Numerical Study of Time-Fractional Delayed Differential Equations

ABSTRACT

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ABSTRACT

In the study of differential calculus, fractional differential equations (FDEs) attract the researchers for their non-local aspects and broad area of applications in various types of models arises in electrical network systems, signal processing systems, diffusion-reaction processes [2, 4, 10, 11]. The delay differential equation, a generalized class of time-fractional PDEs, turns out uniquely in different disciplines of medical sciences, physical sciences, and engineering. Ecology-system, control system, atmospheric models [17] and economic complex macro-dynamics models [5] are the real-life applications of delayed differential equations. In fractional calculus, many computational techniques have been introduced for the appropriate behaviors of models expressed via (non) linear fractional PDEs/integral equations as well, see [1,3,9] and their references. Meanwhile, the waveform relaxation methods and spectral collocation technique [18], pseudo-spectral method [9], and variational iteration method (VIM) [3] have been implemented for the study of proportional delayed functional PDEs. The study of delayed fractional differential equations is a very complicated task, and its study is done via some limited techniques. In the literature, time-fractional PDEs with proportional delay arguments have been simulated via homotopy perturbation method (HPM) [12], fractional VIM [13], FRDTM [14], and integral transform-based HPM [8, 15].In this consequences, Burgers' type delay differential equations is featured as generalisation of Navier-Stoke's equation. As Burgers' equation is combination of nonlinear convection term and diffusive viscosity coefficient performs a frequent turbulence flow. Kinematic wave hypothesis of traffic flow, shock ideology, cosmology,gas dynamics and heat conduction are particular fields of utilization of Burgers' equation [6, 7]. Meanwhile, A new type of delayed differential equation named as pantograph equations appears in different kind of physical phenomena in mechanics and electro-dynamic system by Tayler et al. [16]. "pantograph" means a linkage made of five link joint with pin linkage to generate revolute pairs. A pantograph is made up of parallelograms which are connected in such a way that the movement of one point in tracing an image creates

identical movements in the other point. It has the ability to scale and replicate the image at the same time. In addition, there is a fixed point, a copying point, and the tracing point always follows a straight path. These aptitudes of pantograph has a wide range of engineering applications. It is worth remarking that evaluation of an analytical/numerical result is still a very demanding problem for fractional delayed differential equations (FDDEs). Moreover, accurate solutions to FDDEs are hardly available, and therefore, the advancement of analytical/numerical methods for the study of such types of problems accurately is yet under investigation.

The main objective of this thesis is to present a analytical study of time fractional delayed differential equations by using various existing methods. In the present thesis, an effort has been done to simulate the approximate solutions of time fractional generalised Burgers' equation with proportional delay, pantograph differential equation and integro-differential equations with delay argument. These delayed differential equations have been analysed by using fractional differential transform method, residual power series method, homotopy analysis method, variational iteration technique and decomposition method. Chapter wise summary of the thesis is given below:

In the **Chapter 2**, conformable time fractional nonlinear PDEs (CTF-PDEs) with proportional delay including generalized Burger differential equation with delay argument has been analyzed by using "extended reduced differential transform method (ERC-DTM)". ERC-DTM is an extension of reduced differential transform for conformable fractional derivative. In addition, some new properties of ERC-DTM for a delayed term of the conformable derivative have been demonstrated. Further, numerical implementation of the proposed method determined that the computed analytical solutions converge to the exact solution very rapidly. Accuracy and reliability of the proposed method verified by calculating L_2, L_∞ error norms and convergence theorem. Also, graphical representation and tabular calculation of the approximate solution shows that the estimated errors for different fractional values of α are decreasing speedily with increasing order of approximations. Meanwhile, the advantage of the introduced method is that it does not demand any type of linearization and high computational time in comparison to other developed techniques and produces convergent analytical solutions more quickly. This chapter is a revised version of the article published in *App. Num. Math.*, 157(2020)419–433.

