INFERENCES ON VARIOUS LIFE TESTING MODELS USED IN RELIABILITY THEORY

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Abstract

The history of reliability field may be traced back to the early 1930s, when the probability principles were applied to electric power generation-related problems in the United States. During World War II, Germany applied the basic reliability concepts to improve reliability of their V1 and V2 rockets. Also during World War II, the United States Department of Defence recognized the need for reliability improvement of its equipments. During the period between 1945-1950, it performed various studies concerning the failure of electronic equipment, equipment maintenance and repair cost, etc.

The reliability function $R(t) = P(X > t)$ is defined as a probability of failure free operation until time 't'. Another measure of reliability under the stress-strength set up is $R = P(Y > X)$, which represents the reliability of an item subject to random stress X and random strength Y. Birnbaum (1956) was perhaps one of the first researchers who dealt with the model $P(X < Y)$ in stress-strength content. Later on developed on by Birnbaum and McCarty (1958). Church and Harris (1970) used the term stress-strength for the first time.

A lot of work has been done in the literature of reliability regarding the classical estimation of $R(t)$ and R under complete and censored samples for various lifetime distributions. For a brief review, we may refer to Kelly et al. (1976), Tong (1974), Downton (1973), Woodward and Kelly (1977), Beg and Singh (1979), Awad and Gharraf (1986), Constantine et al. (1986), Iwase (1987), Reiser and Guttman (1986), Voinov (1984), Ury and Wiggins (1979), Brownie (1988), Halperin et al. (1989) and Simonoff et al. (1986). Other authors that have worked on stress-strength reliability are Pugh (1963), Basu (1964), Johnson (1975), Sathe and Shah (1981), Chao (1982), Tyagi and Bhattacharya (1989), Chaturvedi and Rani (1997, 1998), Chaturvedi and Surinder (1999), Chaturvedi and Tomer (2002, 2003), Chaturvedi

et al. (2002), Chaturvedi and Singh (2006, 2008) and others. Recently, Alam and Roohi (2002, 2003) has studied the stress-strength problem from a different angle. They assume the form of the distribution of X and Y to be known and establish the relationship among the parameters of the distributions of stress and strength. Khan and Islam (2007, 2009) worked on stress-strength reliability relationship for Power function distribution and generalized gamma distribution. We have also worked on the same lines in Chapter 5 and Chapter 6 for Gompertz distribution and for a class of lifetime distributions, respectively.

The problem of Sequential analysis arose in the Statistical Research Group, Columbia University in March 1943. The immediate stimulus was questioned about ordinance testing addresed by G. L. Schuyler of the Bureau of Ordinance, Navy Department. It was pointed out by Milton Friedman and W. Allen Wallis that the mere notion of sequential analysis could slightly improve the efficiency of some current most powerful test. G. L. Schuyler to W. Allen Wallis, who together with Milton Friedman brought the problem to Wald's attention. It was subsequently published by Wald (1947).

Hager et al. (1971) studied the robustness and testing procedures for generalized gamma and Weibull model. Barlow and Proschan (1967) studied the robustness and estimation procedures related to exponential distribution. Dantzing (1940) proved the non existence of the fixed sample size procedure to test student's t hypothesis having a preassigned power when the variance is unknown. Some other estimation problems related to exponential distribution were studied by several authors. We may refer to Mukhopadhyay (1992), Mukhopadhyay and Hamdy (1984), Mukhopadhyay and Narayan (1981), Singh and Chaturvedi (1989), Chaturvedi et al. (1993a, b), Chaturvedi (1996), Govindarajulu and Sarkar (1991) and others.

Sequential procedures were developed by Abraham Wald (1947) in response to demand for more efficient testing of anti-aircraft gunnery during the second world war. Moreover, the subject grown considerably especially in the areas of sequential estimation and biostatistics, Sequential statistics is concerned with the treatment of data when the number of observations is not fixed in advance i.e. sample size is a random variable. The experimenter has the option of studying at a sequence of observations one or a fixed number at a time and decides whether to stop sampling and take a decision or to continue sampling and take decision sometime later. The order of sequence of observations which the experimenter will take is specified in advance.

The thesis entitled **Inferences on Various Life Testing Models used in Reliability Theory** comprises of seven Chapters in which Chapter 1 is Introductory which contains the brief discussion about various reliability measures, stress-strength models, their applications and the classical inferential procedures that are used to obtain the estimators of stress-strength reliability. It also contains the brief review of the sequential analysis, its applications and the related work done in this field.

In Chapter 2, we have considered a class of lifetime distributions proposed by Chaturvedi and Rani (1997) which is defined as

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}}{\theta^{ab}\Gamma a}\exp\left(-x^c/\theta^b\right); \quad x,\theta,a,b,c>0,$$

where θ is assumed to be unknown and a, b, c are known constants. On considering different values for a, b and c, the pdf's of different continuous distributions are obtained as specific cases. We have obtained the point and interval estimators for the stress-strength relibility when X and Y follows the class of distribution. Here, we have used the transformation method to obtain the point and interval estimates of $R = P(Y > X)$ and the following results are obtained.

