Contribution to Ordered Random Variables

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Dedicated to

Beloved Parents

For Endless Support, Love and Cooperation

DECLARATION

I, **Ajay Kumar** hereby declare that the research work embodied in this thesis entitled **"Contribution to Ordered Random Variables"** carried out by me under the supervision of **Dr. Surinder Kumar, Professor, Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow (U.P.), India and Co-supervision of Dr. Mohd. Jahangir Sabbir Khan, Assistant Professor, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh (U.P.),** India is an original work and does not contain any work submitted for the award of any degree in this University or any other University.

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CERTIFICATE

This is to certify that the thesis of titled **"Contribution to Ordered Random Variables"** submitted by **Mr. Ajay Kumar,** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other university.

The thesis submitted to **Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow,** India satisfies all the requirements as stipulated in the Doctor of Philosophy (Ph.D.) regulations-1999 as amended in 2008/2010/2013 and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

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ABSTRACT

The present thesis entitled **“Contribution to Ordered Random Variables”** contains seven chapters.

The first chapter is the introduction of those concepts and results, which are subsequently utilized to grasp the idea of ordered random variables. This chapter consist of those ideas like order statistics, generalized order statistics, record values and other various important models of ordered random variables.

In second chapter, we deal with statistical properties of *k-th* upper record values for Muth distribution. This chapter deals with the recurrence relations of single and product moments of *k-th* upper record values established for Muth distribution and obtain its characterization using single moments of *k-th* upper record values. Further, the recurrence relations is established which is useful in computing the moments of higher order for Muth distribution. Its characterization also given through the recurrence relation for single moments of *k-th* upper record values.

In the third chapter, recurrence relations of generalized order statistics have been obtained for Kumaraswamy power function distribution. Characterization of Kumaraswamy power function distribution is also obtained which is based on recurrence relation of single moments of generalized order statistics.

In the fourth chapter, we have developed the recurrence relations for single as well as the product moments from generalized order statistics for Power Lomax distribution. Characterization result is also established for Power Lomax distribution from recurrence relation of single moments for generalized order statistics.

In the sixth chapter, the problem of Stress-Strength reliability model is studied. Here we

consider Erlang-truncated exponential distributed stress. We have dealt with the problem by establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows an Erlang-truncated exponential distribution and Strength follows a power function distribution. Further, these results are explained with an example and are utilized to get optimum cost of any item when the cost function is linear in terms of parameters

In the seventh chapter, the problem of Stress-Strength reliability model is studied for Sushila distributed stress. We have studied the problem by establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows a Sushila distribution and Strength follows a power function distribution.

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Chapter 1
Introduction

CHAPTER 1

INTRODUCTION

In this Chapter, we have introduced those concepts and results of ordered random variables which are used in the subsequent chapters.

1.1. Order statistics (David Nagaraja, 2003)

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (iid) random variables of size n from a continuous population having probability density function (pdf) $f(x)$ and distribution function(df) $F(x)$. If these random variables (rv) are arranged in ascending order of magnitude such that

$$X_{1:n} \le X_{2:n} \le \dots \le X_{n:n}$$

Then $X_{\mathrm{r}:n}$ is called the $r-th$ order statistic and $X_{1:n} = \min(X_1, X_2, ..., X_n)$ and $X_{n:n} = \max(X_1, \mathrm{X}_2, ..., X_n)$ are called extremes order statistics or smallest and largest order statistics respectively.

Order statistics can also be defined by the means of measurable map $T : \Re^n \to \Re^n$ such that

$(X_{1:n}, X_{2:n}, ..., X_{n:n}) = T(X_1, \mathrm{X}_2, ..., X_n)$

(Kamps, 1995 a).

Alternatively, order statistics can also be introduced by the quantile function F_n^{-1} of the empirical distribution function

$$F_n^{-1}(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x)}(X_j), x \in \Re^n,$$

and therefore, order statistics can be introduced as quantiles

$X_{j:n} = F_n^{-1}\left(\frac{j}{n}\right), 1 \le j \le n$ (Kamps, 1995).

The area of order statistics concerned with the properties and applications of those ordered random variables (rv) and the function involving them (David and Nagaraja, 2003). It is different from the rank statistics in which the order of the value of the observation rather than its magnitude is considered.

Order statistics has an important role in model building and in inferential problems. For example, extreme (largest, smallest) values are important in Oceanography, material strength and meteorology.

Another very important application of order statistics is found in life testing models. The $r-th$ order statistics $X_{r:n}$ in a sample of size n represents the life length of a $(n-r+1)$ out of n system. This system consists of n components of the same kind with independently distributed life lengths. All n components starts working simultaneously and the system fails, if r or more components fails. In other words, $n-r+1$ components are necessary for the system to work. For $r=1$ we have a series system and the case $r=n$ corresponds to a parallel system.