In the **Chapter 3**, a comparison has been done to analyze the analytical behavior of nonlinear partial differential equations with a delayed argument for conformable derivative. Two powerful techniques named as new integral decomposition transform method (NIDTM) and optimal homotopy analysis new integral transform method ($_{\text{O}}$HANITM) has been used to approximate the nonhomogeneous pantograph type integro-differential equation and initial valued Burger differential equation with delay argument. Computed output exhibits that both techniques produce convergent solutions and approaches to the exact solution by increasing the order of iteration. Meanwhile, it is evident from tables and figures that NIDTM produces more convergent outcomes with high computational cost than $_{\text{O}}$HANITM for pantograph and pantograph type integro differential equations while both techniques give a similar result at the optimal value of $\hbar$ for Burgers' type nonlinear delayed differential equations. In addition, for time fractional Burgers' type delayed differential equations $_{\text{O}}$HANITM generates more convergent solutions corresponding to NIDTM and other existing methods at an optimal value of $\hbar$ with higher computational cost. This chapter is a revised form of the book chapter *Mathematical Modeling in Intelligent Systems: Theory, Methods & Simulation*, CRC Press, Taylor & Francis Group.

In the **Chapter 4**, a new hybrid residual power series (RPS) method in the space of Laplace transform (in brief, LT)has been proposed for the study of the nonlinear space-time fractional model of partial differential equations with proportional delay arguments (STF-PDEDAs). The main objective of this technique is to convert the target nonlinear STF-PDEDAs into the algebraic equation in LT space. The newly transformed equation is solved by utilizing the revised RPS method to achieve series approximations and refer to this technique as LRPS method. Different kinds of attractive applications of STF-PDEDAs are discussed utilizing proposed LRPSM to reveal its efficiency, effectiveness, and adaptability for constructing series solutions to these STF-PDEDAs. The numerical experiments and graphical findings have been computed for different fractional parameter values to analyze the behavior of LRPSM solutions and to test the authenticity and applicability of the present method in comparison to other existing results obtained by some rigorous techniques. The outcomes concluded that the proposed LRPSM is easy to understand and straightforward to provide results converging to its exact solution. This chapter is a revised form of the article communicated with a reputed journal.

In the **Chapter 5**, the q-homotopy analysis method has been adopted to determine the analytic solution of the nonlinear Caputo time fractional initial valued Burgers' equation with proportional delay. Estimated outcomes demonstrated convergent solutions in terms of absolute error and relative error for different values of α. Moreover, $\hbar$ curves for different fractional orders of α are reported to visualize the $\hbar$ region. $\hbar$-region is a particular region where minimum error lies and one can compute the optimal value of $\hbar$. Computed outcomes show that q-HAM provides convergent results at an optimal value of $\hbar$. Also, q-HAM produces more accurate and fast solutions in comparison to DTM, VIM, HPTM, HPM, and FRDTM. This chapter is a revised form of the article published as conference proceeding in *Series: A.R Proceedings; ISSN: 2582-3922; ISBN: 978-81-942709-6-6 (eBook).*

In the **Chapter 6**, a new integral transform based variational iteration technique (NTVIT) has been proposed to study the behavior of higher-order nonlinear time-fractional delayed differential equations. The NTVIT is a hybrid technique that is developed via the concept of variational theory with the use of the properties of a new integral transform. The stability and convergence of NTVIT are analyzed via Banach's fixed point theory. The effectiveness and validity of NTVIT solutions are demonstrated via the evaluation of error norms: relative/absolute errors for some test-suitable delayed problems of different fractional order. The numerical experiments confirm that NTVIT is capable of producing highly accurate behaviors as compared to some existing techniques. This chapter is revised form of the article accepted in *Nonlinear Engineering-Modeling and Application.*

In the **Chapter 7**, Daftardar-Gejji and Jafari transform method (DGJTM) has been implemented to compute the analytical solution of Caputo time fractional neutral delay differential equations. DGJTM is a new, accessible, and derivative-free technique for nonlinear terms combined with DGM (Daftardar-Gejji and Jafari method) and Laplace transform method. Moreover, DGJTM is an analytical approach that produces a convergent series solution for a large time scale at a less computational cost. In the current chapter, pantograph delayed differential equations including higher order neutral functional delayed differential equations are analyzed and calculated results are compared with other existing methods graphically. The existence and uniqueness theorem together with the convergence theorem were described to assure the convergence of the new technique. This chapter is a revised form of the book chapter *Computing and Simulation for Engineers*, CRC Press, Taylor & Francis Group.

