Stability and Bifurcation Analysis of Some Real Life Problems

Abstract of thesis submitted to the

BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY

(A CENTRAL UNIVERSITY)

LUCKNOW

for the award of the Degree of

DOCTOR OF PHILOSOPHY

in Applied Mathematics

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Enrollment No. 383/13 **Year 2017**

ABSTRACT

God used beautiful mathematics in creating the world.

-Paul Adrien Maurice Dirac

The mathematical modeling of many natural and social phenomena involve one or more continuous variable(s) varying with respect to one or more other variable(s), for example; radioactive decay, population growth, spread of infectious diseases, price of a commodity, rate of change of temperature of a body etc. Differential equations are being used to model these types of real life problems. Mostly problems occur in mathematical biology, mathematical ecology, mathematical economics, mechanical systems etc are multivariable, nonlinear and influenced by many conditions. Consequently, the mathematical models consist of a system of nonlinear differential equations with many parameters. It is well known that analytical solutions of the nonlinear differential equations is not always possible. Moreover, the analytical solutions may not always determine the dynamics of the system. Henri Poincare proposed qualitative approach for systems of nonlinear differential equations. The qualitative approach, a combination of analysis and geometry, study and determine the properties of the solutions of the systems directly from the differential equation without actually solving the system.

A long-term prediction of evolving systems is one of the main aim of the study of mathematical modeling of real life problems. Dynamical system gives a functional description of a mathematical model. In dynamical systems, the independent variable is often referred to as time. Generally, dynamical systems are described by initial-value problems, governed by ordinary or partial differential equations, or by difference equations. The qualitative study includes, aspects of stability and bifurcation analysis, of the dynamical systems, makes the systematic study and deeper understanding of the entirety of processes in systems. Moreover, a long-term prediction of evolving systems become possible.

The human population is increasing gradually and exploitation of renewable natural resources, like marine species, wild life species, has reached to a dangerous level, so there is a global concern to protect the ecosystem. The predator-prey interactions are the building blocks of the ecosystems. The stability and bifurcation analysis of the predator-prey models provides a long-term behaviours of the predator-prey systems.

Heat transfer in fluids due to temperature difference between the fluid layers is known as convection. Thermal instability in porous media plays a very significant role in many areas such as in petroleum industry, chemical engineering and geophysics, etc. External regulation (e.g. thermal, gravity, rotation and magnetic field modulation) of convection can enhance or diminish heat transfer in a convective system. The nature of non-Newtonian fluids is quite different from Newtonian fluids due to their properties like shear stress and shear strain. The basic idea for considering these fluids is due to their oscillatory nature. For example, industrial fluids are basically non-Newtonian. In particular, viscoelastic fluids have been potentially important. Proper understanding of convective motion and its behaviour is necessary for controlling many processes such as geothermal reservoirs, filtration, enhanced oil recovery.

In this thesis, the stability and bifurcation analysis of some predator-prey models under some natural and human phenomena, such as, Allee effect, harvesting, additional food has been investigated. Moreover, stability analysis of internal heating effect on oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation has been investigated.

Chapter 1 is of introductory nature, which provides a brief introduction to interacting populations, basic ecological models, harvesting functions, Allee effect, Additional food. Further, an overview of the mathematical tools for investigating local stability, global stability, various types of bifurcations (codimension one and two) of the steady states of the dynamical models is also provided. Also, convection, porous medium, internal heating, viscoelastic fluid, modulation have been defined and a mathematical model under the Boussinesq approximation is introduced.

Chapter 2 consists of two sections, first section concerned about the stability and the bifurcation analysis of a Leslie-Gower predator-prey model in the presence of Michaelis-Menten type predator harvesting. It is shown that the proposed model exhibits the bi-stability for certain parametric conditions. By means of Dulacs criteria the sufficient conditions for the global stability of the model has been obtained. It is also shown that model exhibits different kinds of bifurcations (e.g., the saddle-node bifurcation, the subcritical and supercritical Hopf bifurcations, Bogdanov-Takens bifurcation, the homoclinic bifurcation) whenever the values of parameters of the model vary. The analytical findings and numerical simulations reveal far richer and complex dynamics in comparison to the models with no harvesting and with constant-yield predator harvesting.

In second section Bogdanov-Takens bifurcations for a Leslie-Gower type predator-prey model in the presence of nonlinear prey harvesting has been examined. The parametric conditions under which the system undergoes a repelling Bogdanov-Takens bifurcation and attracting Bogdanov-Takens bifurcation have been obtained. By numerical simulations and possible phase portrait diagrams, it is shown that for different set of parameter values, the system has stable or unstable trajectories (limit cycle, homoclinic loop) in a small neighbourhood of the cusp of codimension 2.

