given problem. Szemela [37] has calculated approximation high frequencies of circular plate with clamped and simply-supported boundary conditions. Ilanko and Monterrubio [38] have discussed Rayleigh-Ritz method and applied for accurate results of different types of plates like circular, skew, rectangular, square and elliptic plates. Plaut [39] has studied large axisymmetric deflection of a circular plate by generalized Reissner theory. Abbasi et.al. [40] have also obtained semi-analytical solution for static analysis of functionally graded circular plates. This problem has been solved by two different types of methods, i.e solution method, and governing different equation method with various boundary conditions. Wang et.al [41] has computed uni-field solution of circular, annular and sector plate with general boundary conditions.

On the basis of above review of literature, it is recognized by the authors that transcendental thickness variation has not been studied by the researchers although it satisfies the clamped, simply-supported boundary conditions for circular

plate. The main objective of this work is to compute first three frequencies for circular plate with two types of boundary conditions for the transcendental thickness variation. Computed results by varying taper parameter have been compared with the existing result. Convergence of computed results has been expressed by taking significant order of approximations.

III. METHOD OF SOLUTION

Rayleigh-Ritz method is a proximate method used for computation of frequencies for different types of plates. Let us consider a circular plate which is defined in the domain R and represented in following figure1. For free vibrations of circular plate, the displacement ω at time t and point (x,y) is given below.

$$w(x, y, t) = W(x, y) \cos \omega t \tag{1}$$

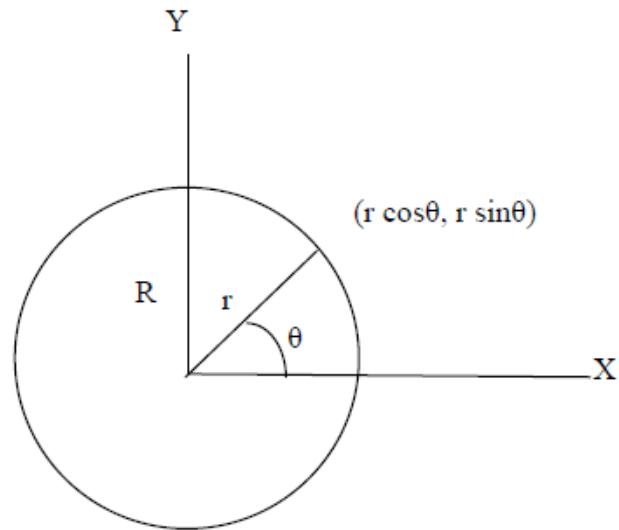


Fig. 1. Representation Circular plate

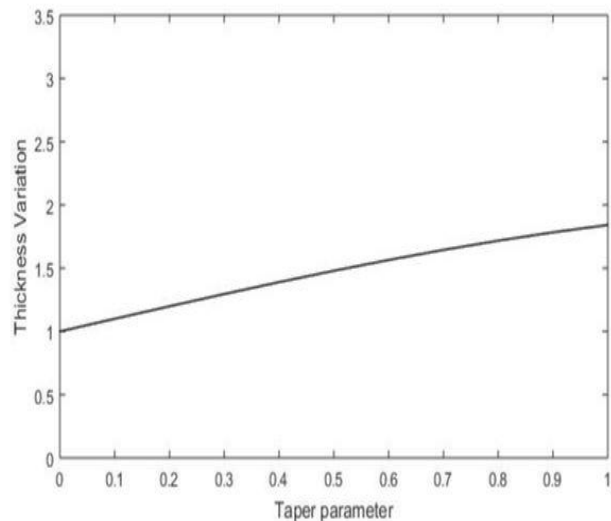


Fig. 2. Circular plate with variable thickness

Where $W(x,y)$ is the maximum displacement at point (x,y) and ω is frequency at the time t of the given plate.

Maximum strain energy and kinetic energy are presented by following equations

$$U_{\max} = \frac{1}{2} \iint_R \frac{Eh^3}{12(1-\nu^2)} \left[\left(\frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right)^2 + 2(1-\nu) \frac{d^2W}{dr^2} \left(\frac{1}{r} \frac{dW}{dr} \right) \right] r dr \quad (2)$$

$$T_{\max} = \frac{1}{2} \omega^2 \iint_R \gamma h W^2 r dr \quad (3)$$

After solving equation (2) and (3), one can obtain Rayleigh quotient which is given by

$$\omega^2 = \frac{E \iint_R \left[\left(\frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right)^2 + 2(1-\nu) \frac{d^2W}{dr^2} \left(\frac{1}{r} \frac{dW}{dr} \right) \right] r dr}{12(1-\nu^2) \iint_R \gamma h W^2 r dr} \quad (4)$$

Where r is the radial distance, γ is the distance of the material of the plate.

The main aim of the Rayleigh-Ritz approach is to optimize the Rayleigh quotient. The two types of boundary condition which are considered in the present work are described

Clamped Boundary Condition

$$\begin{aligned} W &= 0 \\ \frac{dW}{dr} &= 0 \end{aligned} \quad (5)$$

Simply-Supported Boundary Condition

$$\begin{aligned} W &= 0 \\ \frac{d^2W}{dr^2} + \left(\frac{\nu}{r} \right) \frac{dW}{dr} &= 0 \end{aligned} \quad (6)$$

Minimization of the equation (4) leads to determination of natural frequencies i.e λ . Let us consider an approximate solution with displacement ω at time t is given below.

$$W(x, y) = \sum_{j=1}^N C_j \phi_j(x, y) \quad (7)$$

Where C_j^s are the constants $\phi_j(x, y)$ are taken as basis function satisfying the boundary condition and substituting in (4) for minimization of ω^2 over the constants C_j^s , one can lead to

$$\sum_{j=1}^N (a_{ij} - \lambda^2 b_{ij}) C_j = 0 \quad (8)$$

Equation (8) can be rewritten as

$$\sum_{j=1}^N (a_{ij} - \lambda^2 b_{ij}) C_j = 0 \quad j = 1, 2, \dots, N \quad (9)$$

Where

$$a_{ij} = \iint_R f^3 \left(\phi_i \phi_j + \nu (\phi_i \phi_j + \phi_i \phi_j) + \frac{1}{\rho} \phi_i \phi_j \right) d\rho \quad (10)$$

$$b_{ij} = \iint_R f \phi_i \phi_j \rho d\rho \quad (11)$$

$$\lambda^2 = a^4 \omega^2 \rho \quad (12)$$

$$\lambda^2 = \frac{12a^2 \rho (1-\nu^2) \omega^2}{Eh_0} \quad (13)$$

f = thickness variation

Now consider the following non-dimensional variables and parameter $\rho = r/a$, $H = h/h_0$ and h_0 is the thickness of the circular plate at origin and now substituted equation (10) and (11) in equation (9) and solve above equation for λ^2 with constant C_j^s and $j=1,2,3,\dots,N$. The basis function is described below.

$$\phi_i = u^{s+i-1} \quad i = 1, 2, \dots, N \quad (14)$$

$$\phi_i = (1 - \rho^2)^{s+i-1} \quad (15)$$

Where $S=1$ leads to frequencies related to simply supported and $S=2$ is for the clamped circular plate. Transcendental thickness variation is defined by

$$f(\rho) = 1 + \sin \alpha \rho \quad (16)$$

Where f is considered as one parameter and ρ controls radius of the circular plate and S is taken as taper parameter. Solution of a_{ij} and b_{ij} through generalized Jacobi method contains (17) the following type of integrals.

$$\iint_R \sin^m x \cos^n x dx = \frac{\frac{m+1}{2} \frac{n+1}{2}}{2 \frac{m+n+2}{2}} \quad (17)$$

IV. NUMERICAL RESULTS AND DISCUSSION

For computation of results, different parameters have been considered and discussed below in brief:

- 1) Taper parameter which is controlling the entire thickness variation of circular plate is considered from 0 to 1 at the interval difference of 0.1.
- 2) The value of poisson ratio is fixed for isotropic plate and taken as 0.3.
- 3) Approximation order N is taken from 1 to 10 for fast convergence of results.
- 4) The parameters which are controlling the different types of boundary conditions of plate, taken as 2 for clamped circular plate and 1 for simply-supported circular plates.
- 5) All the involved integral in computation are in closed form and computed by equation [17] by the use of Rayleigh-Ritz method, an eigen value problem has been solved by generalized Jacobi method. The first three frequencies are represented in table [1] for the circular plate on different boundary conditions of plate. Similar results are defined in the figure [3] and [4] for simply-supported and clamped plates respectively.

