Analytical Solution of Fractional Differential Equations

ABSTRACT
of
THESIS

Submitted to
Babasaheb Bhimrao Ambedkar University
(A Central University)
Lucknow

for the Award of the Degree of

Doctor of Philosophy
in
APPLIED MATHEMATICS

Under the Supervision of
Dr. BRAJESH KUMAR SINGH

Research Scholar
ANIL KUMAR
Enrolment No. 1058/17

DEPARTMENT OF APPLIED MATHEMATICS
SCHOOL FOR PHYSICAL SCIENCES
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ABSTRACT

Almost all the phenomenons, either the chemical, economical, biological or physical are modelled in terms of ordinary or partial differential equations (ODEs/PDEs). But in the recent years, researchers realized that the phenomenons associated with the memory or hereditary properties can not be well analysed with the ODEs/PDEs because ODEs/ PDEs in terms of the classical derivatives are local in the nature. While the researcher found that the mathematical models analysed in terms of fractional partial differential equations (FPDEs) are more accurate than the classical order ODEs/PDEs due to the non-locality behavior of fractional order derivatives. Also, the fractional order derivatives are defined in terms of integration operators. So they give results over the whole range. While classical order derivative gives the results around the single point. As a result, derivatives of fractional order are the popular choices to simulate realistic physical phenomena more efficiently, and so, FDEs become popular selections in the modelling of complicated realistic phenomena marked by memory/hereditary behaviors, due to their non-local nature (i.e., upcoming-states of a phenomena depends on all its historical states including the present state) of the fractional operators. More precisely, due to local nature, the classical derivatives can describe variations only in a vicinity of a point. On contrary, the fractional derivatives being nonlocal in nature are able to describe variations in a domain/interval. Some of the realistic nonlinear physical activities modelled in terms of differential equations with fractional operators appeared in many branches of science like in modelling of transport and neutron diffusion, biological population, viscoelasticiy, optical fiber, electrochemistry,chemical kinetics, plasma physics, signal processing, fractional-order random Processes etc. Therefore, in comparison with classical derivatives, fractional derivatives carry out these activities in a decent manner. Thus, to study the behavior of the physical phenomena one requires the solutions of these FPDEs. Most of cases, the evaluation of the exact solutions of these kinds of fractional differential equations is not an easy task, and so, there is a need to introduce an analytical/numeric

approach to find a better approximation to the exact solutions for these equations. Many rigourous mathematical techniques for better approximations of these FPDEs have been introduced by many researchers, among them, some techniques are variational iteration technique, homotopy perturbation techniques, homotopy analysis transform techniques, fractional reduced differential transform techniques and many other techniques. Many of these approaches have their inbuilt demerits like evaluation of Adomian's/He's polynomials, Lagrange multipliers, diverging the results and high computational effort.

This thesis aims to present novel hybrid analytical techniques to solve the linear (nonlinear) FPDEs such as the ${}_O$HA$\mathbb{J}$TM, variational iteration $\mathbb{J}$-transform method (VI$\mathbb{J}$TM), homotopy perturbation $\mathbb{J}$-transform method (HP$\mathbb{J}$TM), conformable homotopy transform method (CHTM), CFDT method, differential $\mathbb{J}$-transform method (D$\mathbb{J}$TM). Furthermore, to test the efficiency, validity and reliability of these techniques, we implemented these on some nonlinear FPDEs and found that the proposed techniques are very efficient, accurate, reliable and powerful for finding the analytical solutions of FPDEs.

Chapter 1 This chapter is dedicated to the introduction of the work which includes the definitions and types of ODEs and PDEs, some special kinds of PDEs, history of the fractional calculus, motivation behind the fractional calculus, some analytical techniques and in last preliminaries and definitions have been provided.

