A STUDY OF VARIOUS LIFE TESTING MODELS AND THEIR INFERENTIAL PROCEDURES

THESIS

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BY

PREM LATA GAUTAM

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DR. SURINDER KUMAR
Professor
Department of Applied Statistics
(School for Physical Sciences)
BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY
(A CENTRAL UNIVERSITY, LUCKNOW-226025)

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DECLARATION

I, **Prem Lata Gautam**, Enrolment No. **1209/16**, hereby declare that the work which is being presented in the thesis entitled **"A study of various life testing models and their inferential procedures"** in fulllment of the requirements for the award of the degree of Doctor of Philosophy and submitted in the Department of Applied Statistics of the Babasaheb Bhimrao Ambedkar University(A Central University), Lucknow is an authentic record of my own work carried out during a period from September, 2016 to December, 2020 under the supervision of Dr. Surinder Kumar, Professor, Department of Applied Statistics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma to this or any other University.

This is also declared that the thesis is essentially free from all kinds of plagiarism.

Date:

(Prem Lata Gautam)
Research Scholar,
Department of Applied Statistics,
School for Physical Sciences,
Babasaheb Bhimrao Ambedkar
University,
Lucknow-226025, India.

CERTIFICATE

This is to certify that the thesis titled "**A study of various life testing models and their inferential procedures"** submitted by **Ms. Prem Lata Gautam** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other University.

The thesis submitted to Babasaheb Bhimrao Ambedkar University Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations -1999 as amended in 2008/2010/2013* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

Date: Supervisor

Head of the Department

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List of Publications

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Chapter 1

Introduction

1.1 Introduction to Reliability

Development of science and technology plays an important role in the present scenario. To fulfill the demands of modern society, industries are trying to do more automation in manufactured products. The complexity of industrial systems and their products are increasing day-by-day. Presently main focus is given to the improvement in effectiveness of such complex systems.

The effectiveness of the system is to well perform all the tasks and to increase the efficiency to use it i.e. system must be more durable. The durability of the system is determined by the reliability and quality of the system.

The concept of reliability is as old as man himself. Some questions always come to their mind when they buy or have some products: Will it work satisfactorily? Will it take long? These are some of the familiar questions he asks all the time. To solve such a problem concept of reliability was introduced.

The etymological meaning of reliability is the quality of being trustworthy or of performing consistently well. In Statistics, it is defined as the probability that the unit performs its intended function adequately for a given period of time under the stated operating conditions or environment. Here, unit includes a system, an element, or a part of a system, or similar.

This definition focused on the four field of study namely:

1. Probability
2. Intended Function
3. Time
4. Operating Conditions.

1.2 History of Reliability

In the first quarter of the century, the growth and development of reliability are closely related to quality control and its development. In 1920, a team at Bell Telephone Laboratories developed statistical methods to solve some of their quality control problems. They provided the basis for further growth in the area of statistical quality control (SQC). Afterward, The American Society for Testing and Materials, the American Standards Association, and the American Society for Mechanical Engineers joined Bell Laboratories to promote quality control technology. However, the rate of adoption of these techniques among the enterprise was slow until World War II broke out in 1939.

In World War II, the necessity of reliability increased because of the demands of modern technology and society. Maintenance and improvement were a severe problem for equipment after the war. Failures of equipment and its components like a failure of electronic tubes were common problems at this time. To overcome these difficulties army and navy in the USA set up a joint committee known as the Vacuum Tube Development Committee in 1943. This joint committee introduced quantitative techniques for the measurement of reliability.

After this Universities and Laboratories worked in the same field of problem. The two popular organizations were Bell Laboratories and Aeronautical Radio, Inc, who contributed more than others. Finally, in 1950 the first committee on reliability was set up by the US Department of Defence. This committee was later known as the Advisory Group on the Reliability of Electronic Equipment (AGREE). In 1957, AGREE published its first report.

This report focused on the specifications of reliability such as minimum acceptability limits and reliability test requirements, etc.

Britain and Japan in 1950 were also interested in the principles of reliability for their products. The National Council for Quality and Reliability was established in 1961, to create information about the importance of quality and reliability in the design, manufacture, and use of products. Even if we consider the last two decades, we have seen remarkable progress to use reliability in the organizations and the government departments by the developed and developing countries.

1.3 Measures of Reliability

For any manufacturing product, there are always two possibilities: the first is, it performs well its task without failure, and the second is, it does not perform and fail. The simplest reliability is the possibility that failures may not occur within a given time interval. If 'T' is the time until a unit failure (random variable) occurs, the probability that the unit will not fail in a given environment before time 't' (or its reliability) is

$$R(t) = P_r(T > t)$$

Reliability is always a function of time. It also depends on environmental conditions which may change over time, such as temperature, humidity, vibrations, etc. Probability is an important index of reliability to deal the problems concerning the future prediction of the component or system. Since it is a probability, the value of reliability is between one and zero, i.e.

$$R(0) = 1, \quad R(\infty) = 0$$

and R(t) is a non-increasing function between these limits.

There is another measure of reliability named **Stress-Strength**, is mathematically expressed as $P_r(X < Y)$. This reliability term is pioneered by Church and Harris (1970)[42].

They provide an example of a missile flight where the initial values of the stress corresponds to propulsive force, angles of elevation, atmospheric conditions, etc. It is the assessment of the component, where the random variable X represents the stress and Y represents the strength of the component that can overcome the stress. If the stress exceeds the strength the component would fail i.e. disaster would happen. So the reliability model $P_r(X > Y)$ is known as probability of disaster. Reliability is defined when the probability would not fail.

1.4 Components of Reliability

1.4.1 Hazard Rate

The failure of products or its parts is always a concern for the manufacturer. This failure may be result of such scarcity

1. Design of product (Inherent weak design of product)
2. Quality control problem from the manufacture
3. Use of products by customers (improper use of products)
4. Policies related to the products by consumer

The Hazard rate is the rate of damage at the same time of undamaged products lasting during 't' and is denoted by $h(t)$. Let us consider a test consist of N components placed in operation from time $t = 0$. As the time passes, the components fail. Let after time 't' the number of surviving components are $N_s(t)$ and the failed components are $N_f(t)$. The components fail independently with the probability of failure

$$F(t) = 1 - R(t)$$

where $F(t) = P_r(T \leq t)$, is known as cumulative distribution function and R(t) is the probability of success.

Then,

$$R(t) = \frac{N_s(t)}{N} = 1 - \frac{N_f(t)}{N}$$

The hazard rate which is a measure of instantaneous speed of failures is defined as

$$\begin{aligned} h(t) &= \lim_{t\to\infty} \frac{N_s(t) - N_s(t+\Delta t)}{N_s(t)\Delta t} \\ &= \frac{1}{N_s(t)} \left(-\frac{dN_s(t)}{dt} \right) \end{aligned}$$

Differentiating R(t) with respect to t,

$$\frac{dR(t)}{dt} = \frac{1}{N} \frac{dN_s(t)}{dt}$$

Substituting in terms of the hazard rate,

$$\begin{aligned} \frac{dR(t)}{dt} &= -\frac{1}{N} \frac{dN_s(t)h(t)}{dt} \\ &= -R(t)h(t) \end{aligned}$$

Then,

$$h(t) = -\frac{1}{R(t)} \frac{dR(t)}{dt}$$

Integrating both sides,

$$\int_0^t h(x)dx = -log(R(t))$$

or,

$$R(t) = exp\left[-\int_0^t h(x)dx \right]$$

Then,

$$\begin{aligned} f(t) &= \frac{dF(t)}{dt} \\ &= h(t)exp\left[-\int_0^t h(x)dx\right] \\ &= h(t)R(t) \end{aligned}$$

Therefore,

$$\begin{aligned} h(t) &= \frac{f(t)}{R(t)} \\ &= \frac{f(t)}{1-F(t)} \end{aligned}$$

An idealized shape of the hazard rate of the product is the bathtub curve and is presented in (**Fig. 1.1**). The description of each of the three regions is as follows:

1. Infant Mortality Period: The product population exhibits a hazard rate that decreases during this first period (also known as "infant mortality", "burn-in" or the "debugging period"). The hazard rate stabilizes at some value at time 't_1' when the weak products in the population have failed.

2. Useful Life Period: The product population reaches its lowest hazard rate level and is characterized by an approximately constant hazard rate, which is often referred to as the "constant failure rate".

3. Wear-Out Period: Time 't_2' indicates the end of useful life and the start of the wear-out phase. After this point, the hazard rate increases. When the hazard rate becomes too high, replacement or repair of the population of products should be done.

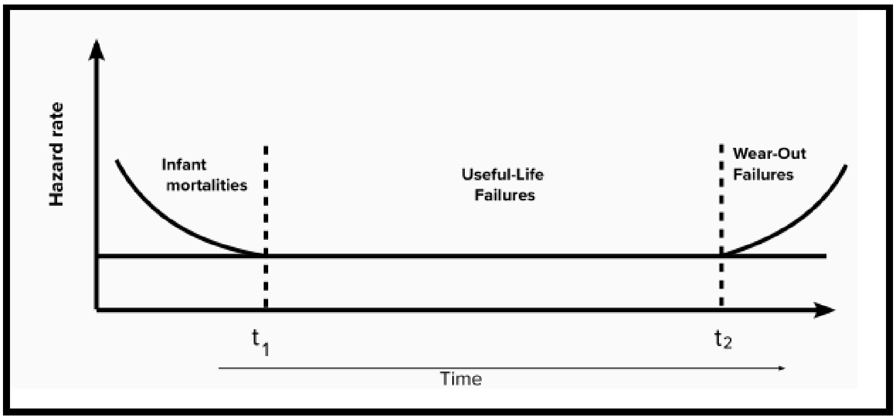


Fig.1.1: Bathtub Curve

1.4.2 Cumulative Hazard function

The cumulative hazard function is not a probability as the hazard function. It is known as the integrated hazard function. It is also defined as a measure of risk i.e., as the value of $H_S(s)$ increases the risk of failure by time 'y' is also increases.

$$H_S(s) = \int_0^y h_Y(t)dt$$

1.4.3 Expected Life or Mean Time of Failure (MTTF)

If $f(t)$ is the probability density function, and T is a random variable, the mean time of failure (MTTF) is the expected value for the time of failure and is defined as

$$E[T] = \int_0^\infty tf(t)dts$$

It can be represented in terms of reliability as,

$$MTTF = \int_0^\infty R(t)dt$$

E[T] is also known as the mean time between failure (MTBF), when the products exhibits a constant hazard rate, i.e. the failure probability density function is exponential. The MTTF is applicable if the failure distribution function is identified because the value of reliability function at a given MTTF depends on the probability distribution function used to model the failure data.

1.5 Maintainability and Availability

Despite the designer's best efforts, no device can be completely reliable. The equipment is likely to fail during operation, which may cost a lot of money and time, and sometimes poses a safety hazard. Therefore, maintenance becomes an important consideration for the long-term performance of the equipment. The equipment requires a preventive cure (to avoid any possible failures) and eliminates malfunctions during its operation (when it occurs). Maintainability is a performance index related to such equipment or systems that perform maintenance operations.

Maintainability is the possibility that a malfunctioning device can be restored to an operable state within a specified time when maintenance is performed under specified conditions. It characterizes the adaptability of the equipment to detect and eliminate faults and prevent faults. It is effective ways of increasing the reliability of a system.

If T is a random variable representing the repair time and 't' is the time when failed equipment will be repaired, then maintainability is defined as

$$M(t) = P_r(T \leq t)$$

Availability is another measure of maintaining equipment performance. It integrates reliability and maintainability parameters and depends on the number of failures that occur and the speed of failures that can be corrected. Long-term or steady-state availability refers to the proportion of time the equipment can be used. Availability can be expressed as

$$\text{Availability} = \frac{\text{Up-time}}{\text{Up-time} + \text{Down-time}}$$

Here, the denominator represents the total time for which the equipment is required to function. Up-time is the availablity of equipment for the period and down-time is delay related to repair.

1.6 Classical Inference on Reliability model $R = P_r(X < Y)$

We have used the transformation method to obtain the classical inference (point estimator and interval estimate) of stress-strength model. Considerations are given to the following methods:

1.6.1 Maximum Likelihood Estimator (MLE) of R

It is a reasonable choice to judge a good estimator. The MLE is parametric point for which the observed sample is most likely. It is also the most popular procedure for the estimation of stress-strength model of reliability due to its flexibility and generality. Casella and Berger (1990)[23] and Lehmann and Casella (1998)[73] briefly described the method of MLE. There is the following step should be taken:

1. First calculate $R = R(\theta) = P_r(X < Y)$ as a function of θ

2. Construction of the MLE $\ddot{\theta}$ of the parameter θ

3. Calculate the MLE of $\ddot{R} = R(\ddot{\theta})$ of R.

1.6.2 Uniformly Minimum Variance Unbiased Estimator (UMVUE) of R

A statistic is said to be UMVUE if it is unbiased and has the smallest variance among all the unbiased estimators. Let us consider $V(\underline{X}, \underline{Y})$ be an unbiased estimator of $R(\theta)$, and T is a sufficient statistics then UMVUE of R is calculated by $\acute{R} = E(V(\underline{X}, \underline{Y})|T)$

$$\acute{R} = \int I(X_1 < Y_1)p(X_1, Y_1|T)dX_1dY_1$$

where, $V(\underline{X}, \underline{Y}) = I(X_1 < Y_1)$ and $p(X_1, Y_1|T)$ is the conditional distribution of (X_1, Y_1) given T.

1.6.3 Interval Estimation of R

In many applications, just knowing a point estimate is not enough. Let us consider a medical application, where X and Y represent the response of the old therapy and the new therapy A and B, respectively. The purpose is to decide whether to abandon the old treatment method and switch to the new treatment method. Let we obtain the point estimator of R and is equal to 0.59. Since we have no information about the variability of R, we still cannot confidently recommend which measures to take. In this case, we must have an interval that is likely to cover the unknown value of R. The confidence interval is
Let the statistics $L(\underline{X}, \underline{Y})$ and $U(\underline{X}, \underline{Y})$ be such that

$$P_r\left(L(\underline{X}, \underline{Y}) < R < U(\underline{X}, \underline{Y})\right) > 1 - \sigma, \quad 0 < \sigma < 1$$

where, $(L(\underline{X}, \underline{Y}), U(\underline{X}, \underline{Y}))$ are the lower bound and upper bound respectively, with confidence coefficient $1 - \sigma$.

1.7 Censoring

It relates to lifetime data analysis for mechanistic or biologic system. Replacement of failed components by new ones prolongs the life of a mechanistic system, while such a replacement (or perfect repair) may not always be feasible in biological system. Censoring is the termination of the observation-life due to some cause other than natural failure to which the system is subjected. In real time situation complete enumeration of data is not possible so to deal with data we have different types of censoring.

1.8 Types of Censoring

1.8.1 Type I censoring

If the experiment is performed with definite number of items and we have to stop the experiment at fixed time, Type I censoring happens. Let 'n' items are put on a life test, say $x_1 < x_2 < ... < x_n$. Assuming that $x_1 < x_2 < ... < x_m$, are the 'm' items that have failed before the pre-assigned time t_0 and $(n-m)$ items have survived beyond 't_0'. In this censoring time of termination 't_0' fixed while 'm', the numbers of items that failed before 't_0' is a random variable.

For example, there is a batch of transistors we put them on a test and record their times to failure. Some transistors may take a long time to burn out. We will not want to wait that long to end the experiment. Therefore we might stop the experiment at a predetermined time.

1.8.2 Type II censoring

It occurs if an experiment is performed with definite number of samples or items and we have to stop the experiment at fixed number of failure observed, Type II censoring happens. Let n items are put on a life test. Assuming the lifetime of first 'r' items that failed, say $x_1 < x_2 < ... < x_r$ and it is apparent that $(n-r)$ items have survived beyond 'x_r'. In this censoring 'r', the number of items that failed before is fixed while 'x_r', the time at which the experiment is terminated is a random variable. The sampling of such type is known as Type II censoring. Type II censoring is almost mandatory in dealing with high cost sophisticated items as colour television tubes.

For example, same experiment is performed as in Type I censoring for the transistors and if we decide to wait until say, r failure (number of failure) of the transistors has burned out.

1.8.3 Informative or non-informative censoring

For example, in a cardiovascular problem, suppose that the study relates to the impact of lowering of the cholesterol level on the risk of heart attacks, and the subjects are divided into a treatment group (where cholesterol level is reduced through some medical plan) and a placebo group (where no such medicine is used). Let there will be higher compliance rate in the treatment group than in placebo group, so the censoring variable may not be distributed independently of the primary response variate, this is referred to as informative censoring. When the variable distributed, independent then non-informative censoring occurs. Non-informative censoring is also known as Random censoring.

1.9 Life-time distribution

Life-time distribution is express as the collection of statistical probabilty distributions that are used in the reliability engineering and life data analysis. They provide the prediction about the life time of products in the population by fitting statistical distribution to life data. Statistical distributions formulated by statisticians, mathematicians and engineers to mathematically model or represent certain behaviour. All statistical distributions have their specific probability density function (pdf) which can be expressed mathematically. Most commonly used distributions are:

1.9.1 The Exponential Distribution

This is commonly used for components that are presented a constant failure rate. This is widely used distribution due to its simplicity. The pdf for 1-parametric exponential distribution is

$$f(t) = \lambda e^{\lambda t}$$

The pdf of 2-parametric exponential distribution is

$$f(t) = \lambda e^{\lambda(t-\gamma)}$$

where, λ is the constant failure rate in failures per unit of measurement and γ is the location parameter.

1.9.2 The Weibull Distribution

The Weibull distribution is a general purpose reliability distribution used to model material strength, times-to failure of electronic and mechanical components and equipment. The pdf of 3-parametric Weibull distribution is

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

where, β, η and γ are the shape, scale and location parameter, respectively.
If the location parameter γ is assumed to be zero. Then the distribution reduces to 2-parameter Weibull distribution with pdf

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

1.9.3 The Normal Distribution

The normal distribution is commonly used for general reliability analysis, times-to-failure of simple electronic and mechanical componentsand equipment. The pdf of the normal distribution is

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

where, μ is the mean and σ is the statndard deviation of the times of failure.

1.9.4 The Lognormal Distribution

The lognormal distribution is commonly used for general reliability analysis, cycles-to-failure in fatigue, material strengths and loading variables in probabilistic design. When the natural logarithms of the times-to-failure are normally distributed, then we say that the data follow the lognormal distribution. The pdf of lognormal distribution is

$$f(t) = \frac{1}{t\sigma'\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t^{'}-\mu^{'}}{\sigma^{'}}\right)^2}$$

$$f(t) \geq 0, t > 0, \sigma^{'} > 0$$

$$t^{'} = in(t)$$

where, $\mu^{'}$ is the mean and $\sigma^{'}$ is the statndard deviation of the times of failure.

1.10 Progress in the reliability analysis

The estimation of $R = P_r(X < Y)$ is implemented in some significant distributions as exponential distribution by Kelly et al. (1976) [67], Tong (1974) [109], in normal distribution by Church and Harris (1970) [42], Downton (1973) [45], Woodward and Kelley (1977) [117], in Pareto distribution by Beg and Singh (1979) [16] and exponential families by Tong (1977)

[110]. Enis and Geisser (1971) [47] presented the Bayes estimation of exponentially or normally distributed. Chao (1982) [44] discussed this model for the exponential family. Balagurusamy (1984) [8] introduced reliability in industrial system for the product manufacturing. Awad and Gharraf (1986) [3] considered the estimation of 'R' for the family of Burr XII distributions. MLE of 'P' when X and Y have bivariate exponential distribution has been considered by Awad et al. (1981) [4]. McCool (1991) [78] obtained MLE of a particular transformation of 'R' and obtained its confidence interval. Chaturvedi and Surinder (1999) [36] revisited the problem of estimation of $R = P_r(X < Y)$ in the case of exponential distriution under Type I and Type II censorings and derived UMVUES. Kundu and Gupta (2006) [70] proposed an approximate maximum likelihood estimator of 'P' for Weibull random variables and obtained asymptotic distribution of the MLE and confidence interval. Sracoglu et al. (2009) [96] discussed an comparative study on estimators for stress-strength reliability in the Gompertz case when one parameter is known. For the exponential Weibull distribution, Chaturvedi and Pathak (2012) [31] obtained the reliability function and Tripathi et al. (2016) [111] investigated the 'R' for the Pareto distribution. .

1.11 Sequential Inference

The sequential test is distinguished from the current test procedures (where the sample size is predetermined) as the number of observations required by sequential tests depends on the outcome of observations and is not predetermined but a random variable. In the sequential estimation problem for the testing of null hypothesis H_0, a rule was given for making one of the three decisions: (1) to accept the hypothesis H_0, (2) to reject the hypothesis H_0, (3) to continue the experiment by making an additional observation. If the first or second decision is made, the process is terminated. If the third decision is made, a second trial is performed. If in the second trial again the third decision is made then repeat it. This process is continued until either the first or the second decision is made.

1.12 Sequential Probability Ratio Test (SPRT)

Wald (1945) [114] constructed the Sequential Probability Ratio Test (SPRT) for the sequential problem of testing. A sequential procedures has two aspects: (1) a stopping rule or a rule which tells us when to stop sampling and (2) an action rule, which tells us what type of inference (or decision) to make after sampling has been stopped i.e. what estimate to adopt for a given parametric function or whether to accept or reject a given hypothesis regarding the parameters. For the stopping rule, considering a sequence of iid random variables $x_1, x_2, ...$ with common probability density function(pdf) $f(x|\theta)$ $\forall x$ and θ.

Let us consider the testing of for simple null hypothesis $H_0 : \theta = \theta_0$ against a simple alternative hypothesis $H_1 : \theta = \theta_1$.

For any positive value m, the probability density that the sample $x_1, x_2, ..., x_m$ are observed is given by

$$p_{0m} = \prod_{i=1}^{m} f(X_i|\theta_0)$$

when H_0 is true and

$$p_{1m} = \prod_{i=1}^{m} f(X_i|\theta_1)$$

when H_1 is true.

The SPRT for H_0 against H_1 is defined as follows: there are two positive constant numbers $\mathcal{A}$ and $\mathcal{B}$ are chosen, such that $0 < \mathcal{B} < 1 < \mathcal{A}$ and at each stage of the experiment, the probability ratio $\frac{p_{1m}}{p_{0m}}$ is computed, and if

$$\mathcal{B} < \frac{p_{1m}}{p_{0m}} < \mathcal{A}$$

the experiment is continued by taking an additional observation and stop taking further observations as soon as one of the inequalities is violate

$$\text{reject } H_0 \text{ if } \tfrac{p_{1m}}{p_{0m}} \geq \mathcal{A}$$
$$\text{accept } H_0 \text{ if } \tfrac{p_{1m}}{p_{0m}} \leq \mathcal{B}$$

For the practical computation, it is more convenient to compute the logarithm of the ratio i.e.,

$$\begin{aligned} ln\left(\frac{p_{1m}}{p_{0m}}\right) &= \sum_{i=1}^{m} ln\left(\frac{f(x_i,\theta_1)}{f(x_i,\theta_0)}\right) \\ &= \sum_{i=1}^{m} z_i \end{aligned}$$

where, $z_i = ln\left(\frac{f(x_i,\theta_1)}{f(x_i,\theta_0)}\right)$. For this we continue taking sample if $ln\mathcal{B} < \sum_{i=1}^{m} z_i < ln\mathcal{A}$ and we reject H_0 if $\sum_{i=1}^{m} z_i \geq ln\mathcal{A}$ and accept H_0 if $\sum_{i=1}^{m} z_i \geq ln\mathcal{B}$.

1.13 Origin of Sequential Inference

H. F. Dodge and H. G. Romig (1929) [44] constructed a double sampling procedure to decide whether a second sample should be drawn or not. This decision depends on the outcome of the first sample. But this method allows only two samples. Walter Bartky (1943) [13] devised multiple sampling schemes for the particular of testing the mean of a binomial distribution. This sampling is closely related to the test procedure that results from the application of the sequential probability ratio test to this particular case. This double sampling and multiple sampling required on an average a smaller number of observations than single sampling.

Hotelling (1942) [62] has given a idea to designing a large scale experiment which may be regarded as a forerunner of sequential analysis. P. C. Mahalanobis (1940) [76] carried out a example of such type of experiment by the sample censuses of area of jute in Bengal. After this sampling census steadily increasing in size. The importance of the development of such large scale survey was considered by Wald (1945) [114].

During World War II in the Statistical Research Group of Columbia University, Milton Friedman and W. Allen Wallis conjectured that a sequential testing procedure might be constructed to control possible errors committed by wrong decisions with exactly same extent as the current procedures based on a predetermined number of observations. This leads the importance of the sequential probability ratio test in the development of military and naval equipment.

1.14 Characteristics for the performance of SPRT

Sequential Probability Ratio Test (SPRT) can be measured by the following two criteria:

1.14.1 Operating Characteristic (OC) Function

Operating Characteristic (OC) fuction denotes the probability that the sequential test will lead to the acceptance of the hypothesis H_0 when θ is the true mean value. It is defined as $L(\theta)$ and relates how well the test procedures achieves its objective of making correct decisions.

If we are interested in the two values of θ as θ_0 and θ_1. Let us consider the null hypothesis $H_0 : \theta = \theta_0$ and the alternative hypothesis $H_1 : \theta = \theta_1$. Then $L(\theta)$ is defined as

$L(\theta)$=**Probability of accepting H_0 when H_0 is the true parameter value**

The OC function is very closely related to the notation of power function in the traditional (fixed sample) theory of tests. For any parameter point θ which is not consistent with the hypothesis H_0, the power is the probability of rejecting H_0 when θ true value is. Hence for any θ not consistent with H_0, the power is $1 - L(\theta)$. $1 - L(\theta)$ is also defined as the probability of Type I error.

An OC function is considered the more favorable the higher value of $L(\theta)$ for θ consistent with H_0 and the lower the value of $L(\theta)$ for θ not consistent with H_0. The mathematical formula for OC functionis

$$L(\theta) = \frac{\mathcal{A}^t - 1}{\mathcal{A}^t - \mathcal{B}^t}$$

$\forall\ \theta$, t is given as $t \neq 0$ and either $t < 0$ or $t > 0$ with $E[e^{z_i}]^t = 1$

1.15 The ideal OC curve

The plots of the points $(\theta, L(\theta))$ for all θ, the corresponding plot is called the operational characteristic curve. Let the hypothesis H_0 to be tested and true parameter point θ lies in a given set ω of the parameter points. Since the probability of accepting H_0 is defined as the OC function $L(\theta)$ and an OC function is considered more desirable the higher value of $L(\theta)$ for any θ in ω. An ideal OC function would be given by a function $L(\theta)$ such that $L(\theta) = 1$ for any θ in ω and $L(\theta) = 0$ for any θ outside ω. Suppose there is only one unknown θ and the hypothesis to be tested is the statement that $\theta \leq \theta_0$. The ideal OC function would be defined as $L(\theta) = 1$ for $\theta \leq \theta_0$ and $L(\theta) = 0$ for $\theta > \theta_0$. An ideal OC curve and typical OC curve is given in the **Fig. 1.2** and **Fig. 1.3**, respectively.

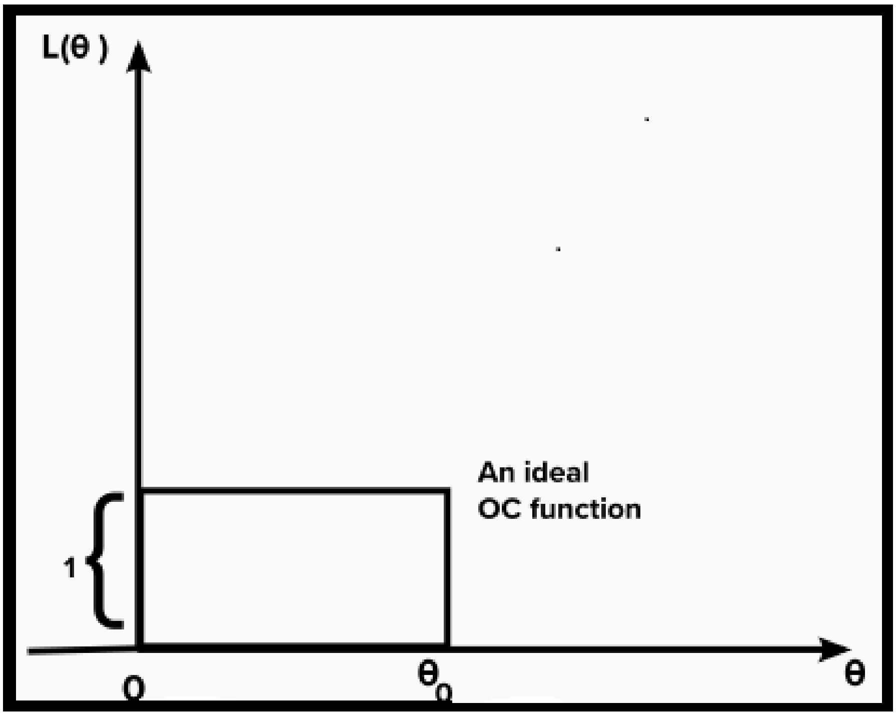


Fig. 1.2: Ideal OC Curve

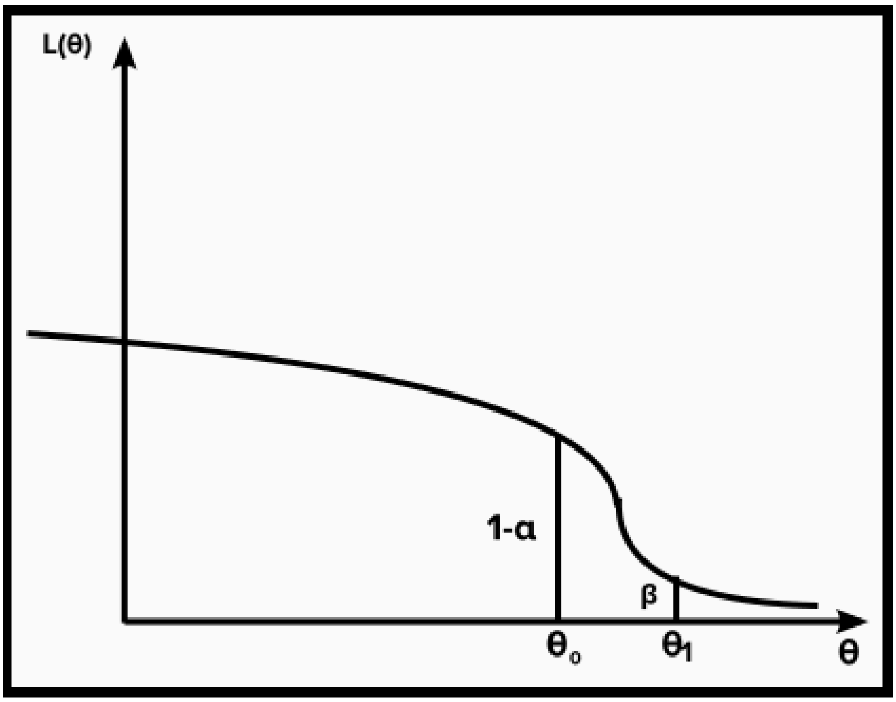


Fig. 1.3: Typical OC Curve

1.16 Average Sample Number (ASN)

The number of observation N required by a sequential procedure to reach a decision is not predetermined but a random variable. If we repeat the same sequential then we shall get different values N. Small value of N on an average is preferable. This average value of N is called the Average Sample Number (ASN). The ASN is given as

$$E_\theta(N) = \frac{L(\theta)log\mathcal{B} + [1 - L(\theta)]log\mathcal{A}}{E_\theta(Z)}$$

The graphical representation of ASN curve is given in **Fig. 1.4**

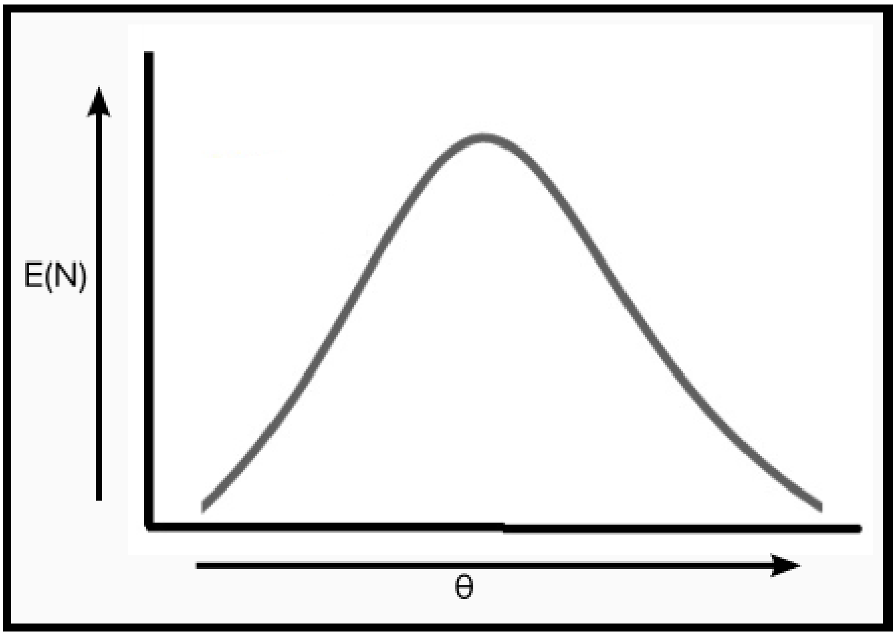


Fig. 1.4: ASN Curve

1.17 Acceptance and Rejection Regions

Considering the sequence of iid random variables $X_1, X_2, ...$ For testing the simple hypothesis $H_0; \theta = \theta_0$ against simple alternative $H_1 : \theta = \theta_1$ with fixed $0 < \alpha, \beta < 1$. Let us consider, $\mathcal{A} = \frac{(1-\beta)}{\alpha}$ and $\mathcal{B} = \frac{\beta}{(1-\alpha)}$ and $W(m) = \sum_{i=1}^{m} Y_i$ and N be the first integer, $m(\geq 1)$ with $\sum_{i=1}^{m} Z_i \leq b$ or $\sum_{i=1}^{m} Z_i \geq a$ where $a = ln(\mathcal{A})$, $b = ln(\mathcal{B})$ and

$$Z_i = ln\left[\frac{f(x_i, |\theta_1)}{f(x_i, |\theta_0)}\right], i = 1, 2, ...$$

having inequality $W(m) \leq e_1 + um$ or $W(m) \leq e_2 + um$ with

$$e_1 = \frac{ln(\mathcal{B})}{\text{coeff. of}\, x_i},\ e_2 = \frac{ln(\mathcal{A})}{\text{coeff. of}\, x_i} \text{ and } u = \frac{\text{intercept form}}{\text{coeff. of}\, x_i}$$

The graphical representation of acceptance and rejection regions is given in **Fig. 1.5**

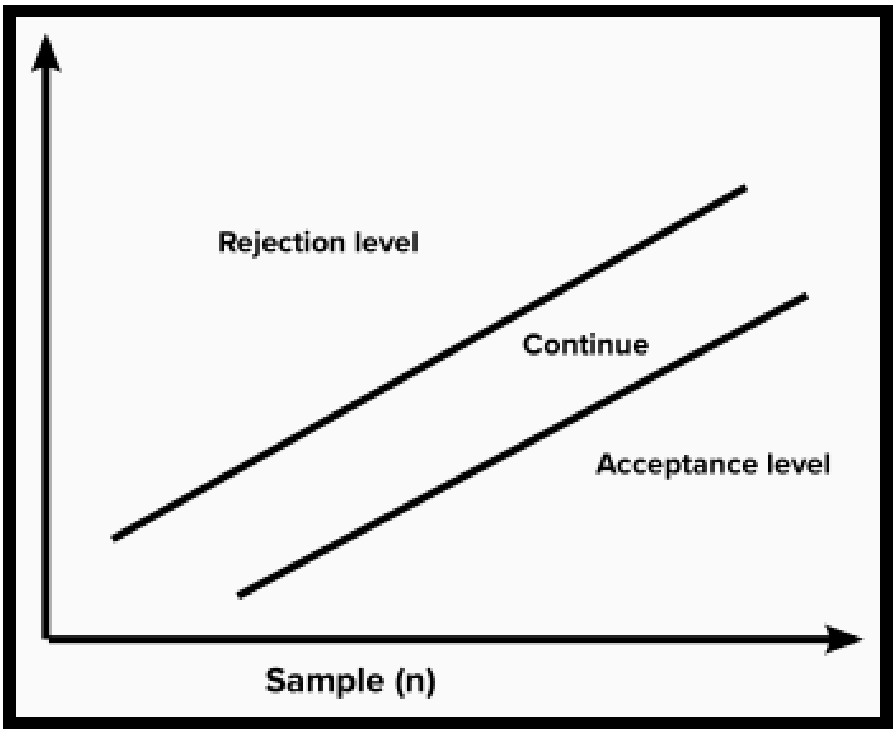


Fig. 1.5: Acceptance and Rejection Region

1.18 Growth in the field of Sequential Analysis

Wald (1947) [114] invented the theoretical expression for the OC and ASN function. Oakland (1950) [85] derived the SPRT for the negative binomial distribution with known dispersion parameter. Epstein and Sobel (1955) [48] develeped SPRT for the scale parameter of an exponential distribution. Phatarfod (1971) [87] developed SPRT for the composite hypotheses for shape parameter of the gamma distribution. Robustness testing and estimation for the exponential distribution studied by Barlow and Proschan (1967) [10] and Hager, Bain and Antle (1971) [58]. Chaturvedi, Kumar and Kumar (1998) [38] derived the robustness of the sequential procedures for the life-time models. Chaturvedi et al. (2000) [39] obtained the SPRT'S for the parameters of a family of distributions, on assigning different values of parameters it covers many distributions. Pandit and Gudaganavar (2010) [86] derived the SPRT for the scale parameter of gamma and exponential distributions. Kharin (2016) [68] investigated the performance and robustness of sequential tests. Ning and Opperman (2019) [83] obtained the SPRT for skew normal distribution.