1.2. Distribution of the order statistics

The pdf of $X_{r:n}$, the $r-th$ order statistics is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x), -\infty < x < \infty \tag{1.2.1}$$

The pdf of smallest and largest order statistics are,

$$f_{1:n}(x) = n\left[1-F(x)\right]^{n-1} f(x); -\infty < x < \infty \tag{1.2.2}$$

$$f_{n:n}(x) = n\left[F(x)\right]^{n-1} f(x); -\infty < x < \infty \tag{1.2.3}$$

The df of $X_{r:n}$ is given by

$$f_{r:n}(x) = P(X_{r:\text{n}} \le x)$$

$= P$ (at least r of $X_1, X_2, \ldots, X_n$) are less than or equal to x)

$= \sum_{i=r}^{n} P$ (exactly i of $X_1, X_2, \ldots, X_n$ are less than or equal to x)

$$= \sum_{i=r}^{n} \binom{n}{i} \left[F(x)\right]^{i} \left[1 - F(x)\right]^{n-1} \tag{1.2.4}$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_{0}^{F(x)} u^{r-1} (1-u)^{n-r} \, du \tag{1.2.5}$$

$$= I_{F(x)}(r, n-r+1) \tag{1.2.6}$$

where $I_p(a,b) = \dfrac{1}{B(a,b)} \displaystyle\int_0^p t^{a-1} (1-t)^{b-1} \, dt$,

and

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt,$$

RHS of (1.2.6) is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991).

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{n-1-i}{r-1} \left[F(x)\right]^{r} \left[1 - F(x)\right]^{n-r-i}; \; -\infty < x < \infty \tag{1.2.7}$$

For continuous case the pdf of $X_{r:n}$ may also be obtained by differentiating (1.2.5) $w.r.t.$ x.

The $k-th$ moment of $X_{r:n}$ is

$$a_{r:n}^{k} = E\left[X_{r:n}^{k}\right] = \int_{-\infty}^{\infty} x^{k} f_{r:n}(x) \, dx \tag{1.2.8}$$

The join pdf of $X_{r:n}, X_{s:n}, 1 \le r < s \le n$ is given

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left[F(x)\right]^{r-1}$$
$$\times\left[F(y)-F(x)\right]^{s-r-1}\left[1-F(y)\right]^{n-s} f(x)\, f(y); \quad -\infty < x < y < \infty \tag{1.2.9}$$

The joint df of $X_{s:n}, 1 \le r < r \le n$ can be obtained as follows:

$F_{r,s:n}(x,y) = P\left(X_{r:n} \le x, X_{s:n} \le y\right) = P$ (at least r of $X_1, X_2, \ldots, X_n$ are at most x and at least s of $X_1, X_2, \ldots, X_n$ are at least y)

$= \sum_{j=s}^{n} \sum_{i=r}^{j} P$ (exactly i of $X_1, X_2, \ldots, X_n$ are at most x and exactly j of $X_1, X_2, \ldots, X_n$ are at most y)

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{\mathrm{i}!(j-i)!(n-j)!}\left[F(x)\right]^{i}\left[F(y)-F(x)\right]^{j-i}\left[1-F(y)\right]^{n-j} \tag{1.2.10}$$

We can write the point df of $X_{r:n}$ and $X_{s:n}$ in (1.2.10) equivalently as:

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_0^{F(x)}\int_0^{F(y)} u^{r-1}(v-u)^{s-r-1} \times (1-v)^{n-s}\, du\, dv$$

$$= I_{F(x),F(y)}(r, s-r, \mathrm{n}-\mathrm{s}+1); -\infty < x < y < \infty, \tag{1.2.11}$$

Which is incomplete bivariate beta function.

It may be noted that for $x \ge y$

$$F_{r,s:n}(x,y) = F_{s:n}(y) \tag{1.2.12}$$

The product of the $j-th$ and $k-th$ order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1 \leq r < s \leq n)$ is given by:

$$\alpha_{r,s:n}^{j,k} = E\left[X_{j:n}^{j} X_{s:n}^{k}\right] = \iint_{-\infty<x<y<\infty} x^j y^k f_{r,s:n}(x,y)\,dx\,dy \tag{1.2.13}$$

In general, the point pdf of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}\left(x_{i_1:n}, x_{i_2:n}, \ldots, x_{i_k:n}\right) = n!\left\{\prod_{j=1}^{k} f\left(x_{i_j}\right)\right\}\prod_{j=0}^{k}\left\{\frac{\left[F\left(x_{i_{j+1}}\right)-F\left(x_{i_j}\right)\right]^{i_{j+1}-i_j-1}}{\left(i_{j+1}-i_j-1\right)!}\right\}$$

$$-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty \tag{1.2.14}$$

where $x_0 = -\infty, x_{k+1} = +\infty, i_0 = 0, i_{k+1} = n+1$.

Remarks.

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.
2. Regarding the probability integral transformation, if $X_{r:n}, 1 \leq r \leq n,$ are the order statistics from a continuous distribution $F(x),$ then the transformation $U_{r:n} = F\left(X_{r:n}\right)$ produces a random variable which is the $r-th$ order statistics from a uniform distribution on $U(0,1)$.
3. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

4. If $X_1, X_2, \ldots, X_n$ be iid random variables from a continuous distribution, then the set of order statistics $\left[X_{1:n}, X_{2:n}, \ldots, X_{n:n}\right]$ is both sufficient and complete (Lehmann, 1986).