In all cases, it is observed that due to increase of taper parameter, the frequencies are increasing. This is because of stiffness of circular plate is increasing by the increasing of taper parameter. When taper parameter is zero then it leads to results related to the uniform thickness variation of circular plates.

All the computed results have been compared up-to the level of five significant digits. These calculated results ensure the convergence of results by varying order of approximate i.e. N and represented below in table [2]. From the table, it is found that results are matching upto five significant digits. In the case of uniform thickness variation the computed results are analyzed with available results and found that results are matching with the existing results.

Table I. First Three Frequencies for clamped and simply-supported circular plate with transcendental thickness variation

| Tape-r Para-mete r | Simple-Support Boundary Condition (S=1) | | | Clamped Boundary Condition (S=2) | | |
|--------------------------|---|-------------|-------------|--|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| α | | | | | | |
| 0 | 4.935 1 | 29.72 | 74.15 6 | 10.21 5 | 39.77 1 | 89.10 4 |
| 0.1 | 5.633 7 | 33.92 4 | 84.64 3 | 11.67 7 | 45.41 4 | 101.7 2 |
| 0.2 | 6.263 5 | 37.70 5 | 94.06 1 | 13.02 | 50.51 7 | 113.0 8 |
| 0.3 | 6.838 5 | 41.14 | 102.6 | 14.26 8 | 55.18 | 123.4 2 |
| 0.4 | 7.367 2 | 44.27 7 | 110.4 | 15.43 6 | 59.46 2 | 132.8 7 |
| 0.5 | 7.854 7 | 47.14 7 | 117.5 1 | 16.53 2 | 63.40 3 | 141.5 3 |
| 0.6 | 8.304 8 | 49.77 3 | 124.0 1 | 17.56 2 | 67.03 | 149.4 5 |
| 0.7 | 8.720 2 | 52.17 3 | 129.9 3 | 18.53 1 | 70.36 5 | 156.7 |
| 0.8 | 9.103 | 54.36 3 | 135.3 2 | 19.44 2 | 73.42 5 | 163.3 1 |
| 0.9 | 9.455 | 56.35 7 | 140.2 2 | 20.29 8 | 76.22 8 | 169.3 3 |
| 1 | 9.777 7 | 58.17 | 144.6 5 | 21.10 2 | 78.79 | 174.7 9 |

Table II. Convergence of the Results for Circular Plate ($\alpha=0, S=1, 2$)

| Reference | λ_1 | λ_2 | λ_3 |
|----------------|-------------|-------------|-------------|
| Present Result | 10.215 | 39.771 | 89.104 |
| [10] | 10.217 | 34.94 | - |
| [12] | 13.9 | 48.48 | 102.77 |
| [13] | 10.21 | - | - |
| [19] | 10.215 | 34.877 | - |
| [20] | 10.26 | - | - |
| [23] | 10.216 | - | - |
| [24] | 10.216 | 39.771 | 89.104 |
| [25] | 10.216 | 39.771 | 89.104 |
| [29] | 10.216 | 34.877 | - |

Table III. Comparisons of the Results for Uniform Circular Plate

| N | Simply-Supported Circular Plate (S=1) | | | Clamped Circular Plate (S=2) | | |
|----|--|-------------|-------------|---------------------------------|-------------|-------------|
| | λ_1 | λ_2 | λ_3 | λ_1 | λ_2 | λ_3 |
| 1 | 5.5856 | - | - | 10.327 | - | - |
| 2 | 4.9405 | 46.706 | - | 10.217 | 43.058 | - |
| 3 | 4.9351 | 30.503 | 163.23 | 10.215 | 39.921 | 109.59 |
| 4 | 4.9351 | 29.736 | 81.896 | 10.215 | 39.773 | 91.157 |
| 5 | 4.9351 | 29.72 | 74.683 | 10.215 | 39.771 | 89.203 |
| 6 | 4.9351 | 29.72 | 74.171 | 10.215 | 39.771 | 89.106 |
| 7 | 4.9351 | 29.72 | 74.156 | 10.215 | 39.771 | 89.104 |
| 8 | 4.9351 | 29.72 | 74.156 | 10.215 | 39.771 | 89.104 |
| 9 | 4.9351 | 29.72 | 74.156 | 10.215 | 39.771 | 89.104 |
| 10 | 4.9351 | 29.72 | 74.156 | 10.215 | 39.771 | 89.104 |

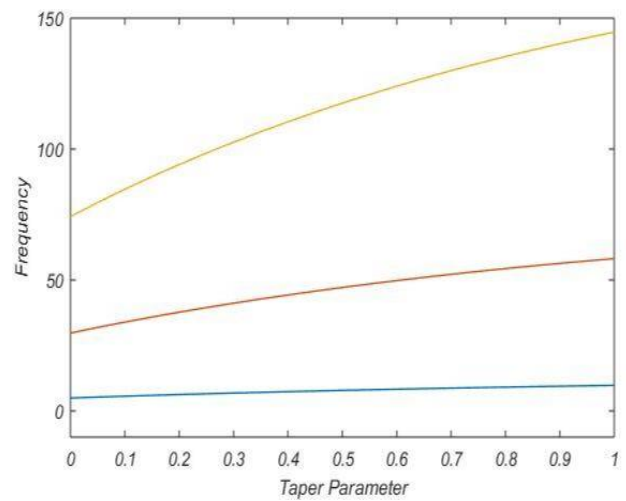


Fig.3. First three frequencies for simply-supported circular plate(S=1)

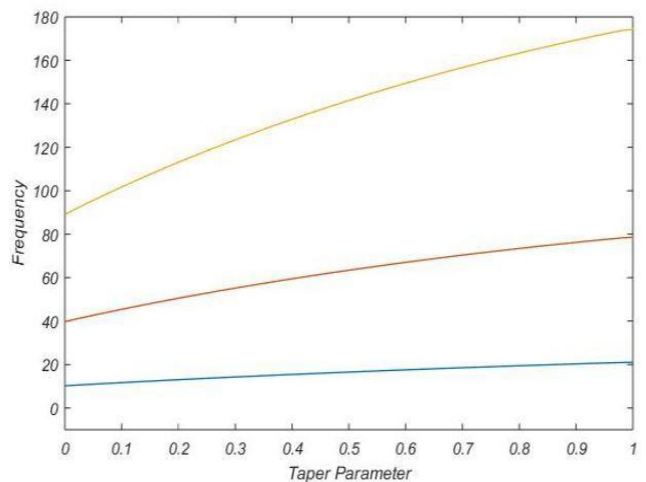


Fig. 4. First three frequencies for clamped circular plate (S=2)

V. CONCLUSION

The current problem investigated the natural frequencies through well known Rayleigh-Ritz technique with two boundary conditions and one taper parameter on the circular plate. When the taper parameter α is increasing from zero to one at the interval (0.1), then frequencies are also increasing of circular plate. The natural frequency has been computed for circular plate upto the five significant digits. Comparison with available result in special cases confirms that accuracy is comparable with best known result.