1.19 Bayesian Inference

In statistical inference, there are two methods to explain probability: Classical and Bayesian. Classical estimation defines probability as the limit of the relative frequency of an event in a large number of experiments, and only in the sense of a well-defined random experiment. When a random process is not associated, Bayesian estimation can impose a probability on each sentence. In Bayesian theory, the probability is a way to express the degree of personal belief. For example, suppose a 30-year-old man has a positive prostate cancer marker blood test. Assume that the accuracy of this test is about 90 percent. If an individual wants to know the possibility that he has prostate cancer. He has a positive result, but the information in front of him is only the possibility of a positive test. This situation is related to Bayesian inference. Application of Bayesian inference has a wide range i.e. science, engineering, philosophy, medicine, sport, and law.

1.19.1 Bayes theorem

Bayes theorem is the primary element of Bayesian statistical methods. Bayes' theorem gives the relativity of two conditional probabilities, which are opposite to other conditional probabilities. The term Bayes' theorem is to commemorate the pastor Thomas Bayes, and is called Bayes' law. The theorem takes the prior probability of A, the prior probability of B, and the conditional probability of B as A to represent the conditional probability or posterior probability of event A after observation of B. It is valid in all probability interpretations. Bayesian formula is how to use data to modify probability statements. The Bayes rule is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The conditional probability is defined as

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$$

1.19.2 Model based on Bayesian inference

The root of Bayesian inference is continued by Byes theorem by replacing event B by the observations y, A by the set of parameter Θ, and the probability by the densities p. Then the result is

$$P(\Theta|y) = \frac{p(y|\Theta)p(\Theta)}{P(y)}$$

where, p(y) is the marginal likelihood of y, $p(\Theta)$ is the set prior distribution of the set of parameter $\Theta(= \theta_1, \theta_2, ..., \theta_j)$ before y is observed, $p(y|\Theta)$ is the likelihood of y underneath a model and $p(\Theta|y)$ is the joint posterior distribution of Θ that expresses uncertainty about parameter set Θ after taking both the prior and data into system.

1.20 The components of Bayesian Inference

Following are the components of Bayesian inference:

- $p(\Theta)$ is the prior distribution for set of parameter Θ, and uses probability as a methods of quantifying uncertainty about Θ before taking the data into system.
- $p(y|\Theta)$ is the function of likelihood which all variables are associated in a full probability model.
- $p(\Theta|y)$ is the joint posterior distribution that shows uncertainty about Θ after taking both the prior and the data into system.

1.21 Prior distribution

The prior distribution is the principle concept of Bayesian inference. It shows the relation for uncertain Θ, is then combined with the probability distribution of the new data to produce the posterior distribution. A posterior distribution used for the future inference about Θ. There is no definite rule for selecting the prior distribution. The choice of prior distribution is subjective and is based on a person's own experience and judgment.

1.22 Types of priors

1.22.1 Informative Prior and Non-informative Prior

An informative prior represents precise and fixed information available for the interest variable say, θ. This accessible information is attached to the posterior distribution. it is to adjust the new posterior distribution more on the current data. This prior applies to those cases where we have few or no data for the parameter θ. If the relevant previous information is not convenient then we use Non-informative.

1.22.2 Proper and Improper Prior

A prior distribution that integrals equal to 1 (i.e the condition of a probability distribution) is a proper prior. Let $p(\theta)$ be the prior distribution for the parameter θ, then

$$\int_0^{\infty} p(\theta)d\theta = 1$$

or,

$$\sum p(\theta)d\theta = 1$$

Sometimes the integral (or summation) of prior distribution is not equal to 1, then such a prior is a Improper Prior. Let $p(\theta)$ be the prior distribution for the parameter θ, then

$$\int_0^{\infty} p(\theta)d\theta \neq 1$$

or,

$$\sum p(\theta)d\theta \neq 1$$

The Non-informative prior frequently leads to an improper prior. The Uniform distribution on an infinite interval and Beta (0, 0) are the example of such priors.

1.22.3 Jeffreys' Non-informative Invariant Prior

Jefferys' constructed a rule to select the non-informative prior g(θ) as : If the sample space Ω has the range $-\infty$ to ∞, then the parameter θ has uniform distribution (g(θ)=constant): if sample space has the range θ to ∞, then log θ has uniform distribution $\left(g(\theta) = \dfrac{1}{\theta}\right)$: If the parameter θ be a real or vector-valued then $\left(g(\theta) = [I(\theta)]^{\frac{1}{2}}\right)$.
where, $I(\theta) = -E\left[\dfrac{\partial^2 log f(x|\theta)}{\partial\theta_i\partial\theta_j}\right]$ is a Fisher information matrix. This prior is also known as non-informative, vague or ignorance prior. **Table 1.1** represents jeffreys' prior for some known distribution.

Table 1.1

Jeffreys' Prior for some known distributions	
Distributions	**Jeffreys' Prior**
Bernoulli (θ)	$(\theta(1-\theta))^{-1/2}$
NBin(r, θ)	$\theta(1-\theta)^{-1/2}$
Poisson(θ)	$\theta^{-1/2}$
U$(0, \theta)$	θ^{-1}
N$(0, \sigma)$	σ^{-1}
N(μ, σ^2)	σ^{-2}
N(θ, θ^2)	θ^{-1}
Weibull (p, θ)	$(\theta p)^{-1}$

1.22.4 Uniform Prior

In Bayesian Inference, Uniform prior is most well known and accepted prior to a state of ignorance. Laplace suggested that in the absence of sufficient reason for assigning unequal probabilities to the values in the parameter space, uniform prior is the best in comparison to other priors. Uniform prior has a property of invariant under linear transformation.

1.22.5 Conjugate priors

The property that the posterior distribution follows the same parametric form as the prior distribution is called conjugacy and a prior which follows this property is Conjugate Prior. This family or prior is mathematically convenient to solve because the posterior distribution follows a known parametric form. **Table 1.2** represents Natural Conjugate prior for some known distribution.

Table 1.2

Natural Conjugate Prior for known distributions	
Distributions	**Natural Conjugate Prior**
Bin(n,θ), n known	Beta(a_1, a_2)
NBin(r, θ), r known	Beta(a_1, a_2)
Poisson(θ)	Gamma(a_1, a_2)
Hypergeometric(a,b,c)	Bin(a_1, a_2)
U(0, θ)	Pareto(a_1, a_2)
N(θ,σ^2), σ^2 known	N(μ,τ^2)
N(θ,σ^2), θ known	Inverted-Gamma(a_1, a_2)
N(θ, r), θ known	Gamma(a_1, a_2)
N(θ,σ^2)	Normal–Inverse Gamma(a, b, c, d)
N((θ, r)	Normal Gamma(a, b, c, d)
Pareto (k, θ), k known	Gamma(a_1, a_2)
Gamma(m, θ), m known	Gamma(a_1, a_2)

1.23 Likelihood

Bayesian inference should be complete when the prior distribution and likelihood must be known and specified. It includes information about the parameter in terms of samples as is established from the joint probability distribution. The concept of likelihood was first introduced by R.A. Fisher (1925) [51] in statistical modeling and inference. Let $f(x|\theta)$ be a joint pdf or pmf of the sample $X = (X_1, X_2, ..., Xn)$. Then the likelihood function for the $X = x_i$, is defined as

$$L(\theta|x) = \prod_{i=1}^{n} f(x|\theta)$$

1.24 Posterior Distribution

The posterior distribution is the product of prior density and the likelihood function. It shows the current or updated knowledge about all the uncertain quantities in the Bayesian inference.

Posterior distribution=Prior distribution* likelihood

1.25 Graphical interpretation of Prior, Likelihood and Posterior Distribution

Fig. 1.6 represents plots for the various values of the parameter θ to the corresponding values of the Prior distribution (green), the Likelihood function (blue) and the Posterior distribution(Red) respectively. From the graph, we see that the posterior distribution is the combination depending on the strength and properties of the prior and quality of data used to derive the likelihood.

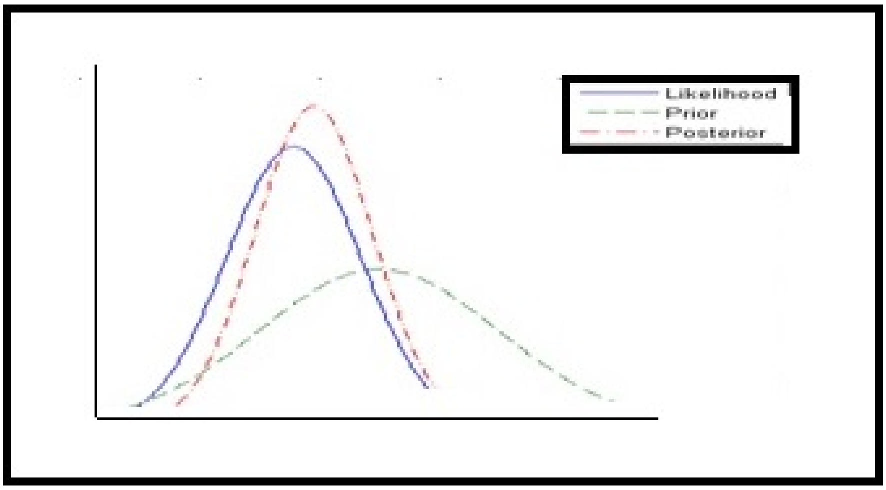


Fig. 1.6: Standard Prior, Likelihood and Posterior distribution Graph

1.26 Loss Functions

In Bayesian inference, the estimation of unknown parameter θ produces the loss function and is denoted by L(θ, a). The loss function reflects the fact that if an action a is close to the parameter θ, then the decision a is reasonable and little loss is incurred and if far from θ, then a large loss incurred. So we can say that the loss function is a non-negative function that is increasing as the distance between 'a' and 'θ' increases.

1.27 Types of Loss Functions

Squared Error Loss Function (SELF) is introduced by Legendre(1805) [72] and Gauss (1810) [52] to develop the least-squares theory. It is the most common loss. It is significant because it gives equal importance to both losses: overestimation and underestimation. The Squared Error Loss Function(SELF) is defined as

$$L(\theta, \theta_{SELF}) = (\theta - \theta_{SELF})^2$$

Quadratic Loss Function (QLF) is the modified form of the squared error loss function. It is the symmetric loss function. The Quadratic Loss Function(QLF) is defined as

$$L(\theta, \theta_{QLF}) = \left(\frac{\theta - \theta_{QLF}}{\theta} \right)^2$$

Precautionary Loss Function (PLF) was introduced by Norstrom (1996) [84]. It is an asymmetric loss function and also derived a class of the precautionary loss functions as a exceptional case. This loss function is useful for the problem of underestimation. The Precautionary Loss Function(PLF) is defined as

$$L(\theta, \theta_{PLF}) = \frac{(\theta - \theta_{PLF})^2}{\theta_{PLF}}$$

Linex Loss Function was introduced by Varian (1975) [112]. It is asymmetrical loss function termed as the Linex Loss function. This loss function is rising exponentially on one side of zero and almost linearly on the other side of zero.

General Entropy Loss Function (GELF) was proposed by Calabria and Plucini (1996) [22]. It is the modified Linex loss function, which is more appropriate for the location parameter not for the scale and other parameters. GELF suggested overcoming this problem of linex loss function.

K-Loss Function (KLF) was proposed by Wasan (1970) [115]. It is precisely defined for the scale parameter of the distribution to compute the inaccuracy of an estimator.

In **Table 1.3**, the mathematically expression for Loss function and Baye estimator are given-

Table 1.3

Different Loss function and their Bayes Estimator		
Loss Functions	**L(θ, a)**	**Bayes Estimator**
SELF	$(\theta - a)^2$	E(θ—x), posterior mean
Weighted SELF	$\dfrac{(\theta - a)}{\theta}$	$(E(\theta^{-1}\|x))^{-1}$
Quadratic	$(1 - \frac{a}{\theta})^2$	$\frac{E(\theta^{-1}\|x)}{\theta^{-2}\|x}$
Precautionary	$\frac{(\theta-a)^2}{a}$	$\sqrt{E(\theta^2\|x)}$
Linex Loss	$exp(c(a-\theta)) - c(a-\theta) - 1$	c $logE(\theta^{-c\theta})\|x)$
General Entropy	$b[(\frac{a}{\theta})^c$-c$log\frac{a}{\theta} - 1]$,c$\neq 0, b > 0$	$[E(\theta^{-c}\|x)]^{-1/c}$
	i) c=1, b$\left(\frac{a}{\theta}\right) - log\frac{a}{\theta} - 1$	$[E(\theta^{-1}\|x)]^{-1}$
	ii) c=-1, b$\left(\frac{\theta}{a}\right) - log\frac{\theta}{a} - 1$	$[E(\theta\|x)]$
K-Loss	$\left(\sqrt{\frac{a}{\theta}} - \sqrt{\frac{\theta}{a}}\right)$	$\sqrt{\frac{E(\theta\|x)}{E(\theta^{-1}\|x)}}$

1.28 Bayes Estimator

Let ‘a’ be a action which is taken to estimate unknown parameter and L(θ, a) represent the loss incurred when the true value of parameter is θ. The Bayes estimator ‘a*’ is that estimator that minimizes the posterior expected loss.

1.29 Types of risk in Bayesian inference

1.29.1 Risk Function

The risk function $R(\theta, \hat{\theta})$ is the average loss that will be incurred if the estimator $\hat{\theta}$ is used at a given value θ. We would be more preferable to the small value of $R(\theta, \hat{\theta})$ for all θ, which is the true value of the parameter. It is used by Neyman, Pearson and Wald to compare the classical estimators and to obtain the best estimator.

1.29.2 Bayes Risk

In Bayesian, to measure the loss function we would use a prior distribution. it is the average loss, where the expectation is for the prior distribution. Here we are interested in the value of estimator that gives the smallest numerical value of Bayes risk.

1.29.3 Posterior Risk

The risk where, expectation is taken with respect to posterior distribution of the parameter such loss is posterior risk.

1.30 Important Aspects

1.30.1 Markov Chain Monte Carlo Method (MCMC) Techniques

This Techniques has become the basis of many modern scientific analyses. It provides a simple method for numerically estimating the values from an unknown distribution and uses those estimating values to perform subsequent analyses. The utility of these methods has as long as original Monte Carlo techniques but their impact on Statistics was started on the 1990. Robert and Casella (2010) [93] founded a comprehensive entry into the history of MCMC methods. In MCMC Techniques sometimes the direct sampling is not easy and we have to obtain the sequence of samples from a probability distribution then we used the Metropolis-Hastings algorithm. The MCMC and Metropolis-Hastings algorithm is mostly used for the sampling from multi-dimensional distributions i.e. when the number of dimensions is high. Gibbs Sampling is another type of MCMC algorithm to obtaining a random samples from a specified multivariate probability distribution, when the direct sampling is not easy. In the statistical inference, it is mostly used in the Bayesian inference to generate the samples from the posterior distributions.

1.30.2 Methods of Random Variate Generation

For the general probability models we have two methods to generate random numbers

(1) Inverse Transformation Method:

A method to generate a random variates from a non-uniform distribution to the uniform variates by applying a transformation is the Inverse Transformation method. Each realization of the non-uniform random variable might be obtained from a single uniform variate or from a sequence of uniforms.

(a) Inverse CDF Method

Let X be a random variable with continuous cumulative distribution function, or CDF P_X, then the random variable $U = P_X(X)$ has a uniform distribution U(0,1). The simple relationship with a uniform random variable U and a random variable X is $X = P_X^{-1}(U)$ is the inverse CDF method. This method does not apply to multivariate distributions.

(2) Accept-Reject Method

This method is one of the most important methods in random number generations. Unlike the inverse CDF method, this method applies to multivariate random variable but they are not efficient in high dimensions. To generate a random variable X with probability density $p_{(X)}$, we chose another random variable Y having probability density $g_Y(x)$, by using some constant 'c' so that $cg_Y(x) \geq p_X(x) \, \forall x$. The density $g_Y(x)$ is known as trial density, proposal density or instrumental density and $cg_Y(x)$ is known as envelop function.

1.31 Advancement in the Bayesian analysis

Bhttacharya (1967) [18], introduced Bayesian analysis in reliability estimation for the uniform, inverted gamma and exponential prior densities under type II censoring for one parameter exponential distribution. Soland (1968) [90] use the Bayes approach to reliability analysis of Weibull distribution. Bayes estimator of reliability model for exponentially or normally distribution was made by Enis and Geisser (1971) [47]. Introduction of Non-parametric empirical Bayes estimator on the reliability model is given by Furguson (1973) [50] and Hollander and Korwar (1976) [61] and the study of system reliability by Bhattacharya and Johnson (1974) [19]. Bayesian estimation in reliability model is done by several authors, for the brief review one may see, Tyagi and Bhattacharya (1989) [111], developed the Bayes estimator for the Maxwells velocity distribution function. For the parameters of Poisson distribution, Alvandi (1990) [2] describe the bayesian estimation under LINEX loss function. Basu and Ebrahimi

(1991) [14], visited the Bayesian inference to the life testing and reliability estimation by using asymmetric loss function. Calabrai and Pulcini (1994) [21], estimated the Bayes estimators for inverse Weibull distribution. Chatruvedi and Rani (1998) [32], introduced the estimation of classical and Bayesian both reliability for the generalized Maxwell failure distribution. Similar work is don by Chatruvedi and Tomer (2002) [37] for the negative binomial distribution and Chatruvedi, Pathak (2013) [30], for binomial and Poisson distributions. Chatruvedi and Singh (2006) [35], developed the Bayesian estimation procedures for a family of lifetime distributions in the case of squared-error and entropy losses. For more review one may see to Martz and Waller (1982) [76], Bansal (2007) [9] and Chatruvedi et al. (2007) [40].

1.32 Content of the Thesis

Chapter 1 of the thesis provides an introductory review of the research topics covered and also state of significance as well as their application.

In the **Chapter 2**, we assume the case when stress X follows the Exponentiated Weibull distribution and the strength Y follows the Power function distribution. Through establishing the relationship among the parameters and on assigning the different possible values for the parameters under study, the probability of disaster is studied and their behaviour is also studied by obtaining the numerical values.

In the **Chapter 3**, we the considered the pdf derived by Chaturvedi and Singh (2008) [25]. The problem of estimating $R(t) = P_r(X > t)$, which is defined as the probability that a system survives until time t and $R = P_r(Y > X)$, which reprsents the stress-strength model are revisited. In order to obatain the maximum likelihood estimators (MLE'S), uniformly minimum variance unbiased estimators (UMVUS'S), interval estimators and the Bayes estimators for the considered model, the technique of transformation method is used.

In the **Chapter 4**, the sequential testing procedures (SPRT) and robustness of SPRT in respect of OC and ASN functions when the distribution under study has undergone a change are derived for the parameters (shape and rate) of Erlang distribution under the two simple hypotheses. The acceptance and rejection regions for simple hypotheses vs simple alternative are derived for rate parameter of the distribution. The mathematical expressions for the robustness of the SPRT of OC and ASN functions for the rate parameter of distribution, when the coefficient of variation is known are also studied. These results are presented through the Tables and Graphs, so that one may see the numerical evaluated departures in respect of OC and ASN functions.

In the **Chapter 5**, we considered a family of lifetime distributions given by Liang (2008) [74]. For the scale parameter of such distribution, Bayes estimators and posterior risk are evaluated for the comparison under the informative priors and non-informative priors. For the comparison on the basis of loss functions: Squared Error Loss Function (SELF), Quadratic Loss Functions (QLF) and Precautionary Loss Functions (PLF) are considered. The performance of the estimator is assessed on the basis of its relative posterior risk. Markov Chain Monte Carlo (MCMC)are used for Simulation to compare the performance of these estimators.

In the **Chapter 6**, risk, posterior risk and Bayes risk is obtained for the parameters of the three parameter Generalised Rayleigh Distribution under type II censoring. Here we are considering the positive and negative powers of the parameters. Bayes estimators of the reliability function $R(t) = P_r(X > t)$ and the stress-strength model $P = P_r(X > Y)$ is also obtained with the help of Bayes estimators. Numerical findings are presented through the Tables and Figures.

Chapter 2

Stress-Strength Relationship Between Exponentiated Weibull Distribution and Power Function Distribution

2.1 Introduction

In every field of manufacturing industry, technology, transportation, maintenance of engineering systems and equipments, etc., the product reliability plays a significant role. Due to complex and complicated mechanism the under and over estimation of the factors may engender great losses. For the purpose of the reliable products and to fulfill the customer's demands of high quality products, the tools of reliability testing are very useful. Reliability tools give the probability of the systems/devices to perform its stated purpose adequately without any failure.

Reliability measure $P_r(X > t)$, where 't' is the time and $P = P_r(Y > X)$ (Stress–Strength models) are the most popular measures of the reliability analysis. For literature review, see Basu (1964) [15], Barlow and Proschan (1967) [10], Church and Harris (1970) [42], Enis and Geisser (1971) [47], Downton (1973) [45], Tong (1974) [108], Kelly et al. (1976) [67], Sinha and Kale (1980) [100], Sathe and Shah (1981) [96], Tong (1977) [110], Chao (1982) [24], Awad and

Gharraf (1986) [3], Constantine et al. (1986) [43], Bain and Engelhardt (1991) [7], Chaturvedi and Surinder (1999) [36], Chaturvedi and Sharma (2007) [34], Surinder and Mayank (2014) [104] and Surinder et al. (2020) [102].

Here, we have considered the problem of measuring $P = P_r(Y > X)$, when the strength Y follows the Power function distribution and stress X follows Exponentiated Weibull distribution $EWD(\chi, \eta)$. Strength is taken as Power function distribution to justify the fact that the strength of the items is always finite. The related probability's of disaster, that occurs when $P_r(X > \tau)$ is studied.

We have considered the $EWD(\chi, \eta)$, proposed by Mudolker and Srivastava (1993) [79] as an extension of the Weibull family obtained by adding a second shape parameter. The pdf and cdf of $EWD(\chi, \eta)$ are, respectively given by

$$\begin{aligned} f(x; \rho, \eta, \chi) &= \chi\eta\rho x^{\eta-1} e^{-\rho x^{\eta}} (1 - e^{-\rho x^{\eta}})^{\chi-1}; \quad x > 0; \rho, \eta, \chi > 0 \\ F(x; \rho, \eta, \chi) &= (1 - e^{-\rho x^{\eta}})^{\chi-1}; \quad x > 0; \rho, \eta, \chi > 0 \end{aligned} \tag{2.1.1}$$

On taking the different values for the parameters in (2.1.1), we have: generalized exponential distribution, when $\eta = 1$, exponential distribution, when $\chi = 1$ and $\eta = 1$ and Burr X Type distribution, when $\eta = 2$. This distribution is an essential model to study the life testing experiment and for fitting the extreme value data.

2.2 The Stress-Strength Reliability, $P_r(X > \tau)$

Let us consider the pdf of Power function distributions, given by

$$g(y; \zeta, \tau) = \frac{\zeta}{\tau}\left(\frac{y}{\tau}\right)^{\zeta-1}; 0 < y < \tau, \zeta > 0 \tag{2.2.1}$$

If the random variable X and Y follows the $EWD(\chi, \eta)$ given at (2.1.1) and Power distribution function given at (2.2.1), respectively, then $\alpha = P_r(X > \tau)$ the probability of disaster is given by

$$\begin{aligned}\alpha &= P_r(X > \tau) \\ &= \int_{\tau}^{\infty} \chi\eta\rho x^{\eta-1} e^{-\rho x^{\eta}} (1 - e^{-\rho x^{\eta}})^{\chi-1} dx\end{aligned} \tag{2.2.2}$$

On substituting $z = (1 - e^{-\rho x^{\eta}})$ in (2.2.2), we get

$$= \chi \int_{(1-e^{-\rho x^{\eta}})}^{1} \quad z^{\chi-1} dt$$

Thus,

$$\alpha = 1 - (1 - e^{-\rho m})^{\chi} \tag{2.2.3}$$

where $m = \tau^{\eta}$

2.3 Numerical study for probability of disaster and m at varying values of 'χ' and 'ρ'

The numerical study of Probability of Disaster $\alpha = P_r(X > \tau)$ for different combinations of m, ρ and χ are computed from (2.2.3). This is evident from Table 2.1 that as the value of 'm' increases the probability of disaster decreases.

Consequently, we may generate the values of 'm' for fixed ρ and χ for different tolerance level of α from (2.2.3), which is presented in Table 2.2. Finally, these obatained values are used to find the optimum cost of the manufactured item at desired tolerance level.

2.4 The Stress-Strength Reliability, $R = P_r(Y > X)$

Let for the Stress-Strength model, random variable X follows the pdf (2.1.1) and random variable Y follows the pdf (2.2.1), respectively. Then R is given by

$$\begin{aligned} R &= \int_0^\tau \chi\eta\rho x^{\eta-1}e^{-\rho x^\eta}(1-e^{-\rho x^\eta})^{\chi-1}\frac{\zeta}{\tau^\zeta}\left(\int_x^\tau y^{\zeta-1}dy\right)dx \\ &= \int_0^\tau \chi\eta\rho x^{\eta-1}e^{-\rho x^\eta}(1-e^{-\rho x^\eta})^{\chi-1}\frac{\zeta}{\tau^\zeta}\left(\frac{v^\zeta - x^\zeta}{\zeta}\right)dx \\ &= \int_0^\tau \chi\eta\rho x^{\eta-1}e^{-\rho x^\eta}\left(1-e^{-\rho x^\eta}\right)^{\chi-1}dx - \frac{1}{\tau^\zeta}\int_0^\tau \chi\eta\rho x^{\eta-1}e^{-\rho x^\eta}(1-e^{-\rho x^\eta})^{\chi-1}dx \end{aligned} \tag{2.4.1}$$

On substituting $(1-e^{-\rho x^\eta}) = z$, in (2.4.1), we get

$$= \int_0^{1-e^{-\rho\tau^\eta}} z^{\chi-1}dt - \frac{\chi}{\tau^\zeta}\int_0^{1-e^{-\rho\tau^\eta}}\left[\frac{-1}{\rho}log(1-z)\right]^{\frac{\zeta}{\eta}} z^{\chi-1}dt \tag{2.4.2}$$

On using the expansion of $ln(1-x) = -x - \dfrac{x^2}{2}.....\mid x \mid\leq 1$, in (2.4.2), we get

$$= \left(1-e^{\rho\tau^\eta}\right)^\chi - \frac{\chi}{\tau^\zeta}\int_0^{1-e^{-\rho\tau^\eta}}\left[\frac{-1}{\rho}(-z)\right]^{\frac{\zeta}{\eta}} z^{\chi-1}dt$$

$$= \left(1-e^{\rho\tau^\eta}\right)^\chi - \frac{\chi}{\tau^\zeta\rho^{\frac{\zeta}{\eta}}\left(\frac{\zeta}{\eta}+\chi\right)}\left(1-e^{-\rho\tau^\eta}\right)^{\left(\frac{\zeta}{\eta}+\chi\right)}$$

$$R = (1-e^{\rho m})^\chi - \frac{\chi}{\left(\frac{\zeta}{\eta}+\chi\right)m^{\mu/\eta}\rho^{\frac{\zeta}{\eta}}}\left(1-e^{-\rho m}\right)^{\left(\frac{\zeta}{\eta}+\chi\right)}; \tag{2.4.3}$$

where $m = \tau^\eta$

For the different values of parameters χ, η and ρ, we can obtained the numerical values for the stress-strength model and are presented in Table 2.3.

2.5 Analytical study

Let for the manufactured item the strength of an item follows Power function distribution and it is desired that the maximum feasible values of 'τ' may have some upper bound say 'τ_0'.

For example, the maximum speed of an engine must not be increased a permissible vibrations capacity. For a fixed α, suppose τ_α is the desired value of τ. For $\tau_\alpha < \tau_0$, one may obtain the desired value of ζ say ζ_α, through utilizing Table 2.3, so that the item manufactured with the strength distribution having parameters $(\zeta_\alpha, \tau_\alpha)$ and subsequently, the desirable strength reliability may be achieved. For $\tau_\alpha > \tau_0$, one have to either reallocate α or have to go for some alternate item.

2.6 Example based on the present study

Let us consider the maximum possible value of 'm' is 6. When $\alpha \leq 0.01$ the value of $m \geq 2$. As m may not exceed 6, thus one require to fix the device in such a way so that $2 \leq m \leq 6$ i.e. $7.84 \leq \tau \leq 16.81$ and consequently, ζ leads to a maximum of $P_r(Y > X)$. In the practical situation we have seen that the cost of manufactured item depends on the mean of strength. In our case, $E(Y) = \dfrac{\zeta\tau}{\zeta+1}$, which implies that the mean strength increases by increasing either of the two parameters 'τ' and 'ζ'.

Let E_1 and E_2 be the costs of adjusting one unit of τ and ζ, respectively.

Minimize $E = E_1\tau + E_2\zeta$ subject to $7.84 \leq \tau \leq 16.81$ and $P_r(Y > X) \geq 0.99$.

Mathematically, the above problem may be stated as:

Considering Table 2.3 where, $m = 2.8$ and 4.1 i.e. $\tau = 7.84$ and 16.81, respectively and we get values of ζ for which $P_r(Y > X) \geq 0.99$ and the cost function for each pair of (τ, ζ) is obtained and are presented in Table 2.4. Thus, in case of our example, the minimum cost lies at $\mathbf{7.84E_1 + 1.2E_2}$ which also further depends upon the values of E_1 and E_2.

Table 2.1

The probability of disaster			
m	$\chi = \mathbf{3.5}, \rho = \mathbf{2.5}$	$\chi = 2.5, \rho = 2.5$	$\chi = 2.5, \rho = 3.5$
1.2	**0.1637**	0.1199	0.0371
1.8	**0.0383**	0.0275	0.0046
2.8	**0.0032**	0.0023	0.000
4.1	**0.0001**	0	0
6.2	**0**	0	0

Table 2.2

For fixed $\chi = 3.5$ and $\rho = 2.5$ and for varying α, the Values of m are						
α	0.05	0.02	**0.01**	0.001	0.0001	0.00001
m=τ^η	1.6921	2.0630	**2.3417**	3.2641	4.1852	5.1063

Table 2.3

For different values of m and ζ, the Stress-Strength reliability of an item when $\chi = 3.5,$, $\rho = 2.5$, $\eta = 0.5$					
$\downarrow m\zeta \rightarrow$	1.2	1.4	1.6	1.8	2.0
1.2	0.80489	0.81774	0.82529	0.82975	0.83239
1.8	0.94663	0.95398	0.95771	0.95963	0.96061
2.8	**0.99128**	**0.99443**	**0.99579**	**0.99637**	**0.99662**
4.1	**0.99765**	**0.99905**	**0.99957**	**0.99976**	**0.99983**
6.2	0.99917	0.99974	0.99991	0.99997	0.99999

Table 2.4

Optimum manufacturing Cost					
τ	ζ	$E = E_1\tau + E_2\zeta$	τ	ζ	$E = E_1\tau + E_2\zeta$
7.84	**1.2**	$\mathbf{7.84E_1 + 1.2E_2}$	16.81	1.2	$16.81E_1 + 1.2E_2$
7.84	1.4	$7.84E_1 + 1.4E_2$	16.81	1.4	$16.81E_1 + 1.4E_2$
7.84	1.6	$7.84E_1 + 1.6E_2$	16.81	1.6	$16.81E_1 + 1.6E_2$
7.84	1.8	$7.84E_1 + 1.8E_2$	16.81	1.8	$16.81E_1 + 1.8E_2$
7.84	2.0	$7.84E_1 + 2.0E_2$	16.81	2.0	$16.81E_1 + 2.0E_2$

Chapter 3

Estimation of $R = P_r(Y > X)$ for the Family of lifetime distributions by Transformation Method

3.1 Introduction

The reliability of an item or system can be defined as a function of time 't' i.e, $R(t) = P_r(X > t)$, which defines the failure free operation of items/components until time 't'. Another important measure of reliability under the stress-strength model is $R = P_r(Y > X)$, which represents the reliability of an item or system for the random strength Y and random stress X.

A lot of work has been done in the literature on the point estiamtion of R. For a brief review literature one may refer to Pugh (1963) [88], Basu (1964) [15], Church and Harris (1970) [42], Enis and Geisser (1971) [47], Downton (1973) [45], Tong (1974) [108], Kelly et al. (1976) [67], Sinha and Kale (1980) [100], Sathe and Shah (1981) [96], Chao (1982) [24], Awad and Gharraf (1986) [3], Chaturvedi and Surinder (1999) [36], Kotz et al. (2003) [69], Rezaei et al. (2010) [91], Chaturvedi and Pathak (2012) [31].

3.2 The Family of lifetime distributions

Chaturvedi and Singh (2008) [25] derived a family of lifetime distributions with the help of Weibull distribution. Let the random variable X follows a family of lifetime distributions, then the pdf is presented as

$$f(x;a,\lambda,\underline{\theta}) = \frac{G^{'}(x;a,\underline{\theta})}{\lambda} exp\left(\frac{-G(x;a,\underline{\theta})}{\lambda}\right); \quad x > a \geq 0, \quad \lambda > 0 \tag{3.2.1}$$

Here, $G(x;a,\underline{\theta})$ is a function of x and may also depend on the parameters a and $\underline{\theta}$. $\underline{\theta}$ may be vector valued. $G^{'}(x;a,\underline{\theta})$ represents the derivative of $G(x;a,\underline{\theta})$ with respect to x.
The presented model (3.2.1) covers the following lifetime distributions as specific cases:

1. For $G(x;a,\underline{\theta}) = x$ and a=0, we get the one-parameter exponential distribution.
2. For $G(x;a,\underline{\theta}) = x^p, (p > 0)$ and a=0, we get the Weibull distribution.
3. For $G(x;a,\underline{\theta}) = x^2$ and a=0, we get the Rayleigh distribution.
4. For $G(x;a,\underline{\theta}) = log(1 + x^b), b > 0$ and a=0, we get the Burr distribution.
5. For $G(x;a,\underline{\theta}) = log(\frac{x}{a})$, we get the Pareto distribution.
6. For $G(x;a,\underline{\theta}) = log(1 + \frac{x}{\nu}), \nu > 0$ and a=0, we get the Lomax distribution.
7. For $G(x;a,\underline{\theta}) = log\left(1 + \frac{x^b}{\nu}\right), b > 0, \nu > 0$ and a=0, we get the Burr distribution with scale parameter $\nu(> 0)$.
8. For $G(x;a,\underline{\theta}) = x^\gamma exp(\nu x), \gamma > 0, \nu > 0$ and a=0, we get the modified Weibull distribution.
9. For $G(x;a,\underline{\theta}) = (x - a) + \frac{\nu}{\lambda} log\left(\frac{x+\nu}{a+\lambda}\right), \nu > 0, \lambda > 0$, we get the generalised Pareto distribution.
10. For $G(x;a,\underline{\theta}) = bx + \frac{\theta}{2}x^2, \theta > 0, b > 0$ and a=0, we get the linear exponential distribution.
11. For $G(x;a,\underline{\theta}) = (1 + x^b)^\theta - 1, \theta > 0, b > 0$ and a=0, we get the generalised power Weibull distribution.

12. For $G(x; a, \underline{\theta}) = \frac{\beta}{b}(e^{bx} - 1), \beta > 0, b > 0$ and a=0, we get the Gompertz distribution.

13. For $G(x; a, \underline{\theta}) = (e^{x^b} - 1), b > 0$ and a=0, we get the Chen distribution.

14. For $G(x; a, \underline{\theta}) = (x - a)$, we get the two-parameter exponential distribution.

3.3 MLE of $R = P_r(Y > X)$

In the following theorem, MLE of R is derived through the transformation method

Theorem 1: The MLE of R is

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y) + \overline{T}(x)} \tag{3.3.1}$$

where, $\overline{T}(y) = \dfrac{1}{n_2}\sum_{j=1}^{n_2} H(y_j; a_2, \theta_2)$ and $\overline{T}(x) = \dfrac{1}{n_1}\sum_{i=1}^{n_1} G(x_i; a_1, \theta_1)$

Proof: Let the random variable X follows a Family of lifetime distribution with pdf

$$f(x; a_1, \lambda_1, \theta_1) = \frac{G^{'}(x; a_1, \theta_1)}{\lambda_1} exp\left(\frac{-G(x; a_1, \theta_1)}{\lambda_1}\right); \quad x > a_1 \geq 0, \quad \lambda_1 > 0 \tag{3.3.2}$$

For the given equation (3.3.2), let us consider the transformation $G(x; a_1, \theta_1) = t$. Then the distribution become

$$f(t; \alpha) = \frac{1}{\alpha} exp\left(\frac{-t}{\alpha}\right) \tag{3.3.3}$$

where, $\alpha = \lambda_1$.