5. Let X be a continuous random variable with $E\left[X_{r:n}\right] = \alpha_{r:n}$

   a) If $\alpha = E(X)$ exists then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, may exist for certain (but not all) values of r, even though α does not exist.

   b) $\alpha_{r:n}$ for all n determine the distribution completely.

1.3. Truncated distribution

Let X be a continuous random variable having pdf $f(x)$ and df $F(x)$ in the interval $[-\infty, \infty]$.

Let $\int\limits_{-\infty}^{Q_1} f(x)\,dx = Q$ and $\int\limits_{-\infty}^{P_1} f(x)\,dx = P$ (1.3.1)

Where Q_1 and P_1 are known constants. Then doubly truncated pdf of X is given by:

$$\frac{f(x)}{P-Q}; x \in (Q_1, P_1) \tag{1.3.2}$$

and the corresponding df is given by

$$\frac{f(x)-Q}{P-Q}; x \in (Q_1, P_1) \tag{1.3.3}$$

The lower and upper truncation points are Q_1 and P_1 respectively; the degrees of truncation are Q (from below) and $1-P$ (from above). If you put $Q = 0$, the distribution will be truncated to the right. Similarly at $P = 1$, the distribution will be truncated to the left, whereas at $Q = 0$ and $P = 1$, we get the non-truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

In the following, we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is truncated from the original population distribution $F(x)$.

Result 1.3.1 (David and Nagaraja, 2003). Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population having $df\ F(x)$ and $pdf\ f(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{r:n} \le \ldots \le X_{\mathrm{n}:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$ given that $X_{\mathrm{s}:n} = y$ for $s > r$ is the same the distribution of the r^{th} order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Result 1.3.2 (David and Nagaraja, 2003). Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with $df\ F(x)$ and $pdf\ f(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{r:n} \le \ldots \le X_{\mathrm{n}:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{\mathrm{s}:n}$ given that $X_{\mathrm{r}:n} = x$ for $r < s,$ is the same the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Result 1.3.3 (David and Nagaraja, 2003). Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with $df\ F(x)$ and $pdf\ f(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{r:n} \le \ldots \le X_{\mathrm{n}:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{\mathrm{s}:n}$ given that $X_{\mathrm{r}:n} = x$ and $X_{k:n} = z$ for $1 \le r < s < k \le n,$ is the same the distribution of the $(s-r)-th$ order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

Remark 1.1. Result 2.1 follows from Result 2.3 by replacing k with $n+1$ with the conversion $z = X_{n+1:n} = \beta,$ where β is the upper range of X and $F(\beta)=1$.

Remark 1.2. Result 2.2 follows from Result 2.3 by letting $r=0$ the convention $X_{0:n} = x = \alpha$ (lower limit).

Remark 1.3. Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$\begin{aligned} f\left(X_{k:n} \middle| X_{1:n} = x_1, \ldots X_{r:n} = x_r, \ldots X_{s:n} = x_s, \ldots X_{n:n} = x_n\right) \\ = f\left(X_{k:n} \middle| X_{r:n} = x_r, X_{s:n} = x_s\right) \end{aligned}$$

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

1.4. Sequential order statistics (Kamps, 1995)

In life testing model, order statistics is used as a model to describe the reliability of k out of n system. A k out of n system consist of n components which start working simultaneously. It works while k components are functioning and breaks down if $n-k+1$ or more components fails. Parallel and series systems are particular cases of k out of n system corresponding to the values $k=1$ and $k=n$ respectively. If a system consists of components of the same kind and without any interactions with respect to life length distributions, are working then the system failure is modeled by an order statistics based on $iid\, rv$, assuring that the failure of one component does not affect the rest of the components. However, in generally it does not happened in real life problems. The failures of certain components can affect the remaining components. This can be thought of as harm caused by the $r-th$ failure system. In this model, the life distribution of remaining components in the system may change after each failure of the components.

If we observe $r-th$ failure at time x, the rest of the components are now supposed to have a possibly different life length distribution. The distribution is truncated on the left-hand side of x to ensure realizations arranged in ascending order of magnitude.

Definition. Let $\left(Y_j^{(i)}\right)_{1\le j\le n-i+1, 1\le i\le n}$ be independent of rv with $\left(Y_j^{(i)}\right)_{1\le j\le n-i+1, 1\le i\le n} \sim F_i, 1\le j\le n,$ where $F_1, F_2, \dots, F_n$ are strictly increasing and continuous distribution functions and

$$F_1^{-1}(1) \le \dots \le F_1^{-1}. \qquad (1)$$

Moreover, let $X_j^{(1)} = Y_j^{(1)}, 1\le j\le n,$

$$X_*^{(1)} = \min\left\{X_1^{(1)}, \dots, X_n^{(1)}\right\}$$

and for $2\le i\le n$

$$X_j^{(i)} = F_i^{-1}\left[F_i\left(Y_j^{(i)}\right)\left(1-F_i\left(X_*^{(i-1)}\right)\right) + F_i\left(X_*^{(i-1)}\right)\right]$$

$$X_*^{(i)} = \min\left\{X_j^{(i)}, \quad 1\le j\le n-i+1\right\}$$

then the rv $X_*^{(1)}, \dots, X_*^{(n)}$ are called sequential order statistics.