In the **Chapter 8**, a system of multi-pantograph differential equations and integro-differential equations with a proportional delay has been studied to acquire new approximations in the domain of Caputo fractional derivative. In the present chapter, fractional differential transform method (FDTM) has been applied to the concerned problems and the efficiency of the adopted method is supported by the sufficient condition of convergence. Also, some important new properties of FDTM have been introduced for fractional integro differential equations with proportional delay to implement FDTM on fractional integral delayed problems successfully. These new properties are generalization of the properties investigated for integer order integro-differential equations. Numerical and graphical outcomes confirm that evaluated series solutions assure convergent results with negligible error and require low computational cost. This chapter is a revised form of the article published in *Math.Meth.In App.Sci.*, (2022) 1- 24.

At the end future plan of presented work is suggested.

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A. Published/Accepted in Journals

1. Singh B.K., **Agrawal S.**, "A new approximation of conformable time fractional partial differential equations with proportional delay", *Applied Numerical Mathematics*, **157 (2020), 419–433, (IF-2.994)**, DOI:10.1016/j.apnum.2020.07.001.

2. Singh B.K., **Agrawal S.**, "Study of time fractional multi-pantograph system and integro differential equations with proportional delay", *Math Meth Appl Sci.*, **(2022),1- 24, (IF-3.007)**, DOI:10.1002/mma.8335.

3. Singh B.K., Awasthi M., **Agrawal S.**, Gupta M., Tomar R., "Stability analysis and new numeric technique for higher order time-fractional delayed differential equations" accepted in *Nonlinear Engineering-Modeling and Application.*

B. Published in eBook

1. Singh B.K., **Agrawal S.**, "A new approximation for conformable time fractional nonlinear delayed differential equations via two efficient methods" in the eBook *Mathematical Modeling in Intelligent Systems: Theory, Methods and Simulation*, by CRC Press (Taylor & Francis), **(2022), 133-157**, DOI: 10.1201/9781003291916-9.

2. Singh B.K., **Agrawal S.**, "Analytical study of higher order time fractional differential equation with proportional delay for large time scale" in the eBook *Computing and Simulation for Engineers* by CRC Press (Taylor & Francis), **(2022), 221-236**, DOI:10.1201/9781003222255-14.

C. Communicated

1. Singh B.K., **Agrawal S.**, Singh N., "Revised RPS method in Laplacian space for space-time fractional PDEs with proportional delayed arguments".

D. Conference Paper

1. Singh B.K., **Agrawal S.**, "Study of nonlinear fractional generalized burger equation with proportional delay via q-HAM", *Series: A.R Proceedings; ISSN: 2582-3922; ISBN: 978-81-942709-6-6 (eBook)*, Doi:10.21467/proceedings.100.15.

E. Papers Presented in International/National Conferences

1. Singh B.K., **Agrawal S.**, “A new application of residual power series methods for solving nonlinear time fractional partial differential equations” presented in *International Conference on Algebra and Applied Analysis (ICAAA-2018)*, organized by Department of Mathematics, Integral University, Lucknow during August 9-11, 2018.

2. Singh B.K., **Agrawal S.**, “Application of RPS method for study of nonlinear time fractional partial differential equations with proportional delay” presented in *International Conference on Recent Advances in Pure and Applied Mathematics*, held at Delhi Technological University, Delhi during October 23-25, 2018.

3. Singh B.K., **Agrawal S.**, “Study of nonlinear fractional generalized burger equation with proportional delay via q-HAM”, presented in *International Conference on Applied Mathematics* & *Computational Sciences*, organized by Dehradun Institute of Technology, Dehradun (UK)-India during October 17-19, 2019.

4. Singh B.K., **Agrawal S.**, “A new approximation for conformable time fractional nonlinear delayed differential equations via two efficient methods” presented in *Advances in Differential Equations and Numerical Analysis*, organized by Department of Mathematics, IIT Guwahati, during October 12-15, 2020.