Now, let us consider Y be a random variable with pdf

$$f(y; a_2, \lambda_2, \theta_2) = \frac{H^{'}(y; a_2, \theta_2)}{\lambda_2} exp\left(\frac{-H(y; a_2, \theta_2)}{\lambda_2}\right); \quad y > a_2 \geq 0, \quad \lambda_2 > 0 \tag{3.3.4}$$

Similarly, let us take the transformation $z = H(y; a_2, \theta_2)$ and $\beta = \lambda_2$, we get

$$f(z; \beta) = \frac{1}{\beta} exp\left(-\frac{z}{\beta}\right) \tag{3.3.5}$$

Let t and z be two independent random variable which follows exponential distribution (3.3.3) and (3.3.5) with parameters α and β, respectively, where $t = G(x; a_1, \theta_1)$ and $z = H(y; a_2, \theta_2)$. The relaibility model is

$$\begin{aligned} R &= P_r(z > t) \\ &= \int_{z=0}^{\infty} \int_{t=0}^{\infty} f(t; \alpha) f(z; \beta) dt dz \\ &= \int_{z=0}^{\infty} \left[1 - exp\left(-\frac{z}{\alpha}\right)\right] \frac{1}{\beta} exp\left(-\frac{z}{\beta}\right) dz \end{aligned}$$

After solving, we get

$$R = \frac{\beta}{\beta + \alpha} \tag{3.3.6}$$

On replacing the α and β by their MLE'S i.e, $\ddot{\alpha} = \overline{t}$ and $\ddot{\beta} = \overline{z}$. The MLE of $R = P_r(z > t)$ is

$$\frac{\overline{z}}{\overline{z} + \overline{t}}$$

where, $\overline{t} = \frac{1}{n_1} \sum_{i=1}^{n_1} t_i$ and $\overline{z} = \frac{1}{n_2} \sum_{j=1}^{n_2} z_j$. Finally, MLE of R is

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y) + \overline{T}(x)}$$

where, $\overline{T}(y) = \frac{1}{n_2} \sum_{j=1}^{n_2} H(y_j; a_2, \theta_2)$ and $\overline{T}(x) = \frac{1}{n_1} \sum_{i=1}^{n_1} G(x_i; a_1, \theta_1)$.

Hence, the theorem follows.

1. Implication

Here we consider the different cases for the distributions to obtain the MLE of $R = P_r(Y > X)$ given in (3.3.1)

Values of parameters for The MLE of $R = P_r(Y > X)$	
Distributions	**Values of Parameter**
The one-parameter exponential distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$
Weibull distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^p$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^p$ for $p > 0$
Rayleigh distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^2$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^2$
Burr distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log(1 + y_j^b)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log(1 + x_i^b)$ for $b > 0$
Pareto distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(\frac{y_j}{a_2}\right)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right)$
Lomax distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(1 + \frac{y_j}{\nu}\right)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{\nu}\right)$ for $\nu > 0$
Burr distribution with scale parameter $\nu(> 0)$	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(1 + \frac{y_j^b}{\nu}\right)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1 + \frac{x_i^b}{\nu}\right)$ for $b > 0, \nu > 0$

The modified Weibull distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^{\gamma} exp(\nu y_j)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{\gamma} exp(\nu x_i)$ for $\gamma > 0, \nu > 0$
The generalised Pareto distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[(y_j - a_2) + \frac{\nu}{\lambda_2} log\left(\frac{y_j+\nu}{a_2+\lambda_2}\right)\right]$ $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[(x_i - a_1) + \frac{\nu}{\lambda_1} log\left(\frac{x_i+\nu}{a_1+\lambda_1}\right)\right]$ for $\lambda_1, \lambda_2 > 0, \quad \nu > 0$
The linear exponential distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[b y_j + \frac{\theta_2}{2} y_j^2\right]$ $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[b x_i + \frac{\theta_1}{2} x_i^2\right]$ for $\theta_1, \theta_2 > 0$ and $b > 0$
The generalised power Weibull distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[(1 + y_j^b)^{\theta_2}\right] - 1$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[(1 + x_i^b)^{\theta_1}\right] - 1$ $\theta_1, \theta_2 > 0$ and $b > 0$
The Gompertz distribution	$\overline{T}(y) = \frac{1}{n_2}\frac{\beta}{b}(e^{b\Pi_{j=1}^{n2} y_j} - 1)$ and $\overline{T}(y) = \frac{1}{n_1}\frac{\beta}{b}(e^{b\Pi_{i=1}^{n_1} x_i} - 1)$ $\beta, b > 0$
Chen distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2}(e^{y_j^b} - 1)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1}(e^{x_i^b} - 1)$ $b > 0$
The two-parameter exponential distribution	$\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2}(y_j - a_2)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1}(x_i - a_1)$

3.4 UMVUE of $R = P_r(Y > X)$

In the following theorem, UMVUE of R is derived through the transformation method

Theorem 2: The UMVUE of R is

$$\acute{R} = \begin{cases} \sum_{i=0}^{n_2-1} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2-i)\Gamma(n_1+i)} \left(\frac{T(x)}{T(y)}\right)^i; & T(x) < T(y) \\ \sum_{i=0}^{n_1-2} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2+i+1)\Gamma(n_1-i-1)} \left(\frac{T(y)}{T(x)}\right)^{i+1}; & T(x) \geq T(y) \end{cases} \tag{3.4.1}$$

where, $T(y) = \sum_{j=1}^{n_2} H(y_j; a_2, \theta_2)$ and $T(x) = \sum_{i=1}^{n_1} G(x_i; a_1, \theta_1)$

Proof: Considering the transfomation $G(x; a_1, \theta_1) = t$ and $z = H(y; a_2, \theta_2)$, we have the transform equations (3.3.3) and (3.3.5). To obtain the measure of reliabilIty estimate $P_r(z > t)$, we required to obtain the UMVUE of $f(t; \alpha)$ and $f(z; \beta)$ i.e, $\acute{f}(t; \alpha)$ and $\acute{f}(z; \beta)$ respectively, which is given by

$$\acute{f}(t; \alpha) = \frac{(n_1 - 1)G'(t; a_1, \theta_1)}{n_1 \bar{t}} \left[1 - \frac{G(t; a_1, \theta_1)}{n_1 \bar{t}}\right]^{n_1-2}; \quad G(t; a_1, \theta_1) < n_1 \bar{t} \tag{3.4.2}$$

and

$$\acute{f}(z; \beta) = \frac{(n_2 - 1)H'(z; a_2, \theta_2)}{n_2 \bar{z}} \left[1 - \frac{H(z; a_2, \theta_1)}{n_2 \bar{z}}\right]^{n_2-2}; \quad H(z; a_2, \theta_1) < n_2 \bar{z} \tag{3.4.3}$$

Now to obtain UMVUE of R we have,

$$\begin{aligned} \acute{R} &= P_r(z > t) \\ &= \int_{t=0}^{\infty} \int_{z=t}^{\infty} \acute{f}(t; \alpha) \acute{f}(z; \beta) dz dt \end{aligned}$$

using (3.4.2) and (3.4.3)

$$\acute{R} = \int_{t=0}^{n_1\bar{t}} \int_{z=t}^{n_2\bar{z}} \frac{(n_1-1)(n_2-1)H^{'}(z;a_2,\theta_2)G^{'}(t;a_1,\theta_1)}{n_1 n_2 \bar{t}\bar{z}} \left[1 - \frac{G(t;a_1,\theta_1)}{n_1\bar{t}}\right]^{n_1-2}$$
$$\left[1 - \frac{H(z;a_2,\theta_1)}{n_2\bar{z}}\right]^{n_2-2} dzdt$$

let $\left[1 - \frac{H(z;a_2,\theta_1)}{n_2\bar{z}}\right] = w$

$$= \int_{t=0}^{min(n_1\bar{t},n_2\bar{z})} \frac{(n_1-1)(n_2-1)G^{'}(t;a_1,\theta_1)}{n_1\bar{t}} \left[1 - \frac{G(t;a_1,\theta_1)}{n_1\bar{t}}\right]^{n_1-2} \left[\frac{w^{n_2-1}}{n_2-1}\right]_0^{1-\frac{H(t;a_2,\theta_1)}{n_2\bar{z}}} dt$$
$$= \int_{t=0}^{min(n_1\bar{t},n_2\bar{z})} \frac{(n_1-1)G^{'}(t;a_1,\theta_1)}{n_1\bar{t}} \left[1 - \frac{G(t;a_1,\theta_1)}{n_1\bar{t}}\right]^{n_1-2} \left[1 - \frac{H(t;a_2,\theta_1)}{n_2\bar{z}}\right]^{n_2-1} dt$$
$$= \int_{t=0}^{min(n_1\bar{t},n_2\bar{z})} \frac{(n_1-1)G^{'}(t;a_1,\theta_1)}{n_1\bar{t}} \left[1 - \frac{G(t;a_1,\theta_1)}{n_1\bar{t}}\right]^{n_1-2} \sum_{i=0}^{n_2-1}(-1)^i \binom{n_2-1}{i} \left[\frac{H(t;a_2,\theta_1)}{n_2\bar{z}}\right]^i dt$$

Now consider the case $n_1\bar{t} < n_2\bar{z}$. Let $1 - \frac{G(t;a_1,\theta_1)}{n_1\bar{t}} = u$, for solving the integral assuming $G(t;a_1,\theta_1) = H(t;a_2,\theta_2)$ i.e., $a_1 = a_2$ and $\theta_1 = \theta_2$.

$$\acute{R} = \int_0^1 (n_1-1) \sum_{i=0}^{n_2-1}(-1)^i \binom{n_2-1}{i} \left[\frac{n_1\bar{t}(1-u)}{n_2\bar{z}}\right]^i u^{n_1-1} du$$
$$= \sum_{i=0}^{n_2-1}(-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2-i)\Gamma(n_1+i)} \left(\frac{n_1\bar{t}}{n_2\bar{z}}\right)^i$$

In a same manner, we tackle the case when $n_1\bar{t} > n_2\bar{z}$:

$$\acute{R} = \sum_{i=0}^{n_1-2}(-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2+i+1)\Gamma(n_1-i-1)} \left(\frac{n_2\bar{z}}{n_1\bar{t}}\right)^{i+1}$$

The UMVUE of $R = P_r(Y > X)$ is obtained by substituting $n_2\bar{z} = T(y) = \sum_{j=1}^{n_2} H(y_j;a_2,\theta_2)$ and $n_1\bar{t} = T(x) = \sum_{i=1}^{n_1} G(x_i;a_1,\theta_1)$.
Hence, the theorem follows.

2. Implication

Here we consider the different cases for the distributions to obtain the UMVUE of $R = P_r(Y > X)$ given in (3.4.1)

Values of parameters for The UMVUE of $R = P_r(Y > X)$	
Distributions	Values of Parameter
The one-parameter exponential distribution	$T(y) = \sum_{j=1}^{n_2} y_j$ and $T(x) = \sum_{i=1}^{n_1} x_i$
Weibull distribution	$T(y) = \sum_{j=1}^{n_2} y_j^p$ and $T(x) = \sum_{i=1}^{n_1} x_i^p$ for $p > 0$
Rayleigh distribution	$T(y) = \sum_{j=1}^{n_2} y_j^2$ and $T(x) = \sum_{i=1}^{n_1} x_i^2$
Burr distribution	$T(y) = \sum_{j=1}^{n_2} log(1 + y_j^b)$ and $T(x) = \sum_{i=1}^{n_1} log(1 + x_i^b)$ for $b > 0$
Pareto distribution	$T(y) = \sum_{j=1}^{n_2} log\left(\frac{y_j}{a_2}\right)$ and $T(x) = \sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right)$
Lomax distribution	$T(y) = \sum_{j=1}^{n_2} log\left(1 + \frac{y_j}{\nu}\right)$ and $T(x) = \sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{\nu}\right)$ for $\nu > 0$
Burr distribution with scale parameter $\nu(> 0)$	$T(y) = \sum_{j=1}^{n_2} log\left(1 + \frac{y_j^b}{\nu}\right)$ and $T(x) = \sum_{i=1}^{n_1} log\left(1 + \frac{x_i^b}{\nu}\right)$ for $b > 0, \nu > 0$

The modified Weibull distribution	$T(y) = \sum_{j=1}^{n_2} y_j^{\gamma} exp(\nu y_j)$ and $T(x) = \sum_{i=1}^{n_1} x_i^{\gamma} exp(\nu x_i)$ for $\gamma > 0, \nu > 0$
The generalised Pareto distribution	$T(y) = \sum_{j=1}^{n_2} \left[(y_j - a_2) + \frac{\nu}{\lambda_2} log\left(\frac{y_j+\nu}{a_2+\lambda_2}\right)\right]$ $T(x) = \sum_{i=1}^{n_1} \left[(x_i - a_1) + \frac{\nu}{\lambda_1} log\left(\frac{x_i+\nu}{a_1+\lambda_1}\right)\right]$ for $\lambda_1, \lambda_2 > 0, \quad \nu > 0$
The linear exponential distribution	$T(y) = \sum_{j=1}^{n_2} \left[by_j + \frac{\theta_2}{2} y_j^2\right]$ $T(x) = \sum_{i=1}^{n_1} \left[bx_i + \frac{\theta_1}{2} x_i^2\right]$ for $\theta_1, \theta_2 > 0$ and $b > 0$
The generalised power Weibull distribution	$T(y) = \sum_{j=1}^{n_2} \left[(1 + y_j^b)^{\theta_2}\right] - 1$ and $T(x) = \sum_{i=1}^{n_1} \left[(1 + x_i^b)^{\theta_1}\right] - 1$ $\theta_1, \theta_2 > 0$ and $b > 0$
The Gompertz distribution	$T(y) = \frac{\beta}{b}(e^{b\Pi_{j=1}^{n_2} y_j} - 1)$ and $T(x) = \frac{\beta}{b}(e^{b\Pi_{i=1}^{n_1} x_i} - 1)$ $\beta, b > 0$
Chen distribution	$T(y) = \sum_{j=1}^{n_2} (e^{y_j^b} - 1)$ and $T(x) = \sum_{i=1}^{n_1} (e^{x_i^b} - 1)$ $b > 0$
The two-parameter exponential distribution	$T(y) = \sum_{j=1}^{n_2} (y_j - a_2)$ and $T(x) = \sum_{i=1}^{n_1} (x_i - a_1)$

3.5 Confidence Interval of $R = P_r(Y > X)$

In the following theorem, confidence interval of R is derived through the transformation method

Theorem 3: The confidence interval of $R = P_r(Y > X)$ is

$$P\left(\frac{n_2\ddot{R}c}{n_1(1-\ddot{R})(1-c)+n_2\ddot{R}c} < R < \frac{n_2\ddot{R}d}{n_1(1-\ddot{R})(1-d)+n_2\ddot{R}d}\right) = 1-\sigma \tag{3.5.1}$$

where, $\ddot{R} = \dfrac{\overline{z}}{\overline{z}+\overline{t}}$ and $0 < c < d$

Proof: From the Theorem 1, the MLE of R is $\frac{\beta}{\beta+\alpha}$ or $\frac{\overline{z}}{\overline{z}+\overline{t}}$. As we know $n_1\overline{t}$ and $n_2\overline{z}$ follows Gamma distribution with parameters (α, n_1) and (β, n_2), respectively. For Confidence Interval of R, we must obtain the exact distribution of the varibale

$$\delta = \frac{\alpha n_1\overline{t}}{\alpha n_1\overline{t}+\beta n_2\overline{z}} \tag{3.5.2}$$

let $\rho = \alpha n_1\overline{t}$ and $\varrho = \beta n_2\overline{z}$ and observe that ρ and ϱ have gamma distribution with the parameters $(1, n_1)$ and $(1, n_2)$ respectively. New set of varible is $\delta = \frac{\rho}{\rho+\varrho}$.
On taking $\psi = \varrho$ and expressing the old variable in terms of new ones $\rho = \frac{\delta\psi}{(1-\delta)}$. The Jacobian of transformation is $J = (1-\delta)^{-2}\psi$. The joint pdf of δ and ψ

$$P_r(\delta, \psi) = \frac{e^{-\left(\frac{\psi}{1-\delta}\right)}\psi^{n_1+n_2-1}\delta^{n_1-1}}{\Gamma(n_1)\Gamma(n_2)(1-\delta)^{n_1+1}} \tag{3.5.3}$$

Intergrating out ψ, we have the maginal distribution of δ

$$P_r(\delta) = [B(n_1, n_2)]^{-1}\,\delta^{n_1-1}(1-\delta)^{n_2-1}; \quad 0 < \delta < 1$$

Here, δ has a beta distribution with the known parameters n_1 and n_2. So we have, for any $0 < c < d$

$$P_r(c < \delta < d) = I_d(n_1, n_2) - I_c(n_1, n_2) \tag{3.5.4}$$

where, $I_x(n_1, n_2) = [B(n_1, n_2)]^{-1} \int_0^x z^{n_1-1}(1-z)^{n_2-1}dz$ is the incomplete beta function. After calculation for the conection of δ and $\ddot{R}$, we have the pivotal quantity

$$\delta = \left[1 + \frac{n_2\ddot{R}(1-R)}{n_2R(1-\ddot{R})}\right]^{-1}$$

where, $R = \frac{\beta}{\beta+\alpha}$ and $\ddot{R} = \frac{\bar{z}}{\bar{z}+\bar{t}}$.

If c and d in (3.5.4) are such that for a given σ

$$I_d(n_1, n_2) - I_c(n_1, n_2) = 1 - \sigma$$

then,

$$P_r\left(c < \left[1 + \frac{n_2\ddot{R}(1-R)}{n_2R(1-\ddot{R})}\right]^{-1} < d\right) = 1 - \sigma \tag{3.5.5}$$

After solving the equation (3.5.5) for R.

$$P_r\left(\frac{n_2\ddot{R}c}{n_1(1-\ddot{R})(1-c)+n_2\ddot{R}c} < R < \frac{n_2\ddot{R}d}{n_1(1-\ddot{R})(1-d)+n_2\ddot{R}d}\right) = 1 - \sigma$$

The above equation is valid for any values of n_1 and n_2, large or small.

Hence, the theorem follows.

3. Implication

Here we consider the different cases for the distributions to obtain the Confidence Interval of $R = P_r(Y > X)$ given in (3.5.1)

Values of parameters for The Confidence Integral of $R = P_r(Y > X)$	
Distributions	Values of Parameter
The one-parameter exponential distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$
Weibull distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^p$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^p,$ $p > 0$
Rayleigh distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^2$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^2$
Burr distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log(1 + y_j^b)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log(1 + x_i^b), \quad b > 0$
Pareto distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(\frac{y_j}{a_2}\right)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right), \quad b > 0$
Lomax distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \ \forall \ \ \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(1 + \frac{y_j}{\nu}\right)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{\nu}\right)$,for $\nu > 0$

Burr distribution with scale parameter $\nu(>0)$

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} log\left(1+\frac{y_j^b}{\nu}\right) \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} log\left(1+\frac{x_i^b}{\nu}\right), \quad \nu>0 \text{ and } b>0$$

The modified Weibull distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} y_j^{\gamma} exp(\nu y_j) \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^{\gamma} exp(\nu x_i), \quad \nu>0 \text{ and } \gamma>0$$

The generalised Pareto distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(y)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[(y_j-a_2)+\frac{\gamma}{\lambda_2} log\left(\frac{y_j+\nu}{a_2+\lambda_2}\right)\right] \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[(x_i-a_1)+\frac{\gamma}{\lambda_1} log\left(\frac{x_i+\nu}{a_1+\lambda_1}\right)\right], \quad \nu>0 \text{ and } \gamma>0$$

The linear exponential distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[by_j+\frac{\theta_2}{2}y_j^2\right] \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[bx_i+\frac{\theta_1}{2}x_i^2\right], \quad \theta_1,\theta_2>0 \text{ and } b>0$$

The generalised power Weibull distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[(1+y_j^b)^{\theta_2}-1\right] \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[(1+x_i^b)^{\theta_1}-1\right], \quad \theta_1,\theta_2>0 \text{ and } b>0$$

The Gompertz distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} \left[\frac{\beta}{b}(e^{by_j}-1)\right] \text{ and}$$
$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1} \left[\frac{\beta}{b}(e^{bx_i}-1)\right], \quad \beta>0 \text{ and } b>0$$

Chen distribution

$$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)} \;\forall\; \overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2} (e^{y_j^b}-1) \text{ and}$$

$$\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1}(e^{x_i^b} - 1), \quad b > 0$$

The two-parameter exponential distribution	$\ddot{R} = \frac{\overline{T}(y)}{\overline{T}(y)+\overline{T}(x)}$ $\forall$ $\overline{T}(y) = \frac{1}{n_2}\sum_{j=1}^{n_2}(y_j - a_2)$ and $\overline{T}(x) = \frac{1}{n_1}\sum_{i=1}^{n_1}(x_i - a_1), \quad a_1, a_2 > 0$

3.6 Bayes Estimator of $R = P_r(Y > X)$

In the following theorem, Bayes estimator of R is derived through the Transformation method

Theorem 4: The Bayes estimator of R is

$$\check{R} = \begin{cases} \frac{\mu^*}{\xi^*+\mu^*}\left(\frac{\eta^*}{\omega^*}\right)^{-\mu^*} {}_2F_1(\mu^*+\xi^*, \mu^*+1, \mu^*+\xi^*+1; B), & \text{for} \quad B < 1 \\ \\ \frac{\mu^*}{\xi^*+\mu^*}\left(\frac{\omega^*}{\eta^*}\right)^{-\xi^*} {}_2F_1(\mu^*+\xi^*, \xi^*, \mu^*+\xi^*+1; \frac{B}{1-B}), & \text{for} \quad B < -1 \end{cases} \tag{3.6.1}$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric series and $B = \frac{\omega^*-\eta^*}{\omega^*} < 1$.

Proof: Let us consider $\underline{t}$ and $\underline{z}$ be the independent samples from the pdfs (3.3.3) and (3.3.5). Here considering the conjugate prior, inverse gamma distributions for α and β with the parameters μ, η, and ξ, ω, respectively. Prior is

$$\pi(\alpha, \beta) \propto \alpha^{-\mu-1}e^{\left(-\frac{\eta}{\alpha}\right)}\beta^{-\xi-1}e^{\left(-\frac{\omega}{\beta}\right)}; \quad \mu, \eta, \xi, \beta > 0 \tag{3.6.2}$$

The likelihood is

$$L(\alpha,\beta|\underline{t},\underline{z}) = \alpha^{-n_1}\beta^{-n_2} \quad exp\left[-\left(\frac{\sum_{i=1}^{n_1} t_i}{\alpha} + \frac{\sum_{j=1}^{n_2} z_j}{\beta}\right)\right] \tag{3.6.3}$$

Applying Bayes formula and using (3.6.2) and (3.6.3). The posterior density of (α,β) is

$$\pi(\alpha,\beta|\underline{t},\underline{z}) \propto \alpha^{-\mu-n_1-1}e^{-\frac{(\eta+n_1\bar{t})}{\alpha}}\beta^{-\xi-n_2-1}e^{-\frac{(\omega+n_2\bar{z})}{\beta}} \tag{3.6.4}$$

Evidently the posterior risk is also the product of gamma pdfs with the updated parameters

$$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + n_1\bar{t}, \quad \xi^* = -(\xi+n_2), \quad \omega^* = \omega + n_2\bar{z}$$

where, $\bar{t}$ and $\bar{z}$ are the sample means.

For posterior pdf of R, we consider a one-to-one transformation $F : R = \frac{\beta}{\beta+\alpha}, \vartheta_R = \alpha+\beta$ with the inverse $Q : \alpha = R\vartheta_R, \beta = R(1-\vartheta_R)$. The Jacobian of transformation is ϑ_R. The joint posterior density of R and ϑ_R becomes

$$\pi^*(R,\vartheta_R|\underline{t},\underline{z}) \propto R^{\mu^*-1}(1-R)^{\xi^*-1} \quad {\vartheta_R}^{\mu^*+\xi^*-1}e^{-\vartheta_R\omega^*(1-BR)}; \quad 0<R<1, \vartheta_R>0 \tag{3.6.5}$$

where $B = \frac{\omega^*-\eta^*}{\omega^*} < 1$
Intergrating the (3.6.5) for ϑ_R

$$\pi_R(R|\underline{t},\underline{z}) = C_R R^{\mu^*-1}(1-R)^{\xi^*-1}(1-BR)^{-(\mu^*+\xi^*)}; \quad 0<R<1 \tag{3.6.6}$$

where, C_R is the normalizing coefficient. For the Baye estimator we have

$$\check{R} = \int R\,\pi_R(R|\underline{t},\underline{z})dR \tag{3.6.7}$$

Using the (3.6.6) and solving (3.6.7), we obtain the bayes estimator of R

$$\check{R} = \begin{cases} \frac{\mu^*}{\xi^*+\mu^*}\left(\frac{\eta^*}{\omega^*}\right)^{-\mu^*} {}_2F_1(\mu^*+\xi^*, \mu^*+1, \mu^*+\xi^*+1; B), & \text{for} \quad B < 1 \\ \\ \frac{\mu^*}{\xi^*+\mu^*}\left(\frac{\omega^*}{\eta^*}\right)^{-\xi^*} {}_2F_1(\mu^*+\xi^*, \xi^*, \mu^*+\xi^*+1; \frac{B}{1-B}), & \text{for} \quad B < -1 \end{cases}$$

where, ${}_2F_1(a,b,c;z) = \sum_{j=1}^{\infty} \frac{a(a+1)...(a+j-1)b(b+1)...(b+j-1)}{c(c+1)...(c+j-1)} \frac{z^j}{j!}$ is the hypergeometric series.

For the Bayes estimator $\ddot{R}$, replacing the parameters as

$$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + n_1\overline{T}(x), \quad \xi^* = -(\xi+n_2), \quad \omega^* = \omega + n_2\overline{T}(y)$$

Hence, the theorem follows.

4. Implication

Here we consider the different cases for the distributions to obtain the Bayes estimators of $R = P_r(Y > X)$ given in (3.6.1)

Values of parameters for The Bayes estimators of $R = P_r(Y > X)$

Distributions	Values of Parameter
The one-parameter exponential distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + n_1\overline{x}$, $\xi^* = -(\xi+n_2), \quad \omega^* = \omega + n_2\overline{y}$
Weibull distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} x_i^p$, $\xi^* = -(\xi+n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} y_j^p, \quad p > 0$
Rayleigh distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} x_i^2$,

$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} y_j^2, \quad p > 0$$

Burr distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} log(1 + x_i^b),$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} log(1 + y_j^b), \quad b > 0$$

Pareto distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} log\left(\frac{x_i}{a_1}\right),$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} log\left(\frac{y_j}{a_2}\right), \quad a_1, a_2 > 0$$

Lomax distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{\nu}\right),$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} log\left(1 + \frac{y_j}{\nu}\right), \quad \nu, b > 0$$

Burr distribution with scale parameter $\nu(> 0)$

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} log\left(1 + \frac{x_i^b}{\nu}\right),$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} log\left(1 + \frac{y_j^b}{\nu}\right), \quad \nu, b > 0$$

The modified Weibull distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} x_i^\gamma exp(\nu x_i),$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} y_j^\gamma exp(\nu y_j), \quad \gamma, \nu > 0$$

The generalised Pareto distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} \left[(x_i - a_1) + \frac{\nu}{\lambda_1} log\left(\frac{x_i + \nu}{a_1 + \lambda_1}\right)\right],$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} \left[(y_j - a_2) + \frac{\nu}{\lambda_2} log\left(\frac{y_j + \nu}{a_2 + \lambda_2}\right)\right],$$
$$\gamma, \nu > 0$$

The linear exponential distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} \left[b x_i + \frac{\theta_1}{2} x_i^2\right],$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} \left[b y_j + \frac{\theta_2}{2} y_j^2\right], \quad b > 0$$

The generalised power Weibull distribution

$$\mu^* = -(n_1 + \mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1} \left[(1 + x_i^b)^{\theta_1} - 1\right],$$
$$\xi^* = -(\xi + n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2} \left[(1 + y_j^b)^{\theta_2} - 1\right], \quad b > 0$$

The Gompertz distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1}\left[\frac{\beta}{b}(e^{bx_i}-1)\right],$ $\xi^* = -(\xi+n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2}\left[\frac{\beta}{b}(e^{by_j}-1)\right], \quad b>0$
Chen distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1}\left[e^{x_i^b}-1\right],$ $\xi^* = -(\xi+n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2}\left[e^{y_j^b}-1\right], \quad b>0$
The two-parameter exponential distribution	$\mu^* = -(n_1+\mu), \quad \eta^* = \eta + \sum_{i=1}^{n_1}(x_i - a_1),$ $\xi^* = -(\xi+n_2), \quad \omega^* = \omega + \sum_{i=1}^{n_2}(y_j - a_2), \quad a_1, a_2 > 0$

Chapter 4

Sequential testing procedure and their Robustness study for Erlang distribution

4.1 Introduction

Wald (1947) [114] originated the concept of sequential estimation for the simple null hypotheses against the simple alternative hypotheses. This concept is influenced by the sequential probability ratio test (SPRT). He also derived the mathmatical expressions for the operating characteristics (OC) and average sample number (ASN) functions for the analysis of SPRT'S performance. Sequential probability ratio test has been applied by several researchers, to tackle with different testing problems, for references, one may referred to Oakland (1950) [85], Epstein and Sobel (1955) [48], Johnson (1966) [64], Phatarford (1971) [87], Bain and Engelhardt (1982) [7], Chaturvedi et al. (2000) [39], Bacanli and Demirhan (2008) [5].

For various life testing models, the robustness of the sequential probability ratio test in respect of OC and ASN functions has been examined by various researchers, when the distribution has undergone a change. For references, one may refer to Harter and Moore (1976)

[59] derived the robustness of the exponential sequential probability ratio test with the help of Monte Carlo method of simulation when the underlying distribution is Weibull distribution. The sampling plans for the reliability of a constant failure rate is also given. For the exponential distribution, the robustness of the SPRT with respect to the risks and the expected sample sizes studied by Montagne and Singpurwalla (1985) [78]. Hubbard and Allen (1991) [63] investigated the sequential probability ratio test for the mean of the negative binomial distribution when the dispersion parameter is known and the robustness of the test to the misspecification of the parameter of distribution is also studied. Chaturvedi et al. (1998) [38] generalised the results given by Montagne and Singpurwalla (1985) [78] with the robustness of the SPRT'S for different parameters of the family of life-testing distributions.

The mathematical expression for the operating characteristic (OC) and the average sample number (ASN) functions for the mean of an inverse Gaussian distribution with known the coefficient of variation (CV) is developed by Joshi and Shah (1990) [65].

4.2 Set-up of the problem

Let X be a random variable following the Erlang distribution presented by the probability density function (pdf)

$$f(x;\rho,k) = \frac{\rho^k x^{k-1} e^{-\rho x}}{(k-1)!}; \quad 0 < x < \infty, \rho \geq 0, k \in N \tag{4.2.1}$$

where 'k' is the shape parameter and 'ρ' is the rate parameter. The Erlang distribution is the sum of 'k' independent exponential random variables each with the same parameter. For a sequence of observations X_1, X_2, X_3, ..., from (4.2.1), testing $H_0 : \rho = \rho_0$ vs $H_0 : \rho = \rho_1$ $(\rho_1 > \rho_0)$ is considered.

For the given probability model in (4.2.1), the sequential probability ratio test (SPRT) and robustness in respect of OC and ASN functions, when the distribution under consideration has undergone a change is calculated in Sections 4.3, 4.4, 4.5, 4.6 and 4.7, respectively. The results are discussed in Remarks 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8. The robustness of the sequential probability ratio test for a misspecified coefficient of variation is also studied in Section 4.7 [see Remarks 4.7]. The acceptance and rejection boundaries for two simple hypotheses in case of the rate parameter 'ρ' are studied in Section 4.8 and shown in the Fig. 4.11. Finally, the deduced results and findings are presented through the Tables and Graphs.

4.3 The sequential probability ratio test for the rate parameter 'ρ'

The sequential probability ratio test for testing the null hypotheses $H_0 : \rho = \rho_0$ vs the alternative hypotheses $H_1 : \rho = \rho_1 (\rho_1 > \rho_0)$ is defined as

$$Z_i = \ln \left[\frac{f(x_i; \rho_1, k)}{f(x_i; \rho_0, k)} \right] \tag{4.3.1}$$

$$Z_i = k \ln \left(\frac{\rho_1}{\rho_0} \right) + x_i(\rho_0 - \rho_1) \tag{4.3.2}$$

Let us choose two numbers $\mathcal{A}$ and $\mathcal{B}$ such that $0 < \mathcal{B} < 1 < \mathcal{A}$. At the m^{th} stage of sampling accept the null hypotheses if $\sum_{i=1}^{m} Z_i \leq \ln \mathcal{B}$, reject the null hypotheses if $\sum_{i=1}^{m} Z_i \geq \ln \mathcal{A}$, otherwise continue sampling by taking one additional observation i.e., the $(m+1)^{th}$ observation. If $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are Type I and Type II errors respectively, then according to Wald (1947) [114], $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} \approx \frac{1-\beta}{\alpha} \text{ and } \mathcal{B} \approx \frac{\beta}{1-\alpha} \tag{4.3.3}$$

The operating characteristic function is defined as

$$L(\theta) = \frac{\mathcal{A}^t - 1}{\mathcal{A}^t - \mathcal{B}^t} \tag{4.3.4}$$

where, 't', the non-zero solution of the following equation

$$E[e^{Z_i}]^t = 1 \tag{4.3.5}$$

or,

$$\int_0^\infty \left[\frac{f(x_i;\rho_1,k)}{f(x_i;\rho_0,k)}\right]^t f(x_i;\rho,k)dx = 1 \tag{4.3.6}$$

From the equations (4.2.1) and (4.3.2), we get

$$E[e^{Z_i}]^t = \frac{\left(\frac{\rho_1}{\rho_0}\right)^{kt}}{\left[1-\frac{t}{\rho}(\rho_0-\rho_1)\right]^k} \tag{4.3.7}$$

From (4.3.7) and (4.3.5), we have

$$\rho = \frac{t(\rho_0-\rho_1)}{1-\left(\frac{\rho_1}{\rho_0}\right)^t} \tag{4.3.8}$$

The above theoretical expression (4.3.8) is not convenient for finding the values of OC and ASN functions. Thus, to handle this problem take logarithm to both the sides and use the expansion $\ln(1-y)$ and retain the term up to 3^{rd} degree in 't'. We get

$$kt\ln\left(\frac{\rho_1}{\rho_0}\right) = k\ln\left[1-\frac{t}{\rho}(\rho_0-\rho_1)\right]$$
$$kt\ln\left(\frac{\rho_1}{\rho_0}\right) = k\left[-t\left(\frac{\rho_0-\rho_1}{\rho}\right)-\frac{t^2}{2}\left(\frac{\rho_0-\rho_1}{\rho}\right)^2-\frac{t^3}{3}\left(\frac{\rho_0-\rho_1}{\rho}\right)^3\right]$$

or,

$$\frac{t^2}{3}\left(\frac{\rho_0-\rho_1}{\rho}\right)^3+\frac{t}{2}\left(\frac{\rho_0-\rho_1}{\rho}\right)^2+\left(\frac{\rho_0-\rho_1}{\rho}\right)+\ln\left(\frac{\rho_1}{\rho_0}\right) = 0 \tag{4.3.9}$$

On solving (4.3.9), we get the roots of 't'. Finally, on substituting the values of 't' in equation (4.3.4) the numerical values of OC function are obtained.

The ASN function is approximately given by

$$E(N \mid \rho) = \frac{L(\rho) logB + [1 - L(\rho)] logA}{E(Z \mid \rho)} \tag{4.3.10}$$

provided $E(Z) \neq 0$, where

$$E(Z \mid \rho) = k \left[\ln \left(\frac{\rho_1}{\rho_0} \right) + \left(\frac{\rho_0 - \rho_1}{\rho} \right) \right] \tag{4.3.11}$$

From (4.3.11) the ASN function under H_0 and H_1 is given by

$$E_0(N) = \frac{(1-\alpha) \ln B + \alpha \ln A}{k \left[\ln \left(\frac{\rho_1}{\rho_0} \right) + \left(\frac{\rho_0 - \rho_1}{\rho} \right) \right]} \tag{4.3.12}$$

and

$$E_1(N) = \frac{\beta \ln B + (1-\beta) \ln A}{k \left[\ln \left(\frac{\rho_1}{\rho_0} \right) + \left(\frac{\rho_0 - \rho_1}{\rho} \right) \right]} \tag{4.3.13}$$

Remarks 4.3: Let us take the testing problem of the simple null hypothesis $H_0 : \rho_0 = 13$ vs the simple alternative $H_1 : \rho_1 = 15$ for fixed $\alpha = \beta = 0.05$. The values for the OC and ASN functions are derived through a computer programme from equation (4.3.9) and (4.3.12) and are presented in Table 4.1 and their curves are depicted in Fig. 4.1 and 4.2, respectively. For $\rho_0 = 13$ and $\rho_1 = 15$ the corresponding values of $L(\rho)$ are **0.9518** and **0.0488**, which proves that the method used for obtaining the real roots of 't' works very satisfactorily.