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Numerical Experiments on Skew Plate with Linear and Exponential Thickness Variations

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Abstract—In the present paper, the Rayleigh-Ritz technique has been employed to compute the transverse vibration of a skew plate with two boundary conditions and two different thickness variations. Clamped and simply-supported boundary conditions have been considered used for computation of the first three frequencies of the two-dimensional plate. The entire boundary of the skew plate is either clamped or simply-supported or two sides are connected or rest two sides are supported. Eigen-value problem is solved by Jacobi method using aspect ratio, taper parameter, and skew angle, the exponential and linear thickness of variation in one direction i.e towards x-axis. All results are presented in the form of a table and frequencies behavior are represented through graphs. The combination of CCCC, SSSS, CSCS has been obtained with an increase in taper parameter. Convergence table is ensured by taking approximation results up to five significant and the comparison table for fixed skew angle, taper parameter, and aspect ratio of the skew plate.

Keywords— Rayleigh-Ritz technique, skew plate, frequencies, boundary conditions, skew angle, taper parameter, convergence table.

I. INTRODUCTION

The study of transverse vibration of skew plates of an exponential and linear variable thickness has been comprehensively studied and a large number of research papers and material are available in this area and some vital research paper are presented Leissa [7-10] is an excellent source of information on the vibration of plates. Dokainish and Kumar [8] have used small deflection theory to obtain the first two natural frequencies of thin orthotropic skew plate for the clamped boundary condition with aspect ratio, skew angle and taper parameter. Nair and Durvasula [9] have given natural frequencies and modes for the skew plate by using Ritz method to consider simply-supported and clamped boundary conditions. Mizusawa [10] has also used a well-known Rayleigh-Ritz method for skew plates with a B-spline function. Liew and Wang [11] have investigated vibrating skew plate with internal line supports by using pb-2 Rayleigh-Ritz method with skew angles and aspect ratios and defined Ritz function. Liew et al. [12] have studied thick skew plates by the pb-2 Rayleigh-Ritz method, which depends on shear deformation plate theory and employed pb-2Ritz polynomial function of two dimensional and other essential functions. Chen et al. [13] have computed upper and lower bounds of the eigenvalue of vibration systems by the Perturbation method with interval parameters. Singh and Chakraverty [14] have also demonstrated the first five natural frequencies of skew plates using the Rayleigh-Ritz method, characteristic orthogonal polynomial and essential boundary conditions. Mathias [15] computed accurate eigenvalues and eigenvectors with the first tridiagonal matrix by Jacobi methods. Singh and Saxena [16] have studied transverse vibrations of skew plates by the Rayleigh-Ritz method with boundary conditions of four edge and linear variable thickness. In 1999, Zithan [17] has used the Rayleigh-Ritz method for rectangular and skew plates of B-Spline trial functions. Reddy and Palaninothan [18] obtained the fundamental frequencies using the consistent mass matrix and transformed element matrix with a simply-supported and clamped of antisymmetric angle-ply, skew laminates and skew angle. Woo, et al. [19] have determined natural frequencies and mode shapes of skew mindlin plates by P-version Finite Element method with skew angle, aspect ratio and the thickness-width ratio of skew plates with and without cuts. Casimir et al. [20] studied Gorman's method and the Finite element method to obtain a dynamic stiffness matrix. In 2006, Dey and Singha [21] studied the finite element approach and Bolotin's method of simply-supported laminated composite skew plates with skew angle thickness to span ratio. Malekzadeh and Karami [22] have investigated different quadrature non-linear analyses of skew composite plates by using first-order shear deformation theory with skew angle and thickness. Zhou et. al.

[23] used the Chebyshev Ritz method on a thick skew plate with three-dimensional linear and small strain elasticity theories with modes as computed with antisymmetric and symmetric of thickness direction. Gao [24] has calculated natural frequency and mode shapes of analysis of structures by Random Factor Method (REM) and Interval Factor Method (IFM) and used Random Factor Method for structural stiffness and mass matrix and Internal factor method for computing lower and upper bounds natural frequency.

In 2007, Malekzaden and Fiouz [25] discussed differential quadrature approaches of thin orthotropic skew plates for extensive deformation analysis of the Differential Quadrature (DQ) method. Singha and Daripa [26] studied the Finite element method and the Galerkin method of laminated composite skew plates. The finite element method solved large amplitude free flexural vibration and the second non-linear matrix amplitude equation solved by the Galerkin method. Lessia [27] has obtained the natural frequency and mode shape of different plates by Rayleigh and Ritz. Zhou and Zheng [28] have also investigated the vibration of the skew plate with the least square Ritz method and large skew angle. Gurses, et.al. [29] have used the Discrete Singular Convolution (DSC) form of a symmetrically laminated skew plate for frequency with skew angle and different geometric parameters and boundaries condition obtained computational space by using geometric transformation and four-node elements. Angeli et. al. [30] have also obtained natural frequency of vibrating systems of polytopic uncertainty with Monte Carlo method or Numerical Optimization. Wang and Cheng [31] discussed Modal Analysis by Free Vibration Response Only (MAFVRO) of cantilever beam and continuous systems with multiple degrees of freedom. Kumar et. al. [32] have performed the analysis of the instability of laminated composite skew plate of different skew angles with different types of linear thickness. In 2014, Srinivasa et al. [33] have also determined the natural frequency of skew plates by QUAD8 finite element with skew angle isotropically laminated, aspect ratio, laminate sequence and fiber orientation angle. Mohazzab and Dozio [34] used the Spectral Collocation Method on isotropic skew plates with boundary conditions, aspect ratio and skew angle. Naghsh and Azhari [35] have applied the Element Free Galerkin (EFG) method for obtaining large amplitude free vibration, Lagrange Multiplier method and Orthogonal Transformation technique for geometrical nonlinearity solution of simple-support boundary condition. Weighted Residual method used for periodic solution and Direct Iteration technique, Dong et al. [36] have introduced a reverberation ray matrix of shear deformation composite laminated beam with various boundary condition, orthotropic ratios and fiber orientations, Poisson's effect.

The main aim of the present work is to compute the first three natural frequencies for skew plates with exponential and linear variable thickness in one direction, with three essential combinations of clamped and simple supported boundary conditions. The three first frequencies have been obtained in three different cases CCCC, SSSS, and CSCS. Convergence and comparison tables are also given for particular cases. All numerical results here obtained by the Rayleigh-Ritz method and all frequencies behavior have been shown through graph.

II. Background

Skew plate is a solid body and bound by four edges with fixed skew angle in two-dimension in domain R. Let us consider $\theta=90^\circ$ then skew plate is mapped into a unit square plate the figure 1.

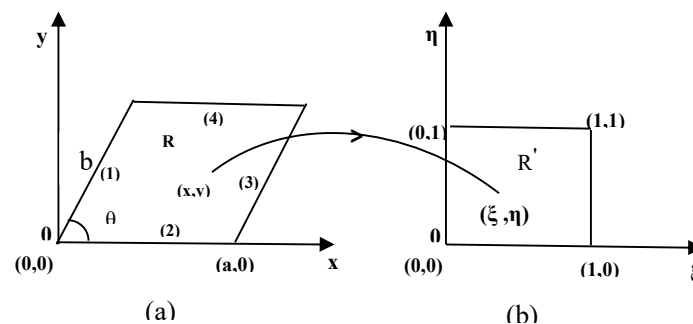


Figure 1. Representation of Skew Plate and Mapped into Square Plate

Kirchhoff plate theory is also known as classical plate theory. This theory is based on few relevant points, which are given below:

1. Rotatory inertia issues are recognized as negligible.
2. The plate is taken negligibly small of normal stresses in the transverse direction.
3. The plate of thickness is express small after the analogy of other dimensions.