If we have absolutely continuous distribution functions $F_1, \dots, F_n$ with densities $f_1, \dots, f_n$ respectively, then the joint pdf of the r sequential order statistics $X_*^{(1)}, \dots, X_*^{(n)}$ is given by

$$f_{X_*^{(1)}, \dots, X_*^{(r)}}\left(x_1, \dots, x_r\right) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left[\frac{1-F_i\left(x_i\right)}{1-F_i\left(x_{i-1}\right)}\right]^{n-i} \frac{f_i\left(x_i\right)}{\left[1-F_i\left(x_{i-1}\right)\right]} r\le n, x_0 = -\infty. \qquad (1.4.1)$$

Sequential order statistics form a Markov chain with transition probabilities

$$P\left(X_*^{(r)} > t \middle| X_*^{(r-1)} = x\right) = \left(\frac{1-F_r(t)}{1-F_r(x)}\right)^{n-r+1}, \ 2 \le r \le n \tag{1.4.2}$$

1.5. Record values (Chandler, 1952)

In our daily life we are there are situations were we observed new record and often interested of recording them frequently interested in observing new records and in recording them. Record values are defined as a model for successive extremes in a sequence of $iid\, rv$. It may also used as a model for continually largest insurance claims in non-life insurance problems, Record values are also used in the life testing models.

Definition. Let $(X_i), i \in N$ be a sequence of iid continuous rv with distribution function $df\ F(x)$ and $pdf\ f(x)$. Let we denote the upper record times by $u(1)=1$ and $u(n)=\min\left\{k > u(n-1): X_k > X_{u(n-1)}\right\}$.

The record value sequence is then defined by $X_{u(n)};\ n=1,2,3,\ldots$

Based on an iid sequence of rv $(X_i),\ i \in N$ with an absolutely continuous $df\ F(x)$ and $pdf\ f(x)$, the joint pdf of first r record values $X_{u(1)},\ldots,X_{u(r)}$ is given by (Ahsanullah, 1995)

$$f_{X_{u(1)},\ldots,X_{u(r)}}(x_1,\ldots,x_r) = \left(\prod_{i=1}^{r-1} \frac{f(x_i)}{1-F(x_i)}\right) f(x_r), \tag{1.5.1}$$

The marginal pdf of $X_{u(r)}$ is

$$f_{X_{u(r)}}(x) = \frac{1}{(r-1)!}\left[-\log \bar{F}(x)\right]^{r-1} f(x) \tag{1.5.2}$$

and the magnitude of df of $X_{u(r)}$ is

$$f_{X_{u(r)}}(x) = 1 - \left[1 - F(x)\right] \sum_{j=0}^{r-1} \frac{1}{j!} \left(\log \frac{1}{1 - F(x)} \right)^{j} \tag{1.5.3}$$

The joint pdf of $X_{u(r)}$ and $X_{u(s)}$, $1 \le r < s,$ is given by

$$\begin{aligned} f_{X_{u(r)}}, X_{u(s)}(x, y) &= \frac{1}{(r-1)!(s-r-1)!} \left[-\log \bar{F}(x)\right]^{r-1} \\ &\times \left[-\log \bar{F}(y) + \log \bar{F}(x)\right]^{s-r-1} \frac{f(x)}{\bar{F}(x)} f(y),\ \alpha < x < y < \beta \end{aligned} \tag{1.5.4}$$

The joint pdf of $X_{u(r)}, X_{u(j)}$ and $X_{u(s)}$ $1 \le r < j < s,$ can similarly be given as

$$f_{X_{u(r)}, X_{u(j)}, X_{u(s)}}(x, t, y) = \frac{1}{(r-1)!(j-r-1)!(s-j-1)!} \left[-\log \bar{F}(x)\right]^{r-1} \left[-\log \bar{F}(t) + \log \bar{F}(x)\right]^{j-r-1}$$

$$\times \left[-\log \bar{F}(y) + \log \bar{F}(t)\right]^{s-j-1} \frac{f(x)}{\left[\bar{F}(x)\right]} \frac{f(t)}{\left[\bar{F}(t)\right]} f(y), \quad \alpha < x < t < y < \beta, \tag{1.5.5}$$

where $\bar{F}(x) = 1 - F(x)$

Hence the conditional pdf of $X_{u(j)}$ given $X_{u(r)} = x$ and $X_{u(s)} = y,$ $1 \le r < j < s$ is

$$f_{X_{u(j)} | X_{u(r)}, X_{u(s)}}(t | x, y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{\left[-\log \bar{F}(t) + \log \bar{F}(x)\right]^{j-r-1} \left[-\log \bar{F}(y) + \log \bar{F}(t)\right]^{s-j-1}}{\left[-\log \bar{F}(y) + \log \bar{F}(x)\right]^{s-r-1}} \frac{f(t)}{\bar{F}(t)},\ \alpha < x < t < y < \beta. \tag{1.5.6}$$

1.6. ***K-th*** Records (Dziubdziela and Kopocinski, 1976)

There are situations in which the record values themselves are regarded as outliers and hence second or third largest values are of special concern.