4.4 Robustness of the sequential probability ratio test for the rate parameter 'ρ' when the shape parameter 'k' has undergone a change

Let us consider that the shape parameter 'k' changes to 'k^*' and the probability distribution in (4.2.1) becomes $f(x; \rho, k^*)$. The robustness of sequential probability ratio test developed for the parameter 'ρ' in Section 4.3 with respect to operating characteristic function, the values of 't' are obtained by evaluating

$$\begin{aligned}
&\int_0^\infty \left[\frac{f(x_i;\rho_1,k)}{f(x_i;\rho_0,k)}\right]^t f(x_i;\rho,k^*)dx = 1 \\
&\left(\frac{\rho_1}{\rho_0}\right)^{kt} \frac{\rho^{k^*}}{\Gamma(k^*)} \int_0^\infty e^{-x[\rho-t(\rho_0-\rho_1)]} x^{(k^*-1)} dx = 1
\end{aligned} \tag{4.4.1}$$

After solving (4.4.1), we get

$$\left(\frac{\rho_1}{\rho_0}\right)^{kt} \left[1 - t\left(\frac{\rho_0-\rho_1}{\rho}\right)\right]^{-k^*} = 1 \tag{4.4.2}$$

Again on taking logarithm and using the expansion of $\ln(1-y)$ and retaining the terms upto 3^{rd} degree in 't', we get quadratic equation

$$\frac{t^2 p}{3}\left(\frac{\rho_0-\rho_1}{\rho}\right)^3 + \frac{tp}{2}\left(\frac{\rho_0-\rho_1}{\rho}\right)^2 + p\left(\frac{\rho_0-\rho_1}{\rho}\right) + \ln\left(\frac{\rho_1}{\rho_0}\right) = 0 \tag{4.4.3}$$

where $p = \dfrac{k^*}{k}$

On solving the (4.4.3), we get the roots of 't'. In order to study the robustness of the sequential probability ratio test with respect to ASN functions replace the denominator of (4.3.10) by

$$\begin{aligned}
E(Z/\rho) &= \int_0^\infty z f(x;\rho,k^*)dx \\
&= k\ln\left(\frac{\rho_1}{\rho_0}\right) + (\rho_0-\rho_1)E(x) \\
&= k\ln\left(\frac{\rho_1}{\rho_0}\right) + (\rho_0-\rho_1)\left(\frac{k^*}{\rho}\right)
\end{aligned}$$

$$E(Z \mid \rho) = k\ln\left(\frac{\rho_1}{\rho_0}\right) - p\left(\frac{\rho_0-\rho_1}{\rho}\right) \tag{4.4.4}$$

Remarks 4.4: For $H_0 : \rho_0 = 13$ vs $H_1 : \rho_1 = 15$, $\alpha = \beta = 0.05$ and for varying values of shape parameter 'k' the roots of 't' obtained from (4.4.3). It is evident from the Fig. 4.3 and Fig. 4.4 that the operating characteristic curve shifts to the left (right) and ASN curve shifts to the left downward (right upward) for $p < 1 (p > 1)$, which shows that the sequential probability ratio test is highly sensitive for any change in 'k'.

4.5 The sequential probability ratio test for the shape parameter 'k'

The sequential probability ratio test for the simple null hypothesis $H_0 : k = k_0$ against the simple alternative hypothesis $H_1 : k = k_1 (k_1 > k_0)$ is defined as:

$$\begin{aligned} Z_i &= \ln\left[\frac{f(x_i;\rho,k_1)}{f(x_i;\rho,k_0)}\right] \\ Z_i &= \ln\left(\frac{\Gamma k_0}{\Gamma k_1}\right) + (k_1 - k_0)\ln\rho + (k_1 - k_0)\ln x_i \end{aligned} \tag{4.5.1}$$

The operating characteristic curve is obtained by the following equation

$$\int_0^\infty \left[\frac{f(x_i;\rho,k_1)}{f(x_i;\rho,k_0)}\right]^t f(x_i;\rho,k)dx = 1 \tag{4.5.2}$$

On using the (4.2.1) and (4.5.1), we get

$$\left(\frac{\Gamma k_0}{\Gamma k_1}\right)^t \frac{1}{\Gamma k}\Gamma[t(k_1 - k_0) + k] = 1 \tag{4.5.3}$$

The theoretical expression (4.5.3) is not convenient for finding the values of OC and ASN functions, thus take the logarithm to both the sides and using the expansion $ln(1 - y)$ and the approximation

$$\ln\Gamma x = \ln\sqrt{2\pi} - x + \left(x - \frac{1}{2}\right)\ln x \tag{4.5.4}$$

we get

$$\frac{t^2}{6}\left(\frac{k_1-k_0}{k}\right)^3(k+1)-\frac{t}{4}\left(\frac{k_1-k_0}{k}\right)^2(2k+1)-\left(k_0-\frac{1}{2}\right)\ln k_0+\left(k_1-\frac{1}{2}\right)\ln k_1$$
$$-\left(1+\ln k-\frac{1}{2k}\right)(k_1-k_0)=0 \tag{4.5.5}$$

On solving,(4.5.5), we get the roots of 't'. Finally, the numerical values of OC function is obtain from equation (4.3.4) and the robustness is studied by replacing the denominator of (4.3.10) by

$$E(Z_i \mid k)=\ln\Gamma k_0-\ln\Gamma k_1+(k_1-k_0)\ln\rho+(k_1-k_0)E(\ln x_i)$$

and

$$E(\ln x_i)=\psi(k)-\ln\rho$$

Use the result (4.5.5) and approximation [see Gradshteyn and Ryzhik (1965, p. 576, 4.352(1)) [56]]

$$\psi(x)=\ln x-\frac{1}{2x} \tag{4.5.6}$$

we get

$$E(Z_i \mid k)=\left(k_0-\frac{1}{2}\right)\ln k_0+\left(k_1-\frac{1}{2}\right)\ln k_1+\left(1+\ln k-\frac{1}{2k}\right)(k_1-k_0) \tag{4.5.7}$$

Remarks 4.5: Let us take the testing problem of the simple null hypothesis $H_0 : k_0 = 13$ vs the simple alternative hypothesis $H_1 : k_1 = 15$ for fixed $\alpha = \beta = 0.05$. The values for the OC and ASN functions are derived through a computer programme from equation (4.5.5) and (4.5.7) and are presented in Table 4.4 and their curves are depicted in Fig. 4.5 and 4.6, respectively. For $k_0 = 13$ and $k_1 = 15$, the corresponding values of $L(k)$ are **0.9583** and **0.0548**, which proves that the method used for obtaining the real roots of 't' works very satisfactorily

4.6 Robustness of the sequential probability ratio test for the shape parameter 'k' when the rate parameter 'ρ' has undergone a change

Let us consider that the rate parameter 'ρ' changes to 'ρ^*' and the probability distribution in (4.2.1) becomes $f(x;\rho^*,k)$. The robustness of sequential probability ratio test developed for the shape parameter 'k' in Section 4.5 with respect to operating characteristic function, the values of 't' are obtained by the evaluating

$$\int_0^\infty \left[\frac{f(x_i;\rho,k_1)}{f(x_i;\rho,k_0)}\right]^t f(x_i;\rho^*,k)dx = 1 \tag{4.6.1}$$

From (4.2.1) and (4.6.1)

$$\left(\frac{\Gamma k_0}{\Gamma k_1}\right)^t \frac{1}{\Gamma k}\rho^{t(k_1-k_0)}\rho^{*k} \int_0^\infty x^{t(k_1-k_0)+k-1}e^{-x\rho^*}dx = 1$$

or,

$$\left(\frac{\Gamma k_0}{\Gamma k_1}\right)^t \frac{1}{\Gamma k}\phi^{t(k_1-k_0)}\Gamma[t(k_1-k_0)+k] = 1 \tag{4.6.2}$$

where $\phi = \dfrac{\rho}{\rho^*}$

Again taking logarithm to both the sides of (4.6.2) and the approximation (4.5.4) then retaining the terms upto 3^{rd} degree in 't', we get quadratic equation

$$\frac{t^2}{6}\left(\frac{k_1-k_0}{k}\right)^3(k+1) - \frac{t}{4}\left(\frac{k_1-k_0}{k}\right)^2(2k+1) - \left(k_0-\frac{1}{2}\right)\ln k_0 + \left(k_1-\frac{1}{2}\right)\ln k_1$$
$$-(k_1-k_0)\ln\phi - \left(1+\ln k-\frac{1}{2k}\right)(k_1-k_0) = 0 \tag{4.6.3}$$

On solving the (4.6.3), we get the roots of 't'. The values of operating characteristic function is now obtain from equation (4.3.4). In order to study the Robustness of the sequential probability ratio test with respect to ASN function replace the denominator of (4.3.10) by

$$E(Z_i \mid k) = \ln \Gamma k_0 - \ln \Gamma k_1 + (k_1 - k_0) \ln \rho + (k_1 - k_0) E(\ln x_i)$$

where

$$E(\ln x_i) = \psi(k) - \ln \rho^*$$

and using the result of (4.5.6) we get

$$E(Z_i \mid k) = \left(k_0 - \frac{1}{2}\right) \ln k_0 + \left(k_1 - \frac{1}{2}\right) \ln k_1 + \left(1 + \ln \phi + \ln k - \frac{1}{2k}\right)(k_1 - k_0) \qquad (4.6.4)$$

Remarks 4.6: For $H_0 : k_0 = 13$ vs $H_1 : k_1 = 15$, $\alpha = \beta = 0.05$ and for varying values of the shape parameter 'k' the roots of 't' are obtained from (4.6.3). It is evident from Table 4.5 and Table 4.6 that the operating characteristic shifts to the right (left) and ASN curve shifts to the right upward (left downward) for $\phi < 1 (\phi > 1)$, which shows that sequential probability ratio test is highly sensitive for any change in the shape parameter 'k'.

4.7 Robustness of the sequential probability ratio test for the rate parameter 'ρ' with known coefficient of variation

We know that the mean and variance for Erlang distribution are $\left(\frac{k}{\rho}\right)$ and $\left(\frac{k}{\rho^2}\right)$ respectively, then the coefficient of variation say, $a \left[= \frac{1}{\sqrt{k}}\right]$. The coefficient of variation is misspecified from 'a' to 'a*', thus the pdf (4.2.1) becomes to $f(x_i; \rho, a^*)$.
The operating characteristic and ASN functions are given by

$$\int_0^{\infty} \left[\frac{f(x_i; \rho_1, a)}{f(x_i; \rho_0, a)}\right]^t f(x_i; \rho, a^*) dx = 1 \qquad (4.7.1)$$

On using (4.3.2) and (4.2.1), we get from the above equation (4.7.1)

$$\left(\frac{\rho_1}{\rho_0}\right)^{\frac{t}{a^2}} \left[\frac{\rho}{\rho - t(\rho_0 - \rho_1)}\right]^{\left(\frac{1}{a^*}\right)^2} = 1 \tag{4.7.2}$$

On taking logarithm and using the expansion of $\ln(1-y)$, we get the following quadratic equation

$$\frac{t^2}{3}Q\left(\frac{\rho_0 - \rho_1}{\rho}\right)^3 + \frac{t}{2}Q\left(\frac{\rho_0 - \rho_1}{\rho}\right)^2 + Q\left(\frac{\rho_0 - \rho_1}{\rho}\right) + \ln\left(\frac{\rho_1}{\rho_0}\right) = 0 \tag{4.7.3}$$

where $Q = \left(\frac{a}{a^*}\right)^2$

From the above (4.7.3), we get the roots of 't'. The values of OC function is obtain from (4.3.4) and the robustness of the sequential probability ratio test with respect to ASN is studied by replacing the denominator of (4.3.10) by

$$E(Z \mid \rho) = \ln\left(\frac{\rho_1}{\rho_0}\right) - Q\left(\frac{\rho_0 - \rho_1}{\rho}\right) \tag{4.7.4}$$

Remarks 4.7: For $H_0 : \rho_0 = 13$ vs $H_1 : \rho_1 = 15$, $\alpha = \beta = 0.05$ for varying values of the rate parameter 'ρ' the roots of 't' are obtained from the equation (4.7.3). It is evident from Fig. 4.9 and Fig. 4.10 that the operating characteristic curve shifts to the left (right) and ASN curve shifts to the left downward (right upward) for $Q < 1 (Q > 1)$, which shows that sequential probability ratio test is highly sensitive for any change in rate parameter 'ρ'.

4.8 Acceptance and rejections regions

One wish to test $H_0 : \rho = \rho_0$ vs $H_1 : \rho = \rho_1 (\rho_1 > \rho_0)$ having pre-assigned α, β such that $0 < \alpha, \beta < 1$. We may defined the ratio Z_i as:

$$Z_i = k \ln\left(\frac{\rho_1}{\rho_0}\right) + x_i(\rho_0 - \rho_1)$$

Let us take, $W(m) = \sum\limits_{i=1}^{m} X_i$ and N= $m \geq 1$ (first integer) for which the inequality $W(m) \leq e_1 + um$ or $W(m) \geq e_2 + um$ holds along with the constants

$$e_1 = \frac{\ln B}{(\rho_0 - \rho_1)}, e_2 = \frac{\ln A}{(\rho_0 - \rho_1)} \text{ and } u = \frac{k \ln\left(\frac{\rho_1}{\rho_0}\right)}{(\rho_0 - \rho_1)} \tag{4.8.1}$$

Remarks 4.8: For the case when $H_0 : \rho_0 = 13$ vs $H_1 : \rho_1 = 15$ for $\alpha = \beta = 0.05$, and $k = 2$ we get the values of $e_1 = 1.472$, $e_2 = -1.472$ and $u = 0.1434$, respectively. We accept H_0 if $W(N) \leq 0.1434N - 1.472$ and accept H_1 when $W(N) \geq 0.1434N + 1.472$ and at the intermediate stages, we continue the sampling by taking the additional observation. The acceptance and rejection regions for the above example are shown in the Fig. 4.11.

Table 4.1

The values of OC and ASN functions when $H_0 : \rho_0 = 13, H_1 : \rho_1 = 15, \alpha = \beta = 0.05$					
ρ	$L(\rho)$	$E(N)$	ρ	$L(\rho)$	$E(N)$
12.20	0.997	140.499	14.20	0.342	413.371
12.40	0.994	159.810	14.40	0.225	384.704
12.60	0.987	183.473	14.60	0.139	346.855
12.80	0.974	212.476	14.80	0.084	307.782
13.00	**0.952**	247.619	15.00	**0.048**	272.012
13.20	0.913	288.858	15.20	0.028	241.207
13.40	0.849	334.096	15.40	0.016	215.433
13.60	0.754	377.757	15.60	0.009	194.095
13.80	0.627	410.695	15.80	0.005	176.433
14.00	0.482	423.684			

Table 4.2

The values of OC functions when $H_0 : \rho_0 = 13, H_1 : \rho_1 = 15, \alpha = \beta = 0.05$					
ρ	$p = 0.96$	$p = 0.98$	$p = 1$	$p = 1.02$	$p = 1.04$
12.20	0.978	0.992	0.997	0.999	0.999
12.40	0.958	0.983	0.994	0.998	0.999
12.60	0.923	0.967	0.987	0.995	0.998
12.80	0.865	0.939	0.974	0.990	0.996
13.00	0.776	0.891	0.952	0.980	0.992
13.20	0.655	0.815	0.913	0.962	0.985
13.40	0.513	0.707	0.849	0.931	0.971
13.60	0.369	0.571	0.754	0.877	0.945
13.80	0.246	0.425	0.627	0.796	0.902
14.00	0.155	0.292	0.483	0.681	0.832
14.20	0.093	0.187	0.342	0.541	0.730
14.40	0.055	0.115	0.225	0.396	0.599
14.60	0.032	0.067	0.139	0.268	0.45
14.80	0.018	0.039	0.084	0.169	0.315
15.00	0.010	0.023	0.048	0.103	0.205
15.20	0.006	0.013	0.028	0.061	0.126
15.40	0.003	0.007	0.016	0.035	0.075
15.60	0.002	0.004	0.009	0.020	0.044
15.80	0.001	0.002	0.005	0.011	0.025

Table 4.3

The values of ASN functions when $H_0 : \rho_0 = 13, H_1 : \rho_1 = 15, \alpha = \beta = 0.05$					
ρ	$p = 0.96$	$p = 0.98$	$p = 1$	$p = 1.02$	$p = 1.04$
12.20	197.155	164.905	140.498	121.890	107.447
12.40	229.719	190.084	159.810	136.889	119.335
12.60	268.275	220.863	183.473	155.076	133.533
12.80	311.330	257.804	212.476	177.310	150.679
13.00	354.294	300.233	247.619	204.585	171.578
13.20	388.949	344.884	288.857	237.844	197.198
13.40	406.003	384.753	334.096	277.499	228.558
13.60	400.708	410.214	377.757	322.424	266.380
13.80	376.304	413.860	410.695	368.488	310.301
14.00	341.219	395.745	423.683	407.624	357.515
14.20	303.608	363.055	413.370	429.762	401.379
14.40	268.501	324.801	384.704	428.445	431.922
14.60	237.965	287.498	346.855	405.538	440.299
14.80	212.299	254.328	307.782	369.438	424.834
15.00	191.018	226.162	272.011	329.259	392.086
15.20	173.408	202.729	241.2071	291.052	351.752
15.40	158.772	183.345	215.4333	257.516	311.445
15.60	146.515	167.273	194.094	229.221	275.164
15.80	136.156	153.858	176.432	205.737	244.185

Table 4.4

The values of OC and ASN functions when $H_0 : k_0 = 13, H_1 : k_1 = 15, \alpha = \beta = 0.05$					
k	$L(k)$	$E(N)$	k	$L(k)$	$E(N)$
12.00	0.999	9.237	14.00	0.491	29.261
12.20	0.998	10.334	14.20	0.350	28.271
12.40	0.996	11.672	14.40	0.234	26.092
12.60	0.990	13.315	14.60	0.148	23.368
12.80	0.979	15.331	14.80	0.091	20.622
13.00	**0.958**	17.770	15.00	**0.054**	18.136
13.20	0.921	20.617	15.20	0.033	16.006
13.40	0.858	23.701	15.40	0.019	14.226
13.60	0.763	26.597	15.60	0.012	12.752
13.80	0.636	28.653	15.80	0.007	11.531

Table 4.5

The values of OC functions when $H_0 : k_0 = 13, H_1 : k_1 = 15, \alpha = \beta = 0.05$					
k	$\phi = 0.96$	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
12.20	0.999945	0.999600	0.998309	0.994513	0.984968
12.40	0.999745	0.998784	0.995741	0.987647	0.968782
12.60	0.999180	0.996855	0.990260	0.974054	0.938909
12.80	0.997805	0.992664	0.979255	0.948609	0.887357
13.00	0.994749	0.984108	0.958327	0.903873	0.805774
13.20	0.988391	0.967552	0.920781	0.831233	0.550266
13.60	0.952350	0.885095	0.763090	0.590112	0.404904
13.80	0.910806	0.802902	0.636174	0.443707	0.276973
14.00	0.842629	0.687288	0.490931	0.309185	0.178940
14.20	0.741441	0.546610	0.350214	0.202464	0.111184
14.60	0.463623	0.273985	0.148072	0.077353	0.040411
14.80	0.326145	0.176626	0.090898	0.046434	0.024058
15.00	0.215002	0.109515	0.054752	0.027661	0.014292
15.20	0.135245	0.066319	0.032657	0.016428	0.008491
15.40	0.082572	0.039648	0.019396	0.009753	0.005050
15.60	0.049562	0.023561	0.011508	0.005796	0.003009
15.80	0.029493	0.013972	0.006832	0.003450	0.001796

Table 4.6

The values of ASN functions when $H_0 : k_0 = 13, H_1 : k_1 = 15, \alpha = \beta = 0.05$					
k	$\phi = 0.96$	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
12.20	8.052826	9.070382	10.33428	11.91780	13.89641
12.40	8.870748	10.11046	11.67214	13.64174	16.08068
12.60	9.848712	11.37437	13.31481	15.75179	18.67666
12.80	11.03152	12.92305	15.33047	18.28361	21.59022
13.00	12.47603	14.82574	20.61763	24.20702	26.90107
13.40	16.42349	19.89954	23.70091	26.85593	28.09234
13.60	19.04171	22.98667	26.59718	28.46045	27.73625
14.00	25.27845	28.58762	29.26052	27.12234	23.55508
14.20	28.18622	29.81917	28.27114	24.74592	20.88337
14.40	30.08768	29.40520	26.09190	22.02627	18.37868
14.80	29.10050	24.91887	20.62176	17.07458	14.35348
15.00	26.67011	22.07522	18.13627	15.10250	12.82553
15.20	23.78829	19.41609	16.00627	13.46138	11.56009
15.40	20.95726	17.10020	14.22656	12.10260	10.50794
15.60	18.42908	15.15175	12.75243	10.97470	9.626742
15.80	16.27716	13.53507	11.53068	10.03223	8.882170

Table 4.7

The values of OC functions when $H_0 : \rho_0 = 13, H_1 : \rho_1 = 15, \alpha = \beta = 0.05$			
ρ	$Q = 0.98$	$Q = 1.00$	$Q = 1.02$
12.40	0.958	0.994	0.999
12.60	0.924	0.987	0.998
12.80	0.867	0.974	0.996
13.00	0.779	0.952	0.993
13.20	0.659	0.913	0.985
13.40	0.517	0.849	0.971
13.60	0.373	0.754	0.946
13.80	0.249	0.627	0.903
14.00	0.095	0.342	0.733
14.40	0.055	0.225	0.602
14.60	0.032	0.139	0.457
14.80	0.018	0.084	0.319
15.00	0.010	0.048	0.207
15.20	0.006	0.028	0.128
15.40	0.003	0.016	0.076
15.60	0.002	0.009	0.025

Table 4.8:

The values of ASN functions when $H_0 : \rho_0 = 13, H_1 : \rho_1 = 15, \alpha = \beta = 0.05$			
ρ	$Q = 0.98$	$Q = 1.00$	$Q = 1.02$
12.40	228.827	159.810	119.027
12.60	267.240	183.473	133.159
12.80	310.231	212.476	150.219
13.00	353.311	247.619	171.008
13.20	388.345	288.858	196.491
13.40	405.996	334.096	227.689
13.60	401.314	377.757	265.342
13.80	377.328	410.695	309.132
14.00	342.404	423.684	356.341
14.20	304.762	413.371	400.440
14.40	269.530	384.704	431.507
14.60	238.844	346.855	440.572
14.80	213.035	307.782	425.706
15.00	191.631	272.012	393.296
15.20	173.921	241.207	353.036
15.40	159.205	194.095	276.207
15.80	136.472	176.433	245.065

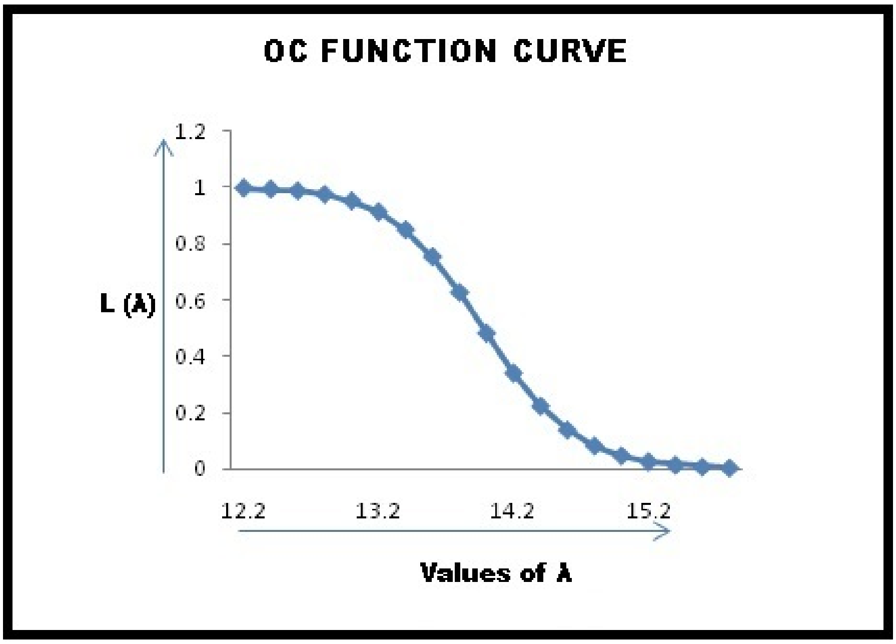

OC FUNCTION CURVE
1.2
1
0.8
L (λ)
0.6
0.4
0.2
0
12.2
13.2
14.2
15.2
Values of λ

Fig: 4.1

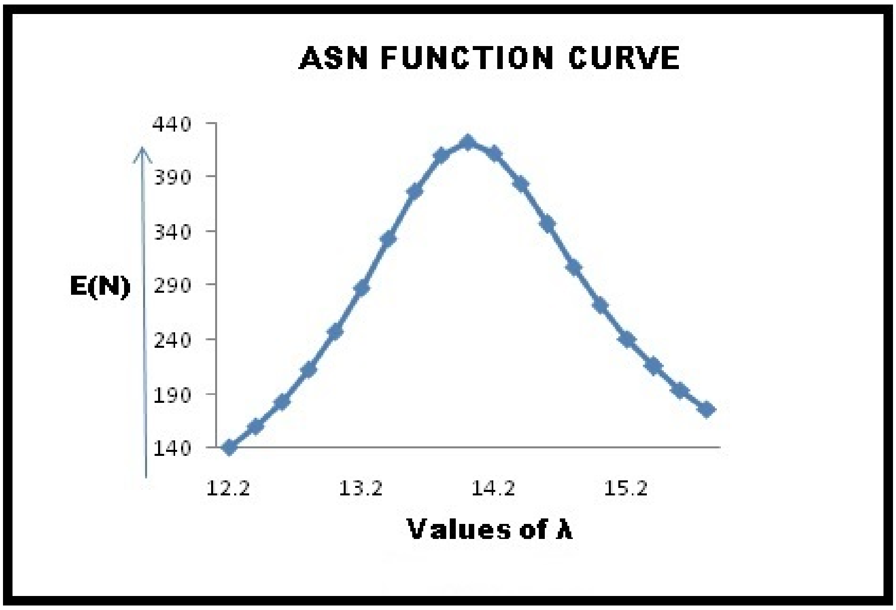

ASN FUNCTION CURVE
440
390
340
E(N)
290
240
190
140
12.2
13.2
14.2
15.2
Values of λ

Fig: 4.2

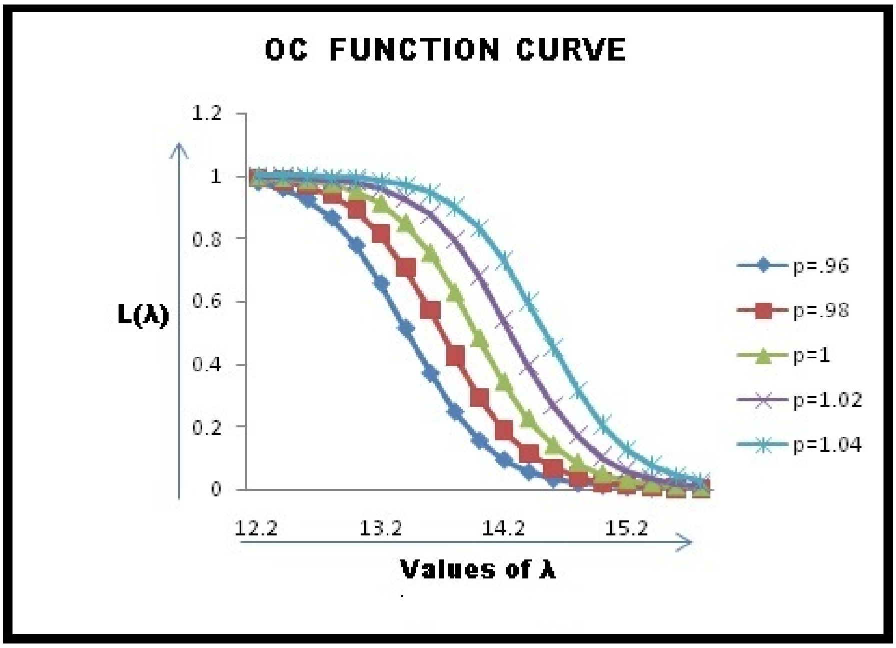

OC FUNCTION CURVE
L(λ)
1.2
1
0.8
0.6
0.4
0.2
0
12.2
13.2
14.2
15.2
Values of λ
p=.96
p=.98
p=1
p=1.02
p=1.04

Fig: 4.3

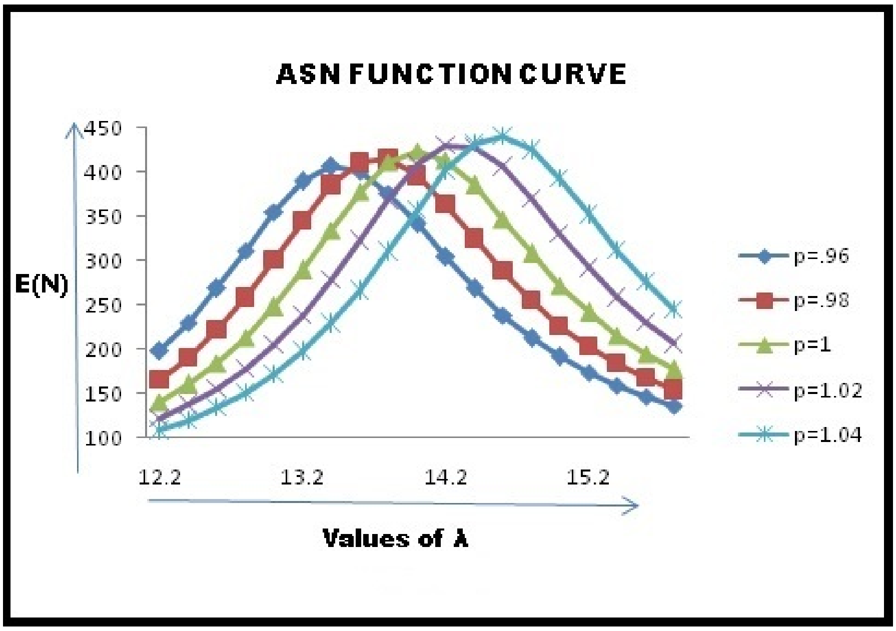

ASN FUNCTION CURVE
E(N)
450
400
350
300
250
200
150
100
12.2
13.2
14.2
15.2
Values of λ
p=.96
p=.98
p=1
p=1.02
p=1.04

Fig: 4.4

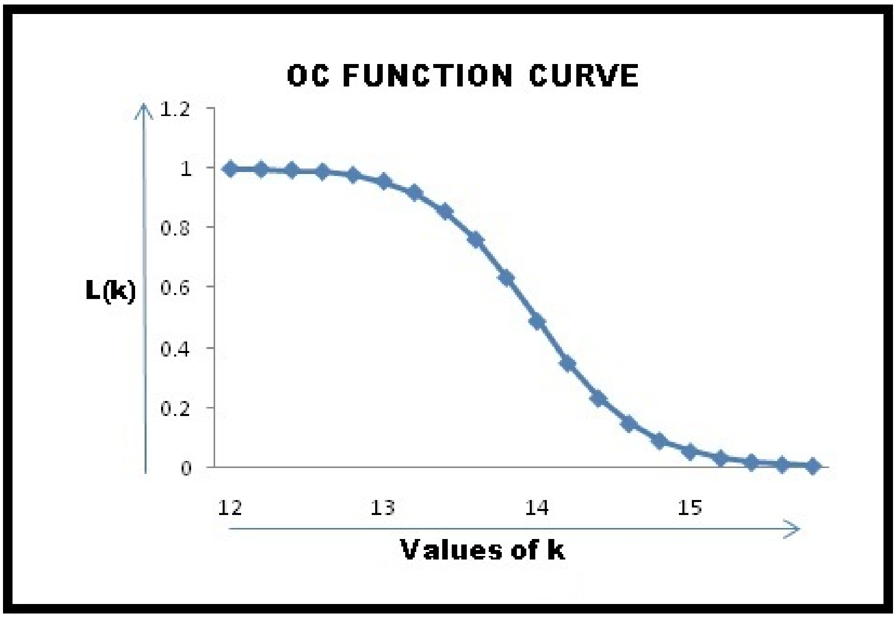

OC FUNCTION CURVE
L(k)
1.2
1
0.8
0.6
0.4
0.2
0
12
13
14
15
Values of k

Fig: 4.5

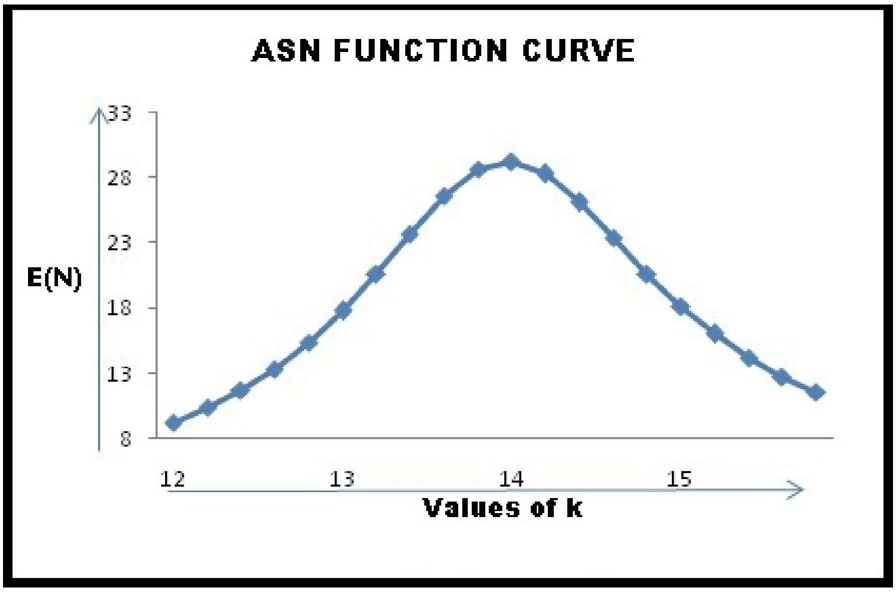
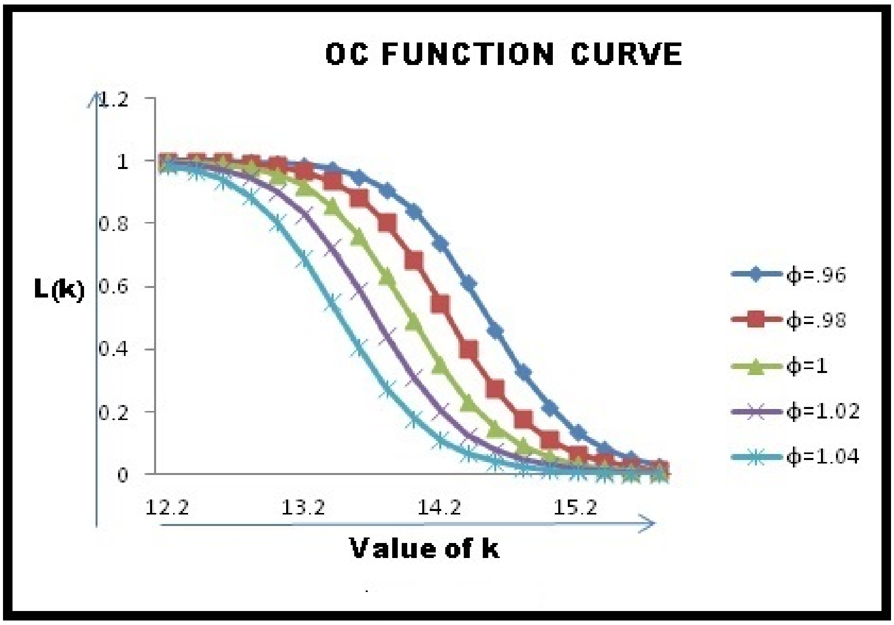
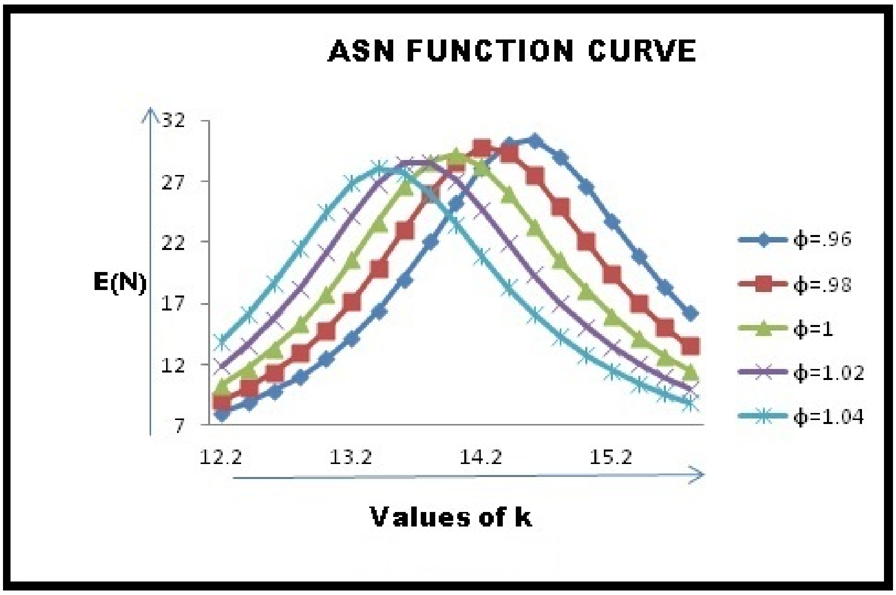