In **Chapter** 3, a Holling-Tanner predator-prey model with ratio-dependent functional response in the presence of nonlinear prey harvesting is considered. The mathematical analysis of the model includes existence and uniqueness of positive solutions, their boundedness and permanence, local stability of all ecological feasible equilibrium points and bifurcation analysis. The nature of the system near origin is studied by using a blow up transformation. The numerical simulations of model have been presented in support of the analytical findings.

In **Chapter** 4, a modified Leslie-Gower predator-prey model with Allee effect II, affecting the functional response has been proposed with the assumption that the extent to which the environment provides protection to both predator and prey is the same. The model has been studied analytically as well as numerically, including stability and bifurcation analysis. Compared with the predator-prey model without Allee effect, it is found that the weak Allee effect II can bring rich and complicated dynamics such as the model undergoes to a series of bifurcations (Homoclinic, Saddle-node and Bogdanov-Takens). The existence of Hopf bifurcation has been shown for models with (without) Allee effect and the local existence and stability of the limit cycle emerging through Hopf bifurcation has also been studied. The phase portrait diagrams are sketched to validate analytical and numerical findings.

In **Chapter** 5, dynamical behaviour of a modified Leslie-Gower predator-prey model with double Allee effect has been analyzed. It has been considered that the extent to which the environment provides protection to both predator and prey is the same. Strong and weak Allee effects have been separately studied. The equilibrium points of the system and their local stability have been studied. It is shown that the dynamics of the system is highly dependent upon the initial conditions. The local bifurcations (hopf, saddle-node, Bogdanov-Takens) have been investigated by considering sufficient parameter(s) as the bifurcation parameter(s). The local existence of the limit cycle emerging through Hopf bifurcation and its stability is studied by means of the first Lyapunov coefficient. The numerical simulations have been done, which supports the analytical findings and also show the emergence of homoclinic loop. The possible phase portraits and parametric diagrams have been depicted.

In **Chapter** 6, a predator-prey model in which the predator is provided with additional food and subjected to Allee effect is analyzed. The existence of equilibrium points and their local stability has been studied. It is found that the model shows bistability, implying that solutions depend highly on the initial values. The occurrence of bifurcations, including Bogdanov-Taken, Hopf-Andronov, Transcritical and Saddle-node, for the system has been shown. Further, the appearance of homoclinic loop, emerging through Hopf-bifurcation has been shown through numerical simulation. Numerical simulations have been proposed to confirm the analytical results.

Chapter 7, investigates the combined effect of internal heating and time periodic gravity modulation in a viscoelastic fluid saturated porous medium by reducing the problem into a complex non-autonomous Ginzgburg-Landau equation. Weak nonlinear stability analysis has been performed by using power series expansion in terms of the amplitude of gravity modulation, which is assumed to be small. The Nusselt number is obtained in terms of the amplitude for oscillatory mode of convection. Influence of viscoelastic parameters on heat transfer has been discussed. Gravity modulation is found to have destabilizing effect at low frequencies and stabilizing effect at high frequencies. Finally, it is found that overstability advances the onset of convection, more with internal heating. The conditions for which the complex Ginzgburg-Landau equation undergoes Hopf bifurcation and the amplitude equation undergoes supercritical pitchfork bifurcation are studied.

Each chapter contains introduction, conclusion and references relevant to discipline. All the references are placed in alphabetical order of last name of the first order, in a separate section **BIBLIOGRAPHY**, at the end of the thesis.

Portion of above works are published/communicated in different journals, the list is as follows:

1. Manoj Kumar Singh, B.S. Bhadauria, B. K. Singh, Qualitative analysis of a Leslie-Gower predator-prey system with nonlinear harvesting in predator. Ain Shams Engineering Journal (Elsevier), http://dx.doi.org/10.1016/j.asej.2016.07.007. (In press)

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3. B.S. Bhadauria, Manoj Kumar Singh, Ajay Singh, Palle Kiran, B.K. Singh, Stability analysis and internal heating effect on oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation. International Journal of Applied Mechanics and Engineering (De Gruyter), 21(4), 785-803 (2016).

4. Manoj Kumar Singh, B.S. Bhadauria, Qualitative analysis of an additional food provided predator-prey model in the presence of Allee effect. (Communicated)

5. Manoj Kumar Singh, B.S. Bhadauria, Qualitative analysis of modified Leslie-Gower predator-prey model with weak Allee effect type II. (Communicated)

6. Manoj Kumar Singh, B.S. Bhadauria, Bogdanov-Takens bifurcations for a predator-prey system with nonlinear harvesting in prey. (Communicated)

7. Manoj Kumar Singh, B.S. Bhadauria,Qualitative analysis and optimum harvesting of a ratio-dependent Holling-Tanner predator-prey model with nonlinear prey harvesting. (Communicated)