4. Normal to the undeformed and deformed middle surface of remains straight normal and unstretched due to transverse shear.

All boundary conditions are based on the boundary value problem and defined below briefly:

Clamped boundary condition

This is a geometric (essential) boundary condition. In this condition, displacement and slope are both considered to be zero.

$$\omega = 0 \tag{1}$$

$$\frac{\partial \omega}{\partial \xi} = 0 \tag{2}$$

Where ω =displacement, ξ =normal to the plate

Simple-Supported boundary condition

This is a mixed type of boundary condition i.e, one geometric and other natural condition both are included. In this case, displacement and bending moment must be zero.

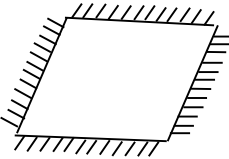
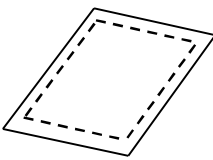
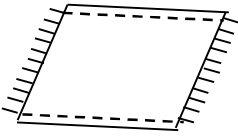
$$\omega = 0 \tag{3}$$

$$M_{\xi} = 0 \tag{4}$$

For the skew plate, the following combinations are represented in the table 1 alongwith selected combination of boundary condition in the present work.

Table 1. Representation of Boundary Conditions with Different Cases

| Case | Edge1 | Edge2 | Edge 3 | Edge4 |
|------|-------|-------|--------|-------|
| 1 | C | C | C | S |
| 2 | S | S | S | S |
| 3 | C | S | C | S |

| | |
|------|--|
| CCCC |  |
| SSSS |  |
| CSCS |  |

III. Mathematical Formulation

Rayleigh-Ritz method is based on the Rayleigh method and both are used for continuous systems. The Rayleigh technique is used for a single function and Rayleigh-Ritz considers the linear combination of several functions. The selected function must satisfy the boundary conditions in domain R. The plate is defined by three numbers a, b, and θ , are represented in figure 1 and $\theta=90^\circ$ is a special case of skew plate transferred into the square plate in domain R' shown by following transformation:

$$\xi = ax + b(\cos \theta)y \tag{5}$$

$$\eta = b(\sin \theta)y \tag{6}$$

Where x and y are generated new-co-ordinates shown in (5) let us consider the thickness h at point (x,y) of R be given

$$h = ah_0e^{\alpha x} \tag{7}$$

In the expansion of $e^{\alpha x}$, when two terms are considered, then it is converted into the linear expansion of thickness variation as $h = ah_0(1 + \alpha x)$,

Where α =taper parameter

h_0 =non-dimensional thickness at (0,0)

The exponential and linear thickness variations are shown below in figure 2 and 3, respectively.

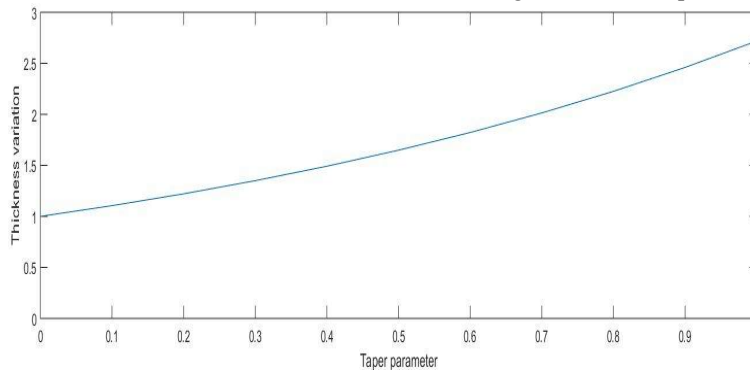


Figure 2. Skew Plates with Exponential Thickness

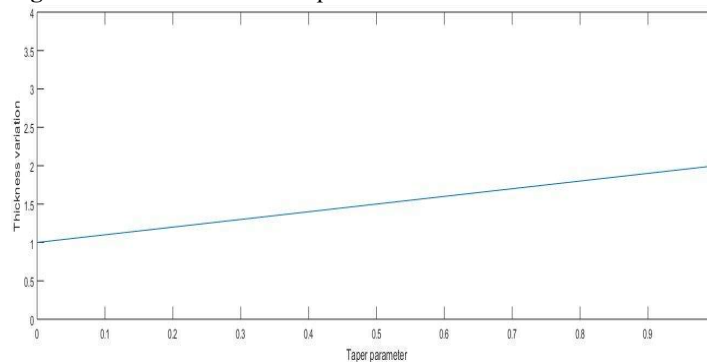


Figure 3. Skew Plates with Linear Variable Thickness

Now consider a displacement of plate is

$$w(x, y, t) = W(x, y) \sin \omega t \tag{8}$$

Where, $W(x, y)$ =Maximum displacement at time t

ω = angular frequency

Let us consider kinetic energy T and maximum kinetic T_{max} obtain by using equation (8) and both equation given below.

$$T = \frac{1}{2} \iint_R h \rho \dot{w}(x, y, t) dx dy \tag{9}$$

$$T_{\max} = \frac{1}{2} \omega^2 \iint_R \rho h W^2(x, y) dx dy \tag{10}$$

Now consider strain energy as

$$U = \frac{1}{2} \iint_R D \left[(\nabla^2 w)^2 + 2(1-\nu) \left\{ (w^{xy})^2 - w^{xx} w^{yy} \right\} \right] dx dy \tag{11}$$

Maximum strain energy may also be obtain by using equation (8) and can be written as

$$U_{\max} = \frac{1}{2} \iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ (W^{xy})^2 - W^{xx} W^{yy} \right\} \right] dx dy \tag{12}$$

Maximum kinetic and strain energies both are equal, then

$$T_{\max} = U_{\max} \tag{13}$$

$$\frac{1}{2} \omega^2 \iint_R \rho h W^2(x, y) dx dy = \frac{1}{2} \iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ (W^{xy})^2 - W^{xx} W^{yy} \right\} \right] dx dy \tag{14}$$

$$\omega^2 = \frac{\iint_R D \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ (W^{xy})^2 - W^{xx} W^{yy} \right\} \right] dx dy}{\iint_R \rho h W^2(x, y) dx dy} \tag{15}$$

Equation (15) can be written as

$$\omega^2 = \frac{D \iint_R \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ (W^{xy})^2 - W^{xx} W^{yy} \right\} \right] dx dy}{\rho h \iint_R W^2(x, y) dx dy} \tag{16}$$

Where $D = \frac{E}{12\rho(1-\nu^2)}$ (17)

Now value of D is substituting in equation (16)

$$\omega^2 = \frac{E \iint_R \left[(\nabla^2 W)^2 + 2(1-\nu) \left\{ (W^{xy})^2 - W^{xx} W^{yy} \right\} \right] dx dy}{12\rho(1-\nu^2) \iint_R W^2(x, y) dx dy} \tag{18}$$

Where
 D= flexural rigidity
 E= young modulus
 ν =Poisson ratio
 ρ =density of skew plate

Let us consider the N-term approximation.

$$W(x, y) = \sum_{j=1}^N c_j \phi_j(x, y) \tag{19}$$

Consequently, c_j 's are arbitrary constants and ϕ_j basis functions, satisfying the essential and mixed boundary conditions. Change the variable (x, y) to (ξ, η) and minimizing Rayleigh quotient as a function of the c_j 's ($j=1, 2, 3, \dots$). Then substituting (19) in to the equation (18) and then eigenvalue problem is given by

$$\sum_{j=1}^N (a_{ij} - \lambda^2 b_{ij}) c_j = 0 \quad \text{i.e.} \quad i = 1, 2, 3, \dots, N \tag{20}$$