ASN FUNCTION CURVE
E(N)
33
28
23
18
13
8
12
13
14
15
Values of k

Fig: 4.6



OC FUNCTION CURVE
L(k)
1.2
1
0.8
0.6
0.4
0.2
0
12.2
13.2
14.2
15.2
Value of k
ϕ=.96
ϕ=.98
ϕ=1
ϕ=1.02
ϕ=1.04

Fig: 4.7



ASN FUNCTION CURVE
E(N)
32
27
22
17
12
7
12.2
13.2
14.2
15.2
Values of k
ϕ=.96
ϕ=.98
ϕ=1
ϕ=1.02
ϕ=1.04

Fig: 4.8

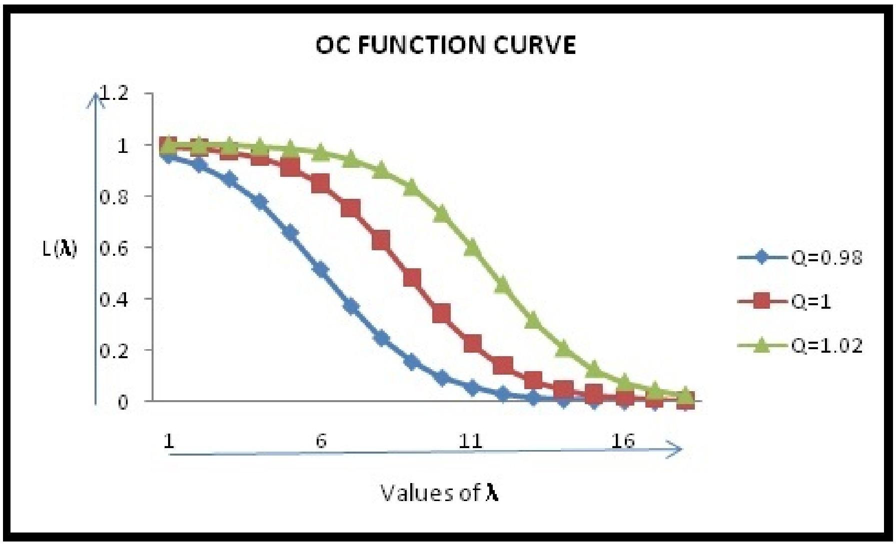


Fig: 4.9

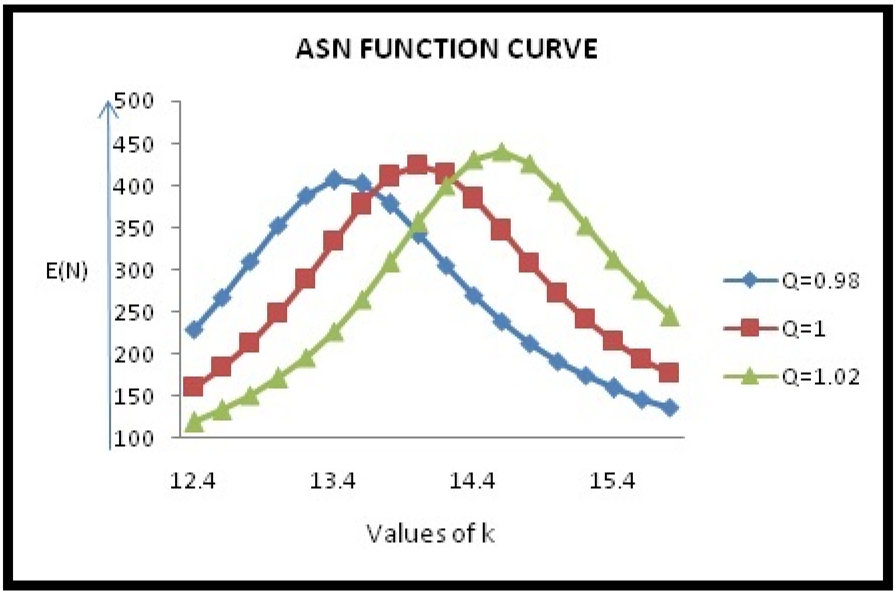


Fig: 4.10

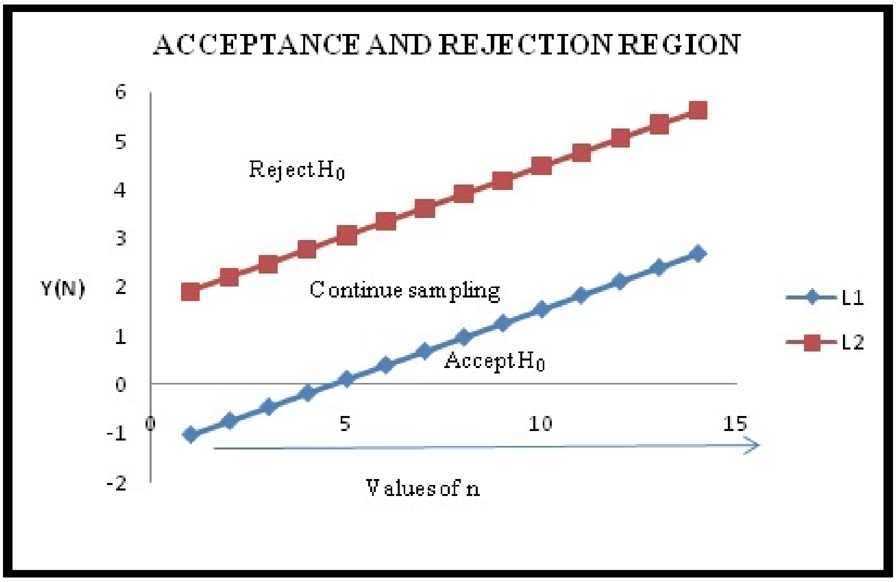

ACCEPTANCE AND REJECTION REGION
6
5
4
3
2
1
0
-1
-2
Y(N)
0
5
10
15
Reject H0
Continue sampling
Accept H0
Values of n
L1
L2

Fig: 4.11

Chapter 5

Bayesian Approach for the scale parameter of Positive exponential family of distributions

5.1 Introduction

Bayesian Decision theory plays a very important role as the prediction of the future, based on the prior knowledge. The loss function takes over the responsibility of obtaining the good estimator. The prior distribution and the loss function together enhance the performance of the Bayes estimator. Legendre (1805) [72] and Gauss (1810) [52] proposed the most widely used loss function known as the Squared Error Loss Function (SELF). The Qudratic loss function used by Taguchi (1986) [106] is another type or the modified form of SELF. For the situation where, overestimation is considered more serious as compare to underestimation, it is preferred to use the asymmetric loss function in place of squared error loss function, for a brief study one may see to Ferguson (1967) [49], Varian (1975) [112], Berger (1980) [17], Zellner (1986) [118] to study such cases. Precautionary Loss Function (PLF) was introduced by Norstrom(1996) [84]. It is an assymmetric loss function which is very simple to operate.

5.2 Set-up of the problem

Let a random variable X follows the Positive Exponential Family of Distribution defined as

$$f(x;\lambda,\gamma,\phi)=\frac{\lambda x^{\lambda\gamma-1}e^{-\left(x^{\lambda}/\phi\right)}}{\Gamma\gamma\phi^{\gamma}};x>0,\lambda,\gamma,\phi>0 \tag{5.2.1}$$

where, ϕ is assumed to be unknown and λ,γ are known constants. On assigning the different values to λ and γ, we get different distribution. For example: on taking $\lambda=\gamma=1$, the pdf (5.2.1) becomes one-parameter exponential distribution, when $\lambda=1$ it becomes gamma distribution, when $\lambda>0$, it becomes Generalized gamma distribution, when γ=1, it becomes Weibull distribution, when $\gamma>0$, $\lambda=1$, it becomes Erlang distribution, when $\gamma>1/2$, $\lambda=2$, it becomes half - normal distribution, when $\gamma>1/2$, $\lambda>0$, it becomes Maxwell Failure distribution, when $\gamma>m/2$, $\lambda=2$, it becomes chi-distribution, when $\gamma=1$, $\lambda=2$, it becomes Rayleigh distribution and when $\gamma=p+1$, $\lambda=2$, it becomes Generalized Rayleigh distribution.

The motive of this chapter is to obtained the Bayes estimators and the Posterior risks for the scale parameter of the PEFD through various loss functions and priors. The different Sections of this chapter are organized as: the posterior distributions under uniform and inverted gamma priors are obtained in Section 5.3. For these posterior distributions, the respective Bayes estimators are evaluated through various loss functions in Section 5.4. In Section 5.5, the posterior risk is obtained from the Bayes estimators under SELF, QLF and PLF for both the priors. A simulation study is done through a computer programme in R and Tables and Graphs given in Section 5.6. Finally, results are presented in Section 5.7.

5.3 Posterior distributions under the assumption of different priors

The section provides the posterior distribution in case of uniform (i.e. non-informative) prior and inverted gamma (i.e. conjugate) prior which are as follows:

5.3.1 Posterior distribution for the Uniform prior

The posterior distribution in case of uniform prior can be formulated as:

$$P_U(\phi \mid x) = \frac{P(\phi)L(\phi)}{\int_0^\infty P(\phi)L(\phi)d\phi} \tag{5.3.1}$$

where $P(\phi)$ is the uniform prior i.e.$P(\phi) = k$ and

$$L(\phi \mid \ x) = \left(\frac{\alpha}{\Gamma\gamma}\right)^n \frac{1}{\phi^{n\gamma}} \prod_{i=1}^{n} x_i^{\alpha\gamma-1} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \tag{5.3.2}$$

$$L(\phi \mid \ x) \propto \phi^{(-n\gamma)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \tag{5.3.3}$$

From (5.3.3) in (5.3.1), we get

$$P_U(\phi \mid x) = \frac{\phi^{(-n\gamma)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right)}{\int_0^\infty \phi^{(-n\gamma)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) d\phi} \tag{5.3.4}$$

On solving (5.3.4) we get the posterior distribution under the uniform prior which is as follows

$$P_U(\phi \mid x) = \frac{\phi^{(-n\gamma)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \sum_{i=1}^{n} x_i^\alpha{}^{n\gamma-1}}{\Gamma n\gamma - 1} \tag{5.3.5}$$

5.3.2 Posterior distribution for the Inverted Gamma prior

For inverted gamma prior, the same procedure will be used to obtain the posterior distribution which is as follows

$$P_{IG}(\phi \mid x) = \frac{G(\phi)L(\phi)}{\int_0^\infty G(\phi)L(\phi)d\phi} \tag{5.3.6}$$

The prior is defined as G(ϕ)

$$G(\phi) = \frac{\alpha\delta^{\gamma/\alpha}\phi^{-(\gamma+1)}exp(-\delta/\phi)}{\Gamma\gamma/\alpha} \tag{5.3.7}$$

On substituting the value of the prior G(ϕ) from (5.3.7) and L(ϕ | x) from (5.3.3) in (5.3.6) we get

$$P_{IG}(\phi \mid x) = \frac{\phi^{-(n\gamma+v+1)}exp - \left(\frac{\delta + \sum_{i=1}^{n} x_i^\alpha}{\phi}\right)}{\int_0^\infty \phi^{-(n\gamma+v+1)}exp - \left(\frac{\delta + \sum_{i=1}^{n} x_i^\alpha}{\phi}\right) d\phi} \tag{5.3.8}$$

The posterior distribution in case of the inverted gamma prior is given as

$$P_{IG}(\phi \mid x) = \frac{\phi^{-(n\gamma+v+1)}exp - \left(\frac{\delta + \sum_{i=1}^{n} x_i^\alpha}{\phi}\right)\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{(n\gamma+v)}}{\Gamma(n\gamma + v} \tag{5.3.9}$$

5.4 Bayes estimation under the assumption of different priors for various loss functions

5.4.1 Bayes estimation under Uniform Prior

The Bayes estimates are obtained for the above mentioned loss functions under the uniform prior are given as:

For squared error loss function

In case of SELF, the Bayes estimates can be obtained as follows:

$$L(\phi, \phi_{SELF}) = (\phi - \phi_{SELF})^2 \tag{5.4.1}$$

$$[\phi^{1/\alpha}]_{SELF} = E_U[\phi^{1/\alpha} \mid x] \tag{5.4.2}$$

$$[\phi^{1/\alpha}]_{SELF} = \int_0^\infty \phi^{1/\alpha} P_U(\phi \mid x) dx \tag{5.4.3}$$

$$= \int_0^\infty \frac{\phi^{\left(-n\gamma+\frac{1}{\alpha}\right)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \sum_{i=1}^{n} x_i^{\alpha\ n\gamma-1}}{\Gamma(n\gamma-1)} d\phi \tag{5.4.4}$$

On evaluating (5.4.4), we get

$$[\phi^{1/\alpha}]_{SELF} = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{1/\alpha} \Gamma(n\gamma + \frac{1}{\alpha} - 1)}{\Gamma(n\gamma-1)} \tag{5.4.5}$$

For qudratic loss function

In case of QLF, the Bayes estimates can be evaluated as follows:

$$[\phi^{1/\alpha}]_{QLF} = \frac{E_U[\phi^{-1/\alpha}]}{E_U[\phi^{-2/\alpha}]} \tag{5.4.6}$$

$$E_U[\phi^{-1/\alpha}] = \int_0^\infty \frac{\phi^{-\left(n\gamma+\frac{1}{\alpha}\right)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \sum_{i=1}^{n} x_i^\alpha{}^{n\gamma-1}}{\Gamma(n\gamma-1)} d\phi \tag{5.4.7}$$

$$E_U[\phi^{-1/\alpha}] = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{-1}{\alpha}} \Gamma(n\gamma-1+\frac{1}{\alpha})}{\Gamma(n\gamma-1)} \tag{5.4.8}$$

$$E_U[\phi^{-2/\alpha}] = \int_0^\infty \frac{\phi^{-\left(n\gamma+\frac{2}{\alpha}\right)} exp\left(-\sum_{i=1}^{n} x_i^\alpha/\phi\right) \sum_{i=1}^{n} x_i^\alpha{}^{n\gamma-1}}{\Gamma(n\gamma-1)} d\phi \tag{5.4.9}$$

$$E_U[\phi^{-2/\alpha}] = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{-2}{\alpha}} \Gamma n\gamma-1+\frac{2}{\alpha}}{\Gamma(n\gamma-1)} \tag{5.4.10}$$

Substituting the values from (5.4.8) and (5.4.10) in (5.4.6)

$$[\phi^{1/\alpha}]_{QLF} = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \Gamma(n\gamma-1+\frac{1}{\alpha})}{\Gamma(n\gamma-1+\frac{2}{\alpha})} \tag{5.4.11}$$

For precautionary loss function

The Bayes estimates in case of the uniform prior under PLF is given as

$$[\phi^{1/\alpha}]_{PLF} = \left[E_U\left(\phi^{2/\alpha} \mid x\right)\right]^{1/2} \tag{5.4.12}$$

$$[\phi^{1/\alpha}]_{PLF} = \int_0^\infty \frac{\phi^{-\left(n\gamma-\frac{2}{\alpha}\right)} exp - \left(\frac{\sum_{i=1}^{n} x_i^\alpha}{\phi}\right) \left(\sum_{i=1}^{n} x_i^\alpha\right)^{n\gamma-1}}{\Gamma n\gamma-1} d\phi \tag{5.4.13}$$

On solving (5.4.13), we get

$$[\phi^{1/\alpha}]_{PLF} = \left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \left[\frac{\Gamma(n\gamma-1-\frac{2}{\alpha})}{\Gamma(n\gamma-1)}\right]^{1/2} \tag{5.4.14}$$

5.4.2 Bayes estimation under Inverted Gamma Prior

The Bayes estimates are obtained for the above mentioned loss functions under the inverted gamma prior are given as

For squared error loss function

In case of SELF, the Bayes estimates can be evaluated as follows:

$$[\phi^{1/\alpha}]_{SELF} = \int_0^\infty \phi^{1/\alpha} P_{IG}(\phi \mid x)dx \tag{5.4.15}$$

$$[\phi^{1/\alpha}]_{SELF} = \int_0^\infty \frac{\phi^{-\left(n\gamma+v+1-\frac{1}{\alpha}\right)} exp - \left(\frac{\delta+\sum_{i=1}^{n} x_i^\alpha}{\phi}\right) \delta + \sum_{i=1}^{n} x_i^\alpha{}^{n\gamma+v}}{\Gamma(n\gamma+v)} d\phi \tag{5.4.16}$$

On further evaluating (5.4.16) we get

$$[\phi^{1/\alpha}]_{SELF} = \frac{\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \Gamma(n\gamma + v - \frac{1}{\alpha})}{\Gamma n\gamma + v} \tag{5.4.17}$$

For qudratic loss function

In case of QLF, the Bayes estimates can be obtained as follows:

$$[\phi^{1/\alpha}]_{QLF} = \frac{E_{IG}[\phi^{-1/\alpha}]}{E_{IG}[\phi^{-2/\alpha}]} \tag{5.4.18}$$

$$E_{IG}[\phi^{-1/\alpha}]_{QLF} = \int_0^\infty \phi^{-1/\alpha} P(\phi \mid x)d\phi \tag{5.4.19}$$

$$E_{IG}[\phi^{-1/\alpha}]_{QLF} = \int_0^\infty \frac{\phi^{-\left(n\gamma+v+1+\frac{1}{\alpha}\right)} exp - \left(\frac{\delta+\sum_{i=1}^{n} x_i^\alpha}{\phi}\right) \left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{n\gamma+v}}{\Gamma(n\gamma+v)} d\phi \tag{5.4.20}$$

$$E_{IG}[\phi^{-1/\alpha}] = \frac{\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{-1}{\alpha}} \Gamma(n\gamma + v + \frac{1}{\alpha})}{\Gamma(n\gamma + v)} \tag{5.4.21}$$

$$E_{IG}[\phi^{-2/\alpha}] = \frac{\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{-2}{\alpha}} \Gamma(n\gamma + v + \frac{2}{\alpha})}{\Gamma(n\gamma + v)} \tag{5.4.22}$$

On substituting the values from (5.4.21) and (5.4.22) in (5.4.18) we get

$$[\phi^{1/\alpha}]_{QLF} = \frac{\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \Gamma(n\gamma + v + \frac{1}{\alpha})}{\Gamma(n\gamma + v + \frac{2}{\alpha})} \tag{5.4.23}$$

For precautionary loss function

The Bayes estimates in case of the inverted gamma prior under PLF is given as

$$[\phi^{1/\alpha}]_{PLF} = \int_0^\infty \frac{\phi^{-\left(n\gamma+v+1-\frac{2}{\alpha}\right)} exp - \left(\frac{\delta + \sum_{i=1}^{n} x_i^\alpha}{\phi}\right) \left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{n\gamma+v}}{\Gamma(n\gamma + v)} d\phi \tag{5.4.24}$$

Evaluating the above integral we get

$$[\phi^{1/\alpha}]_{PLF} = \left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \left[\frac{\Gamma(n\gamma + v - \frac{2}{\alpha})}{\Gamma(n\gamma + v)}\right]^{1/2} \tag{5.4.25}$$

5.5 The Posterior risks in case of different loss functions

In this section, the posterior risk is evaluated by using the Bayes estimates from the earlier section.

5.5.1 Posterior risks for different loss functions under Uniform Prior

For squared error loss function

The posterior risk in case of SELF is formulated as follows:

$$P_U[\phi_{SELF}^{1/\alpha}] = E_U(\phi^{2/\alpha}|\underline{x}) - [E_U(\phi^{1/\alpha}|\underline{x})]^2 \tag{5.5.1}$$

$$E_U(\phi^{1/\alpha} \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}} \Gamma n\gamma - 1 - \frac{1}{\alpha}}{\Gamma(n\gamma - 1)} \tag{5.5.2}$$

$$E_U(\phi^{2/\alpha} \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{2}{\alpha}} \Gamma n\gamma - 1 - \frac{2}{\alpha}}{\Gamma(n\gamma - 1)} \tag{5.5.3}$$

On substituting the values from (5.5.2) and (5.5.3) in (5.5.1), we get

$$P_U[\phi_{SELF}^{1/\alpha}] = \frac{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{2}{\alpha}}}{\Gamma n\gamma - 1}\left[\left(\Gamma n\gamma - 1 - \frac{2}{\alpha}\right) - \frac{\left(\Gamma n\gamma - 1 - \frac{1}{\alpha}\right)^2}{\Gamma(n\gamma - 1)}\right] \tag{5.5.4}$$

For qudratic loss function

The posterior risk in case of QLF is evaluated using the following expressions:

$$P_U[\phi_{QLF}^{1/\alpha}] = 1 - \frac{\left[E_U(\phi^{-1/\alpha}|\underline{x})\right]^2}{E_U(\phi^{-2/\alpha}|\underline{x})} \tag{5.5.5}$$

Using (5.4.8) and (5.4.10) in the above expression, we get

$$P_U[\phi_{QLF}^{1/\alpha}] = 1 - \frac{\left(\Gamma n\gamma - 1 + \frac{1}{\alpha}\right)^2}{\Gamma(n\gamma - 1)\Gamma\left(n\gamma - 1 + \frac{2}{\alpha}\right)} \tag{5.5.6}$$

For precautionary loss function

The posterior risk in case of PLF is formulated as follows:

$$P_U[\phi^{1/\alpha}_{PLF}] = 2\left[\phi^{1/\alpha}_{PLF} - E_U(\phi^{-1/\alpha}|\underline{x})\right] \tag{5.5.7}$$

Using (5.4.8) and (5.4.11) in the above expression, we get

$$P_U[\phi^{1/\alpha}_{PLF}] = 2\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}}\left[\left(\frac{\Gamma n\gamma - 1 + \frac{1}{\alpha}}{\Gamma n\gamma - 1}\right)^{1/2} - \frac{\Gamma n\gamma - 1 - \frac{1}{\alpha}}{\Gamma n\gamma - 1}\right] \tag{5.5.8}$$

5.5.2 Posterior risks for different loss functions under Inverted Gamma Prior

For squared error loss function

The posterior risk in case of SELF is evaluated using the following expressions:

$$P_{IG}[\phi^{1/\alpha}_{SELF}] = E_{IG}(\phi^{2/\alpha}|\underline{x}) - [E_{IG}(\phi^{1/\alpha}|\underline{x})]^2 \tag{5.5.9}$$

$$E_{IG}(\phi^{1/\alpha} \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}} \Gamma n\gamma + v - \frac{1}{\alpha}}{\Gamma(n\gamma + v)} \tag{5.5.10}$$

$$E_{IG}(\phi^{2/\alpha} \mid \underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{2}{\alpha}} \Gamma n\gamma + v - \frac{2}{\alpha}}{\Gamma(n\gamma + v)} \tag{5.5.11}$$

Substituting the values from (5.5.10) and (5.5.11) in (5.5.9), we get

$$P_{IG}[\phi^{1/\alpha}_{SELF}] = \frac{\left(\delta + \sum_{i=1}^{n} x_i^\alpha\right)^{\frac{2}{\alpha}}}{\Gamma n\gamma + v}\left[\left(\Gamma n\gamma + v - \frac{2}{\alpha}\right) - \frac{\left(\Gamma n\gamma + v - \frac{1}{\alpha}\right)^2}{\Gamma n\gamma + v}\right] \tag{5.5.12}$$

For qudratic loss function

The posterior risk in case of QLF is formulated as follows:

$$P_{IG}[\phi^{1/\alpha}_{QLF}] = 1 - \frac{\left[E_{IG}(\phi^{-1/\alpha}|\underline{x})\right]^2}{E_{IG}(\phi^{-2/\alpha}|\underline{x})} \tag{5.5.13}$$

Substituting the values from (5.4.21) and (5.4.22) in (5.5.13), we have

$$P_{IG}[\phi^{1/\alpha}_{QLF}] = 1 - \frac{\left(\Gamma n\gamma + v + \frac{1}{\alpha}\right)^2}{\left(\Gamma n\gamma + v\right)\left(\Gamma n\gamma + v + \frac{2}{\alpha}\right)} \tag{5.5.14}$$

For precautionary loss function

The posterior risk in case of PLF is evaluated using the following expressions:

$$P_{IG}[\phi^{1/\alpha}_{PLF}] = 2\left[\phi^{1/\alpha}_{PLF} - E_{IG}(\phi^{-1/\alpha}|\underline{x})\right] \tag{5.5.15}$$

Using (5.4.25) and (5.4.21), we get

$$P[\phi^{1/\alpha}_{PLF}] = 2\left(\delta + \sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}} \left[\left(\frac{\Gamma n\gamma + v + \frac{1}{\alpha}}{\Gamma n\gamma + v}\right)^{1/2} - \frac{\Gamma n\gamma + v - \frac{1}{\alpha}}{\Gamma n\gamma + v}\right] \tag{5.5.16}$$

5.6 Simulation Study

In our model (5.2.1), as the cdf is not in a close form and we are unable to use the method of inverse transformation to generate the random numbers, so as to overcome this problem we have used another well known method known as the Accept and Reject method. This method used to generate the random numbers of the pdf $g(x)$ [also known as target density]. For this we choose another probability density $g(y)$ with the conditions: $g(x) > 0$, when $g(y) > 0$) and $g(x)/g(y) \leq H, \forall x$, where H is normalising constant. The numerical values are obtained on R software with Markov chain Monte Carlo method of simulation and are presented in Fig. 5.1.

For the random number generation and simulation in the model (5.2.1). We have generated 5000 random samples on assigning different values to ϕ, α and γ. To get our objective, the

Baye estimators and respective Posterior risks under 10000 replications and averages of the 10000 outputs are induced. The numerical values of Bayes estimators and posterior risk are shown in the Tables (Table 5.1 -Table 5.10) and are presented in the Figures (Fig. 5.2-Fig. 5.21), respectively.

5.7 Results and Conclusion

To acheive the objective, comparision is made between the priors as well as for the different loss functons on the basis of posterior risk for the various values of ϕ.

- For the uniform prior as the value of ϕ increases, the value of posterior risk is also increases under SELF and PLF for all the distributions.
- For the inverted gamma prior as the value of ϕ increases, the value of posterior risk is decreases under SELF and PLF for all the distributions.
- Values of posterior risk are same under QLF for the uniform and inverted gamma prior respectively, for a particular distribution.
- From the Table 5.1 and Table 5.10, we see that QLF has minimum and PLF has maximum value of posterior risk in most of the distributions.

Table 5.1

The values of Bayes estimator and (Posterior risk) for one-parameter exponential distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.2617 (0.2499)	1.0094 (0.1000)	1.3488 (0.1742)
	20	1.1237 (0.0780)	1.0113 (0.0500)	1.1563 (0.0651)
	30	1.0815 (0.0447)	1.0094 (0.0333)	1.1013 (0.0396)
	40	1.0609 (0.0311)	1.0078 (0.0250)	1.0751 (0.0284)
	50	1.0484 (0.0238)	1.0064 (0.0200)	1.0595 (0.0221)
1.5	10	1.9021 (0.5681)	1.5216 (0.1000)	2.0333 (0.2626)
	20	1.6765 (0.1737)	1.5088 (0.0500)	1.7251 (0.0972)
	30	1.6226 (0.1007)	1.5145 (0.0333)	1.6524 (0.0595)
	40	1.5942 (0.0703)	1.5145 (0.0250)	1.6156 (0.0427)
	50	1.5761 (0.0539)	1.5131 (0.0200)	1.5927 (0.0333)
2.0	10	2.5168 (0.9941)	2.0134 (0.1000)	2.6906 (0.3475)
	20	2.2454 (0.3111)	2.0209 (0.0500)	2.3105 (0.1301)
	30	2.1595 (0.1782)	2.0155 (0.0333)	2.1991 (0.0792)
	40	2.1183 (0.1242)	2.0124 (0.0250)	2.1468 (0.0568)
	50	2.1047 (0.0961)	2.0205 (0.0200)	2.1271 (0.0445)
2.5	10	3.0786 (1.4921)	2.4629 (0.1000)	3.2912 (0.4251)
	20	2.7448 (0.4663)	2.4703 (0.0500)	2.8244 (0.1591)
	30	2.6376 (0.2669)	2.4618 (0.3333)	2.6861 (0.0968)
	40	2.5958 (0.1868)	2.4661 (0.0250)	2.6306 (0.0696)
	50	2.5698 (0.1434)	2.4671 (0.0200)	2.5969 (0.0543)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.0913 (0.1528)	0.9015 (0.0869)	1.1537 (1.4118)
	20	1.0398 (0.0612)	0.9431 (0.0465)	1.0675 (1.1779)
	30	1.0211 (0.0377)	0.9562 (0.0317)	1.0388 (0.1041)
	40	1.0151 (0.0274)	0.9662 (0.0240)	1.0282 (1.0763)
	50	1.0077 (0.0213)	0.9685 (0.0194)	1.0181 (0.0505)
1.5	10	0.7624 (0.0745)	0.6298 (0.0869)	0.8061 (0.6881)
	20	0.7144 (0.0288)	0.6479 (0.0465)	0.7334 (0.5557)
	30	0.6998 (0.0177)	0.6554 (0.0317)	0.7121 (0.5189)
	40	0.6943 (0.0128)	0.6608 (0.0241)	0.7032 (0.5035)
	50	0.6866 (0.0099)	0.6601 (0.0194)	0.6937 (0.4883)
2.0	10	0.5775 (0.0425)	0.4771 (0.0869)	0.6106 (0.3932)
	20	0.5373 (0.0163)	0.4874 (0.0465)	0.5517 (0.3141)
	30	0.5273 (0.0101)	0.4938 (0.0317)	0.5365 (0.2944)
	40	0.5202 (0.0071)	0.4951 (0.0241)	0.5269 (0.2824)
	50	0.5152 (0.0055)	0.4952 (0.0194)	0.5205 (0.2748)
2.5	10	0.4727 (0.0284)	0.3905 (0.0869)	0.4997 (0.2625)
	20	0.4333 (0.0106)	0.3931 (0.0465)	0.4448 (0.2041)
	30	0.4212 (0.0064)	0.3944 (0.0317)	0.4285 (0.1877)
	40	0.4161 (0.0046)	0.3959 (0.0240)	0.4213 (0.1806)
	50	0.4132 (0.0035)	0.3971 (0.0194)	0.4174 (0.1767)

Table 5.2

The values Bayes estimator and (Posterior risk) for Gamma distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.0085 (0.1241)	0.8404 (0.0833)	1.0631 (0.1091)
	20	0.9132 (0.0417)	0.8317 (0.0416)	0.9347 (0.0429)
	30	0.8919 (0.0249)	0.8424 (0.0277)	0.9054 (0.0268)
	40	0.8795 (0.0176)	0.8428 (0.0208)	0.8892 (0.0194)
	50	0.8694 (0.0135)	0.8404 (0.0166)	0.8771 (0.0152)
1.5	10	1.5036 (0.2762)	1.2531 (0.0833)	1.5849 (0.1626)
	20	1.3777 (0.0949)	1.2628 (0.0416)	1.4101 (0.0648)
	30	1.3336 (0.0556)	1.2598 (0.0277)	1.3539 (0.0401)
	40	1.3166 (0.0394)	1.2617 (0.0208)	1.3311 (0.0291)
	50	1.3003 (0.0302)	1.2569 (0.0166)	1.3116 (0.0227)
2.0	10	2.0114 (0.4935)	1.6762 (0.0833)	2.1202 (0.2176)
	20	1.8316 (0.1675)	1.6789 (0.0416)	1.8747 (0.0862)
	30	1.7767 (0.0987)	1.6781 (0.0277)	1.8034 (0.0534)
	40	1.7501 (0.0697)	1.6771 (0.0208)	1.7694 (0.0386)
	50	1.7392 (0.0541)	1.6812 (0.0166)	1.7544 (0.0303)
2.5	10	2.4654 (0.7468)	2.0545 (0.0833)	2.5988 (0.2667)
	20	2.2439 (0.2522)	2.0569 (0.0416)	2.2967 (0.1056)
	30	2.1818 (0.1493)	2.0606 (0.2777)	2.2147 (0.0656)
	40	2.1408 (0.1045)	2.0515 (0.0208)	2.1644 (0.0473)
	50	2.1281 (0.0811)	2.0572 (0.0167)	2.1467 (0.0371)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	0.9043 (0.0847)	0.7703 (0.0741)	0.9463 (0.9526)
	20	0.8644 (0.0347)	0.7966 (0.0392)	0.8834 (0.8086)
	30	0.8501 (0.0216)	0.8047 (0.0266)	0.8622 (0.7614)
	40	0.8425 (0.0156)	0.8085 (0.0202)	0.8516 (0.7391)
	50	0.8401 (0.0122)	0.8127 (0.0162)	0.8472 (0.7285)
1.5	10	0.6295 (0.0411)	0.5363 (0.0741)	0.6588 (0.4612)
	20	0.5941 (0.0164)	0.5474 (0.0392)	0.6071 (0.3822)
	30	0.5801 (0.0101)	0.5491 (0.0267)	0.5884 (0.3551)
	40	0.5743 (0.0072)	0.5511 (0.0202)	0.5805 (0.3433)
	50	0.5722 (0.0057)	0.5536 (0.0162)	0.5771 (0.3381)
2.0	10	0.4762 (0.0234)	0.4056 (0.0741)	0.4984 (0.2634)
	20	0.4466 (0.0092)	0.4115 (0.0392)	0.4564 (0.2153)
	30	0.4376 (0.0057)	0.4143 (0.0267)	0.4439 (0.2021)
	40	0.4317 (0.0041)	0.4142 (0.0202)	0.4363 (0.1939)
	50	0.4284 (0.0031)	0.4145 (0.0162)	0.4321 (0.1895)
2.5	10	0.3887 (0.0155)	0.3311 (0.0741)	0.4068 (0.1751)
	20	0.3603 (0.0060)	0.3320 (0.0392)	0.3682 (0.1401)
	30	0.3496 (0.0036)	0.3310 (0.0266)	0.3547 (0.1288)
	40	0.3457 (0.0046)	0.3317 (0.0240)	0.3494 (0.1806)
	50	0.3428 (0.0021)	0.3317 (0.0162)	0.3457 (0.1214)

Table 5.3

The values of Bayes estimator and (Posterior risk) for Generalised Gamma distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	0.6396 (0.0214)	0.5765 (0.0492)	0.6580 (0.0391)
	20	0.6033 (0.0101)	0.5736 (0.0245)	0.6113 (0.0175)
	30	0.5933 (0.0062)	0.5739 (0.0163)	0.5984 (0.0114)
	40	0.5866 (0.0044)	0.5722 (0.0122)	0.5903 (0.0084)
	50	0.5856 (0.0035)	0.5741 (0.0098)	0.5886 (0.0068)
1.5	10	0.9549 (0.0582)	0.8607 (0.0492)	0.9823 (0.0596)
	20	0.9029 (0.0227)	0.8585 (0.0245)	0.9148 (0.0268)
	30	0.8911 (0.0141)	0.8618 (0.0163)	0.8986 (0.0175)
	40	0.8834 (0.0102)	0.8617 (0.0122)	0.8891 (0.0131)
	50	0.8763 (0.0079)	0.8591 (0.0098)	0.8807 (0.0103)
2.0	10	1.2607 (0.1024)	1.1421 (0.0492)	1.3034 (0.0802)
	20	1.2051 (0.0405)	1.1458 (0.0245)	1.2211 (0.0363)
	30	1.1844 (0.0249)	1.1456 (0.0163)	1.1945 (0.0236)
	40	1.1726 (0.0697)	1.1438 (0.0208)	1.1801 (0.0386)
	50	1.1687 (0.0141)	1.1458 (0.0098)	1.1746 (0.0141)
2.5	10	1.5531 (0.1548)	1.3997 (0.0492)	1.5975 (0.0993)
	20	1.4768 (0.0611)	1.4041 (0.0245)	1.4963 (0.0449)
	30	1.4527 (0.0376)	1.4051 (0.0163)	1.4652 (0.0292)
	40	1.4369 (0.0271)	1.4016 (0.0122)	1.4461 (0.0217)
	50	1.4274 (0.0211)	1.3994 (0.0098)	1.4346 (0.0173)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	0.6042 (0.0211)	0.5491 (0.0455)	0.6201 (0.3269)
	20	0.5811 (0.0091)	0.5535 (0.0236)	0.5884 (0.2756)
	30	0.5751 (0.0057)	0.5566 (0.0159)	0.5798 (0.2589)
	40	0.5721 (0.0041)	0.5582 (0.0121)	0.5756 (0.2499)
	50	0.5695 (0.0032)	0.5585 (0.0096)	0.5723 (0.2436)
1.5	10	0.4189 (0.0101)	0.3807 (0.0455)	0.4299 (0.1572)
	20	0.4006 (0.0042)	0.3816 (0.0236)	0.4056 (0.1308)
	30	0.3943 (0.0026)	0.3817 (0.0159)	0.3976 (0.1219)
	40	0.3900 (0.0019)	0.3806 (0.0121)	0.3924 (0.1162)
	50	0.3886 (0.0015)	0.3811 (0.0096)	0.3906 (0.1134)
2.0	10	0.3178 (0.0058)	0.2888 (0.0455)	0.3261 (0.0901)
	20	0.3011 (0.0024)	0.2868 (0.0236)	0.3048 (0.0738)
	30	0.2963 (0.0015)	0.2869 (0.0159)	0.2988 (0.0687)
	40	0.2918 (0.0010)	0.2847 (0.0121)	0.2936 (0.0651)
	50	0.2917 (0.0008)	0.2891 (0.0096)	0.2932 (0.0639)
2.5	10	0.2591 (0.0038)	0.2354 (0.0455)	0.2659 (0.0596)
	20	0.2429 (0.0015)	0.2314 (0.0236)	0.2461 (0.0481)
	30	0.2376 (0.0009)	0.2301 (0.0159)	0.2396 (0.0442)
	40	0.2351 (0.0007)	0.2295 (0.0121)	0.2365 (0.0422)
	50	0.2331 (0.0005)	0.2285 (0.0096)	0.2341 (0.0407)