Observing successive $k-th$ largest value in a sequence, Dziubdziela and Kopocinski (1976) proposed the following model of $k-th$ record values.

Definition. Let $(X_i), i \in N$ be iid rv with a df $F(x)$ and let k be a positive integer. The random variable $u_{(k)}(n)$ given by

$$u_{(k)}(1) = 1,$$

$$u_{(k)}(n+1) = \min\left\{ j \in N; X_{j:j+k-1} > X_{u_{(k)}(n):u_{(k)}(n)+k-1} \right\}, j \in N$$

are called $k-th$ record times and the quantities $X_{u_{(k)}(n):u_{(k)}(n)+k-1}$, which we denoted by $X_{u_{(k)}(n)}, n \in N$ are termed as $k-th$ record values. Obviously, we obtain ordinary record values in the case $k=1$

Moreover, Nagaraja (1988) points out that k-records with an underlying df $F(x)$ can be viewed as ordinary record values $(k=1)$ based on the df $G(x)$ (minimum distribution) with

$$G(x) = 1 - (1 - F(x))^k$$

Based on the $(X_i), i \in N$, of $iid\,rv$ possessing an absolutely continuous $df\ F(x)$ and $df\ f(x)$, the joint pdf of k-records $X_{u_{(k)}(1)}, \ldots, X_{u_{(k)}(r)}$

$$f_{X_{u_{(k)}}(1),\ldots\ldots,X_{u_{(k)}(r)}}(x_1, \ldots, x_r) = k^r \left(\prod_{i=1}^{r=1} \frac{f(x_i)}{1 - f(x_i)} \right) \left[1 - f(x_r) \right]^{k-1} f(x_r), \tag{1.6.1}$$

The marginal pdf of $X_{u_{(k)}(r)}$ is

$$f_{X_{u_{(k)}(r)}}(x) = \frac{k^r}{(r-1)!} \left[-\log \bar{F}(x) \right]^{r-1} \left[\bar{F}(x) \right]^{k-1} f(x) \tag{1.6.2}$$

and the marginal pdf of $X_{u_{(k)}(r)}$ is

$$F_{X_{u(k)(r)}}(x)=1-\left[1-F(x)\right]^k\sum_{j=0}^{r-1}\frac{1}{j!}\left(k\log\frac{1}{\{1-F(x)\}}\right)^j \tag{1.6.3}$$

1.7. Progressive Type-II right censoring (Balakrishnan and Aggrawala, 2000)

Progressive censoring is used in life testing problems. In this censoring life testing devices and components can be removed at different stages during the experiment which may reduce the cost and the time of experiment (Cohen, 1963; Sen, 1986). Under this scheme of censoring, from a total of n units placed on a life test, only m are completely observed until failure. At the time of the first failure, R_1 of $n-1$ working units are randomly withdrawn (or censored) from experiment. At the time of the next failure, R_2 of $n-2-R_1$ working units are censored and so on. Finally at the time of the m^{th} failure, all the remaining $R_m = n-m-R_1-\ldots R_{m-1}$ working units are censored.

Suppose n iid units are placed on a life test with the corresponding failure times $X_1, X_2, \ldots, X_n$ being identically distributed with a continuous df $F(x)$ and pdf $f(x)$. Suppose further that the prefixed number of failures to be observed is m and that the progressive Type-II right censoring scheme is $(R_1, R_2, \ldots, R_m)$. Then we shall denote the m completely observed failure times by $X_{1:m:n}, X_{2:m:n}, \ldots, X_{\mathrm{m}:m:n}$ where $X_{i:m:n}$ $, i=1,2,\ldots,m$ denotes the $i-th$ failure time in the progressive type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$ bearing in mind that these still depend on the particular choice of $(R_1, R_2, \ldots, R_m)$ used. In a particular case when $R_1 = R_2 = \ldots = R_m = 0,$ i.e. no withdrawals are carried out, then this model reduces to ordinary type-II censoring.

Above model is described as progressively Type-II right-censored order statistics from $f(x)$ arising from a sample of size n with the censoring scheme $(R_1, R_2, \ldots, R_m)$.

The pdf of all m progressively Type-II right-censored order statistics is

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1,\ldots,x_m) = C\prod_{i=1}^{m} f(x_i)\left[1-F(x_i)\right]^{R_i}; \quad x_1 < x_2 < \ldots < x_n \tag{1.7.1}$$

where

$$C = n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{m-1}-m+1).$$

1.8. Generalized order statistics (Kamps, 1995)

Definition. Let $n \in N, k \geq 1, m_1, m_2, \ldots, m_{n-1} \in \Re^{n-1}$, $M_r = \prod_{j=r}^{n-1} m_j, 1 \leq r \leq n-1,$ be parameters such that $\gamma_r = k+n-r+M_r \geq 1$ for all $r \in \{1,\ldots,\mathrm{n}-1\}$, and let $\tilde{m} = (m_1,\ldots,m_{n-1})$ and if $n \geq 2$, then $X\left(r,n,\tilde{m},k\right)$, $r = 1,2,\ldots,n$ are called generalized order statistics (gos) if their joint pdf is given by

$$\begin{aligned} f_{X\left(1,n,\tilde{m},k\right),\ldots,X\left(\mathrm{n},n,\tilde{m},k\right)}(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left(1-F(x_i)\right)^{m_i} f(x_i)\right) \\ \times\left(1-F(x_n)\right)^{k-1} f(x_n) \end{aligned} \tag{1.8.1}$$

on the cone

$$F^{-1}(0) < x_1 \leq \ldots\ldots \leq x_n < F^{-1}_{(1)}$$

We will consider two cases.