(i) The MLE of $R = P(Y > X)$ is given by

$$\tilde{R} = \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon}+a_1\bar{\eta}}\right)^{a_1}\frac{1}{B(a_1,a_2)}\,{}_2F_1\left(a_1, 1-a_1; a_1; \frac{a_1\bar{\eta}}{a_2\bar{\varepsilon}+a_1\bar{\eta}}\right),$$

where, $\bar{\varepsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X(\text{say})$ and, $\bar{\eta} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y(\text{say})$.

(ii) The UMVUE of $R = P(Y > X)$ is given by

$$\hat{R} = \begin{cases} \dfrac{B\left[(n_2-1)a_2+i+1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum\limits_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \\ \times \sum\limits_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{a_1+j} & ; \text{if } T_Y < T_X \\ \text{where } 0 \le i \le a_1 - 1 < \infty \text{ and } 0 \le j \le (n_1-1)a_1 - 1 < \infty & \\ & \\ \dfrac{B\left[(n_1-1)a_1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum\limits_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \\ \times \sum\limits_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} & ; \text{if } T_X < T_Y \\ \text{where } 0 \le i \le a_2 - 1 < \infty \text{ and } 0 \le j \le (n_2-1)a_2 + i < \infty & \end{cases}$$

where, $T_X = \sum\limits_{i=1}^{n_1} X_i^{c_1}$ and, $T_Y = \sum\limits_{j=1}^{n_2} Y_j^{c_2}$.

(iii) The Confidence interval for $R = P(Y > X)$ is given by

$$P\left(I_{\left(\frac{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{1-\gamma_2}}{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{1-\gamma_2}+1}\right)}(a_1, a_2) < R < I_{\left(\frac{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{\gamma_1}}{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{\gamma_1}+1}\right)}(a_1, a_2) \right) = 1 - \gamma$$

where, $\bar{T}_X = \dfrac{1}{n_1}\sum\limits_{i=1}^{n_1} X_i^{c_1}$ and, $\bar{T}_Y = \dfrac{1}{n_2}\sum\limits_{j=1}^{n_2} Y_j^{c_2}$.

In Chapter 3, we have considered the Nakagami distribution given by M. Nakagami (1960) which is defined as

$$f(x;\lambda,\beta) = \frac{2\lambda^\lambda}{\Gamma\lambda\beta^\lambda} x^{2\lambda-1} e^{-\dfrac{\lambda}{\beta}x^2}; \quad x, \lambda, \beta > 0,$$

where, λ is a shape parameter and β is scale parameter.

Sequential probability ratio test (SPRT) is developed for the scale parameter of Nakagami Distribution and the robustness of scale parameter is studied when the shape parameter has undergone a change, for testing the hypothesis regarding the parameter of Nakagami Distribution. The expression for the Operating Characteristic (OC) and Average Sample

Number (ASN) functions are derived and the results are presented through Graphs and Tables. In this chapter, we conclude that for the present model, the SPRT for testing the hypothesis regarding β, is highly non-robust for changes in λ.

In Chapter 4, the problem of stress-strength reliability model is studied for a class of lifetime distributions proposed by Chaturvedi and Rani (1970). Here, deviating from the conventional techniques of obtaining the UMVUE of $R = P(Y > X)$, we have obtained the UMVUE of $R = P(Y > X)$ by using the estimate of the reliability function.

In Chapter 5, the problem of stress-strength reliability model is studied for Gompertz distributed stress. We have studied the problem by establishing the relationship among the parameters of the distributions of stress and strength of the manufacturing items. It is considered that the stress follows a Gompertz distribution and strength follows a Power function distribution. Further, these results are explained with an example and are utilized to get optimum cost of any item when the cost function is linear in terms of parameters. The technique used in this Chapter can be helpful in manufacturing an effective and reliable system that ensures the customer satisfaction.

In Chapter 6, The problem of stress-strength reliability model is studied through establishing the relationship among the parameters of the distributions of stress and strength of the manufacturing items. It is considered that the stress follows a class of lifetime distributions and strength follows a Power function distribution. Further, these results are used to obtain optimum cost when the cost function is linear in terms of parameters. The technique used in this Chapter could serve as a useful tool for the design and manufacturing of a reliable system.

In Chapter 7, the problem of stress-strength reliability model is studied for exponential distribution under complete sample case, Type I and Type II censoring. Here, we have obtained the MLE and UMVUE of $R = P(Y > X)$. In order to obtain the MLE and UMVUE, first we obtain the estimate of reliability function R(t). Further, the estimates of reliability function R(t) are used to find the MLE and UMVUE for $R = P(Y > X)$.

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