Table 5.4

The values of Bayes estimator and (Posterior risk) for weibull distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.3638 (0.1199)	1.2331 (0.0473)	1.4031 (0.1278)
	20	1.2827 (0.0436)	1.2235 (0.0229)	1.2988 (0.0651)
	30	1.2617 (0.0264)	1.2234 (0.0151)	1.2718 (0.0466)
	40	1.2517 (0.0189)	1.2233 (0.0112)	1.2591 (0.0373)
	50	1.2449 (0.0147)	1.2224 (0.0089)	1.2507 (0.0316)
1.5	10	2.0347 (0.2661)	1.8397 (0.0473)	2.0932 (0.2326)
	20	1.9218 (0.0978)	1.8332 (0.0229)	1.9459 (0.1192)
	30	1.8937 (0.0597)	1.8362 (0.0151)	1.9089 (0.0858)
	40	1.8772 (0.0427)	1.8348 (0.0112)	1.8883 (0.0686)
	50	1.8611 (0.0331)	1.8274 (0.0089)	1.8696 (0.0577)
2.0	10	2.7286 (0.4797)	2.4671 (0.4732)	2.8071 (0.3615)
	20	2.5515 (0.1727)	2.4338 (0.0229)	2.5834 (0.1825)
	30	2.5162 (0.1055)	2.4398 (0.0151)	2.5364 (0.1315)
	40	2.4896 (0.0751)	2.4333 (0.0112)	2.5043 (0.1048)
	50	2.4791 (0.0585)	2.4343 (0.0089)	2.4905 (0.0888)
2.5	10	3.3467 (0.7239)	3.0259 (0.0473)	3.4430 (0.4917)
	20	3.1526 (0.2645)	3.0072 (0.0229)	3.1921 (0.2511)
	30	3.0921 (0.1594)	2.9981 (0.0151)	3.1168 (0.1792)
	40	3.0747 (0.1147)	3.0052 (0.0112)	3.0929 (0.1441)
	50	3.0549 (0.0891)	2.9998 (0.0089)	3.0691 (0.1216)
	Inverted Gamma Prior			
ϕ	n	SELF	QLF	PLF
1.0	10	1.2248 (0.0802)	1.1237 (0.0408)	1.2543 (0.3599)
	20	1.2101 (0.0357)	1.1583 (0.0212)	1.2241 (0.2591)
	30	1.2061 (0.0229)	1.1713 (0.0143)	1.2153 (0.2193)
	40	1.1996 (0.0167)	1.1734 (0.0108)	1.2064 (0.1945)
	50	1.1954 (0.0132)	1.1745 (0.0087)	1.2008 (0.1779)
1.5	10	0.8481 (0.0383)	0.7780 (0.0408)	0.8684 (0.1718)
	20	0.8338 (0.0171)	0.7981 (0.0212)	0.8434 (0.1232)
	30	0.8248 (0.0107)	0.8009 (0.0143)	0.8311 (0.1026)
	40	0.8218 (0.0078)	0.8039 (0.0108)	0.8265 (0.0914)
	50	0.8217 (0.0062)	0.8074 (0.0087)	0.8255 (0.0841)
2.0	10	0.6498 (0.0223)	0.5962 (0.0408)	0.6655 (0.1002)
	20	0.6243 (0.0095)	0.5976 (0.0212)	0.6315 (0.0688)
	30	0.6186 (0.0061)	0.6007 (0.0143)	0.6233 (0.0576)
	40	0.6167 (0.0044)	0.6033 (0.0108)	0.6202 (0.0514)
	50	0.6132 (0.0064)	0.6024 (0.0087)	0.6159 (0.0468)
2.5	10	0.5298 (0.0148)	0.4861 (0.0408)	0.5426 (0.0665)
	20	0.5061 (0.0062)	0.4844 (0.0212)	0.5119 (0.0452)
	30	0.4971 (0.0038)	0.4827 (0.0143)	0.5008 (0.0372)
	40	0.4935 (0.0028)	0.4828 (0.0108)	0.4963 (0.0329)
	50	0.4923 (0.0022)	0.4837 (0.0087)	0.4946 (0.0301)

Table 5.5

The values of Bayes estimator and (Posterior risk) for half normal distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	2.3627 (0.5371)	2.0674 (0.0604)	2.4628 (0.4748)
	20	2.1619 (0.1568)	2.0418 (0.0273)	2.1959 (0.2236)
	30	2.1018 (0.0891)	2.0267 (0.0176)	2.1221 (0.1613)
	40	2.0861 (0.0626)	2.0311 (0.0130)	2.1005 (0.1332)
	50	2.0695 (0.0479)	2.0264 (0.0103)	2.0808 (0.1152)
1.5	10	3.5622 (1.2146)	3.1169 (0.0604)	3.7130 (1.0737)
	20	3.2314 (0.3497)	3.0519 (0.0273)	3.2823 (0.4984)
	30	3.1482 (0.1996)	3.0358 (0.0176)	3.1786 (0.3616)
	40	3.1205 (0.1401)	3.0384 (0.0131)	3.1422 (0.2981)
	50	3.1082 (0.1079)	3.0435 (0.0103)	3.1252 (0.2594)
2.0	10	4.7062 (2.1298)	4.1179 (0.0604)	4.9055 (1.8827)
	20	4.2929 (0.6181)	4.0544 (0.0273)	4.3604 (0.8809)
	30	4.1901 (0.3541)	4.0404 (0.0176)	4.2305 (0.6414)
	40	4.1405 (0.2468)	4.0315 (0.0131)	4.1693 (0.5253)
	50	4.1047 (0.1884)	4.0192 (0.0103)	4.1271 (0.4532)
2.5	10	5.8323 (3.2899)	5.1032 (0.0604)	6.0793 (2.9082)
	20	5.2985 (0.9454)	5.0041 (0.0273)	5.3819 (1.3475)
	30	5.1918 (0.5441)	5.0063 (0.0176)	5.2419 (0.9856)
	40	5.1338 (0.3798)	4.9987 (0.0131)	5.1696 (0.8086)
	50	5.1071 (0.2922)	5.0007 (0.0103)	5.1349 (0.7027)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.9427 (0.2375)	1.7661 (0.0444)	1.9973 (0.5109)
	20	1.9573 (0.1078)	1.8641 (0.0235)	1.9832 (0.3347)
	30	1.9651 (0.0697)	1.9017 (0.0159)	1.9821 (0.2667)
	40	1.9595 (0.0509)	1.9117 (0.0121)	1.6721 (0.2258)
	50	1.9711 (0.0408)	1.9324 (0.0097)	1.9811 (0.2024)
1.5	10	1.3562 (0.1158)	1.3562 (0.0444)	1.3943 (0.2491)
	20	1.3474 (0.0511)	1.2832 (0.0235)	1.3652 (0.1587)
	30	1.3496 (0.0329)	1.3061 (0.0159)	1.3613 (0.1261)
	40	1.3493 (0.0242)	1.3163 (0.0121)	1.3579 (0.1073)
	50	1.3457 (0.0191)	1.3194 (0.0097)	1.3526 (0.0944)
2.0	10	1.0312 (0.0663)	0.9375 (0.0443)	1.0602 (0.1426)
	20	1.0124 (0.0288)	0.9642 (0.0235)	1.0258 (0.0893)
	30	1.0102 (0.0184)	0.9776 (0.0159)	1.0189 (0.0704)
	40	1.0083 (0.0134)	0.9837 (0.0121)	1.0147 (0.0597)
	50	1.0087 (0.0106)	0.9889 (0.0097)	1.0138 (0.0531)
2.5	10	0.8471 (0.0444)	0.7701 (0.0443)	0.8708 (0.0956)
	20	0.8214 (0.0189)	0.7823 (0.0235)	0.8323 (0.0588)
	30	0.8117 (0.0119)	0.7855 (0.0159)	0.8187 (0.0455)
	40	0.8102 (0.0087)	0.7904 (0.0121)	0.8154 (0.0386)
	50	0.8043 (0.0022)	0.7885 (0.0087)	0.8084 (0.3375)

Table 5.6

The values of Bayes estimator and (Posterior risk) for Chi-Square distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	2.3519 (0.5312)	2.0579 (0.0604)	2.4515 (0.4695)
	20	2.1572 (0.1562)	2.0374 (0.0273)	2.1912 (0.2227)
	30	2.1097 (0.0897)	2.0343 (0.0176)	2.1301 (0.1625)
	40	2.0798 (0.0622)	2.0251 (0.0131)	2.0943 (0.1326)
	50	2.0693 (0.0478)	2.0262 (0.0103)	2.0805 (0.1151)
1.5	10	3.5527 (1.2098)	3.1086 (0.0604)	3.7031 (1.0694)
	20	3.2371 (0.3512)	3.0531 (0.0273)	3.2881 (0.5006)
	30	3.1556 (0.2006)	3.0429 (0.0176)	3.1861 (0.3635)
	40	3.1297 (0.1409)	3.0474 (0.0131)	3.1516 (0.3001)
	50	3.1012 (0.1074)	3.0366 (0.0103)	3.1181 (0.2583)
2.0	10	4.7061 (2.1266)	4.1178 (0.0604)	4.9054 (1.8799)
	20	4.2755 (0.6131)	4.0381 (0.0273)	4.3428 (0.8739)
	30	4.1832 (0.3528)	4.0338 (0.0176)	4.2236 (0.6391)
	40	4.1488 (0.2481)	4.0396 (0.0131)	4.1778 (0.5281)
	50	4.1083 (0.1889)	4.0227 (0.0103)	4.1307 (0.4542)
2.5	10	5.8111 (3.2755)	5.0847 (0.0604)	6.0571 (2.8955)
	20	5.3171 (0.9513)	5.0217 (0.0273)	5.4008 (1.3559)
	30	5.1827 (0.5422)	4.9976 (0.0176)	5.2327 (0.9822)
	40	5.1541 (0.3831)	5.0184 (0.0131)	5.1899 (0.8154)
	50	5.0944 (0.2905)	4.9883 (0.0103)	5.1222 (0.6986)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.9553 (0.2408)	1.7775 (0.0443)	2.0102 (0.5181)
	20	1.9517 (0.1072)	1.8588 (0.0235)	1.9776 (0.3328)
	30	1.9558 (0.0691)	1.8927 (0.0159)	1.9728 (0.2641)
	40	1.9619 (0.0511)	1.9141 (0.0121)	1.9745 (0.2265)
	50	1.9675 (0.0406)	1.9289 (0.0097)	1.9776 (0.2017)
1.5	10	1.3548 (0.1155)	1.2316 (0.0443)	1.3928 (0.2485)
	20	1.3481 (0.0513)	1.2839 (0.0235)	1.3659 (0.1591)
	30	1.3465 (0.0328)	1.3031 (0.0159)	1.3581 (0.1255)
	40	1.3524 (0.0243)	1.3195 (0.0121)	1.3611 (0.1077)
	50	1.3484 (0.0191)	1.3219 (0.0097)	1.3553 (0.0949)
2.0	10	1.0318 (0.0661)	0.9381 (0.0443)	1.0608 (0.1423)
	20	1.0117 (0.0287)	0.9635 (0.0235)	1.0251 (0.0891)
	30	1.0092 (0.0184)	0.9766 (0.0159)	1.0179 (0.0703)
	40	1.0065 (0.0134)	0.9819 (0.0121)	1.0129 (0.0596)
	50	1.0082 (0.0106)	0.9884 (0.0097)	1.0133 (0.0529)
2.5	10	0.8444 (0.0442)	0.7676 (0.0443)	0.8681 (0.0951)
	20	0.8196 (0.0188)	0.7805 (0.0235)	0.8304 (0.0585)
	30	0.8125 (0.0119)	0.7863 (0.0159)	0.8195 (0.0456)
	40	0.8069 (0.0086)	0.7872 (0.0121)	0.8121 (0.0383)
	50	0.8083 (0.0068)	0.7925 (0.0097)	0.8124 (0.0341)

Table 5.7

The values of Bayes estimator and (Posterior risk) for Maxwell Failure distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.1797 (0.0301)	1.1376 (0.0176)	1.1911 (0.0543)
	20	1.1671 (0.0129)	1.1469 (0.0085)	1.1723 (0.0342)
	30	1.1684 (0.0082)	1.1551 (0.0056)	1.1718 (0.0271)
	40	1.1646 (0.0061)	1.1547 (0.0042)	1.1671 (0.0229)
	50	1.1684 (0.0047)	1.1605 (0.0033)	1.1704 (0.0204)
1.5	10	1.7626 (0.0671)	1.6997 (0.0176)	1.7797 (0.1214)
	20	1.7471 (0.0288)	1.7169 (0.0085)	1.7549 (0.0765)
	30	1.7495 (0.0185)	1.7295 (0.0056)	1.7546 (0.0607)
	40	1.7429 (0.0134)	1.7281 (0.0042)	1.7467 (0.0514)
	50	1.7457 (0.0106)	1.7339 (0.0033)	1.7487 (0.0456)
2.0	10	2.3261 (0.1171)	2.2429 (0.1769)	2.3485 (0.2120)
	20	2.3148 (0.0507)	2.2749 (0.0085)	2.3251 (0.1346)
	30	2.3167 (0.0325)	2.2904 (0.0056)	2.3235 (0.1067)
	40	2.3152 (0.0238)	2.2956 (0.0042)	2.3202 (0.0908)
	50	2.3133 (0.0188)	2.2977 (0.0033)	2.3173 (0.0803)
2.5	10	2.8924 (0.1814)	2.7891 (0.0176)	2.9203 (0.3286)
	20	2.8668 (0.0782)	2.8174 (0.0085)	2.8797 (0.2075)
	30	2.8684 (0.0498)	2.8359 (0.0056)	2.8768 (0.1683)
	40	2.8698 (0.0366)	2.8455 (0.0042)	2.8760 (0.1398)
	50	2.8771 (0.0291)	2.8576 (0.0033)	2.8821 (0.1243)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.1080 (0.0236)	1.0722 (0.0159)	1.1176 (0.0903)
	20	1.1261 (0.0114)	1.1076 (0.0081)	1.1309 (0.0620)
	30	1.1251 (0.0074)	1.1128 (0.0054)	1.1283 (0.0494)
	40	1.1294 (0.0055)	1.1201 (0.0041)	1.1318 (0.0426)
	50	1.1330 (0.0044)	1.1255 (0.0033)	1.1349 (0.0381)
1.5	10	0.7696 (0.0113)	0.7448 (0.0159)	0.7763 (0.0435)
	20	0.7707 (0.0053)	0.7581 (0.0081)	0.7740 (0.0291)
	30	0.7751 (0.0035)	0.7667 (0.0054)	0.7773 (0.0235)
	40	0.7740 (0.0026)	0.7676 (0.0041)	0.7756 (0.0201)
	50	0.7725 (0.0020)	0.7674 (0.0033)	0.7738 (0.0177)
2.0	10	0.5843 (0.0064)	0.5654 (0.0159)	0.5894 (0.0247)
	20	0.5801 (0.0031)	0.5705 (0.0081)	0.5824 (0.0164)
	30	0.5817 (0.0019)	0.5753 (0.0054)	0.5833 (0.0132)
	40	0.5796 (0.0014)	0.5748 (0.0041)	0.5808 (0.0112)
	50	0.5779 (0.0011)	0.5740 (0.0033)	0.5788 (0.0099)
2.5	10	0.4783 (0.0043)	0.4629 (0.0159)	0.4824 (0.0165)
	20	0.4682 (0.0019)	0.4605 (0.0081)	0.4702 (0.0106)
	30	0.4654 (0.0012)	0.4603 (0.0054)	0.4668 (0.0084)
	40	0.4658 (0.0009)	0.4619 (0.0041)	0.4668 (0.0072)
	50	0.4643 (0.0007)	0.4612 (0.0033)	0.4651 (0.0064)

Table 5.8

The values of Bayes estimator and (Posterior risk) for Rayleigh distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.4882 (0.0780)	1.4055 (0.0273)	1.5116 (0.1112)
	20	1.4511 (0.0311)	1.4129 (0.0131)	1.4612 (0.0663)
	30	1.4441 (0.0194)	1.4191 (0.0085)	1.4505 (0.0515)
	40	1.4372 (0.0140)	1.4188 (0.0063)	1.4420 (0.0433)
	50	1.4353 (0.0111)	1.4206 (0.0051)	1.4390 (0.0381)
1.5	10	2.2324 (0.1752)	2.1084 (0.0273)	2.2675 (0.2497)
	20	2.1795 (0.0702)	2.1221 (0.0131)	2.1946 (0.1494)
	30	2.1650 (0.0436)	2.1277 (0.0085)	2.1747 (0.1156
	40	2.1568 (0.0315)	2.1292 (0.0063)	2.1639 (0.0975)
	50	2.1558 (0.0248)	2.1338 (0.0050)	2.1614 (0.0861)
2.0	10	2.9692 (0.3105)	2.8043 (0.0273)	3.0161 (0.4426)
	20	2.8774 (0.1224)	2.8017 (0.0131)	2.8974 (0.2606)
	30	2.8683 (0.0766)	2.8188 (0.0085)	2.8811 (0.2031)
	40	2.8581 (0.0555)	2.8214 (0.0063)	2.8675 (0.1714)
	50	2.8528 (0.0435)	2.8237 (0.0050)	2.8602 (0.1509)
2.5	10	3.6600 (0.4756)	3.4567 (0.0273)	3.7176 (0.6779)
	20	3.5915 (0.1912)	3.4970 (0.0130)	3.6166 (0.4072)
	30	3.5646 (0.1185)	3.5032 (0.0085)	3.5806 (0.3143)
	40	3.5529 (0.0859)	3.5074 (0.0063)	3.5646 (0.2652)
	50	3.5356 (0.0668)	3.4995 (0.0050)	3.5448 (0.2319)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.3603 (0.0545)	1.2955 (0.0235)	1.3783 (0.1693)
	20	1.3805 (0.0259)	1.3468 (0.0121)	1.3893 (0.1151)
	30	1.3847 (0.0169)	1.3621 (0.0081)	1.3906 (0.0921)
	40	1.3884 (0.0126)	1.3713 (0.0061)	1.3928 (0.0794)
	50	1.1375 (0.0099)	1.3738 (0.0049)	1.3911 (0.0703)
1.5	10	0.9505 (0.0266)	0.9052 (0.0235)	0.9631 (0.0826)
	20	0.9472 (0.0122)	0.9241 (0.0121)	0.9533 (0.0542)
	30	0.9472 (0.0079)	0.9317 (0.0081)	0.9512 (0.0432)
	40	0.9502 (0.0059)	0.9835 (0.0061)	0.9532 (0.0371)
	50	0.9504 (0.0046)	0.9409 (0.0049)	0.9528 (0.0331)
2.0	10	0.7218 (0.0151)	0.6874 (0.0235)	0.7313 (0.0471)
	20	0.7121 (0.0069)	0.6947 (0.0121)	0.7166 (0.0305)
	30	0.7106 (0.0044)	0.6990 (0.0081)	0.7136 (0.0242)
	40	0.7128 (0.0033)	0.7040 (0.0061)	0.7151 (0.0209)
	50	0.7089 (0.0026)	0.7019 (0.0049)	0.7107 (0.0183)
2.5	10	0.5928 (0.0101)	0.5646 (0.0235)	0.6006 (0.0316)
	20	0.5765 (0.0045)	0.5625 (0.0121)	0.5802 (0.0201)
	30	0.5728 (0.0029)	0.5635 (0.0081)	0.5753 (0.0157)
	40	0.5715 (0.0021)	0.5644 (0.0061)	0.5733 (0.0134)
	50	0.5691 (0.0016)	0.5634 (0.0049)	0.5705 (0.0118)

Table 5.9

The values of Bayes estimator and (Posterior risk) for Generalised Rayleigh distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	1.1801 (0.0299)	1.1379 (0.0176)	1.1914 (0.0543)
	20	1.1678 (0.0129)	1.1477 (0.0085)	1.1730 (0.0343)
	30	1.1631 (0.0081)	1.1499 (0.0056)	1.1665 (0.0269)
	40	1.1678 (0.0061)	1.1579 (0.0042)	1.1703 (0.0231)
	50	1.1647 (0.0047)	1.1568 (0.0033)	1.1667 (0.0203)
1.5	10	1.7628 (0.0669)	1.6999 (0.0176)	1.7799 (0.1212)
	20	1.7598 (0.0293)	1.7295 (0.0085)	1.7677 (0.0778)
	30	1.7411 (0.0183)	1.7213 (0.0056)	1.7462 (0.0602)
	40	1.7480 (0.0135)	1.7332 (0.0042)	1.7518 (0.0517)
	50	1.7456 (0.0106)	1.7338 (0.0033)	1.7486 (0.0457)
2.0	10	2.3254 (0.1166)	2.2424 (0.0176)	2.3479 (0.2112)
	20	2.3212 (0.0512)	2.2812 (0.0085)	2.3316 (0.1358)
	30	2.3219 (0.0326)	2.2955 (0.0056)	2.3287 (0.1072)
	40	2.3175 (0.0238)	2.2978 (0.0042)	2.3225 (0.0912)
	50	2.3186 (0.0188)	2.3029 (0.0033)	2.3225 (0.0806)
2.5	10	2.9027 (0.1829)	2.7991 (0.0176)	2.9308 (0.3313)
	20	2.8775 (0.0788)	2.8279 (0.0085)	2.8904 (0.2089)
	30	2.8710 (0.0500)	2.8384 (0.0056)	2.8794 (0.1642)
	40	2.8743 (0.0368)	2.8499 (0.0042)	2.8805 (0.1403)
	50	2.8788 (0.0291)	2.8594 (0.0033)	2.8837 (0.1245)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	1.0999 (0.0232)	1.0644 (0.0159)	1.1094 (0.0889)
	20	1.1191 (0.0112)	1.1007 (0.0081)	1.1238 (0.0612)
	30	1.1268 (0.0074)	1.1144 (0.0054)	1.1301 (0.0496)
	40	1.1302 (0.0055)	1.1208 (0.0041)	1.1326 (0.0427)
	50	1.1328 (0.0044)	1.1253 (0.0033)	1.1347 (0.0381)
1.5	10	0.7650 (0.0112)	0.7403 (0.0159)	0.7716 (0.0429)
	20	0.7688 (0.0053)	0.7562 (0.0081)	0.7721 (0.0291)
	30	0.7749 (0.0035)	0.7664 (0.0054)	0.7771 (0.0235)
	40	0.7781 (0.0026)	0.7717 (0.0041)	0.7797 (0.0202)
	50	0.7746 (0.0020)	0.7695 (0.0033)	0.7759 (0.0178)
2.0	10	0.5882 (0.0065)	0.5692 (0.0159)	0.5933 (0.0251)
	20	0.5787 (0.0030)	0.5692 (0.0081)	0.5812 (0.0163)
	30	0.5793 (0.0019)	0.5729 (0.0054)	0.5809 (0.0131)
	40	0.5794 (0.0014)	0.5746 (0.0041)	0.5806 (0.0112)
	50	0.5802 (0.0011)	0.5763 (0.0033)	0.5811 (0.0100)
2.5	10	0.4789 (0.0043)	0.4634 (0.0159)	0.4831 (0.0165)
	20	0.4707 (0.0019)	0.4631 (0.0081)	0.4727 (0.0108)
	30	0.4661 (0.0012)	0.4609 (0.0054)	0.4674 (0.0084)
	40	0.4645 (0.0009)	0.4606 (0.0041)	0.4654 (0.0072)
	50	0.4643 (0.0007)	0.4612 (0.0033)	0.4651 (0.0064)

Table 5.10

The values of Bayes estimator and (Posterior risk) for Erlang distribution under Uniform Prior				
ϕ	n	SELF	QLF	PLF
1.0	10	0.6521 (0.0322)	0.5776 (0.0571)	0.6742 (0.0442)
	20	0.6099 (0.0122)	0.5750 (0.0285)	0.6194 (0.0189)
	30	0.5973 (0.0074)	0.5745 (0.0191)	0.6033 (0.0121)
	40	0.5934 (0.0053)	0.5764 (0.0142)	0.5978 (0.0088)
	50	0.5895 (0.0041)	0.5760 (0.0114)	0.5931 (0.0069)
1.5	10	0.9745 (0.0722)	0.8631 (0.0571)	1.0075 (0.0661)
	20	0.9135 (0.0273)	0.8613 (0.0285)	0.9277 (0.0283)
	30	0.8978 (0.0168)	0.8636 (0.0191)	0.9028 (0.0181)
	40	0.8933 (0.0122)	0.8678 (0.0142)	0.9001 (0.0132)
	50	0.8854 (0.0094)	0.8652 (0.0114)	0.8907 (0.0104)
2.0	10	1.2988 (0.1276)	1.1504 (0.0571)	1.3429 (0.0880)
	20	1.2223 (0.0489)	0.1524 (0.0285)	1.2412 (0.0379)
	30	1.1934 (0.0296)	1.1480 (0.0191)	1.2054 (0.0239)
	40	1.1831 (0.0213)	1.1493 (0.0142)	1.1919 (0.0175)
	50	1.1801 (0.0168)	1.1531 (0.0114)	1.1871 (0.0139)
2.5	10	1.5916 (0.1931)	1.4097 (0.0571)	1.5161 (0.1079)
	20	1.4929 (0.0731)	1.4076 (0.0285)	2.8904 (0.0462)
	30	1.4665 (0.0451)	1.4106 (0.0191)	1.4812 (0.0294)
	40	1.4506 (0.0322)	1.4092 (0.0142)	1.4614 (0.0215)
	50	1.4410 (0.0250)	1.4081 (0.0114)	1.4495 (0.0171)
		under Inverted Gamma Prior		
ϕ	n	SELF	QLF	PLF
1.0	10	0.6066 (0.0250)	0.5427 (0.0526)	0.6252 (0.4191)
	20	0.5889 (0.0108)	0.5566 (0.0273)	0.5976 (0.3716)
	30	0.5812 (0.0068)	0.5597 (0.0185)	0.5869 (0.3536)
	40	0.5765 (0.0049)	0.5604 (0.0139)	0.5807 (0.3442)
	50	0.5745 (0.0039)	0.5616 (0.0112)	0.5778 (0.3395)
1.5	10	0.4234 (0.0122)	0.3789 (0.0526)	0.4365 (0.2045)
	20	0.4038 (0.0051)	0.3817 (0.0273)	0.4098 (0.1747)
	30	0.3974 (0.0031)	0.3827 (0.0185)	0.4013 (0.1655)
	40	0.3947 (0.0023)	0.3837 (0.0139)	0.3976 (0.1614)
	50	0.3923 (0.0018)	0.3835 (0.0112)	0.3946 (0.1583)
2.0	10	0.3237 (0.0071)	0.2896 (0.0526)	0.3336 (0.1189)
	20	0.3039 (0.0030)	0.2872 (0.0273)	0.3084 (0.0988)
	30	0.2980 (0.0017)	0.2870 (0.0185)	0.3009 (0.0929)
	40	0.2944 (0.0012)	0.2862 (0.0139)	0.2966 (0.0898)
	50	0.2932 (0.0010)	0.2866 (0.0112)	0.2949 (0.0884)
2.5	10	0.2646 (0.0047)	0.2368 (0.0526)	0.2728 (0.0763)
	20	0.2449 (0.0018)	0.2314 (0.0273)	0.2485 (0.0641)
	30	0.2393 (0.0011)	0.2304 (0.0185)	0.2416 (0.0599)
	40	0.2367 (0.0008)	0.2301 (0.0139)	0.2384 (0.0580)
	50	0.2346 (0.0006)	0.2294 (0.0112)	0.2360 (0.0566)

