CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS AND ITS MOMENTS THROUGH ORDERED RANDOM VARIABLES

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Abstract

Several models of ordered random variables, including order statistics and record values, are contained in the model of generalized order statistics introduced by Kamps (1995). The concept of generalized order statistics provides a large class of models with many interesting and useful properties for both description and analysis of practical problems. Applications are multifarious in a variety of disciplines and in particular in reliability.

The model of generalized order statistics can be easily applicable in practice problems except when $F(x)$ is so called inverse distribution function (inverse exponential, inverse Weibull and inverse Pareto, etc). For this, the concept of lower generalized order statistics was given by Pawlas and Szynal (2001b). Later Burkschat *et al.* (2003) introduced it as dual generalized order statistics so as to enable a common approach to descending ordered random variables like reversed order statistics and lower record values.

Characterization is a condition involving certain properties of a random variables $X = (X_1, X_2, \ldots, X_n)$, which identifies the associated distribution function $F(x)$. The property that uniquely determines $F(x)$ may be based on a function of random variables whose joint distribution is related to that of $X = (X_1, X_2, \ldots, X_n)$. A Characterization can be of use in the construction of goodness of fit tests and in the examination of the consequences of modeling assumptions made by an applied scientist. For example, the independence of spacings of order statistics of a random sample from a continuous distribution implies the distribution function is exponential, and thus can be used to construct a goodness-of-fit test, even with a censored sample (Arnold *et al.*, 1998).

The only method of finding distribution function $F(x)$ exactly, which avoids the subjective choice, is a characterization theorem. A theorem is on a characterization of a distribution function if it concludes that a set of conditions is satisfied by $F(x)$ and only by $F(x)$.

Characterization of distributions through linearity of regression of order statistics, record values and generalized order statistics have been considered by many in the literature. Ferguson (1967) introduced the characterization of distributions based on the linearity of regression of adjacent order statistics $E(X_{r+1:n}|X_{r:n} = x)$ and its dual $E(X_{r:n}|X_{r+1:n} = x)$, where $X_{r:n}$ is the r^{th} order statistics. Shanbhag (1970) characterized exponential and geometric distributions in terms of conditional expectations for single order gap. Khan and Khan (1987) characterized Burr type XII distribution through linear regression for single order gap. Khan and Abu-Salih (1989) characterized a general class of distributions through conditional expectation of function of order statistics:

$$E\left[h(X_{r+1:n})|X_{r:n} = x\right] = a^* h(x) + b^*$$

and

$$E\left[h(X_{r:n})|X_{r+1:n} = x\right] = a_1^* h(x) + b_1^*$$

Wesolowski and Ahsanullah (1997) characterized the distributions by the regression of non-adjacent order statistics through the relation

$$E\left(X_{r+2:n}|X_{r:n} = x\right) = ax + b$$

Characterization of distributions via linearity of regression of order statistics when gap is higher is considered by Khan and Ali (1987), Franco and Ruiz (1997), Dembińska and Wesolowski (1998) and Lopez-Bláquez and Moreno-Rebollo (1997). Whereas, Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the generalized form of distributions through higher order gap. Khan and Athar (2004) also characterized some continuous distributions through linearity of regression when conditioning is done on a pair of order statistics. Using the result of Rao and Shanbhag (1994) dealing with an extended version of the integrated Cauchy functional equation Dembińska and Wesolowski (1998) and Athar *et al.* (2003) characterized the distributions by means of the regression equation

$$E\left(X_{r+i:n}|X_{r:n}=x\right)=ax+b$$

For record values Nagaraja (1977) characterized continuous distributions by using the relation

$$E\left(X_{U(r+1)}|X_{U(r)}=x\right)=ax+b$$

Nagaraja (1988b) also characterized distributions by considering the equation

$$E\left(X_{U(r)}|X_{U(r+1)}=x\right)=ax+b$$

Franco and Ruiz (1996, 1997) obtained the distribution function from the conditional expectations

$$E\left[h(X_{U(n-1)})|X_{U(n)}=x\right]$$

where $'h'$ is a real, continuous and strictly monotonic function. Ahsanullah and Wesolowski (1998) extended the result of Nagaraja (1977) and characterized the distributions for double order gap. Lopez-Bláquez and Moreno-Rebollo (1997), Dembińska and Wesolowski (2000) and Athar *et al.* (2003) extended the result of Nagaraja(1988b) and characterized the continuous distributions conditioned on non-adjacent records. Characterization of continuous distributions conditioned on a pair of adjacent records was investigated by Bairamov *et al.* (2005). They characterized the exponential distribution and continuous distributions by taking the monotone transformations.