Where

$$a_{ij} = \frac{1}{\sin^4(\theta)} \iint_R f^3 \left[\begin{aligned} &\phi_i^{xx} \phi_j^{xx} - 2\mu \cos(\theta) (\phi_i^{xy} \phi_j^{xx} + \phi_i^{xx} \phi_j^{xy}) + \mu^2 (\nu \sin^2(\theta) + \cos^2(\theta)) (\phi_i^{yy} \phi_j^{xx} + \phi_i^{xx} \phi_j^{yy}) \\ &+ 2\mu^2 (1 + \cos^2(\theta) - \nu \sin^2(\theta)) \phi_i^{xy} \phi_j^{xy} - 2\mu^3 \cos(\theta) (\phi_i^{yy} \phi_j^{xy} + \phi_i^{xy} \phi_j^{yy}) + \mu^4 \phi_i^{yy} \phi_j^{yy} \end{aligned} \right] dx dy \tag{21}$$

$$A_1 = \frac{-2\mu \cos(\theta)}{\sin^4(\theta)} \tag{22}$$

$$A_2 = \frac{\mu^2 (\nu \sin^2(\theta) + \cos^2(\theta))}{\sin^4(\theta)} \tag{23}$$

$$A_3 = \frac{2\mu^2 (1 + \cos^2(\theta) - \nu \sin^2(\theta))}{\sin^4(\theta)} \tag{24}$$

$$A_4 = \frac{-2\mu^3 \cos(\theta)}{\sin^4(\theta)} \tag{25}$$

$$A_5 = \frac{\mu^4}{\sin^4(\theta)} \tag{26}$$

$$\lambda^2 = \frac{a^4 \omega^2 \rho h}{D} = \frac{12a^4 \omega^2 \rho (1 - \nu^2)}{Eh_0^2} \tag{27}$$

$$\mu = a/b \tag{28}$$

where

$$a_{ij} = \iint_R f^3 \left[\phi_i^{xx} \phi_j^{xx} + A_1 (\phi_i^{xy} \phi_j^{xx} + \phi_i^{xx} \phi_j^{xy}) + A_2 (\phi_i^{yy} \phi_j^{xx} + \phi_i^{xx} \phi_j^{yy}) + A_3 \phi_i^{xy} \phi_j^{xy} + A_4 (\phi_i^{yy} \phi_j^{xy} + \phi_i^{xy} \phi_j^{yy}) + A_5 \phi_i^{yy} \phi_j^{yy} \right] dx dy \tag{29}$$

$$b_{ij} = \iint_R f \phi_i \phi_j dx dy \tag{30}$$

Let us consider a basis function $\phi_i(x, y)$ that satisfied all essential boundaries conditions

$$\phi_i(x, y) = x^p y^q (1-x)^r (1-y)^s \quad (1, x, y, x^2, xy, y^2, \dots) \tag{31}$$

$$f(x, y) = e^{\alpha x} \tag{32}$$

Now f is considered as exponential function with one parameter (α) and skew angle of the skew plate.

Now putting f and ϕ_i in equations (29) and (30) into calculating a_{ij}, b_{ij} expressions for the integration by Jacobi Method with integration involved as.

$$\iint_R x^p y^q (1-x)^r (1-y)^s dx dy = \frac{p!q!r!s!}{(p+r+1)!(q+s+1)!} \tag{33}$$

IV. Numerical Results and Discussion

Numerical results for computation of various parameters have been discussed and given below;

1. The parameter of aspect ratio $\mu = a/b$ ether 1.0 or 2.0;
2. The approximation of matrix order N is taken 1 to 28 for convergence up to 5 significant digits of all value of the parameter;
3. The value of Poisson's ratio has been fixed as 0.3 for all calculation;
4. Taper parameter α which controls the thickness of the skew plate is 0 to 1 with interval 0.1;

- Each integrals use p, q, r and s which have taken values 2 and 1 for clamped and simply-supported boundary conditions, respectively. When the skew plate is considered CCCC then 1, 2, 3, and 4 side of the plate are clamped, Similarly SSSS means 1, 2, 3, and 4 side are simply-supported and CSCS means 1, 2, 3, 4 means that sides 1 and 3 are clamped and rest are simply-supported boundary conditions;

The results are obtained for CCCC, SSSS and CSCS with poisson's ratio as 0.3, $\mu = a/b = 1.0$, presented in tables 3 and 4. All frequencies are decreasing when taper parameter α is increased from 0 to 1 with fixed skew angle 60° . When $\alpha=0$ is special case of results as uniform thickness variation of given plate. The computing results in table 3 are using exponential thickness variation while table 4 represents linear thickness variation.

Table 2. First Three Frequencies ($\mu=1.0$) with Exponential Thickness Variation

| $\theta=60^\circ$ | α | λ_1 | λ_2 | λ_3 |
|-------------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 46.108 | 81.602 | 105.52 |
| | 0.1 | 42.720 | 76.109 | 99.482 |
| | 0.2 | 41.465 | 74.513 | 97.869 |
| | 0.3 | 40.763 | 73.600 | 96.974 |
| | 0.4 | 40.301 | 72.971 | 96.377 |
| | 0.5 | 39.966 | 72.497 | 95.940 |
| | 0.6 | 39.710 | 72.120 | 95.603 |
| | 0.7 | 39.505 | 71.810 | 95.332 |
| | 0.8 | 39.338 | 71.548 | 95.109 |
| | 0.9 | 39.197 | 71.324 | 94.921 |
| 1.0 | 39.077 | 71.128 | 94.761 | |
| SSSS | 0.0 | 25.162 | 52.659 | 72.713 |
| | 0.1 | 22.597 | 47.122 | 66.461 |
| | 0.2 | 21.463 | 45.342 | 64.751 |
| | 0.3 | 20.814 | 44.422 | 63.869 |
| | 0.4 | 20.387 | 43.838 | 63.308 |
| | 0.5 | 20.082 | 43.425 | 62.910 |
| | 0.6 | 19.852 | 43.112 | 62.609 |
| | 0.7 | 19.671 | 42.865 | 62.371 |
| | 0.8 | 19.524 | 42.664 | 62.177 |
| | 0.9 | 19.402 | 42.495 | 62.014 |
| 1.0 | 19.300 | 42.352 | 61.876 | |
| CSCS | 0.0 | 37.067 | 64.388 | 93.577 |
| | 0.1 | 32.815 | 60.394 | 84.747 |
| | 0.2 | 31.275 | 58.966 | 82.622 |
| | 0.3 | 30.411 | 58.157 | 81.432 |
| | 0.4 | 29.837 | 57.613 | 80.617 |
| | 0.5 | 29.420 | 57.214 | 80.007 |
| | 0.6 | 29.099 | 56.904 | 79.525 |
| | 0.7 | 28.841 | 56.654 | 79.130 |
| | 0.8 | 28.629 | 56.446 | 78.799 |
| | 0.9 | 28.450 | 56.270 | 78.516 |
| 1.0 | 28.297 | 56.118 | 78.271 | |