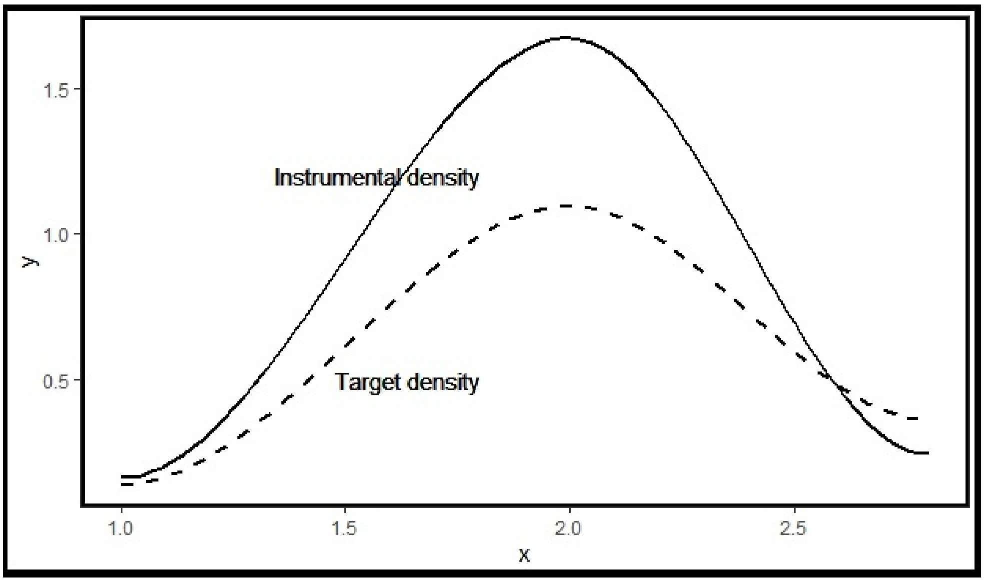


Fig. 5.1: Graph of Instrumental and Target density

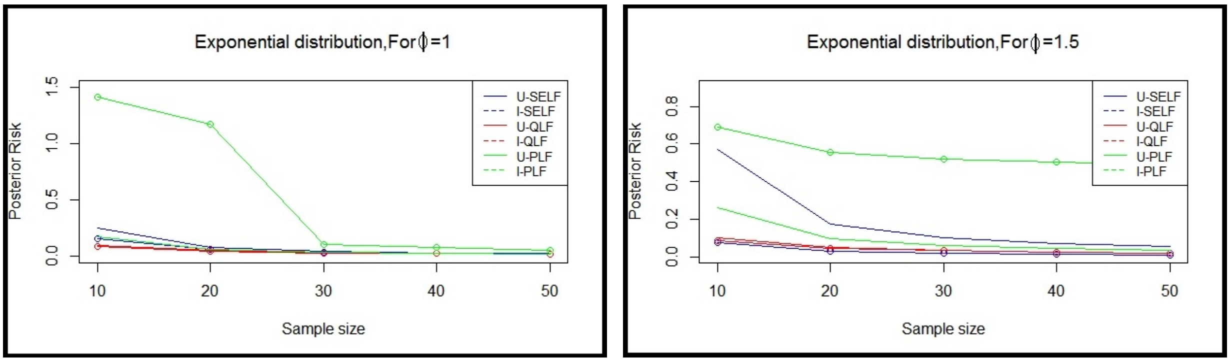


Fig. 5.2

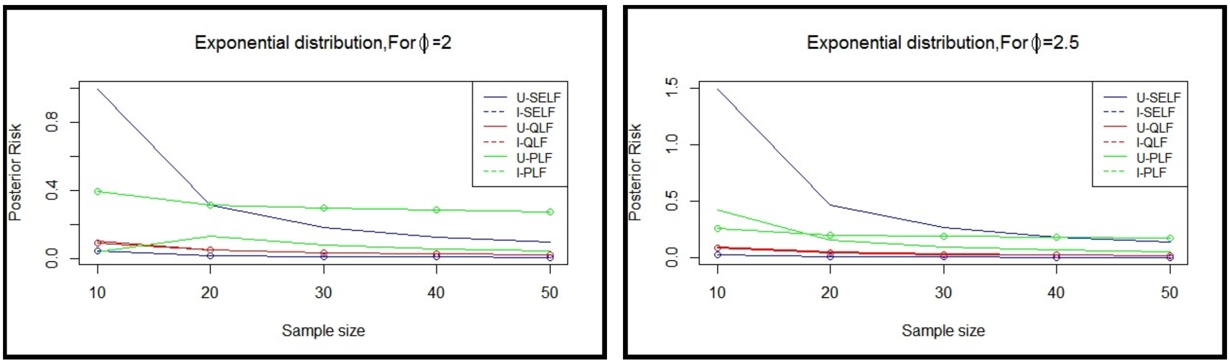


Fig. 5.3

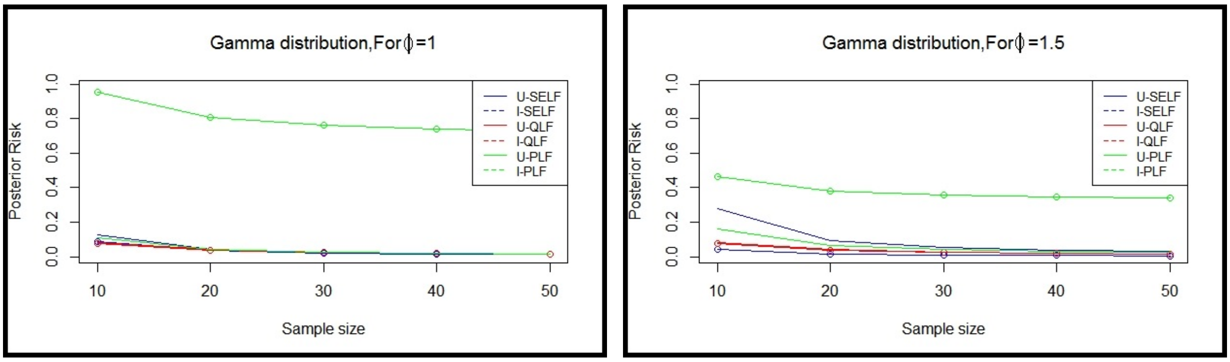


Fig. 5.4

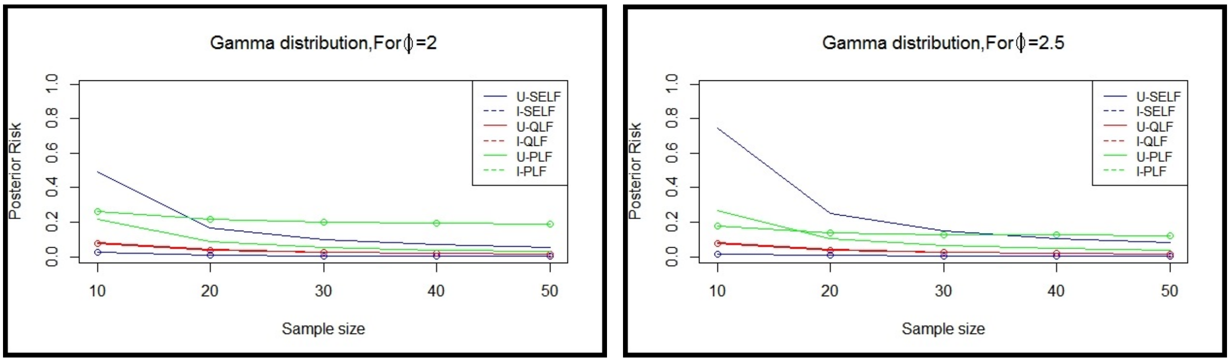


Fig. 5.5

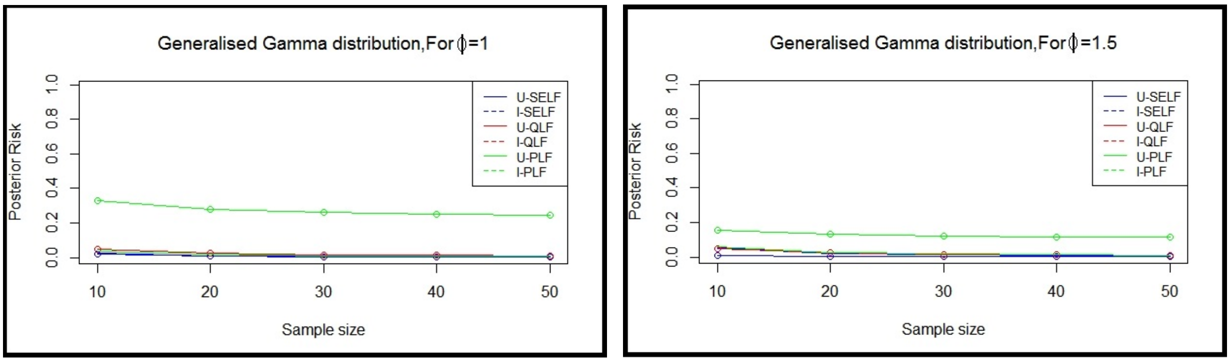


Fig. 5.6

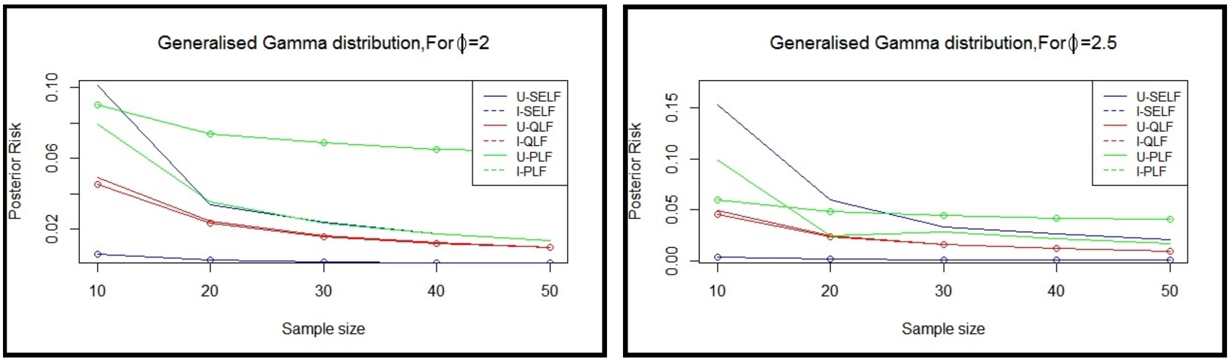


Fig. 5.7

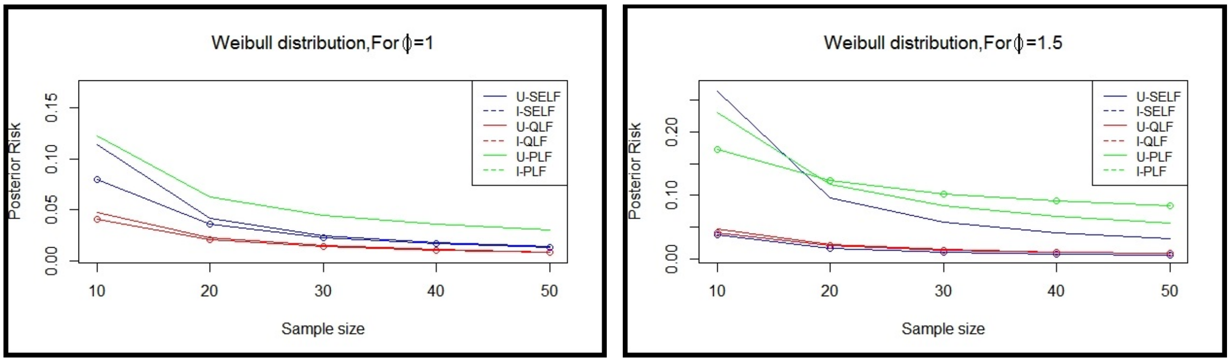


Fig. 5.8

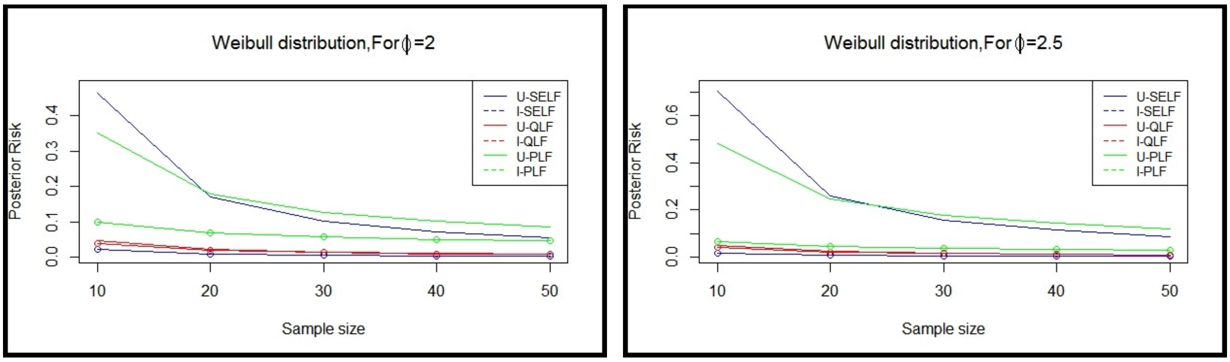


Fig. 5.9

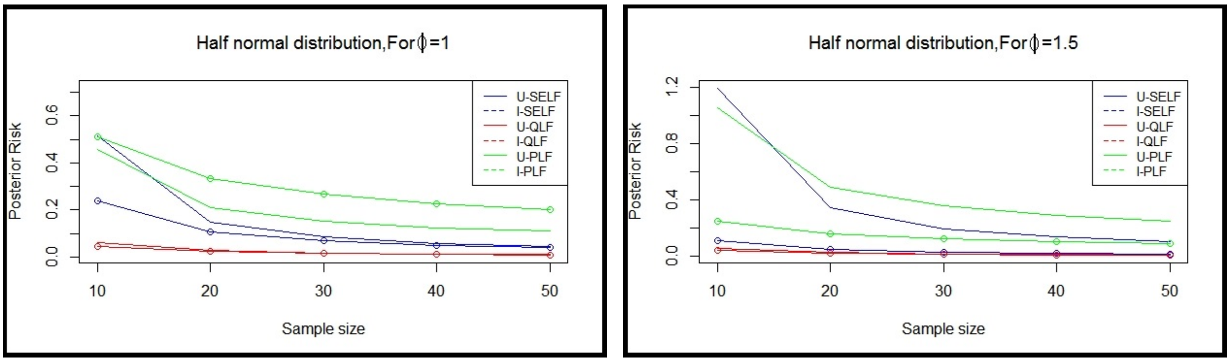


Fig. 5.10

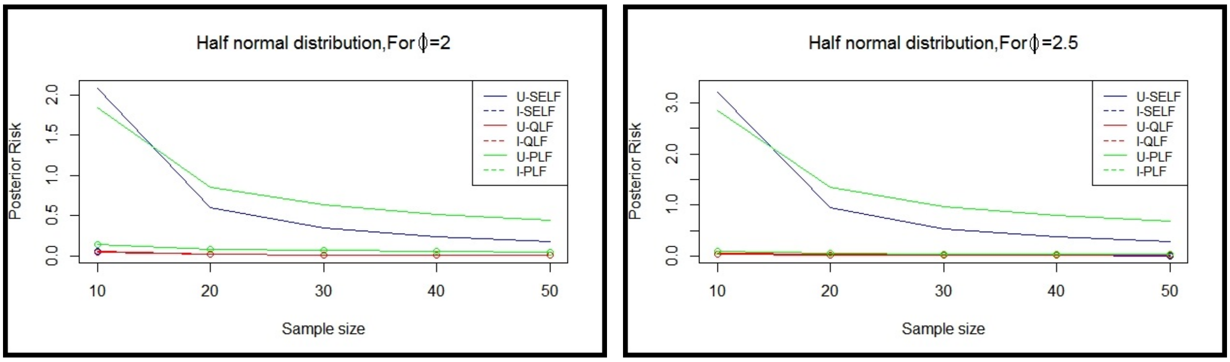


Fig. 5.11

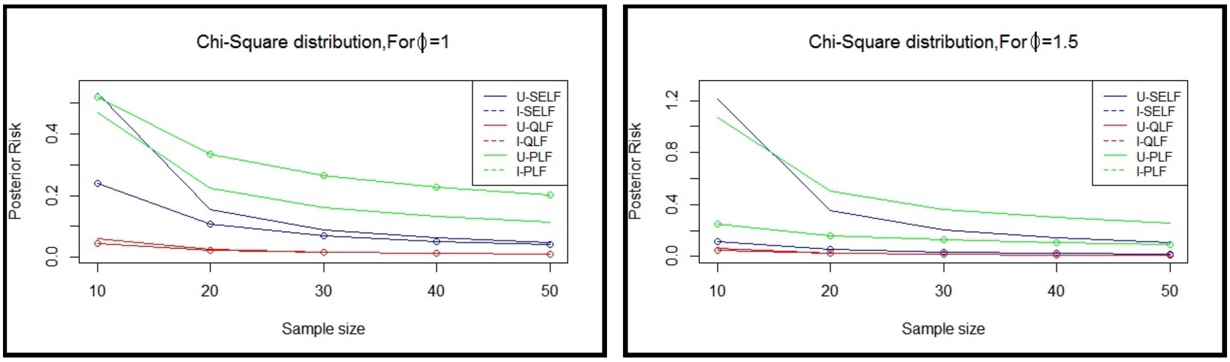


Fig. 5.12

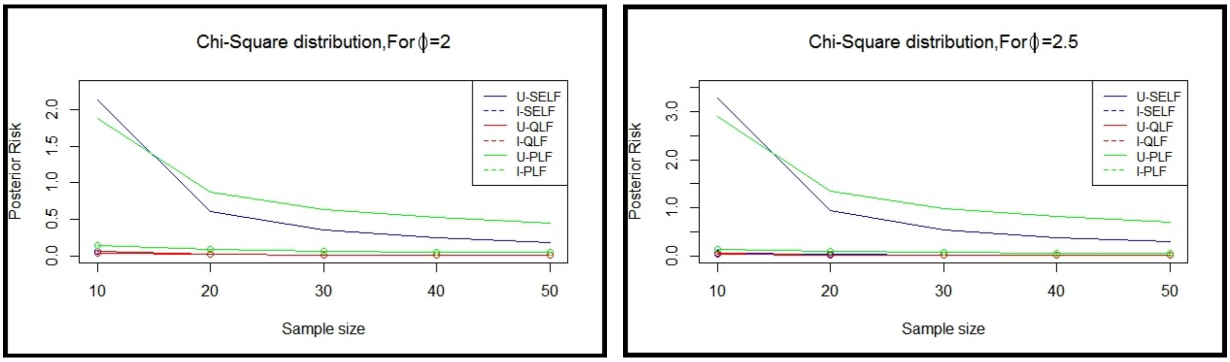


Fig. 5.13

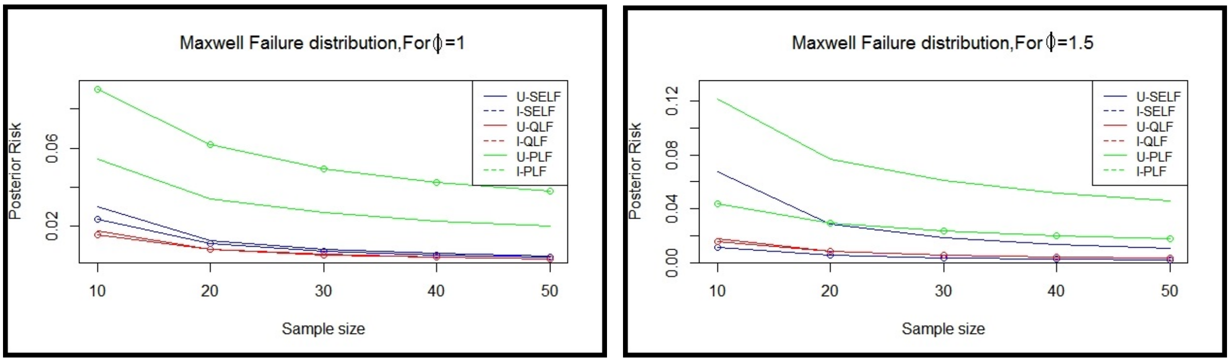


Fig. 5.14

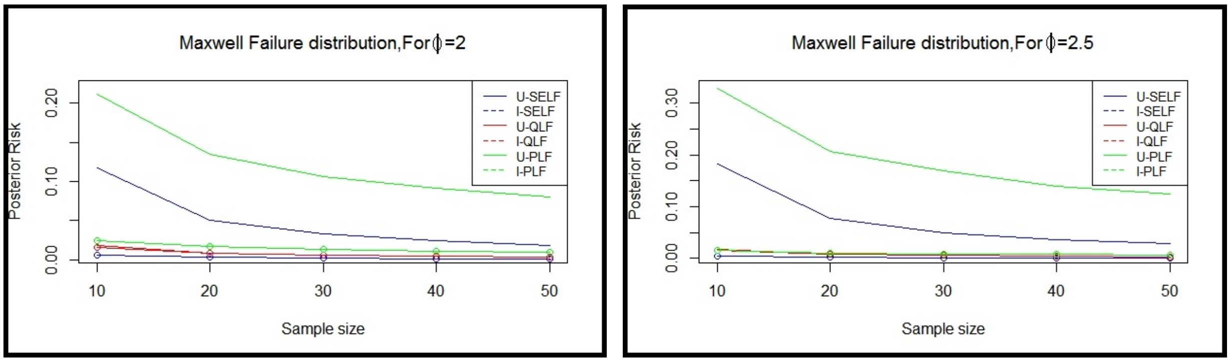


Fig. 5.15

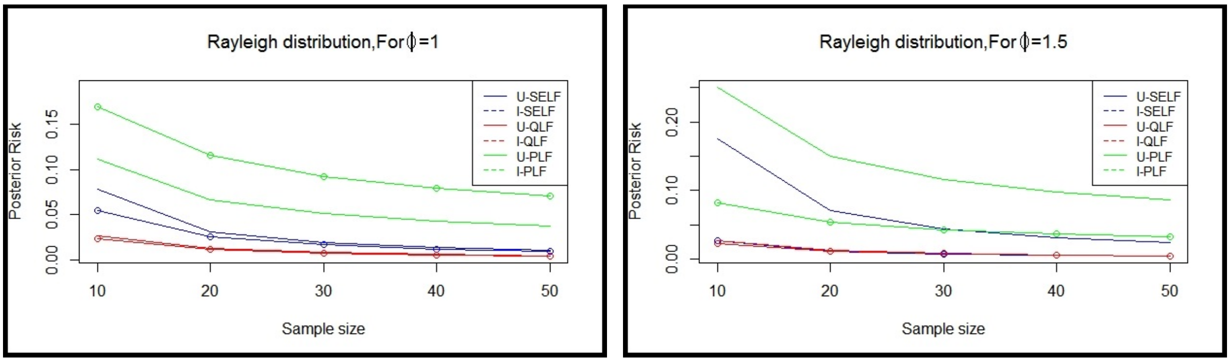


Fig. 5.16

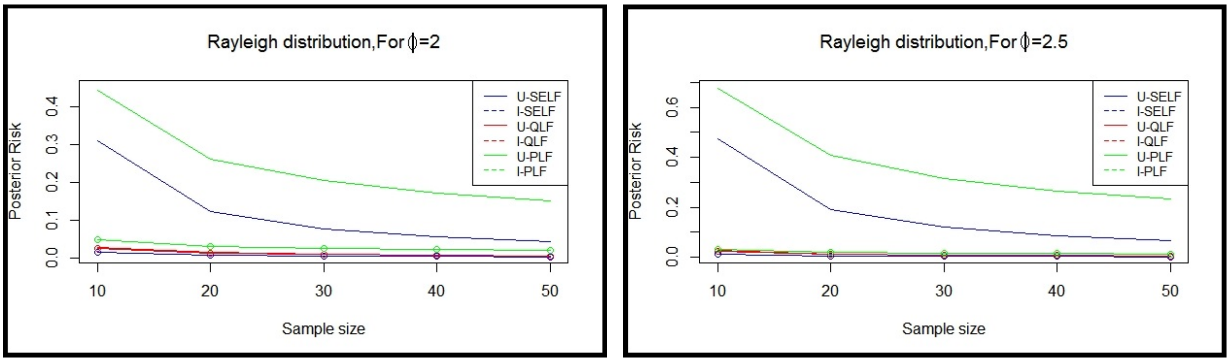


Fig. 5.17

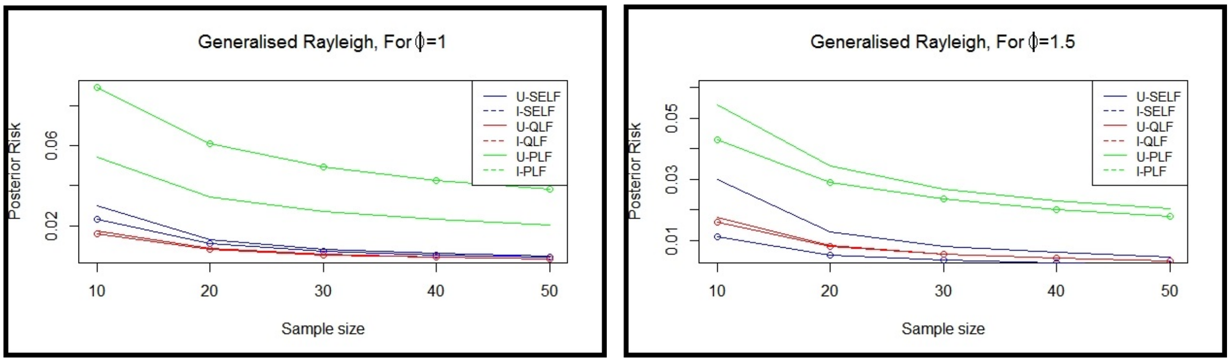


Fig. 5.18

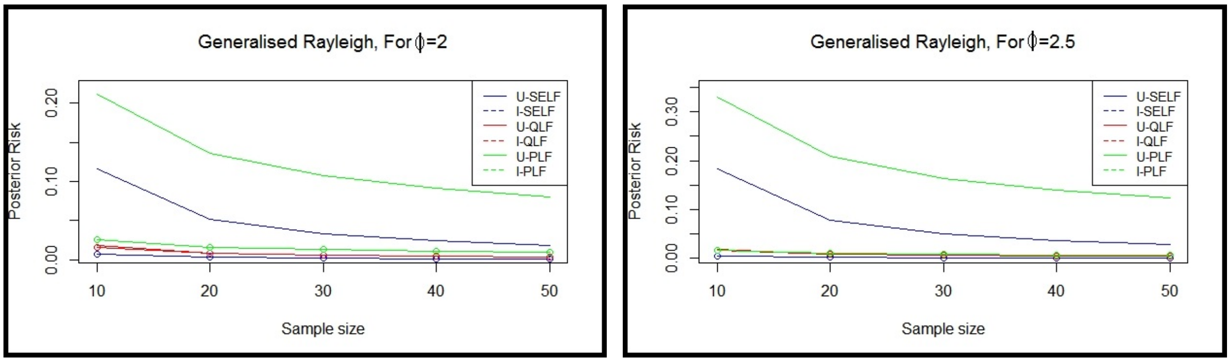


Fig. 5.19

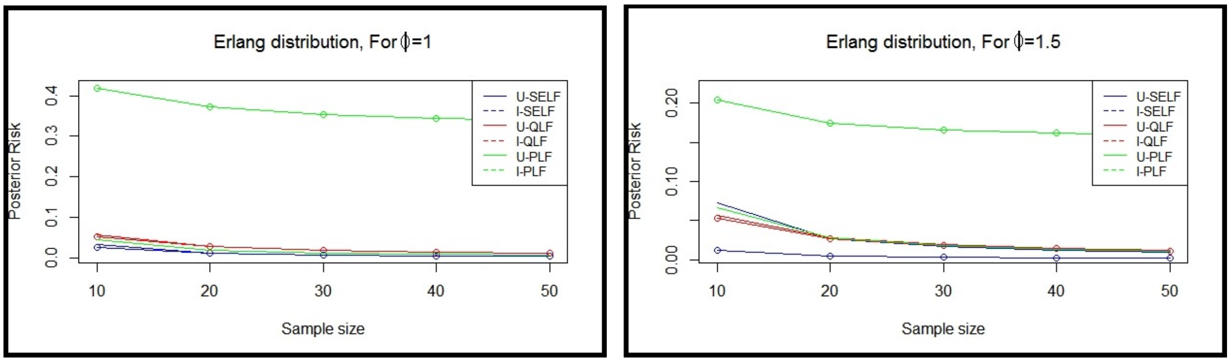


Fig. 5.20

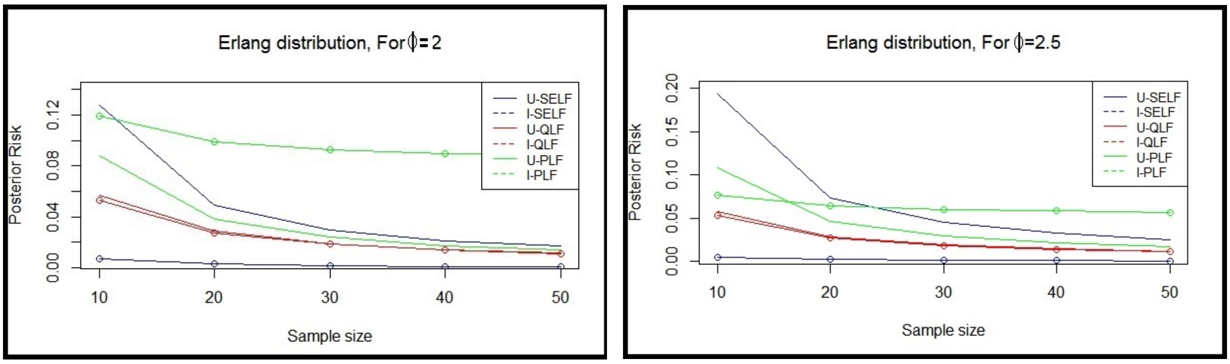


Fig. 5.21

Chapter 6

Bayesian Estimation Procedures under Type II censoring for three parameter Generalised Rayleigh Distribution

6.1 Introduction

Kundu and Raqab (2015) [71] introduce a three parameter generalized Rayleigh distribution with the location or threshold parameter ν. The cdf and pdf is

$$F(y;\beta,\rho,\nu)=\left(1-e^{-\rho(y-\nu)^2}\right)^{\beta};\qquad y>\nu,-\infty<\nu<\infty \tag{6.1.1}$$

$$f(y;\beta,\rho,\nu)=2\beta\rho(y-\nu)e^{-\rho(y-\nu)^2}\left(1-e^{-\rho(y-\nu)^2}\right)^{\beta-1};\qquad y>\nu,-\infty<\nu<\infty \tag{6.1.2}$$

and the reliability function is

$$R(t)=1-\left(1-e^{-\rho(y-\nu)^2}\right)^{\beta};\qquad y>\nu,-\infty<\nu<\infty \tag{6.1.3}$$

where, $\beta>0$ and $\rho>0$ are the shape and scale parameter.

In this chapter, a comparitive study is done for the various loss functions: SELF, QLF and GELF in the case of Conjugate prior distribution. Here, firstly we obtained the Bayes estimator of the positive and negative powers of the parameters

6.2 Set up of the Problem

For the given pdf assuming that β is unknown but ρ and ν are known parameters. Let us arranged the life time of first 'l' units as $0 < Y_{(1)} < Y_{(2)} < \ldots < Y_{(l)}, \quad 0 < l < n$ where it is evident that $(n-l)$ items are survived until $Y_{(l)}$. Then likelihood is

$$L(\beta|y) = \beta^l exp(-\beta s_l) \tag{6.2.1}$$

where,

$$s_l = -\left[\sum_{i=1}^{l} log\left\{1 - e^{-\rho(y_{(i)}-\nu)^2}\right\} + (n-l)log\left\{1 - e^{-\rho(y_{(l)}-\nu)^2}\right\}\right]$$

Here, we see from (6.2.1) and considering that for the parameter β, conjugate prior is gamma distribution with the following pdf presented as

$$\pi(\beta) = \frac{w^v}{\Gamma\nu}\beta^{\nu-1}exp(-w\beta); \qquad \beta, w, \nu > 0 \tag{6.2.2}$$

On using the (6.2.1), (6.2.2) and from the Bayes theorem, posterior distribution for β is given as

$$h(\beta|s_l) = \frac{(s_l+w)^{l+\nu}}{\Gamma(l+\nu)}\beta^{l+\nu-1}exp\left\{-\beta(s_l+w)\right\} \tag{6.2.3}$$

In this chapter, estimation of $P = P_r(X > Y)$ is also considered. For this problem, let us suppose that 'n' items of Y and 'm' items of G with their pdfs $f(y;\beta_1,\rho_1,\nu_1)$ and $f(g;\beta_2,\rho_2,\nu_2)$, respectively, are put together on a life test with l and k being their truncation numbers. Let us consider

$S_l = -\left[\sum_{i=1}^{l} log\left\{1 - e^{-\rho_1(y_{(i)}-\nu_1)^2}\right\} + (n-l)log\left\{1 - e^{-\rho_1(y_{(l)}-\nu_1)^2}\right\}\right]$ and

$T_k = -\left[\sum_{j=1}^{k} log\left\{1 - e^{-\rho_2(y_{(j)}-\nu_2)^2}\right\} + (m-k)log\left\{1 - e^{-\rho_2(y_{(k)}-\nu_2)^2}\right\}\right].$

For the Bayes estimator conjugate prior for the parameters β_1 and β_2 as in (6.2.2) with known values of the parameters (ν_1, w_1) and (ν_2, w_2) are taken.

For further calculation let us take the transformation on pdf (6.1.2) as $V = -log\left\{1 - e^{-\rho(y-\nu)^2}\right\}$ and see that V follows exponential distribution with

$$f(v;\beta) = \beta e^{-\beta v}; \qquad v > 0$$

Again, consider the transformation on (6.1.2) as $Z_i = (n-i+1)\left\{V_{(i)} - V_{(i-1)}\right\}$, $i = 1, 2, \dots l$, then we see that each $Z_i's$ are iid random variable with exponential distribution. As we know that $\sum_{i=1}^{l} Z_i = S_l$ and using the additive property of exponential distribution, pdf of S_l is given by

$$h(s_l;\beta) = \frac{\beta^l}{\Gamma l} s_l^{l-1} exp(-\beta s_l); \qquad s_l > 0 \tag{6.2.4}$$

On using (6.2.2) and (6.2.4), the marginal pdf of S_l is given by

$$f(s_l) = \frac{w^\nu {s_l}^{l-1}}{B(l,v)(s_l + w)^{l+v}}; \qquad s_l > 0 \tag{6.2.5}$$

Let $\hat{\phi}_*$ and $L\left(\hat{\phi}_*, \phi\right)$, the Bayes estimator of $\phi = \psi(\beta)$ and loss for estimating ϕ by $\hat{\phi}$, respectively, the risk is defined as

$$R_*(\hat{\phi}_*) = E_{S_l|\phi}\left\{L(\hat{\phi}_*, \phi)\right\} \tag{6.2.6}$$

The posterior risk is defined as

$$R_{p*}(\hat{\phi}_*) = E_{\phi|S_l}\left\{L(\hat{\phi}_*, \phi)\right\} \tag{6.2.7}$$

The Bayes risk is defined as

$$R_{B*}(\hat{\phi}_*) = E_{S_l}\left[E_{\phi|S_l}\left\{L(\hat{\phi}_*, \phi)\right\}\right] \tag{6.2.8}$$

Here, we see for all risk as: risk of Bayes estimator of β is a function of ϕ and is independent of sample data, the posterior risk is a function of the sample data S_l and prior parameters and is independent of ϕ, and the Bayes risk is only a function of priors parameters. The relation between these risk is defined as

$$R_{B*}(\hat{\phi}_*) = E_\phi \left\{ R_*(\hat{\phi}_*) \right\} \tag{6.2.9}$$

and

$$R_{B*}(\hat{\phi}_*) = E_{S_l} \left\{ R_{p*}(\hat{\phi}_*) s \right\} \tag{6.2.10}$$

We are setting * by SELF, QLF and GELF, respectively.

6.3 Bayes Estimators for the powers of β, under various loss function i.e., SELF, QLF and GELF

For the SELF, QLF and GELF, Bayes estimators are obtained in the following Theorems (Theorem 3.1, 3.2 and 3.3).

Theorem 3.1: Bayes estimators for the positive and negative powers of β i.e. β^q and β^{-q} under squared error loss function are denoted by

$$\hat{\beta}^q_{SB} = \frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}(s_l+w)^{-q} \tag{6.3.1}$$

and

$$\hat{\beta}^{-q}_{SB} = \frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu)}(s_l+w)^{q} \tag{6.3.2}$$

Proof: Under squared error loss function, the Bayes estimator is the posterior mean i.e.,

$$E(\beta^q|y) = \int_0^\infty \beta^q h(\beta|s_l) d\beta$$

on using (6.2.3), we get

$$\hat{\beta}_{SB}^{q} = \frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}(s_l+w)^{-q}$$

and

$$\hat{\beta}_{SB}^{-q} = \frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu)}(s_l+w)^{q}$$

Hence, the theorem follows.

Theorem 3.2: Bayes estimators for the positive and negative powers of β i.e. β^q and β^{-q} under quadratic loss function are denoted by

$$\hat{\beta}_{QB}^{q} = \frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu-2q)}(s_l+w)^{-q} \tag{6.3.3}$$

$$\hat{\beta}_{QB}^{-q} = \frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu+2q)}(s_l+w)^{q} \tag{6.3.4}$$

Proof: Under quadratic loss function, the Bayes estimator is given by

$$\hat{\beta}_{QB}^{q} = \frac{E(\beta^{-q}|y)}{E(\beta^{-2q}|y)}$$

On using (6.2.3), we get

$$\hat{\beta}_{QB}^{q} = \frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu-2q)}(s_l+w)^{-q}$$

$$\hat{\beta}_{QB}^{-q} = \frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu+2q)}(s_l+w)^{q}$$

Hence, the theorem follows.

Theorem 3.3: Bayes estimators for the positive and negative powers of β i.e. β^q and β^{-q} under General Entropy linex loss function are denoted by

$$\beta_{GB}^{\hat{q}} = \left\{\frac{\Gamma(l+\nu-aq)}{\Gamma(l+\nu)}\right\}^{-1/a}(s_l+w)^{-q} \tag{6.3.5}$$

and

$$\hat{\beta^{-q}_{GB}} = \left\{\frac{\Gamma(l+\nu+aq)}{\Gamma(l+\nu)}\right\}^{-1/a}(s_l+w)^q \tag{6.3.6}$$

Proof: Under General Entropy loss function, Bayes estimator is given by

$$\beta^{\hat{q}}_{GB} = \left[E\left\{(\beta^q)^{-a}\right\}\right]^{-1/a}$$

On using (6.2.3), we get

$$\beta^{\hat{q}}_{GB} = \left\{\frac{\Gamma(l+\nu-aq)}{\Gamma(l+\nu)}\right\}^{-1/a}(s_l+w)^{-q}$$

and

$$\hat{\beta^{-q}_{GB}} = \left\{\frac{\Gamma(l+\nu+aq)}{\Gamma(l+\nu)}\right\}^{-1/a}(s_l+w)^q$$

Hence, the theorem follows.

6.4 Risks, Posterior risks and Bayes risks under different loss function

In the following Theorems (i.e. Therorem 4.1, 4.2 and 4.3), mathematical expressions for risks, posterior risk and Bayes risks of Bayes estimator for the positive and negative powers of β are evaluated under various loss functions.

Therorem 4.1: Risk, posterior risk and Bayes risks of Bayes estimator of powers of β i.e., β^q and β^{-q}, under SELF are given by

$$R_{SELF}ELF(\hat{\beta}^q_{SB}) = \left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\}^2\frac{\beta^{2q}}{\Gamma l}\int_0^\infty\frac{e^{-z}z^{l-1}dz}{(z+\beta w)^{2q}}+\beta^{2q}-2\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\}\frac{\beta^{2q}}{\Gamma l}\int_0^\infty\frac{e^{-z}z^{l-1}dz}{(z+\beta w)^q} \tag{6.4.1}$$

$$R_{PSELF}(\hat{\beta}^q_{SB}) = \frac{(s_l+w)^{-2q}}{\Gamma(l+\nu)}\left[\Gamma(l+\nu+2q)-\frac{\{\Gamma(l+\nu+q)\}^2}{\Gamma(l+\nu)}\right] \tag{6.4.2}$$

$$R_{BSELF}(\hat{\beta}_{SB}^{q}) = w^{-2q}\frac{\Gamma(\nu+2q)}{\Gamma\nu}\left[1-\frac{\{\Gamma(l+\nu+q)\}^2}{\Gamma(l+\nu+2q)\Gamma(l+\nu)}\right] \tag{6.4.3}$$

$$R_{SELF}(\hat{\beta}_{SB}^{-q}) = \left\{\frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu)}\right\}\frac{\beta^{-2q}}{\Gamma l}\int_0^\infty (z+\beta w)^{2q}e^{-z}z^{l-1}dz\left[\left\{\frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu)}\right\}-2\right]+\beta^{-2q} \tag{6.4.4}$$

$$R_{PSELF}(\hat{\beta}_{SB}^{-q}) = \frac{(s_l+w)^{2q}}{\Gamma(l+\nu)}\left[\Gamma(l+\nu-2q)-\frac{\{\Gamma(l+\nu-q)\}^2}{\Gamma(l+\nu)}\right] \tag{6.4.5}$$

$$R_{BSELF}(\hat{\beta}_{SB}^{-q}) = w^{2q}\frac{\Gamma(\nu-2q)}{\Gamma\nu}\left[1-\frac{\{\Gamma(l+\nu-q)\}^2}{\Gamma(l+\nu)\Gamma(l+\nu-2q)}\right] \tag{6.4.6}$$

Proof: For the squared error loss function, risk is given by

$$R_{SELF}(\hat{\beta}_{SB}^{q}) = E_{S_l/\beta}\left[(\beta^q-\hat{\beta}_{SB}^{q})^2\right]$$

On using (6.3.1), we get

$$R_{SELF}(\hat{\beta}_{SB}^{q}) = \left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\}^2 E_{S_l/\beta}(s_l+w)^{-2q}+\beta^{2q}-2\beta^q\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\}E_{S_l/\beta}(s_l+w)^{-q} \tag{6.4.7}$$

From (6.2.4), we get

$$E_{S_l/\beta}(s_l+w)^{-2q} = \frac{\beta^{2q}}{\Gamma l}\int_0^\infty \frac{e^{-z}z^{l-1}dz}{(z+\beta w)^{2q}}$$

and

$$E_{S_l/\beta}(s_l+w)^{-q} = \frac{\beta^{q}}{\Gamma l}\int_0^\infty \frac{e^{-z}z^{l-1}dz}{(z+\beta w)^{q}}$$

Substituting above values in (6.4.7), we get the required result.

Posterior risk for squared error loss function is defined as

$$R_{PSELF}(\hat{\beta}_{SB}^{q}) = E_{\beta/S_l}\left\{\beta^{2q}\right\}-\left[E_{\beta/S_l}\left\{\beta^q\right\}\right]^2 \tag{6.4.8}$$

On using (6.2.3), we get

$$E_{\beta/S_l}\left\{\beta^{2q}\right\} = \frac{\Gamma(l+\nu+2q)}{\Gamma(l+\nu)}(s_l+w)^{-2q}$$

and

$$E_{\beta/S_l}\left\{\beta^{q}\right\} = \frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}(s_l+w)^{-q}$$

Substituting above values in (6.4.8), we get the posterior risk under squared error loss function.

On using (6.2.10) and (6.4.2), the Bayes risk under squared error loss function is

$$R_{BSELF}(\hat{\beta}^q_{SB}) = \frac{E_{Sl}\left[(s_l+w)^{-2q}\right]}{\Gamma(l+\nu)}\left[\Gamma(l+\nu+2q) - \frac{\{\Gamma(l+\nu+q)\}^2}{\Gamma(l+\nu)}\right] \tag{6.4.9}$$

using (6.2.5), we get

$$E_{Sl}\left[(s_l+w)^{-2q}\right] = \frac{w^{-2q}\Gamma(\nu+2q)\Gamma(l+\nu)}{\Gamma(l+\nu+2q)\Gamma\nu}$$

Substituting above value in (6.4.9), we get the required result.

Similarly, we can get risk, posterior risk and Bayes risk (6.4.4), (6.4.5) and (6.4.6), respectively, for the negative powers of q under squared error loss function. Hence, the theorem follows.