Further, Yanev *et al.* (2008), Yanev and Ahsanullah (2009) and Khan and Khan (2009) characterized the continuous distributions conditioned on a pair of non-adjacent records. Recently Noor and Athar (2014) characterized the continuous distributions by taking the conditional expectation

$$g_{r,s}^{p}(x) = E[\{\Psi(X_{U(s)}) - \Psi(X_{U(r)})\}^{p} | X_{U(r)} = x]$$

Concept of *gos* was given by Kamps (1995). Since, many ordered variables like order statistics, record values and k -record values are special cases of *gos*, therefore, characterization through *gos* is of special interest. Keseling (1999) characterized the continuous distributions by taking the conditional expectations.

$$E[h(X(r+1, n, \tilde{m}, k)) | X(r, n, \tilde{m}, k) = x]$$

where $h(.)$ is a real strictly monotonic function. They also characterized the continuous distributions by taking the conditional expectations

$$E[X(r, n, \tilde{m}, k) | X(r+2, n, \tilde{m}, k) = x]$$

Bieniek and Syznal(2003) investigated the characterization of the continuous distributions by considering the conditional expectations

$$E[X(r+l, n, \tilde{m}, k) | X(r, n, \tilde{m}, k) = x], l \geq 2.$$

Samuel (2008) characterized the conditional expectation through the relation

$$E\left[h\left(X(r+1,n,m,k)\right)|X(r,n,m,k)=x\right]=a^{*}h(x)+b^{*}$$

Khan and Alzaid (2004) characterized a general class of distribution $\bar{F}(x)=[ax+b]^{c}$ through linear regression of generalized order statistics using Rao and Shanbhag's (1994) result. They characterized the distributions by means of relation.

$$E\left[X(s,n,m,k)|X(r,n,m,k)=x\right]=a^{*}x+b^{*}$$

Khan *et al.* (2006), Beg and Ahsanullah (2006) have characterized the distribution functions through the relation

$$E\left[\xi\left\{X(s,n,m,k)\right\}|X(r,n,m,k)=x\right]=g_{s|r}(x)$$

and its dual

$$E\left[\xi\left\{X(r,n,m,k)\right\}|X(s,n,m,k)=x\right]=g_{r|s}(x)$$

Further, Ahsanullah and Beg (2008) characterized the continuous distribution functions conditioned on a pair of adjacent gos through the relation

$$E\left[\xi\left\{X(r+1,n,m,k)\right\}|X(r,n,m,k)=x,\;\;X(r+2,n,m,k)=x\right]=g_{r+1|r,\,r+2}(x)$$

Bieniek (2007) characterized continuous distributions based on conditional expectations, through the relation

$$E\left[g\left(X(r+1,n,m,k)\right)|X(r,n,m,k)=x\right]=h(x)$$

using Meijer's G-Function.

Characterization of continuous distributions conditioned on nonadjacent *gos* was considered by Cramer *et al.* (2004a), Raqab and Abu-Lawi (2004). In these literature, the authors are mainly concerned in finding the distribution function when the regression lines are linear. Bieniek (2009), Khan and Khan (2011) have characterized the continuous distribution functions conditioned on non-adjacent *gos* using Meijer's G-Function. Later on Khan *et al.* (2012) extended the result of Bieniek (2009), Khan and Khan (2011) and characterized the continuous distributions conditioned on a pair of non-adjacent *gos*, *i.e.* by taking the conditional expectation

$$E[h\{X(j,n,\tilde{m},k)\}|X(r,n,\tilde{m},k)=x, X(s,n,\tilde{m},k)=y], \quad 1 \leq r < j < s \leq n,$$

here $h(x)$ is considered as monotonic and differentiable function of x. Recently, Noor *et al.* (2014) characterized the continuous distributions by taking the conditional expectation

$$g_{r,s,p} = E[\{\psi(X(s,n,\tilde{m},k)) - \psi(X(r,n,\tilde{m},k))\}^p \mid X(r,n,\tilde{m},k)=x]$$

Using the concept of *gos*, Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) that enables a common approach to descendingly ordered random variables like reversed ordered order statistics, lower record values etc. The various developments on *dgos* and related topic have been studied by Ahsanullah (2004), Mbah and Ahsanullah (2007), Khan *et al.* (2009), Khan *et al.* (2010 a,b), Faizan and Khan (2011), Tavangar (2011) among the others. Khan *et al.* (2009) have characterized continuous distributions through conditional expectation of *dgos*, conditioned on a pair of non-adjacent *dgos*. Recently, Khan and Khan (2012) have characterized continuous distribution functions conditioned on non-adjacent *dgos* using Meijer's G-Function.