Table 3. First Three Frequencies ($\mu=1.0$) with Linear of Thickness Variation

| $\theta=600$ | α | λ_1 | λ_2 | λ_3 |
|--------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 46.108 | 81.602 | 105.52 |
| | 0.1 | 40.486 | 66.394 | 94.154 |
| | 0.2 | 36.841 | 59.923 | 88.696 |
| | 0.3 | 34.793 | 57.346 | 85.628 |
| | 0.4 | 33.566 | 55.952 | 83.857 |
| | 0.5 | 32.764 | 55.046 | 82.715 |
| | 0.6 | 32.203 | 54.394 | 81.919 |
| | 0.7 | 31.789 | 53.896 | 81.333 |
| | 0.8 | 31.472 | 53.501 | 80.885 |
| | 0.9 | 31.222 | 53.182 | 80.533 |
| SSSS | 0.0 | 30.983 | 73.756 | 91.796 |
| | 0.1 | 26.427 | 63.309 | 82.133 |
| | 0.2 | 22.974 | 63.991 | 80.925 |
| | 0.3 | 20.790 | 67.861 | 82.626 |
| | 0.4 | 19.444 | 70.861 | 86.854 |
| | 0.5 | 18.588 | 72.434 | 92.875 |
| | 0.6 | 18.020 | 73.226 | 99.604 |
| | 0.7 | 17.626 | 73.669 | 106.45 |
| | 0.8 | 17.340 | 73.943 | 113.20 |
| | 0.9 | 17.127 | 74.127 | 119.75 |
| CSCS | 0.0 | 37.067 | 64.388 | 93.577 |
| | 0.1 | 29.676 | 56.662 | 74.941 |
| | 0.2 | 25.531 | 52.582 | 69.553 |
| | 0.3 | 23.198 | 50.403 | 67.054 |
| | 0.4 | 21.717 | 49.013 | 65.527 |
| | 0.5 | 20.690 | 48.026 | 64.461 |
| | 0.6 | 19.932 | 47.277 | 63.663 |
| | 0.7 | 19.347 | 46.683 | 63.037 |
| | 0.8 | 18.881 | 46.198 | 62.534 |
| | 0.9 | 18.500 | 45.792 | 62.119 |
| 1.0 | 18.183 | 45.447 | 61.773 | |

Similarly results of table 5 and 6 include results for CCCC, SSSS, CSCS with aspect ratio $\mu = \frac{a}{b} = 2.0$. The behavior of frequencies is the same as in tables 2 and 3.

Table 4. First Three Frequencies ($\mu=2.0$) with Exponential Thickness Variation

| $\theta=600$ | α | λ_1 | λ_2 | λ_3 |
|--------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 128.80 | 159.72 | 213.68 |
| | 0.1 | 125.91 | 146.26 | 186.97 |
| | 0.2 | 123.49 | 142.74 | 182.34 |
| | 0.3 | 122.10 | 141.35 | 180.45 |
| | 0.4 | 121.25 | 140.52 | 179.34 |
| | 0.5 | 120.68 | 139.92 | 178.60 |
| | 0.6 | 120.27 | 139.46 | 178.05 |
| | 0.7 | 119.97 | 139.08 | 177.63 |
| | 0.8 | 119.73 | 138.77 | 177.29 |
| | 0.9 | 119.54 | 138.51 | 177.02 |
| | 1.0 | 119.39 | 138.29 | 176.80 |
| SSSS | 0.0 | 64.891 | 126.46 | 227.12 |
| | 0.1 | 60.798 | 106.26 | 209.25 |
| | 0.2 | 56.928 | 96.241 | 242.61 |
| | 0.3 | 54.073 | 90.840 | 264.68 |
| | 0.4 | 51.905 | 87.691 | 266.02 |
| | 0.5 | 50.157 | 85.827 | 265.36 |
| | 0.6 | 48.685 | 84.771 | 264.66 |
| | 0.7 | 47.412 | 84.254 | 264.08 |
| | 0.8 | 46.290 | 84.110 | 263.61 |
| | 0.9 | 45.288 | 84.232 | 263.24 |
| | 1.0 | 44.386 | 84.547 | 262.94 |
| CSCS | 0.0 | 69.927 | 114.45 | 177.00 |
| | 0.1 | 63.939 | 96.270 | 147.70 |
| | 0.2 | 59.854 | 91.204 | 142.06 |
| | 0.3 | 57.533 | 89.020 | 139.55 |
| | 0.4 | 56.089 | 87.667 | 137.96 |
| | 0.5 | 55.111 | 86.686 | 136.82 |
| | 0.6 | 54.406 | 85.920 | 135.95 |
| | 0.7 | 53.874 | 85.298 | 135.25 |
| | 0.8 | 53.458 | 84.780 | 134.67 |
| | 0.9 | 53.124 | 84.342 | 134.19 |
| | 1.0 | 52.849 | 83.966 | 133.78 |

Table 5. First Three Frequencies ($\mu=2.0$) with Linear of Thickness Variation

| $\theta=600$ | α | λ_1 | λ_2 | λ_3 |
|--------------|----------|-------------|-------------|-------------|
| CCCC | 0.0 | 128.80 | 159.72 | 213.68 |
| | 0.1 | 126.80 | 153.59 | 202.67 |
| | 0.2 | 125.93 | 152.17 | 200.77 |
| | 0.3 | 125.44 | 151.37 | 199.70 |
| | 0.4 | 125.12 | 150.81 | 198.95 |
| | 0.5 | 124.89 | 150.40 | 198.37 |
| | 0.6 | 124.72 | 150.06 | 197.89 |
| | 0.7 | 124.58 | 149.79 | 197.50 |
| | 0.8 | 124.47 | 149.56 | 197.17 |
| | 0.9 | 124.37 | 149.37 | 196.88 |
| SSSS | 0.0 | 63.859 | 96.553 | 147.96 |
| | 0.1 | 61.025 | 88.516 | 133.76 |
| | 0.2 | 59.698 | 86.587 | 131.20 |
| | 0.3 | 58.940 | 85.652 | 130.07 |
| | 0.4 | 58.443 | 85.064 | 129.39 |
| | 0.5 | 58.089 | 84.642 | 128.91 |
| | 0.6 | 57.821 | 84.318 | 128.56 |
| | 0.7 | 57.611 | 84.057 | 128.27 |
| | 0.8 | 57.441 | 83.840 | 128.03 |
| | 0.9 | 57.301 | 83.656 | 127.83 |
| CSCS | 0.0 | 69.927 | 114.45 | 177.00 |
| | 0.1 | 66.469 | 106.40 | 165.02 |
| | 0.2 | 65.170 | 104.50 | 162.79 |
| | 0.3 | 64.455 | 103.44 | 161.52 |
| | 0.4 | 63.989 | 102.70 | 160.62 |
| | 0.5 | 63.656 | 102.15 | 159.93 |
| | 0.6 | 63.402 | 101.71 | 159.37 |
| | 0.7 | 63.202 | 101.34 | 158.89 |
| | 0.8 | 63.038 | 101.04 | 158.49 |
| | 0.9 | 62.901 | 100.78 | 158.14 |
| 1.0 | 62.785 | 100.55 | 157.83 | |

Table 6 represents the convergence of result upto five significant digits and the order of approximation N is 1 to 28 for finding the accuracy of the results. Comparison table 7 is also prepared for the special case of uniform thickness variation in which $\alpha=0$. It is observed that the presented results are in close agreement with the available results.