Case I. $m_1 = \ldots = m_{n-1} = m$ and we write gos in form $X(r,n,m,k)$, $r \geq 1$. Then in this case the marginal density function of the $r-th$ gos based on an absolutely continuous df $F(x)$ with pdf $f(x)$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}\left(1-\bar{F}(x)\right)^{\gamma_r - 1} g_m^{r-1}\left(F(x)\right) f(x),\ \alpha < x < \beta \tag{1.8.2}$$

And the joint pdf of the gos $X(r,n,m,k)$ and $X(\mathrm{s},n,m,k)$, where $1 \leq r < s \leq n,$ based on the absolutely continuous df $F(x)$ with pdf $f(x)$ is given by

$$\begin{aligned} f_{X(r,n,m,k),X(\mathrm{s},n,m,k)}(x,y) &= \frac{c_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^m g_m^{r-1}\left(F(x)\right) \\ &\times\left[h_m\left(F(y)\right)-h_m F(x)\right]^{s-r-1}\left[\bar{F}(y)\right]^{s-r-1} f(x)\, f(y) \quad \alpha < x < y < \beta \end{aligned} \tag{1.8.3}$$

The joint pdf of $X(r,n,m,k)$, $X(\mathrm{j},n,m,k)$ and $X(\mathrm{s},n,m,k), 1 \leq r < j < s \leq n,$ can similarly be given as

$$\begin{aligned} f_{X(\mathrm{r},n,m,k),X(\mathrm{j},n,m,k),X(\mathrm{s},n,m,k)}(x,t,y) &= C_{r,j,s:n}\left[\bar{F}(x)\right]^m g_m^{r-1}\left(F(x)\right) \\ &\times\left[h_m\left(F(t)\right)-h_m\left(F(x)\right)\right]^{j-r-1}\left[h_m\left(F(y)\right)-h_m\left(F(t)\right)\right]^{s-j-1} \\ &\left[\bar{F}(t)\right]^m\left[\bar{F}(y)\right]^{\gamma_s - 1} f(x)\, f(t)\, f(y),\ \alpha < x < t < y < \beta \end{aligned} \tag{1.8.4}$$

where

$$\bar{F}(x) = 1 - F(x)$$

$$C_{r,\,s:n} = \frac{\mathrm{c}_{s-1}!}{(r-1)!(s-r-1)!}$$

$$C_{r,\,j,\,s:n} = \frac{\mathrm{c}_{s-1}!}{(r-1)!(j-r-1)!(s-j-1)!}$$

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i \tag{1.8.5}$$

$$g_m(x) = h_m(x) - h_m(0) \tag{1.8.6}$$

$$= \int_0^x (1-t)^m \, dt$$

and
$$h_m(x) = \begin{cases} -\frac{(1-x)^{m+1}}{m+1} & m \neq -1 \\ \log \frac{1}{(1-x)} & m = -1 \end{cases} , x \in (0,1) \tag{1.8.7}$$

Thus, conditional distribution of $X(j,n,m,k)$ given $X(\mathrm{r},n,m,k) = x$ and $X(\mathrm{s},n,m,k) = y$ is given by

$$f_{X(j,n,m,k)|X(\mathrm{r},n,m,k)=x, X(\mathrm{s},n,m,k)=y}(t|x,y) = \frac{(s-r-1)!(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{\left[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(t)\}^{m+1}\right]^{j-r-1} \left[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}\right]^{s-j-1}}{\left[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}\right]^{s-r-1}} \left[\bar{F}(t)\right]^m f(t); \ \alpha < x < t < y < \beta \tag{1.8.8}$$

Case II. For the case II, when $\gamma_i \neq \gamma_j, \ i \neq j,$ the pdf of $X(r,n,\tilde{m},k)$ is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i - 1} f(x) \tag{1.8.9}$$

and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(\mathrm{s},n,\tilde{m},k)$ $1 \leq r < s \leq n,$ is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)=c_{s-1}\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}\gamma_i\right]$$

$$\left[\sum_{i=1}^{r}a_i(r)\left(\bar{F}(x)\right)^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)};\ \alpha<x<\beta \qquad (1.8.10)$$