Chapter 2, In the present chapter, two new hybrid efficient techniques: namely optimal homotopy analysis $\mathbb{J}$-transform method (${}_O$HA$\mathbb{J}$TM) and variational iteration $\mathbb{J}$-transform method (VI$\mathbb{J}$TM) are proposed for solving multi-dimensional space-time fractional telegraph equation

1. One-space dimensional STF-HTE

$$ {}_{\tau}\mathcal{D}_C^{2\alpha}\varphi(\sigma_1,\tau) + 2a\,{}_{\tau}\mathcal{D}_C^{\alpha}\varphi(\sigma_1,\tau) + b^2\varphi(\sigma_1,\tau) = {}_{\sigma_1}\mathcal{D}_C^{2\beta}\varphi(\sigma_1,\tau) + g(\sigma_1,\tau), \quad (0.0.1) $$

2. Two space dimensional TF-HTE

$$
{}_{\tau}\mathcal{D}_{C}^{2\alpha}\varphi(\sigma_1,\sigma_2,\tau)+2a\ {}_{\tau}\mathcal{D}_{C}^{\alpha}\varphi(\sigma_1,\sigma_2,\tau)+b^2\varphi(\sigma_1,\sigma_2,\tau)=\frac{\partial^2}{\partial\sigma_1^2}\varphi\left(\sigma_1,\sigma_2,\tau\right)
+\frac{\partial^2}{\partial\sigma_2^2}\varphi\left(\sigma_1,\sigma_2,\tau\right)+g_1(\sigma_1,\sigma_2,\tau), \tag{0.0.2}
$$

3. Three space dimensional TF-HTE

$$
{}_{\tau}\mathcal{D}_{C}^{2\alpha}\varphi+2a\ {}_{\tau}\mathcal{D}_{C}^{\alpha}\varphi+b^2\varphi=\frac{\partial^2\varphi}{\partial\sigma_1^2}+\frac{\partial^2\varphi}{\partial\sigma_2^2}+\frac{\partial^2\varphi}{\partial\sigma_3^2}+g_2(\sigma_1,\sigma_2,\sigma_3,\tau), \tag{0.0.3}
$$

where g, g_1, g_2 are the external forces, ${}_{\tau}\mathcal{D}_{C}^{\alpha}(\cdot)$ and ${}_{\sigma}\mathcal{D}_{C}^{\beta}(\cdot)$ are the time and space Caputo-FD respectively.

appearing in modelling of numerous real-world phenomena like-electrical signal-propagation, random walk, propagation of waves and so forth. The ${}_{O}$HA$\mathbb{J}$TM is developed via utilizing properties of newly developed $\mathbb{J}$-transform to optimal homotopy analysis method while VI$\mathbb{J}$TM is based upon the concept of variational iteration theory and the properties of the $\mathbb{J}$-transform. Banach fixed point theory is utilized to analyze the stability of VI$\mathbb{J}$TM. Both methods: ${}_{O}$HA$\mathbb{J}$TM and VI$\mathbb{J}$TM produces stable solutions converging to the exact solutions, which is illustrated by considering five different test examples of multi-dimensional space-time fractional telegraph equation. In addition, the computed approximate results are expressed in the compact form of Mittag-Leffler function. The numerical findings demonstrate that the developed techniques perform better for multi-dimensional space-time fractional telegraph equation, and at a fixed iteration: ${}_{O}$HA$\mathbb{J}$TM produces better accuracy as compared to VI$\mathbb{J}$TM and recently developed techniques.

Chapter 3 presents a comparative analysis of the conformable space-time fractional Fokker-Planck equation (CSTF-FPE) (0.0.4)

$$
\mathcal{T}[\varphi(\sigma,\tau)]={}_{\tau}\mathcal{D}_{\alpha}^{C}\varphi(\sigma,\tau)-\left[-{}_{\sigma}\mathcal{D}_{\beta}^{C}P(\sigma,\tau,\varphi)+{}_{\sigma}\mathcal{D}_{2\beta}^{C}Q(\sigma,\tau,\varphi)\right]\varphi(\sigma,\tau)=0,
\varphi(\sigma,0)=f(\sigma),\ \sigma,\ \tau>0, \tag{0.0.4}
$$

where $\varphi(\sigma,\tau)$ a unidentified function to be computed, α, β $(0 < \alpha, \beta \leq 1)$ are the orders of time and space conformable derivatives, respectively; $P(\sigma,\tau,\varphi)$ stand for the drift coefficient and $Q(\sigma,\tau,\varphi)$ denotes diffusion coefficient whereas f be given smooth function.