Table 6. Convergence of the Results ($\alpha=0.5, \theta=600, \mu=2.0$)

| Boundary Condition | N | λ_1 | λ_2 | λ_3 |
|--------------------|---|-------------|-------------|-------------|
| CCCC | 1 | 130.57 | 0 | 0 |
| | 2 | 128.25 | 181.67 | 0 |
| | 3 | 127.94 | 175.52 | 351.58 |
| | 4 | 126.29 | 158.12 | 281.75 |
| | 5 | 122.55 | 153.16 | 271.08 |

| | | | | |
|--|----|--------|--------|--------|
| | 6 | 121.74 | 153.03 | 270.51 |
| | 7 | 121.74 | 148.87 | 214.90 |
| | 8 | 121.64 | 143.54 | 197.36 |
| | 9 | 121.63 | 142.36 | 196.35 |
| | 10 | 121.63 | 142.30 | 196.31 |
| | 11 | 121.57 | 142.29 | 190.68 |
| | 12 | 120.97 | 141.69 | 187.15 |
| | 13 | 120.92 | 141.31 | 185.79 |
| | 14 | 120.91 | 141.31 | 185.59 |
| | 15 | 120.91 | 141.31 | 185.59 |
| | 16 | 120.91 | 141.26 | 185.51 |
| | 17 | 120.91 | 140.23 | 181.66 |
| | 18 | 120.84 | 140.08 | 180.08 |
| | 19 | 120.84 | 140.08 | 180.05 |
| | 20 | 120.84 | 140.07 | 180.05 |
| | 21 | 120.84 | 140.07 | 180.05 |
| | 22 | 120.84 | 140.07 | 180.04 |
| | 23 | 120.79 | 139.96 | 178.73 |
| | 24 | 120.69 | 139.96 | 178.60 |
| | 25 | 120.68 | 139.92 | 178.60 |
| | 26 | 120.68 | 139.92 | 178.60 |
| | 27 | 120.68 | 139.92 | 178.60 |
| | 28 | 120.68 | 139.92 | 178.60 |

Table 7. Comparison of Results for Skew Plate ($\alpha=0.0$)

| Case | Reference | λ_1 | λ_2 | λ_3 |
|-------------------------------------|-----------|-------------|-------------|-------------|
| $\theta=60^\circ, \mu=1.0$ CCCC, | Present | 46.108 | 81.602 | 105.52 |
| | [11] | 46.490 | 69.930 | - |
| | [16] | 46.166 | 81.602 | 105.52 |
| $\theta=60^\circ, \mu=1.0$ SSSS | Present | 25.162 | 52.659 | 72.713 |
| | [11] | 25.310 | 53.320 | - |
| | [16] | 25.314 | 52.660 | 72.714 |
| | [35] | - | 52.389 | 80.776 |
| $\theta=60^\circ, \mu=1.0$ CSCS | Present | 37.067 | 64.388 | 93.577 |
| | [16] | 37.193 | 64.389 | 93.577 |
| $\theta=60^\circ, \mu=2.0$ CCCC | Present | 128.80 | 159.72 | 213.68 |
| | [14] | 128.74 | 159.72 | 213.38 |
| | [16] | 128.90 | 159.73 | 215.29 |
| | [28] | 128.42 | 159.96 | - |
| $\theta=60^\circ, \mu=2.0$ SSSS | Present | 63.859 | 96.553 | 147.96 |
| | [14] | 64.069 | 96.558 | 153.76 |
| | [16] | 64.069 | 96.553 | 153.76 |
| | [28] | 61.778 | 104.96 | 147.30 |
| $\theta=60^\circ, \mu=2.0$ CSCS | Present | 69.927 | 114.45 | 177.00 |
| | [14] | 70.165 | 114.46 | 179.91 |
| | [16] | 70.166 | 114.46 | 176.91 |
| | [28] | - | 115.35 | 187.94 |

The behavior of the frequencies is also represented in the form of graphs as shown in figures 4, 5, 6 and 7. Similar behavior is also seen in the graphs.

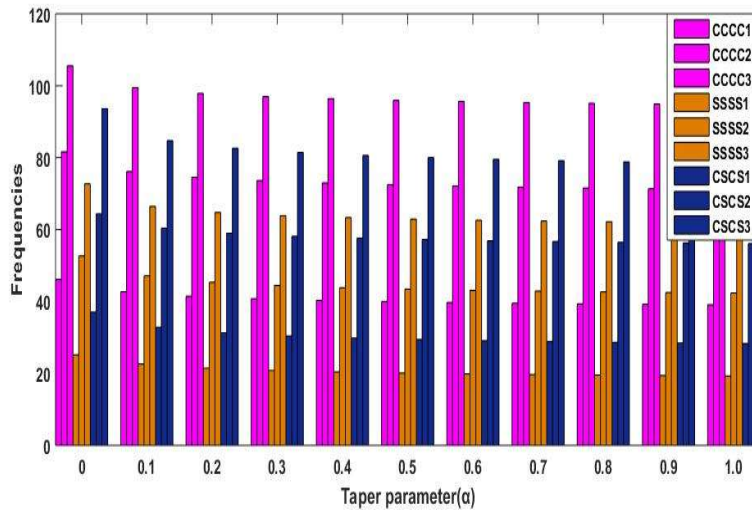


Figure 4. Representation of First Three Frequencies ($\mu = 1.0$) for Exponential Thickness Variation

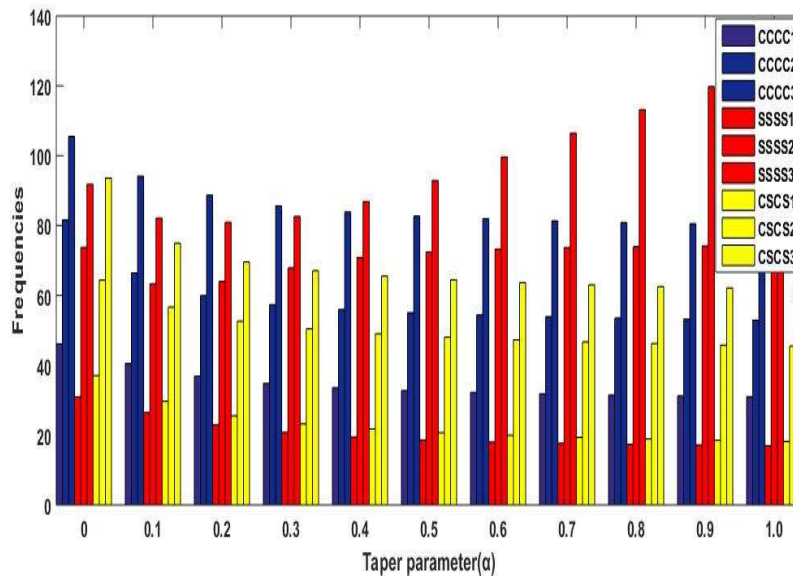


Figure 5. Representation of First Three Frequencies ($\mu = 1.0$) for Linear Thickness Variation

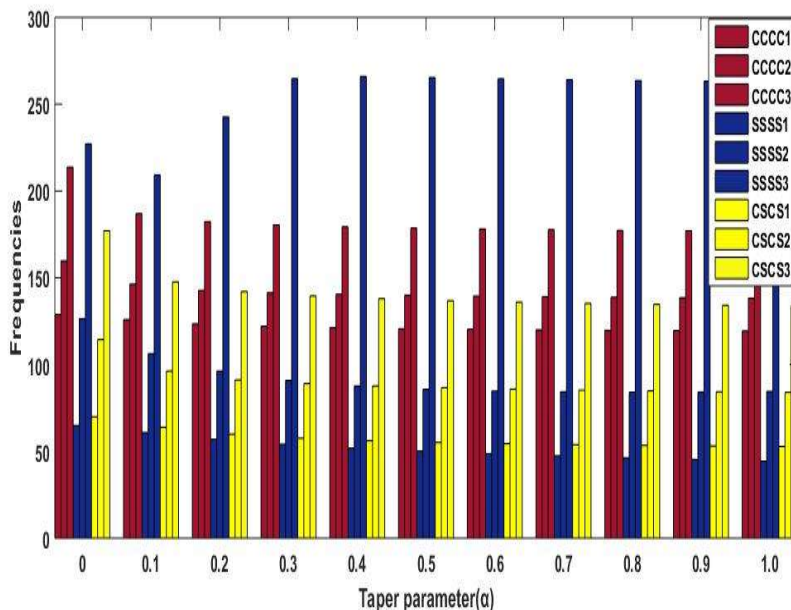


Figure 6. Representation of First Three Frequencies ($\mu = 1.0$) for Exponential Thickness Variation