Order statistics and their moments have received attention from the beginning of this century. Since, Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics. The moments of order statistics, assumed considerable importance in the statistics literature and have been numerically tabulated extensively for several distributions. For example one can refer to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold and Balakrishnan (1989), Arnold *et al.* (1992) for details. There are mainly three reasons due to which recurrence relations and identities have attained importance:

i. Reduces the amount of direct computation and hence reduces the time and labour.

ii. They express the higher order moments in terms of lower order moments and hence make the evaluation of higher order moments easy.

iii. Provide some simple checks to test the accuracy of computation of moments of order statistics.

Shah (1966, 1970), Tarter (1966) have obtained moments of order statistics from logistic distribution. Malik (1967) has obtained recurrence relations for the moments of order statistics from power function distribution. Lieblien (1955), Balakrishnan and Joshi (1981) have obtained recurrence relations for moments of order statistics from Weibull distribution.

Saleh *et al.* (1975), Joshi (1978, 1979) have given recurrence relations for the moments of order statistics from exponential and truncated exponential distributions. Balakrishnan and Joshi (1983, 1984) obtained recurrence relations for single and product moments of order statistics from symmetrically truncated logistic distribution and doubly truncated exponential distributions.

Khan *et al.* (1983) developed general results for finding the k^{th} moment of order statistics without considering any particular distribution. Further, these results were utilized to obtain recurrence relations for doubly truncated and non-truncated distributions, thus unifying all the known results on recurrence relations for moments of order statistics.

Khan *et al.* (1984) obtained the inverse moments of order statistics for Weibull distribution whereas Ali and Khan (1996) obtained the ratio and inverse moments of order statistics from Weibull and exponential distribution. Unifying earlier results Khan and Athar (2000) established the relations for ratio and product moments of order statistics from doubly truncated Weibull distribution.

Khan and Khan (1987) obtained recurrence relations for single and product moments of order statistics for doubly truncated Burr distribution (Burr type XII) and utilized the relations to characterize the distribution. Athar *et al.* (2011) has obtained moments of order statistics from extended type-I generalized logistic distribution.

Kamps (1995) investigated recurrence relations for moments of generalized order statistics based on non-identically distributed random variables, which contains order statistics and record values as special cases.

Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases.

Pawlas and Szynal (2001a) derived recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Khan *et al.* (2007) obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution.

Athar and Islam (2004) established some recurrence relations between expectation of function of single and joint generalized order statistics from a general class of distribution. Further, Athar *et al.* (2009) generalized the result of Athar and Islam (2004) and established the relations for the expectation of function of *gos* for truncated distributions. Athar *et al.* (2007) obtained the ratio and inverse moments of generalized order statistics from Weibull distribution. Further, Kumar and Khan (2013) have obtained relations for generalized order statistics from doubly truncated generalized Exponential distribution and its characterization.

The thesis entitled **Characterization of Probability Distributions and its Moments Through Ordered Random Variables** comprises six chapters, in which Chapter 1 is an introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

In Chapter 2 of the thesis, we have characterized a families of continuous probability distributions by considering conditional expectation of difference of $p^{th}, \ (p \geq 1)$ power of two records values conditioned on a pair of non-adjacent records and and the following two results are obtained:

(i) For $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$

$$\begin{aligned} g^{p}_{l,s}(x,y) &= E[\{\Psi(X_{U(j)}) - \Psi(X_{U(l)})\}^p | X_{U(l)} = x, X_{U(s)} = y] \\ &= [\Psi(y) - \Psi(x)]^p \frac{\Gamma(s-l)\Gamma(p+j-l)}{\Gamma(j-l)\Gamma(p+s-l)} \\ & 1 \leq l < j < s \leq n, \ l = r, r+1 \end{aligned}$$

if and only if

$$F(x) = 1 - e^{-[a\Psi(x)+b]}, \qquad \alpha \leq x \leq \beta,$$

where $g^{p}_{r,s}(x,y)$ is a finite and differentiable function of x and $\Gamma(.)$ is a gamma function.