The two efficacious techniques: Fractional homotopy analysis transform method (FHATM) and conformable homotopy transform method (CHTM) have been used to solve (0.0.4). The efficaciousness and validity of the proposed techniques are represented in terms of three different examples of CSTF-FPEs by computing various error norms: L_2 and absolute error. Convergence analysis of both techniques have been studied theoretically and verified graphically as well. The findings demonstrate that for a given order of approximation CHTM requires less computational time in comparison to FHATM while FHATM is more general and converges faster than CHTM. The obtained information is confirmed graphically for different numerical values of fractional order.

Chapter 4 presents the study on the two new hybrid methods: variational iteration $\mathbb{J}$-transform technique (VI$\mathbb{J}$TM) and $\mathbb{J}$-transform method with optimal homotopy analysis ($_O$HA$\mathbb{J}$TM) for analytical assessment of space-time fractional Fokker-Planck equations (STF-FPE), appearing in many realistic physical situations, e.g., in ultraslow kinetics, Brownian motion of particles, anomalous diffusion, polymerases of Ribonucleic acid, deoxyribonucleic acid, continuous random movement and formation of wave patterns. $_O$HA$\mathbb{J}$TM is developed via optimal homotopy analysis after implementing the properties of $\mathbb{J}$-transform while (VI$\mathbb{J}$TM) is produced by implementing properties of the $\mathbb{J}$-transform and the theory of variational iteration. The Banach approach has been utilized to analyze the convergence of these methods. In addition, it is demonstrated that VI$\mathbb{J}$TM is T-stable. Computed new approximations are reported as closed-form expression of Mittag-Leffler function, and in addition, effectiveness/validity of the proposed new approximations is demonstrated via three test problems of STF-FPE by computing the error norms: L_2 and absolute errors. The numerical assessment demonstrates that the developed techniques perform better for STF-FPE and for a given iteration, and $_O$HA$\mathbb{J}$TM produces new approximations with better accuracy as compared to VI$\mathbb{J}$TM as well as the techniques developed recently. The fractional model of concerned equation can be taken in the form

$$ {}_{\tau}\mathcal{D}_C^{\alpha}\varphi(\sigma,\tau) = \mathcal{T}[\varphi(\sigma,\tau)], \qquad \varphi(\sigma,0) = g(\sigma), \quad \sigma > 0,\ \tau > 0, \tag{0.0.5} $$

where $\mathcal{T}[\varphi(\sigma,\tau)] = \left[-{}_{\sigma}\mathcal{D}_C^{\beta}P(\sigma,\tau,\varphi) + {}_{\sigma}\mathcal{D}_C^{2\beta}Q(\sigma,\tau,\varphi)\right]\varphi(\sigma,\tau)$, $\varphi(\sigma,\tau)$ a unknown function to evaluate, $\alpha,\beta(0 < \alpha,\beta \leq 1)$ denote the orders of time and space Caputo-FD respectively; $P(\sigma,\tau,\varphi)$ is the drift coefficient while $Q(\sigma,\tau,\varphi)$ the diffusion coefficient, g-smooth function, and the operator ${}_{\tau}\mathcal{D}_C^{\alpha}$ stands for Caputo-FD.

Chapter 5 deals with the study of the conformable fractional differential transform (CFDT) method to compute the numerical solution of space-time fractional Fokker-Planck equation with conformable fractional derivative. The computed results are compared with the existing results in the literature and also depicted graphically for $\alpha = \beta = 1$. The accuracy of the computed results for different values of α and $\beta = 1$ is measured in terms of L_2 error norms. The findings show that the present results agreed well with the results of various well-known methods such as adomian decomposition method (ADM), variational iteration method (VIM), fractional variational iteration method (FVIM) and fractional reduced differential transform method (FRDTM) and so forth. The proposed results converge to the exact solutions.