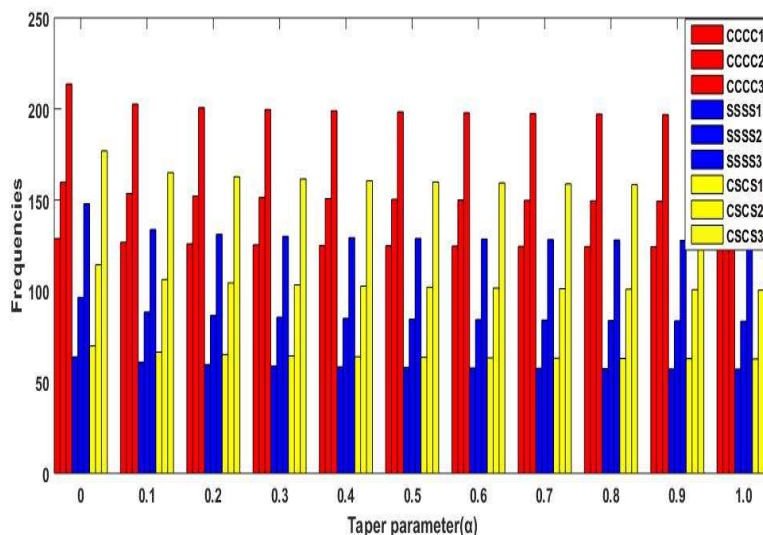


Figure 7. Representation of First Three Frequencies ($\mu = 1.0$) for Linear Thickness Variation

V. Concluding Remarks

In this work, it is found that the first three natural frequencies of skew plate by using Rayleigh-Ritz technique and skew angle 60° have been accurately computed for different combinations of boundary conditions and thickness variation of given system handled by single program. When the taper parameter for skew plate is increasing then the behaviour of first three frequencies are in decreasing form. Convergence results represents that the presented results are in excellent convergence form. In special cases of uniform thickness, the results are in close agreement with the existing results. The presented work can be extended for other combinations of boundary conditions and other categories of thickness variations.

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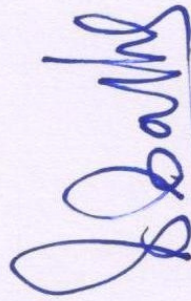
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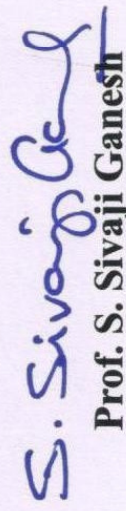
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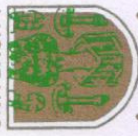
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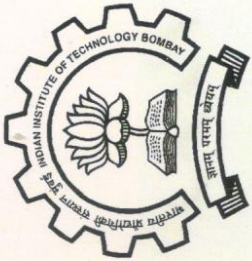
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E - Certificate

This is to certify that **Mrs. Neetu Singh** participated in the Six Days Faculty Development Programme on "**Applications of Mathematics in Diverse Fields**" from 29.06.2020 to 04.07.2020 organized by the Department of Mathematics, Anand Institute of Higher Technology, Kazhipattur, Chennai.



Dr.S.Meena

Head, Department of Mathematics



Dr.P.Suresh Mohan Kumar

Principal



Jaypee Institute of Information Technology, Noida (Declared Deemed to be University under Section 3 of UGC Act)

विद्यया तत्त्व ज्योतिषमः

Reference No.: MA/FDP/SCTA/2020/009

Certificate of Participation

This is to certify that **Neetu Singh** of **Babasaheb Bhimrao Ambedkar University Lucknow** has participated in one week online Faculty Development Programme (FDP) on **Soft Computing Techniques and their Applications (SCTA 2020)** organized by the Department of Mathematics of Jaypee Institute of Information Technology, Noida from July 13, 2020 to July 18, 2020.

Neha Singhal
18/07/2020

Dr. Neha Singhal
FDP Coordinator

Aggarwal
18/07/2020

Prof. A. K. Aggarwal
FDP Coordinator

Alka Tripathi

Prof. Alka Tripathi
Head of Department

Rupla

Prof. Hari Om Gupta
Director



Arul Anandar College (Autonomous)

Reaccredited by NAAC at 'A' Grade with a CGPA of 3.66

DST - FIST Sponsored College

Karumathur 625514, Madurai Dt.

Department of Mathematics

Participation Certificate

This is to certify that

Neetu Singh

Babasaheb Bhimrao Ambedkar University Lucknow

participated in the Online Short Term Training Programme (STTP) on

“MATLAB and MATHEMATICA for Scientific Research” organised

by the Department of Mathematics from 27.07.2020 to 29.07.2020.

Dr M Gilbert Rani

Coordinator

Dr M Sajan Joseph

Coordinator

Mr J Robert Dhiliban

Coordinator

Dr J Xavier Adaikalaraj

Convener

Rev Fr U Godwin Rufus SJ

Principal

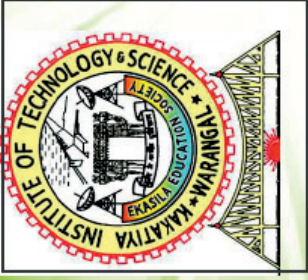
KAKATIYA INSTITUTE OF TECHNOLOGY AND SCIENCE

(An Autonomous Institute Under Kakatiya University)

Warangal - 506 015, Telangana, India.

NAAC- 'A' Grade Accredited with 3.21 CGPA

DEPARTMENT OF MATHEMATICS AND HUMANITIES



Certificate of Participation

This certificate is awarded to

Neetu Singh.

Ph.D. of

Babasaheb Bhimrao Ambedkar University Lucknow

for his/her active participation in the **One Week Online FDP on Mathematical Modeling & Numerical Techniques 2020** held from 27/7/2020 to 31/7/2020 by the Faculty of Mathematics, Department of Mathematics and Humanities,

Kakatiya Institute of Technology and Science, Warangal.

Dr. V. Anand

Asst. Prof. of Mathematics

Dr. K. Shiva Shanker

Assoc. Prof & HoD
Dept. of M & H

Dr. K. Ashoka Reddy

Principal, KITS- Warangal



BBDNIIT

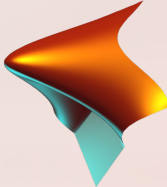
BABU BANARASI DAS

NORTHERN INDIA INSTITUTE OF TECHNOLOGY, LUCKNOW

Affiliated to Dr. A.P.J. Abdul Kalam Technical University (AKTU) College Code : 056)

Approved by All India Council of Technical Education(AICTE)

NATIONAL LEVEL FACULTY DEVELOPMENT PROGRAMME ON



**"MATLAB TOOLS & APPLICATIONS"
CERTIFICATE OF PARTICIPATION**

This is to certify that

Neetu Singh

from Babasaheb Bhimrao Ambedkar University Lucknow has successfully attended **National Level Faculty Development Programme (FDP)** on **"MATLAB TOOLS & APPLICATIONS"** organized by Department of Mathematics, Babu Banarasi Das Northern India Institute of Technology (AKTU College Code: 056), Lucknow from **August 07-08 & 10, 2020.**

Dr. Amresh Kumar
Head-Mathematics



Date: August 10, 2020
ID : OJCK5A-CE000032

Dr. V. K. Singh
Director-Engineering





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Departments of Computer Science, BCA & IQAC

Certificate of Participation

This is to certify that.....
Mrs. Neetu Singh , Research Scholar.....of

Neetu Singh

.....has participated in National

Level Webinar on **"LaTeX and its Applications"**, organized by the Departments of Computer Science, BCA & IQAC,

SBRR Mahajana First Grade College(Autonomous), Mysuru, Karnataka, India, on 19-08-2020.

Certificate No : SBRR-MFGC-IQAC-059

Dr. B.R. Jayakumari
Principal

Smt. Shruthy Poonacha
HoD, Computer Science

Sri. Manjunath K.S
HoD, BCA