5. Singh B.K., **Agrawal S.**, “ A novel technique for approximation of nonlinear time fractional partial differential equations with delay argument” presented in *Recent Advancement in Physical Sciences*, jointly organized by Department of Chemistry, Department of Physics & Department of Mathematics, National Institute of Technology, Uttarakhand during December 19-20, 2021.

F. Workshop/ Webinar

1. *Workshop cum Minter School on 'Methods for Nonlinear Dynamical Systems & Chaos* organized by Department of Mathematics, NIT, Uttarakhand from 23-12-2019 to 27-12-2019 at Satellite Campus, MNIT Jaipur.

2. Online quiz on *Math Quiz 2k20 on Pure Mathematics* organized by Vel Tech High Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Chennai during June, 2020.

3. *Workshop on research methodology* organized by IGNTU, Amarkantak (MP) from May 26-30,2020.

Bibliography

[1] Abazari R, Ganji M, Extended two-dimensional DTM and its application on nonlinear PDEs with proportional delay, *Int. J. Comput. Math.*, 88 (8) (2011) 1749-1762.

[2] Caputo M., Mainardi F, Linear models of dissipation in anelastic solids, *Rivista del Nuovo Cimento*,1971;1:161-98. 161-98.

[3] Chen X., Wang L., The variational iteration method for solving a neutral functional-differential equation with proportional delays, *Comput. Math. Appl.*, 59,2696-2702,(2010).

[4] He JH., Nonlinear oscillation with fractional derivative and its applications,*Int. Conf. Vibr. Engg'98,Dalian*,1998;288-91.

[5] Keller AA., Contribution of the delay differential equations to the complex economic macrodynamics,*WSEAS Transactions on Systems*, 9(4) (2010) 258-271.

[6] Jordan PM., A note on Burgers' equation with time delay,Instability via finite-time blow-up, *Physics Letters A*,372(2008) 6363–6367.

[7] Kolmanovskii V., Myshkis A., Introduction to the theory and applications of functional differential equations, *Kluwer, Dordrecht,The Netherlands.*,(1999).

[8] Kumar M., A Hybrid Method to Solve Time-Space Fractional PDEs with Proportional Delay, *Int. J. Appl. Comput. Math.*, https://doi.org/10.1007/s40819-022-01277-6.

[9] Mead J., Zubik-Kowal B., An iterated pseudospectral method for delay partial differential equations, *Appl. Numer. Math.*, 55 (2005) 227–250.

[10] Miller KS., Ross B, An Introduction to the Fractional Calculus and Fractional Differential Equations, *Wiley, New York*; 1993.

[11] Podlubny I., Fractional Differential Equations, *Academic Press, San Diego*;1999.

[12] Sakar MG., Uludag F., Erdogan F., Numerical solution of time-fractional non-linear PDEs with proportional delays by homotopy perturbation method, *Appl. Math. Modelling*, (2016), http://dx.doi.org/10.1016/j.apm.2016.02.005.

[13] Singh BK., Kumar P., Fractional variational iteration method for solving fractional partial differential equations with proportional delay, *Int. J. Differ. Eqns.*,2017, (2017), Article ID 5206380, 11 pages, https://doi.org/10.1155/2017/5206380.

[14] Singh BK., Kumar P., Extended fractional reduced differential transform for solving fractional partial differential equations with proportional delay, *Int. J. Appl. Comput. Math.*, 3 (2017) 631 - 649. https://doi.org/10.1007/s40819-017-0374-9.

[15] Singh BK., Kumar P., Homotopy perturbation transform method for solving fractional partial differential equations with proportional delay, *SeMA Journal*,75,111-125,(2018), https://doi.org/10.1007/s40324-017-0117-1.

[16] Tayler AB., Ockendon JR., The dynamics of a current collection system for an electric locomotive, *Proc. R. Soc. Lond. Ser. A.*,322,447-468,(1971).

[17] Wu J., Theory and Applications of Partial Functional Differential Equations, *Springer–Verlag, New York*, 1996 .

[18] Zubik-Kowal B., Jackiewicz, Z., Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations, *Appl. Numer. Math.*, 56(3-4),433-443,(2006).