Chapter 6 This chapter addresses the numerical simulation of the Caputo-time fractional model of Klein-Fock Gordon (FKFG) equations which exhibit the behavior of spinless particle-like Higgs Boson. We have taken the homotopy perturbation $\mathbb{J}$-transform method(HP$\mathbb{J}$TM) for the numerical solution of the FKFG equation in form of convergent series. The convergence analysis of the adopted model has also been provided. In order to test the validity/effectiveness of HP$\mathbb{J}$TM, three different test examples of FKFG equations have been studied numerically, and the evaluated approximate behaviors are also depicted graphically. The outcomes clarify that the proposed HP$\mathbb{J}$TM is very efficient and easy to implement. The theoretical demonstration reveals that evaluated series solutions via HP$\mathbb{J}$TM are comparatively more accurate as compared to the other existing techniques and converge very fast to the exact solution.

Chapter 7, This chapter proposes an analytical new hybrid approach so-called differential $\mathbb{J}$-transform method (D$\mathbb{J}$TM) to evaluate the behavior of n-space dimensional fractional-nonlinear hyperbolic-like wave equations, where time-fractional derivative is considered in Caputo format. The D$\mathbb{J}$TM is the hybrid method in which projected differential transform is implemented after imposing recently introduced integral transform so-called $\mathbb{J}$-transform. The efficiency and applicability of the proposed D$\mathbb{J}$TM has been tested by considering three different test examples of the Caputo time-fractional nonlinear hyperbolic-like wave equations in terms of absolute error norms and the different order D$\mathbb{J}$TM solutions are compared with exact solution behaviors and the existing results, for the large time level $\tau \in [0, 10]$. In addition, the convergence analysis of D$\mathbb{J}$TM is studied theoretically and verified it numerically as well as graphically, which confirms that the numerical experiments via D$\mathbb{J}$TM for distinct fractional orders support the theoretical findings excellently, and the presented D$\mathbb{J}$TM results converge to their exact solution behavior, very fast. The evaluated series approximations are expressed in the compact form of Mittag-Leffler functions.

At the end future scope of presented work is suggested.

List of Publications

A. Published/Accepted

1. Singh B. K., **Kumar, A.**, & Gupta M. Efficient new approximations for space-time fractional multi-dimensional Telegraph equation. International Journal of Applied and Computational Mathematics. 8(5):1-36 (2022).

2. Singh, B. K., **Kumar, A.**, New approximate series solutions of conformable time–space fractional Fokker–Planck equation via two efficacious techniques. Partial Differential Equations in Applied Mathematics. 6:100451(2022).

3. Singh, B. K., **Kumar, A.**, & Gupta, M. New approximations of space-time fractional Fokker-Planck equations. Computational Methods for Differential Equations. (2022).

4. Singh, B. K., **Kumar, A.**, Rai, S. N., & Prakasha, D. G. Study of nonlinear time-fractional hyperbolic-like equations with variable coefficients via semi-analytical technique: Differential $\mathbb{J}$-transform method. International Journal of Modern Physics B, 2450001 (2023).

Book Chapters

1. Singh B. K. and **Kumar, A.**, (2018, December) Numerical study of conformable space and time fractional Fokker–Planck equation via CFDT method. In International Conference on Recent Advances in Pure and Applied Mathematics (pp. 221-233). Springer, Singapore.

2. Singh B. K. and **Kumar, A.**, "New Approximations of fractional Klein-Fock Gordon equations via Homotopy Perturbation J-transform Method" has been accepted in "2nd International Conference on Computational Sciences-Modelling, Computing and Soft Computing" conducted online by the Department of Mathematics of Manipal Institute of Technology, Manipal, India during March 28-30, 2022

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