The joint pdf of $X(r,n,\tilde{m},k)$, $X(j,n,\tilde{m},k)$, and $X(s,n,\tilde{m},k)$, $1\le r<j<s\le n$, may similarly be given as

$$f_{X(r,n,\tilde{m},k),X(j,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,t,y)$$

$$=c_{s-1}\left[\sum_{i=1}^{r}a_i(r)\left(\bar{F}(x)\right)^{\gamma_i}\right]\left[\sum_{i=r+1}^{j}a_i^{(r)}(j)\left\{\frac{\bar{F}(t)}{\bar{F}(x)}\right\}^{\gamma_i}\right]$$

$$\times\left[\sum_{i=j+1}^{s}a_i^{(j)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(t)}\right\}^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(t)}{\bar{F}(t)}\frac{f(y)}{\bar{F}(y)};\ \alpha<x<y<\beta \qquad (1.8.11)$$

Hence the conditional pdf of $X(j,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$ and $X(s,n,\tilde{\mathrm{m}},\mathrm{k})=y$ $1\le r<j<s\le n,$ is given by

$$f_{X(j,n,\tilde{m},k)/X(r,n,\tilde{m},k)=x,X(s,n,\tilde{m},k)=y}(t/x,y)$$

$$=\frac{\left[\sum_{i=r+1}^{j}a_i^{(r)}(j)\left\{\frac{\bar{F}(t)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=j+1}^{s}a_i^{(j)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(t)}\right\}^{\gamma_i}\right]\frac{f(t)}{\bar{F}(t)}}{\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]};\ \alpha<x<t<y<\beta \qquad (1.8.12)$$

where

$$a_i(r)=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \qquad \gamma_j\neq\gamma_i \qquad 1\leq i\leq r\leq n$$

$$a_i^{(j)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{(\gamma_j-\gamma_i)}, \qquad \gamma_j\neq\gamma_i \qquad r+1\leq i\leq s\leq n.$$

gos involves all those models which are related to ordered random variables. For instance It includes Ordinary order statistics [David and Nagaraja, 2003; Lawless, 1982; Arnold et al, 1992; Balakrishnan and Rao, 1998 a, b], order statistics with non-integral sample size [Stigler, 1977; Rohatgi and Saleh, 1988], records [Arnold et al, 1998; Nevzorov, 2001], $k-th$ records (Dziubdziela and Kopocinski, 1976), sequential order statistics [Kamps, 1995; Cramer and Kamps, 2003], Progressive Type-II right censoring (Balakrishnan and Aggrawala, 2000).

Table 1.1. Variants of the generalized order statistics

		$\gamma_n=k$	γ_r	m_r
i)	Sequential order Statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r-\gamma_{r+1}-1)$
ii)	Ordinary- order Statistics	1	$n-r+1$	0
iii)	Record values	1	1	-1
iv)	Progressively type II censored order Statistics	R_n+1	$n-r+1+\sum_{j=r}^{n}R_j$	R_r
v)	Pfeifer's record Values	β_n	β_r	$\beta_r-\beta_{r+1}-1$

1.9. Moments and recurrence relations.

Order statistics and their moments have received consideration since the nineteenth century Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics, moments of order statistics and have been numerically tabulated extensively for several distributions. For more review one may refers to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold and Balakrishnan (1989), Arnold *et al.* (1992).

Following are the three important reasons of recurrence relations and identities which attracted the researchers:

i) Reduces the amount of direct computation and hence saves the time and labour.

ii) They may define the higher order moments in terms of lower order moments and hence make the valuation of higher order moments easier.

iii) Give some simple checks to test the accuracy of evaluation of moments of order statistics.

Since the developments of this theory, several researcher are working on this topic due to its importance and popularity in the field of ordered random variables theory. For a brief review one may go through the findings of the following researchers

Shah (1966, 1970), Tarter (1966) Krishnaiah and Rizvi (1967), Joshi (1979b), Joshi (1982) and Malik (1967) has obtained recurrence relations for the moments of order statistics from power function distribution.

Lieblien (1955), Balakrishnan and Joshi (1981) etc. have considered various continuous life testing models namely logistic distribution, gamma distribution, exponential and truncated Exponential distribution, Weibull distribution, etc. have obtained recurrence relation for moments of order statistics. Saleh *et al.* (1975), Joshi (1978, 1979a).

Malik (1966), Kabe (1972), Huang (1975), Balakrishnan and Joshi (1982) Balakrishnan and Joshi (1983, 1984), Khan *et al.* (1983a) established general results for the $k-th$ moment of order statistics without assuming any particular distribution.

Khan *et al.* (1983b) obtained the result for doubly truncated and non- truncated distributions. Khan *et al.* (1984) obtained the result for inverse moments of order statistics from exponential distribution. Later, Ali and Khan (1996) deduced the result for ratio and inverse moments of order statistics of Weibull and exponential distribution. This results were extended by Khan and Athar (2000) for ratio and product moments of order statistics from doubly truncated Weibull distribution.

Khan and Khan (1987) obtained recurrence relations for single and product moments of order statistics for Burr type XII and obtained the characterizations of distribution.

Further Khan *et al.* (1987) established the relations for logistic distribution. Ali and Khan (1987) obtained the recurrence relations between moments of order statistics for log-logistic distribution whereas Balakrishnan and Kocherlakota (1986) and Al-Shboul *et al.* (1989) obtained moments of order statistics for doubly truncated log-logistic distribution whereas Balakrishnan *et al.* (1988) obtained recurrence relations and identities for moments of order statistics for some specific continuous distributions.

