ON SOME APPLICATIONS OF STOCHASTIC ORDERS IN RELIABILITY THEORY

THESIS

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Abstract

In the probability theory and statistics, stochastic orders work as a tool to compare the probability distributions of random variables/vectors. They can be classified into three categories: univariate, joint, and multivariate. Univariate stochastic orders are used to compare any two random variables whereas the joint stochastic orders are used to compare two components of a random vector. Multivariate stochastic orders are extensions of univariate stochastic orders and they are used to compare two random vectors of the same dimensions. The role of stochastic ordering usually arises when the measures of central tendencies and dispersion (e.g., mean, median, and variance) of two random variables are not very informative for the comparison purposes. Because such comparisons are based on only two numbers, and they may not exist in some cases.

Stochastic orders are partial orders defined on the space of distributions. Instead of comparing merely two numbers, they mostly deal with two functions to select the random variable which is bigger than another random variable according to location, magnitude, dispersion, residual lifetimes, concentration, etc. For more information about various stochastic orders, we refer the reader to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), Belzunce *et al.* (2016), and references cited therein. Also, for some newly defined stochastic orders relevant to the thesis, we refer the reader to Misra *et al.* (2020a,b).

In many different fields of probability theory and statistics, the theory of stochastic orders has been used extensively. Such areas include reliability theory, engineering, survival analysis, biological sciences, operations research, and economics. In reliability theory, operations research, engineering, and related areas, there is a need to compare lifetimes of different systems/components. Such comparisons of the lifetimes of systems are often arises in the allocation of redundant component(s) or spare(s) to systems. In biological sciences, the lifetimes (or the residual lifetimes) of two living organisms or two classes

of living organisms may be compared. For instance, comparison can be made between the class of individuals who consume a drug with the class of individuals who do not consume the drug to know the impact of that specific drug. In economics, one can compare different income distributions using stochastic orders.

In reliability engineering, one often targets to increase the system reliability by allocating some extra components (called spares or redundant components) to the system. There are generally two types of redundancies that are widely used, called active redundancy and standby redundancy. In the case of active redundancy (also known as hot redundancy), the spares are placed parallel to the system's original components and keep working with those components simultaneously. In the case of standby redundancy (also known as cold redundancy), the spares are placed on the system's original components in such a manner that a spare starts working immediately after the component to which it is attached has failed. Several researchers have studied the problem of allocating redundancies in a system to achieve optimum configurations by stochastic comparisons between the lifetimes of the resulting systems using different stochastic orders (see, for example, Boland *et al.* (1992), Shaked and Shanthikumar (1992), Singh and Misra (1994), Valdés and Zequeira (2003), Valdés and Zequeira (2006), Li and Hu (2008), Valdés *et al.* (2010), Misra *et al.* (2011a,b), Zhao *et al.* (2012), Zhao *et al.* (2016), Yan and Luo (2018), Yan *et al.* (2019), and the references cited therein).

In reliability and survival analysis, lifetime distributions play a significant role. In this thesis, we mainly consider a lifetime distribution, namely, Topp-Leone generated family of distributions proposed by Rezaei *et al.* (2017). The Topp-Leone generated family of distributions exhibit the bathtub-shaped hazard rates and can be used for lifetime modeling. For some recent developments based on this family of distributions, we refer the reader to Aryal *et al.* (2017), Sebastian *et al.* (2019), and Shekhawat and Sharma (2020).

In this thesis, we present some of the applications of stochastic orders in the field of reliability theory. The present thesis addresses following specific problems:

- We stochastically compare the lifetimes of two series and parallel systems with components having lifetimes from the Topp-Leone generated family of distributions. Also, we investigate some reliability measures or characteristics (see, for example, Barlow and Proschan (1975) and Lai and Xie (2006)) for this family of distributions.
- We present real data applications to compare the fits of different models of Topp-

Leone generated family of distributions.

- We also deal with the problems related to the allocation of active and standby redundancies in series systems.

In Chapter 1, we have provided an introduction about stochastic orders and its applications in different fields of probability and statistics. Also, we discussed some univariate and bivariate stochastic orders related to our study. We have presented an introduction about redundancy allocations and a lifetime distribution, namely, Topp-Leone generated family of distributions, and reviewed the literature related to these problems. This chapter contains some basic notation, definitions, and useful lemmas relevant to the thesis.

One of the main results of Chapter 2 provides the condition under which the reversed hazard rate function of any random variable from Topp-Leone generated family of distributions, is decreasing. Chapter 2 also includes the stochastic comparison results of two random variables from the Topp-Leone generated family of distributions. In this chapter, we also consider a particular case of this family of distributions, namely, Topp-Leone exponential distribution. We study few reliability characteristics of this distribution, such as the hazard rate function, the reversed hazard rate function, the mean residual life function, and the expected inactivity time. Renyi entropy measure for the Topp-Leone exponential distribution has also been discussed. Moreover, we define the Topp-Leone generated log-logistic distribution, and the Topp-Leone generated Lomax distribution using the base-line distributions as log-logistic and Lomax distributions, respectively.

Chapter 3 presents real data applications to discuss the importance of the Topp-Leone generated family of distributions. In this chapter, we compare the fits of the Topp-Leone generated Weibull, Topp-Leone generated log logistic, and Topp-Leone generated Lomax distributions using three real data sets. Also, we provide the applications of the Topp-Leone exponential distribution with two real data sets, and compare the fit of this distribution with the Lomax and Burr-XII distributions.

In Chapter 4, we stochastically compare the lifetimes of two series and parallel systems with components having lifetimes from the Topp-Leone generated family of distributions. We present the comparison results with heterogeneity in one parameter while another is fixed. We compare the lifetimes of two series systems with respect to the hazard rate order, and with the help of a counterexample, we show that the hazard rate order cannot be extended to the likelihood ratio order for this comparison. Also, we provide the comparison results for parallel systems with respect to the usual stochastic order and the likelihood ratio order. We show that the usual stochastic order cannot be extended to the hazard rate order using a counterexample. Moreover, we derive the results when this family of distributions has different base-line distributions. All the comparison results we study with the help of vector majorization technique.

In Chapter 5, we study the problem of allocating one active/standby redundant spare in the n-component series system using the residual stochastic precedence and the inactivity stochastic precedence orders. The comparison results based on these orders have special concern as they take care of the dependence structure between the residual lifetimes (inactivity times) of the random variables. We also compare two parallel systems in terms of the inactivity stochastic precedence order.

Chapter 6 summarizes the work done in the thesis.

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