(ii) For $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$.

$$\begin{aligned} \xi^{p}_{r,l}(x,y) &= E[\{\Psi(X_{U(l)}) - \Psi(X_{U(j)})\}^p | X_{U(r)} = x, X_{U(l)} = y] \\ &= [\Psi(y) - \Psi(x)]^p \frac{\Gamma(l-r)\Gamma(p+l-j)}{\Gamma(l-j)\Gamma(p+l-r)} \\ & 1 \leq r < j < l \leq n, \ l = s-1, s \end{aligned}$$

if and only if

$$F(y) = 1 - e^{-[a\Psi(y)+b]}, \qquad \alpha \leq y \leq \beta,$$

provided that $\xi^{p}_{r,s}(x,y)$ is a finite and differentiable function of y and there exists a $q \in (\alpha, \beta)$ such that

$$q = \inf \left[x : x \geq F^{-1}\left(\frac{e-1}{e}\right)\right]$$

Examples of various distributions are given by properly choosing parameters $\Psi(x)$, a and b.

In Chapter 3 of the thesis, we have obtained results based on conditional expectation of single dual generalized order statistics conditioned on a pair of non-adjacent dual generalized order statistics using Meijer's G-Function. Thus, extending the result of Khan and Khan (2012) conditioned on a non-adjacent *dgos* and the following two results are obtained:

(i) For $\psi(t)$ be a monotonic and differentiable function of t.

$$g_{j|l,s}(x,y) = E[\psi(X^*(j,n,\tilde{m},k)) \mid X^*(l,n,\tilde{m},k) = x, X^*(s,n,\tilde{m},k) = y]$$

$$1 < r+1 < j < s \leq n, \quad l = r, r+1$$

exist, then

$$(\gamma_{r+1}-1)\frac{f(x)}{F(x)} - \frac{\frac{\partial}{\partial x} G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]}$$

and

$$\frac{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}-\gamma_{r+1}+1, ..., \gamma_s-\gamma_{r+1}+1\right)}{G_{s-r}(F(y) \mid \gamma_{r+1}-\gamma_{r+1}+1, ..., \gamma_s-\gamma_{r+1}+1)} = \exp\left(-\int_x^\beta D_1(t,y)dt\right)$$

where $g()$ is a finite and differentiable function of x, and

$$D_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]}$$

(ii) For $\psi(t)$ be a monotonic and differentiable function of t.

$$\xi_{j|r,l}(x,y) = E[\psi(X^*(j,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k) = x, X^*(l,n,\tilde{m},k) = y],$$
$$1 \leq r < j < s-1 < n, \quad l = s-1, s$$

exist, then

$$(\gamma_s - 1)\frac{f(y)}{F(y)} - \frac{\dfrac{\partial}{\partial y} G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]}$$

$$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\right) = a_s^{(r)}(s) \exp\left[\int_y^\beta D_2(x,t)dt\right],$$
$$\gamma_i > \gamma_s, \quad i = r+1, ..., s-1$$

and for $\gamma_{r+1} = \cdots = \gamma_s$,

$$\frac{1 + \log\{F(y)\}}{1 + \log\{F(x)\}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_y^q D_2(x,t)dt\right]$$

where

$$p \in (\alpha, \beta) \text{ such that } \quad -\log F(p) = 1$$

and

$$D_2(x,y) = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]}$$

Further, from these results lower records values are obtained and the results obtained by Khan *et al.* (2009), Khan *et al.* (2010a) and Khan and Khan (2012) are discussed.

In Chapter 4 of the thesis, we have obtained characterization of conditional expectation of difference of p^{th}, $(p \geq 1)$ power of two generalized order statistics conditioned on a pair of non-adjacent generalized order statistics using Meijer's G-Function. Further, some of its important deductions are discussed and some examples are obtained based on the deductions.

In Chapter 5 of the thesis, we have obtained explicit expressions for single and product moment of order statistics from Lindley distribution $F(x) = 1 - \left[1 + \frac{\theta x}{1+\theta}\right] e^{-\theta x}, \ x > 0, \theta > 0$. Further, means and covariance of order statistics from Lindley distributions are obtained.

In Chapter 6 of the thesis, we have obtained recurrence relations for single and product moments of generalized order statistics from New Weibull Pareto distribution $F(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \ 0 < x < \infty; \beta > 0, \delta > 0, \ \theta > 0$. Further, the distribution is characterized by a recurrence relation of single moments. Also, some deductions and particular cases are given.

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