Theorem 4.2: Risk, posterior risk and Bayes risks of Bayes estimator of β i.e., β^q and β^{-q}, under QLF is given by

$$R_{QLF}(\hat{\beta}^q_{QB}) = 1+\left\{\frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu-2q)}\right\}^2\frac{1}{\Gamma l}\int_0^\infty\frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{2q}} - 2\left\{\frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu-2q)}\right\}\frac{1}{\Gamma l}\int_0^\infty\frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{q}} \tag{6.4.10}$$

$$R_{PQLF}(\hat{\beta}^q_{QB}) = 1 - \frac{\{\Gamma(l+\nu-q)\}^2}{\Gamma(l+\nu)\Gamma(l+\nu-2q)} \tag{6.4.11}$$

$$R_{BQLF}(\hat{\beta}^q_{QB}) = 1 - \frac{\{\Gamma(l+\nu-q)\}^2}{\Gamma(l+\nu)\Gamma(l+\nu-2q)} \tag{6.4.12}$$

$$
\begin{aligned}
R_{QLF}(\hat{\beta}_{QB}^{-q}) =&1+\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu-2q)}\right\}^2 \frac{1}{\Gamma l}\int_0^\infty \frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{-2q}} \\
&-2\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu-2q)}\right\} \frac{1}{\Gamma l}\int_0^\infty \frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{-q}}
\end{aligned}
\tag{6.4.13}
$$

$$
R_{PQLF}(\hat{\beta}_{QB}^{-q}) = 1-\frac{\{\Gamma(l+\nu+q)\}^2}{\Gamma(l+\nu)\Gamma(l+\nu+2q)} \tag{6.4.14}
$$

$$
R_{BQLF}(\hat{\beta}_{QB}^{-q}) = 1-\frac{\{\Gamma(l+\nu+q)\}^2}{\Gamma(l+\nu)\Gamma(l+\nu+2q)} \tag{6.4.15}
$$

Proof: For the quadratic loss function, risk is given as

$$
R_{QLF}(\hat{\beta}_{QLFB}^{q}) = E_{S_l/\beta}\left[\left(1-\frac{\hat{\beta}_{QG}^{q}}{\beta^q}\right)^2\right]
$$

On using (6.3.3), we get

$$
R_{QLF}(\hat{\beta}_{QB}^{q}) = 1+\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\}^2 E_{S_l/\beta}\left\{\beta(s_l+w)\right\}^{-2q}-2\left\{\frac{\Gamma(l+\nu+q)}{\Gamma(l+\nu)}\right\} E_{S_l/\beta}\left\{\beta(s_l+w)\right\} \tag{6.4.16}
$$

From (6.2.4), we get

$$
E_{S_l/\beta}\left\{\beta(s_l+w)\right\}^{-2q} = \frac{1}{\Gamma l}\int_0^\infty \frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{2q}}
$$

$$
E_{S_l/\beta}\left\{\beta(s_l+w)\right\}^{-q} = \frac{1}{\Gamma l}\int_0^\infty \frac{z^{l-1}e^{-z}dz}{(z+\beta w)^{q}}
$$

Substituting above values in (6.4.16), we get the required result.

Posterior risk under quadratic loss function is defined as

$$
R_{PQLF}(\hat{\beta}_{QB}^{q}) = 1-\frac{\left\{E_{\beta/S_l}(\beta^{-q})\right\}^2}{E_{\beta/S_l}(\beta^{-2q})} \tag{6.4.17}
$$

using (6.2.3), we have

$$
E_{\beta/S_l}(\beta^{-q}) = \frac{\Gamma(l+\nu-q)}{\Gamma(l+\nu)}(s_l+w)^q
$$

and

$$E_{\beta/S_l}(\beta^{-2q}) = \frac{\Gamma(l+\nu-2q)}{\Gamma(l+\nu)}(s_l+w)^{2q}$$

Substituting above values in (6.4.17), we get the required result. For quadratic loss function, Bayes risk and Posterior risk are same.

Similarly, we can get risk, posterior risk and Bayes risk (6.4.13), (6.4.14) and (6.4.15), respectively, for the negative powers of q under quadratic loss function. Hence, the theorem follows.

Theorem 4.3: Risk, posterior risk and Bayes risks of Bayes estimator of β i.e, β^q and β^{-q}, under GELF are given by

$$\begin{aligned} R_{GELF}(\hat{\beta}^q_{GB}) = & \left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\}\frac{1}{\Gamma l}\int_0^\infty \frac{e^{-z}z^{l-1}dz}{(z+\beta w)^{2q}} - 1 \\ & - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\} + \frac{aq}{\Gamma l}\int_0^\infty log(z+\beta w)e^{-z}z^{l-1}dz \end{aligned} \tag{6.4.18}$$

$$R_{PGELF}(\hat{\beta}^q_{GB}) = aq\psi(l+\nu) - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\} \tag{6.4.19}$$

$$R_{BGELF}(\hat{\beta}^q_{GB}) = aq\psi(l+\nu) - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\} \tag{6.4.20}$$

and

$$\begin{aligned} R_{GELF}(\hat{\beta}^{-q}_{GB}) = & \left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu+aq)}\right\}\frac{1}{\Gamma l}\int_0^\infty \frac{e^{-z}z^{l-1}dz}{(z+\beta w)^{-2q}} - 1 \\ & - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu+aq)}\right\} - \frac{aq}{\Gamma l}\int_0^\infty log(z+\beta w)e^{-z}z^{l-1}dz \end{aligned} \tag{6.4.21}$$

$$R_{PGELF}(\hat{\beta}^{-q}_{GB}) = -\left[aq\psi(l+\nu) - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu+aq)}\right\}\right] \tag{6.4.22}$$

$$R_{BGELF}(\hat{\beta}^{-q}_{GB}) = -\left[aq\psi(l+\nu) - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu+aq)}\right\}\right] \tag{6.4.23}$$

Proof: For the general entropy loss function, risk is given by

$$R_{GELF}(\hat{\beta}^q_{GB}) = E_{S_l/\beta}\left[\left(\frac{\hat{\beta}^q_{GB}}{\beta^q}\right)^a - a\left(\frac{\hat{\beta}^q_{GB}}{\beta^q}\right) - 1\right]$$

On using (6.3.5), we get

$$\begin{aligned} R_{GELF}(\hat{\beta}^q_{GB}) =& E_{S_l/\beta}\left[\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\}\{\beta(s_l+w)\}^{-aq} - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\}\right] \\ &+ aqlog\{\beta(s_l+w)\} - 1 \end{aligned} \tag{6.4.24}$$

From (6.2.4), we get

$$E_{S_l/\beta}\{\beta(s_l+w)\}^{-aq} = \frac{\beta^{-aq}}{\Gamma l}\int_0^\infty \frac{e^{-z}z^{l-1}dz}{\left(\frac{z}{\beta}+w\right)^q}$$

$$E_{S_l/\beta}\left[log\{\beta(s_l+w)\}\right] = \frac{1}{\Gamma l}\int_0^\infty log(z+\beta w)e^{-z}z^{l-1}dz$$

substituting above values in (6.4.24), we get the requires result.

Posterior risk for general entropy loss function is given by

$$\begin{aligned} R_{PGELF}(\hat{\beta}^q_{GB}) =& E_{\beta/S_l}\left[\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\}\{\beta(s_l+w)\}^{-aq} - log\left\{\frac{\Gamma(l+\nu)}{\Gamma(l+\nu-aq)}\right\}\right] \\ &+ aqlog(\beta) + aqlog\{\beta(s_l+w)\} - 1 \end{aligned} \tag{6.4.25}$$

On using (6.2.3), we get

$$E_{\beta/S_l}\{\beta(s_l+w)\}^{-aq} = \frac{(s_l+w)^{-aq}}{\Gamma(l+\nu)}\Gamma(l+\nu-aq)$$

$$E_{\beta/S_l}\{log(\beta)\} = \frac{\Gamma(l+\nu)}{(s_l+w)^{-aq}}\left[\psi(l+\nu) - log(s_l+w)\right]$$

Substituting above values in (6.4.25), we get the required result. For GELF, Bayes risk and posterior risk are same.

Similarly, we can get risk, posterior risk and Bayes risk (6.4.21), (6.4.22) and (6.4.23), respectively, for the negative powers of q under general entropy loss function. Hence, the theorem follows.

6.5 Bayes estimator of the cdf, pdf and Reliability function $R(t)$

In the following theorems, we obtain Bayes estimator of the cdf, pdf and Reliability function under SELF, QLF and GELF.

Theorem 5.1: Bayes estimator for the cdf given in (6.1.1) at a point 'y' for the powers of β under SELF, QLF and GELF are defined as

$$\hat{F}_{SELF}(y;\beta,\rho,\nu) = \left[1 + \frac{log\left\{1 - e^{\rho(y-\nu)^2}\right\}}{(s_l + w)}\right]^{l+\nu} \tag{6.5.1}$$

$$\hat{F}_{QLF}(y;\beta,\rho,\nu) = \left[1 + \frac{log\left\{1 - e^{\rho(y-\nu)^2}\right\}}{(s_l + w)}\right]^{l+\nu-2} \tag{6.5.2}$$

$$\hat{F}_{GELF}(y;\beta,\rho,\nu) = \left[1 + \frac{log\left\{1 - e^{\rho(y-\nu)^2}\right\}}{(s_l + w)}\right]^{l+\nu-1} \tag{6.5.3}$$

Proof: The cdf (6.1.1) is defined as

$$\begin{aligned}
\hat{F}(y;\beta,\rho,\nu) &= \left\{1 - e^{-\rho(y-\nu)}\right\}^{\beta} \\
&= exp\left[\beta log\left\{1 - e^{-\rho(y-\nu)}\right\}\right] \\
&= \sum_{i=0}^{\infty} \frac{\left[log\left\{1 - e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\hat{\beta}^i
\end{aligned}$$

Using Lemma 1, given by Chaturvedi and Tomar (2002) [37], we have

$$\hat{F}_{SELF}(y;\beta,\rho,\nu)=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\hat{\beta}_{SB}^i$$

On using (6.3.1),

$$\begin{aligned}\hat{F}_{SELF}(y;\beta,\rho,\nu)&=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\frac{\Gamma(l+\nu+i)}{\Gamma(l+\nu)}\left(s_l+w\right)^{-i}\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(l+\nu+i\right)}{\Gamma\left(l+\nu\right)}\right\}\left[\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^i\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\frac{(l+\nu+i-1)!}{(l+\nu-1)!}\left[\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^i\end{aligned}\tag{6.5.4}$$

On solving (6.5.4), we get the required result i.e, equation (6.5.1).

Similarly, in the case of quadratic loss function, cdf is given by

$$\hat{F}_{QLF}(y;\beta,\rho,\nu)=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{1!}\hat{\beta}_{QB}^i$$

On using (6.3.3),

$$\begin{aligned}\hat{F}_{QLF}\left(y;\beta,\rho,\nu\right)&=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\frac{\Gamma\left(l+\nu-i\right)}{\Gamma\left(l+\nu-2i\right)}\left(s_l+w\right)^{-i}\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(l+\nu-i\right)}{\Gamma\left(l+\nu-2i\right)}\right\}\left[\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^i\end{aligned}\tag{6.5.5}$$

On solving (6.5.5), we get the required result i.e, equation (6.5.2)

The cdf under general entropy loss function is

$$\hat{F}_{GELF}(y;\beta,\rho,\nu)=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{1!}\hat{\beta}^i_{GB}$$

From (6.3.5), we get

$$\begin{aligned}\hat{F}_{GELF}(y;\beta,\rho,\nu)&=\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{1!}\frac{\Gamma\left(l+\nu\right)}{\Gamma\left(l+\nu-i\right)}\left(s_l+w\right)^{-i}\\&=\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(l+\nu-1\right)}{\Gamma\left(l+\nu-i-1\right)}\right\}\left[\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^i\end{aligned}\tag{6.5.6}$$

On solving (6.5.6), we get the required result i.e, equation (6.5.3). Hence, the theorem follows.

Theorem 5.2: Bayes estimator for the pdf given on (6.1.2) at a point 'y' for the powers of β under SELF, QLF and GELF are defined as

$$\hat{f}_{SELF}(y;\beta,\rho,\nu)=\frac{2\rho\left(l+\nu+1\right)\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left(s_l+w\right)\left\{1-e^{-\rho(y-\nu)^2}\right\}}\left[1+\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^{l+\nu}\tag{6.5.7}$$

$$\hat{f}_{QLF}(y;\beta,\rho,\nu)=\frac{2\rho\left(l+\nu-3\right)\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left(s_l+w\right)\left\{1-e^{-\rho(y-\nu)^2}\right\}}\left[1+\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^{l+\nu-3}\tag{6.5.8}$$

$$\hat{f}_{GELF}(y;\beta,\rho,\nu)=\frac{2\rho\left(l+\nu-1\right)\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left(s_l+w\right)\left\{1-e^{-\rho(y-\nu)^2}\right\}}\left[1+\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{\left(s_l+w\right)}\right]^{l+\nu-2}\tag{6.5.9}$$

Proof: As we know that

$$\frac{d}{dy}\hat{F}\left(y;\beta,\rho,\nu\right)=\hat{f}(y;\beta,\rho,\nu)$$

$$\hat{f}(y;\beta,\rho,\nu)=\frac{d}{dy}\left[\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\hat{\beta}^i\right]$$

$$=\sum_{i=0}^{\infty}\frac{i\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^{i-1}}{i!}\frac{2\rho\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left\{1-e^{-\rho(y-\nu)^2}\right\}}\hat{\beta}^i$$

$$\hat{f}(y;\beta,\rho,\nu)=\frac{2\rho\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left\{1-e^{-\rho(y-\nu)^2}\right\}}\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\hat{\beta}^{i+1} \tag{6.5.10}$$

On using (6.3.1), we get

$$\hat{\beta}^{i+1}=\frac{\Gamma(l+\nu+i+1)}{\Gamma(l+\nu)}\left(s_l+w\right)^{-(i+1)}$$

$$\hat{f}(y;\beta,\rho,\nu)=\frac{2\rho\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left\{1-e^{-\rho(y-\nu)^2}\right\}}\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\rho(y-\nu)^2}\right\}\right]^i}{i!}\frac{\Gamma(l+\nu+i+1)}{\Gamma(l+\nu)}\left(s_l+w\right)^{-(i+1)}$$

$$=\frac{2\rho\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left\{1-e^{-\rho(y-\nu)^2}\right\}\left(s_l+w\right)}\sum_{i=0}^{\infty}\frac{(l+\nu+i)!}{(l+\nu-1)}\left[\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^i$$

$$\hat{f}_{SELF}(y;\beta,\rho,\nu)=\frac{2\rho\left(l+\nu+1\right)\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left(s_l+w\right)\left\{1-e^{-\rho(y-\nu)^2}\right\}}\left[1+\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^{l+\nu}$$

Similarly for the case of quadratic loss function and using (6.3.3)

$$\hat{\beta}^{i+1}=\frac{\Gamma(l+\nu+i+1)}{\Gamma(l+\nu-2i-2)}\left(s_l+w\right)^{-(i+1)}$$

On solving as above, we get

$$\hat{f}_{QLF}(y;\beta,\rho,\nu)=\frac{2\rho\left(l+\nu-3\right)\left(y-\nu\right)e^{-\rho(y-\nu)^2}}{\left(s_l+w\right)\left\{1-e^{-\rho(y-\nu)^2}\right\}}\left[1+\frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^{l+\nu-3}$$

For general entropy loss function using (6.3.5) as

$$\hat{\beta}^{i+1} = \frac{\Gamma(l+\nu+1)}{\Gamma(l+\nu-q+1)} (s_l+w)^{-(i+1)}$$

Hence, the theorem follows.

Theorem 5.3: Bayes estimator of the reliability function R(t) given in (6.1.3) at a point 'y' for the powers of β under SELF, QLF and GELF are defined as

$$\hat{R}_{SELF}(y;\beta,\rho,\nu) = 1 - \left[1 + \frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^{l+\nu} \tag{6.5.11}$$

$$\hat{R}_{QLF}(y;\beta,\rho,\nu) = 1 - \left[1 + \frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^{l+\nu-2} \tag{6.5.12}$$

$$\hat{R}_{GELF}(y;\beta,\rho,\nu) = 1 - \left[1 + \frac{log\left\{1-e^{-\rho(y-\nu)^2}\right\}}{(s_l+w)}\right]^{l+\nu-1} \tag{6.5.13}$$

Proof: As we know that

$$\hat{R}(y;\beta,\rho,\nu) = 1 - \hat{F}(y;\beta,\rho,\nu)$$

Results can be obtained with the help of (6.5.1), (6.5.2) and (6.5.3).

6.6 Bayes estimator of $P = P_r(X > Y)$

Theorem 61: Bayes estimator of P under SELF, QLF and GELF are given as

$$\begin{aligned}\hat{P}_{SELF} =& 1 - \frac{2(k+\nu_2-1)}{(t_k+w_2)} \int_0^\infty \frac{(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{\{1-e^{-\rho_2(y-\nu_2)^2}\}} \left\{1 + \frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1} \\ &\times \left\{1 + \frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right\}^{k+\nu_2} dy\end{aligned} \tag{6.6.1}$$

$$\hat{P}_{QLF} = 1 - \frac{2(k+\nu_2-1)}{(t_k+w_2)} \int_0^\infty \frac{(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{\{1-e^{-\rho_2(y-\nu_2)^2}\}} \left\{1 + \frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1-2} \times \left\{1 + \frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right\}^{k+\nu_2-3} dy \tag{6.6.2}$$

$$\hat{P}_{GELF} = 1 - \frac{2(k+\nu_2-1)}{(t_k+w_2)} \int_0^\infty \frac{(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{\{1-e^{-\rho_2(y-\nu_2)^2}\}} \left\{1 + \frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1-1} \times \left\{1 + \frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right\}^{k+\nu_2-2} dy \tag{6.6.3}$$

Proof: For the reliability model we know that

$$\hat{P} = \int_{y=0}^\infty \int_{y=0}^\infty \hat{f}(y;\beta_1,\rho_1,\nu_1)\hat{f}(y;\beta_2,\rho_2,\nu_2)dy$$

$$\hat{P} = \int_{y=0}^\infty \hat{R}(y;\beta_1,\rho_1,\nu_1)\hat{f}(y;\beta_2,\rho_2,\nu_2)dy \tag{6.6.4}$$

Using (6.5.7) and (6.5.11)

$$\hat{P}_{SELF} = \int_0^\infty \left[1 - \left\{1 + \frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1}\right] \frac{2(k+\nu_2-1)\rho_2(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{(t_k+w_2)\{1-e^{-\rho_2(y-\nu_2)^2}\}} \times \left[1 + \frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2} dy$$

$$\hat{P}_{SELF} = \int_0^\infty \frac{2(k+\nu_2-1)\rho_2(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{(t_k+w_2)\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}} \left[1+\frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2} dy$$
$$- \int_0^\infty \frac{2(k+\nu_2-1)\rho_2(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{(t_k+w_2)\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}} \left\{1+\frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1}$$
$$\times \left[1+\frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2} dy$$

$$\hat{P}_{SELF} = 1 - \frac{2(k+\nu_2-1)\rho_2}{(t_k+w_2)} \int_0^\infty \frac{(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}} \left\{1+\frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}^{l+\nu_1}\right\}$$
$$\times \left[1+\frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2} dy$$

On using, (6.5.8) and (6.5.12), Bayes estimator of P under quadratic loss function is

$$\hat{P}_{QLF} = \int_0^\infty \left[1-\left\{1+\frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1-2}\right] \frac{2(k+\nu_2-1)\rho_2(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{(t_k+w_2)\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}$$
$$\times \left[1+\frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2} dy \tag{6.6.5}$$

On solving (6.6.5), we have

$$\hat{P}_{QLF} = 1 - \frac{2(k+\nu_2-1)}{(t_k+w_2)} \int_0^\infty \frac{(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}} \left\{1+\frac{log\left\{1-e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l+w_1)}\right\}^{l+\nu_1-2}$$
$$\times \left\{1+\frac{log\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right\}^{k+\nu_2-3} dy$$

Similarly, for the general entropy loss function, Bayes estimator of P is

$$\hat{P}_{GELF} = \int_0^\infty \left[1 - \left\{1 + \frac{log\left\{1 - e^{-\rho_1(y-\nu_1)^2}\right\}}{(s_l + w_1)}\right\}^{l+\nu_1-1}\right] \frac{2(k+\nu_2-1)\rho_2(y-\nu_2)e^{-\rho_2(y-\nu_2)^2}}{(t_k+w_2)\left\{1-e^{-\rho_2(y-\nu_2)^2}\right\}}$$
$$\times \left[1 + \frac{log\left\{1 - e^{-\rho_2(y-\nu_2)^2}\right\}}{(t_k+w_2)}\right]^{k+\nu_2-2} dy \tag{6.6.6}$$

On solving (6.6.6), we get the required result. Hence, the theorem follows.

6.7 Simulation Studies

In this chapter our obejective is to comparing the performance of the Bayes estimators and different type of risks by the simulation process. Firstly, we generate a sample of size 10 for the given cdf in (6.1.1), with $\rho = 0.5$, $\nu = 1$ and $\beta = 1.5$ which is given in Data Set I. From these random samples we get $S_l = 3.807633$. For the Data set I, the numerical values of Bayes estimator, Posterior risk and Bayes risk under the different loss functions are obtained in Table 6.1 for q=1. For the prior and posterior parameters $d = 3$ and $\nu = 5$, the prior and posterior densities are presented in Fig. 6.1.

Data Set I								
2.144675	3.599558	2.782179	3.743291	3.795892	2.094704	2.667774	1.673280	3.334993
3.772203								

For further simulation we have generated another set of random number (Data Set II), for $n = 20$. These values are used for the comparisons for the actual and estimated values of $R(t)$ and are presented in Table 6.2.

Data Set II								
2.318244	2.628341	4.753797	2.151703	2.625649	2.820270	1.860507	1.979054	2.492621
2.210561	2.487871	3.369444	2.157737	1.629553	3.075879	2.528517	3.402918	2.946452
2.527474	2.149112							

For obtaining the values of $\hat{P}_{SELF}$, $\hat{P}_{QLF}$ and $\hat{P}_{GELF}$, we have generated two sets of random number (Data Set III and Data Set IV) of sizes $l = 30$ ans $k = 35$ with parameters ($\beta_1 = 2.5, \nu_1 = 1.5$ and $\rho_1 = 1$) and ($\beta_2 = 2.5, \nu_2 = 0.5$ and $\rho_2 = 1$), respectively, these values are presented in Table 6.3.

Data Set III								
2.810702	2.418058	2.184799	2.471951	2.439273	2.254269	2.297142	1.989514	2.590317
3.075284	2.577875	2.519324	2.186240	2.745431	2.705603	3.233207	2.546441	2.462568
2.530647	3.073059	3.275108	3.408853	2.786113	3.015170	3.381099	2.430881	3.019297
2.355386	2.648049	2.155254						

Data Set IV								
1.177279	2.718798	1.954938,	2.040666	1.359097	1.663967	2.034732	1.307619	1.174362
1.883247	1.607426	1.021445	1.309567	1.984419	2.109319	1.589474	1.590095	1.448158
1.412161	2.271597	1.998776	2.361582	1.422357	2.314836	0.796465	1.294174	1.560182
1.522881	1.673208	2.020792	1.626318	2.150253	1.328395	1.901145	1.005523	

6.8 Conclusion and Interpretation

1. One can enumerate for the complete sample by putting l=n and k=m.

2. From Table 6.1, it is depicts that the values of posterior risk under GELF has smaller value than the QLF and SELF for the positive power of β. For the negative power of β, values under SELF for posterior risk has smaller value.

3. From Table 6.2, we observe that as 't' increases, $R(t)$ and estimates of R i.e., $\hat{R}_{SELF}(t)$, $\hat{R}_{QLF}(t)$, and $\hat{R}_{GELF}(t)$ are decreases. The Bayes estimator of $\hat{R}_{QLF}(t)$ are less than the corresponding estimates of SELF and GELF for all the values of 't'. $R(t)$ has maximum value in all the cases.

4. Table 6.3 shows the values of P for different loss functions and we see the slight change in the obtained values.

5. Fig. 6.1, shows satisfactory result. The posterior distribution has maximum peak and taking almost 50 percent of the likelihood and prior distribution.

6. From the Fig. 6.2, Fig. 6.3 and Fig. 6.4, we see that as the value of 'l' increases, the graph of estimated $\hat{f}(y;\beta,\rho,\nu)$ come close to the graph of $f(y;\beta,\rho,\nu)$ in the case of all loss functions i.e. SELF, QLF and GELF, respectively.

7. From the Fig. 6.5, Fig. 6.6 and Fig. 6.7, we see that as the value of 'l' increases, the graph of estimated $\hat{F}(y;\beta,\rho,\nu)$ come close to the graph of $F(y;\beta,\rho,\nu)$ in the case of all loss functions i.e. SELF, QLF and GELF, respectively.

6.9 Tables and Graphs

Table 6.1

Numerical values of Bayes estimator Posterior risk for the positive and and negative powers of β						
Loss Function	Bayes Estimator		Posterior Risk		Bayes Risk	
	Positive	Negative	Positive	Negative	Positive	Negative
SELF	1.978329	0.539175	0.244612	0.0207650	0.0988235	0.083333
QLF	1.731038	0.4757431	1	1	1	1
GELF	1.854684	0.5054771	0.0332885	0.031250	0.0332885	0.031250

Table 6.2

Values of R(t) and estimated values i.e, $\hat{R}_{SELF}(t)$, $\hat{R}_{QLF}(t)$ and $\hat{R}_{GELF}(t)$				
t	$R(t)$	$\hat{R}_{SELF}(t)$	$\hat{R}_{QLF}(t)$	$\hat{R}_{GELF}(t)$
1.2	0.9972136	0.9942070	0.9893808	0.9921567
1.3	0.9907697	0.9810937	0.9698450	0.9761228
1.4	0.9786818	0.9587415	0.9399665	0.9502316
1.5	0.9597214	0.9270383	0.9007230	0.9148917
1.6	0.9331413	0.8866256	0.8535301	0.8711360
1.7	0.8987078	0.8386328	0.8000076	0.8203553
1.8	0.8566917	0.7844912	0.7418447	0.7641298
1.9	0.8078184	0.7257914	0.6807072	0.7041067
2.0	0.7531880	0.6641698	0.6181697	0.6419076
2.1	0.6941722	0.6012162	0.5556660	0.5790568
2.2	0.6323026	0.5384037	0.4944538	0.5169283
2.3	0.5691581	0.4770354	0.4355920	0.4567087
2.4	0.5062630	0.4182106	0.3799305	0.3993754
2.5	0.4450026	0.3628060	0.3281094	0.3456877

Table 6.3

Values of $\hat{P}$		
$\hat{P}_{SELF}$	$\hat{P}_{QLF}$	$\hat{P}_{GELF}$
0.9827967	0.9889845	0.9911976

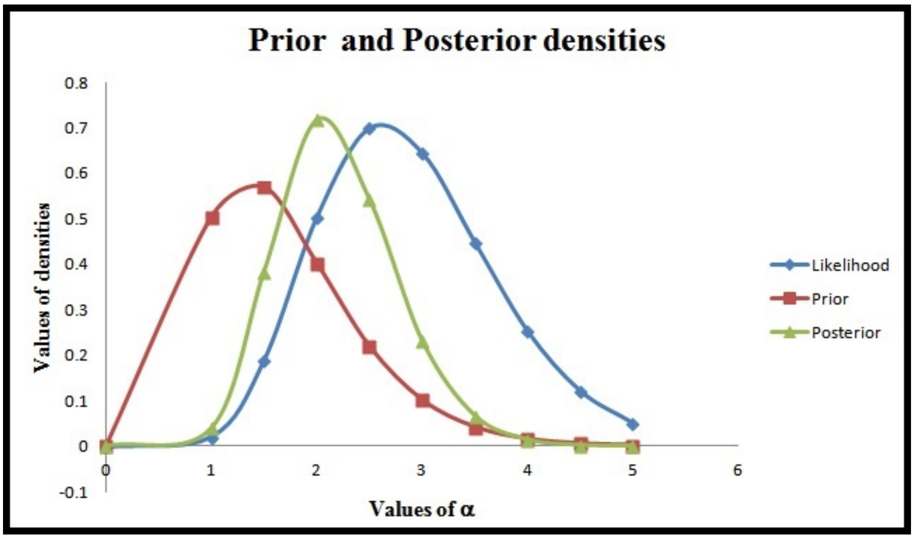


Fig. 6.1: Prior and Posterior densities

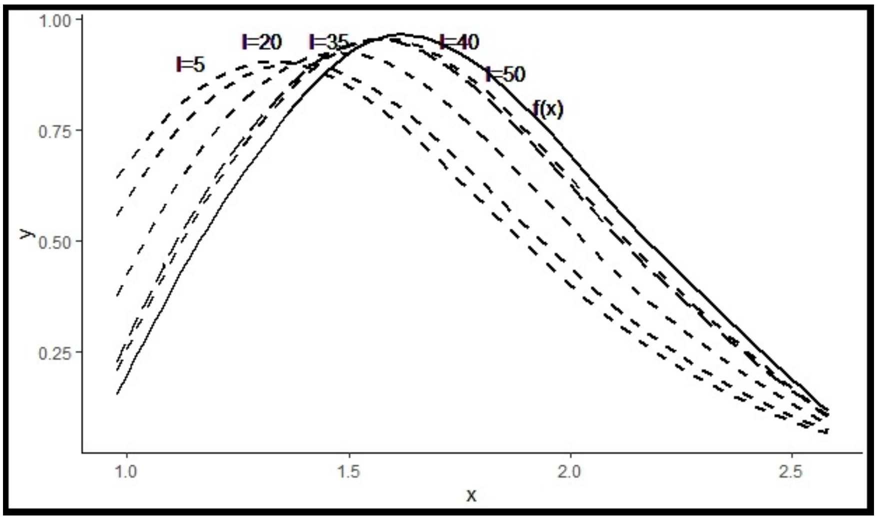


Fig. 6.2: The curve of $f(y;\beta,\rho,\nu)$ and $\hat{f}(y;\beta,\rho,\nu)_{SELF}$

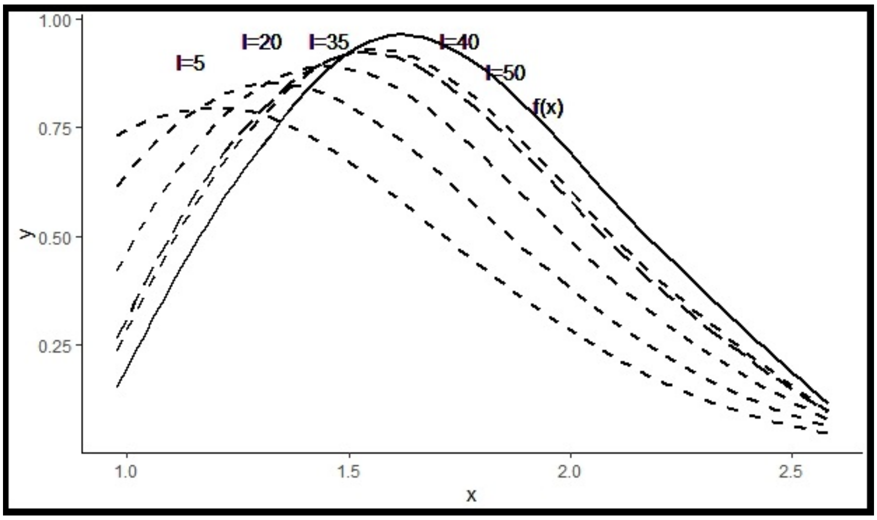


Fig. 6.3: The curve of $f(y;\beta,\rho,\nu)$ and $\hat{f}(y;\beta,\rho,\nu)_{QLF}$

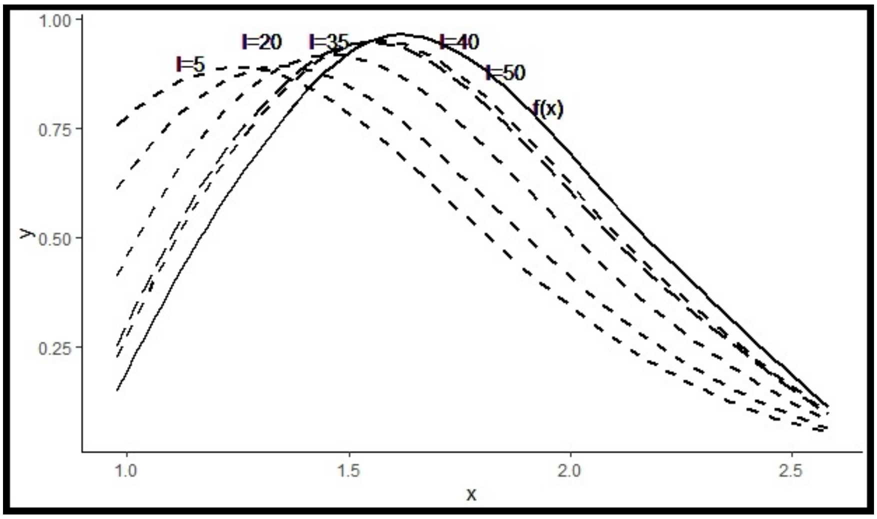


Fig. 6.4: The curve of $f(y;\beta,\rho,\nu)$ and $\hat{f}(y;\beta,\rho,\nu)_{GELF}$

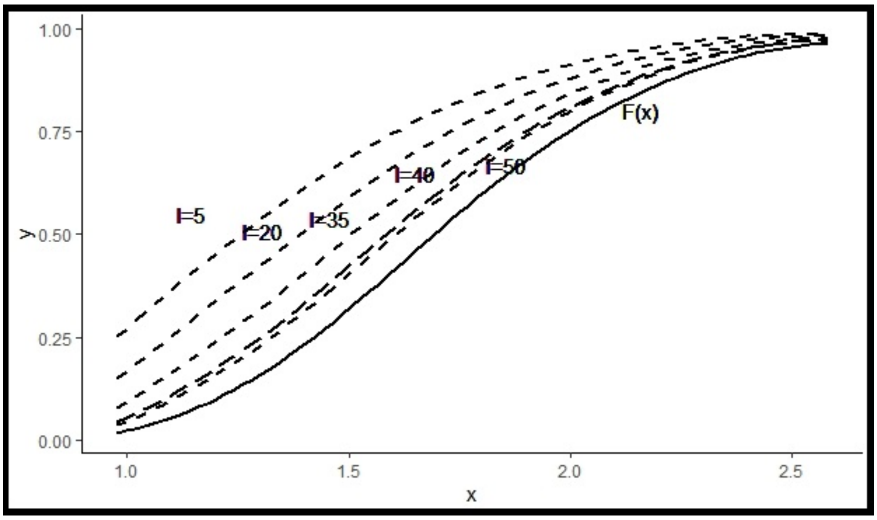


Fig. 6.5: The curve of $F(y;\beta,\rho,\nu)$ and $\hat{F}(y;\beta,\rho,\nu)_{SELF}$

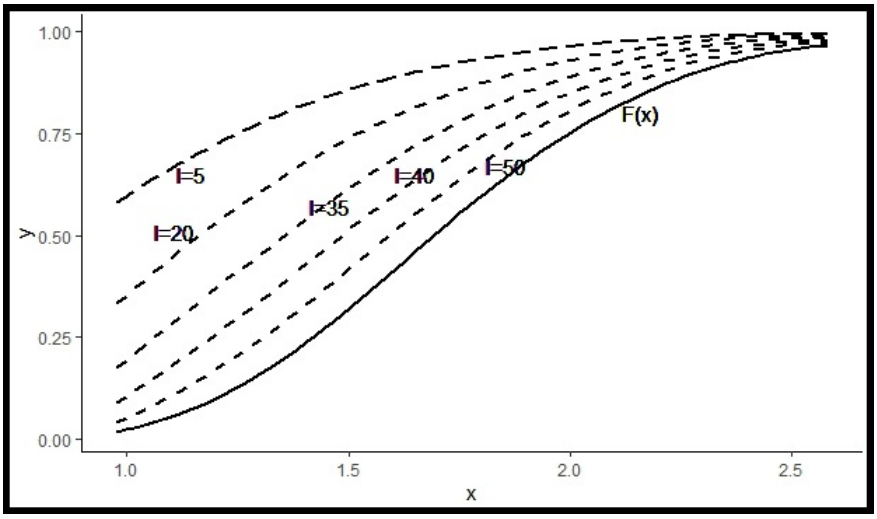


Figure 6.6: The curve of $F(y;\beta,\rho,\nu)$ and $\hat{F}(y;\beta,\rho,\nu)_{QLF}$

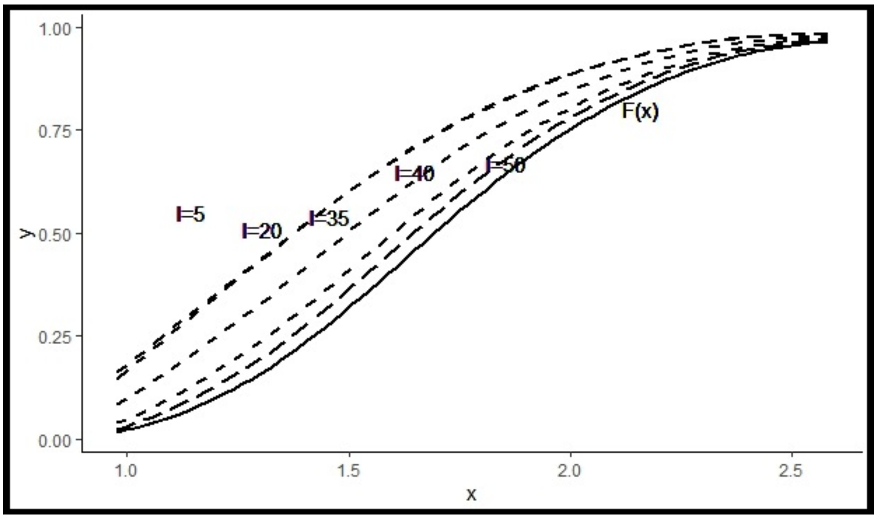


Figure 6.7: The curve of $F(y;\beta,\rho,\nu)$ and $\hat{F}(y;\beta,\rho,\nu)_{GELF}$

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