Ali and Khan (1995) have obtained ratio and product moments of two order statistics of different order from Burr distribution. Further they have deduced the moments and inverse moments of single order statistics from the product moments.

Balakrishnan and Aggarwala (1996) obtained the relationships for moments of order statistics from the right-truncated generalized half logistic distribution while Ali and Khan (1997) established recurrence relations for the expectations of a function of single order statistics from a general class of distribution. Further Ali and Khan (1998) also established recurrence relations for expected values of certain functions of two order statistics.

Saran and Pushkarana (1999) established the recurrence relations for single and product moments of order statistics from doubly truncated generalized exponential distribution. Related results may also be found in Khan *et al.* (1987) and Khan and Abu-Salih (1989).

Balakrishnan *et al.* (1992, 1993), Balakrishnan and Ahsanullah (1994a, b, 1995) have established some recurrence relations for the single and product moments of record values for Gumble, generalized extreme value, generalized Pareto, Lomax and exponential distributions, respectively.

Pawlas and Szynal (1998) developed relations for single and product moments of $k-th$ record values from exponential and Gumbel distributions, whereas Pawlas and Szynal (1999) established relations for single and product moments of $k-th$ record values from Pareto, generalized Pareto and Burr distributions. Raqab and Ahsanullah (2000) obtained relations for marginal and joint moment generating function of record values from power function distribution.

Kamps (1995) investigated recurrence relations for moments of generalized order statistics based on non-identically distributed random variables, which contains order statistics and record values as special cases.

Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases.

Pawlas and Szynal (2001a) derived recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Khan *et al.*

(2007) obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution.

Athar and Islam (2004) established some recurrence relations between expectation of function of single and joint generalized order statistics from a general class of distribution. Further Athar *et al.* (2008) generalized the result of Athar and Islam (2004) and established the relations for the expectation of function of lower gos for truncated distributions.

Athar *et al.* (2007) obtained the ratio and inverse moments of generalized order statistics from Weibull distribution. Khan *et al.* (2010a) obtained marginal and joint moment generating functions of generalized order statistics from Gompertz distribution.

Pawlas and Szynal (2001b) obtained the recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. Khan and Kumar (2010, 2011a) derived the recurrence relations for single and product moments of lower generalized order statistics from the exponentiated Pareto and exponentiated gamma distributions respectively.

1.10. Some Continuous distributions

I. Muth distribution

A random variable X is said to have a Muth distribution with shape parameter α if its *pdf* is given by

$$f(x;\,\alpha) = \left(e^{\alpha x} - \alpha\right)\exp\left\{\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right\}, \qquad x > 0, \alpha \in (0,1]$$

and corresponding *cdf* is given by

$$F(x;\,\alpha) = 1 - \exp\left[\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right], \qquad x > 0, \alpha \in (0,1]$$

relation between *pdf* and *cdf* is given as

$$f\left(x;\,\alpha\right)=\left(e^{\alpha x}-\alpha\right)\left(1-F\left(x\right)\right)$$

If we take $\alpha\to 0$ we get standard exponential distribution. This distribution has great applications in reliability theory

II. The Power Muth distribution

If we take the transformation $X=\beta Y^{\frac{1}{\gamma}}$, where $\beta>0,\gamma>0$ where Y follows Muth distribution with parameter $\alpha=1.$ we get the Power Muth distribution. The *pdf* of the Power Muth distribution is given as

$$f\left(x;\beta,\gamma\right)=\frac{\gamma}{\beta^{\gamma}}x^{\gamma-1}\left(e^{\left(\frac{x}{\beta}\right)^{\gamma}}-1\right)\exp\left\{\left(\frac{x}{\beta}\right)^{\gamma}-\left(e^{\left(\frac{x}{\beta}\right)^{\gamma}}-1\right)\right\},\ x>0,\beta>0,\gamma>0$$

and corresponding *cdf* is given by

$$F\left(x;\beta,\gamma\right)=1-\exp\left\{\left(\frac{x}{\beta}\right)^{\gamma}-\left(e^{\left(\frac{x}{\beta}\right)^{\gamma}}-1\right)\right\},\ x>0,\beta>0,\gamma>0$$

Where β is a scale parameter and γ is a shape parameter.

III. Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f\left(x;\theta\right)=\theta e^{-\theta x},\ 0\le x<\infty,\theta>0$$

Corresponding *cdf* is given by

$$f\left(x;\theta\right)=1-e^{-\theta x}$$

The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P\left[X > s+t \,/\, X > s\right] = P\left[X > t\right] \text{ for all } s \text{ and } t.$$

Then X will have exponential distribution. This property is also known as memory less property

This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc. This distribution is extensively use in reliability theory and reliability engineering.

IV. Kumarswamy distribution

A random variable X is said to have an Kumaraswami distribution if its pdf is given by

$$f\left(x;a,b\right) = abx^{a-1}\left(1-x^{a}\right)^{b-1}, \;\; 0 \le x < 1, \; a,b > 0$$

Corresponding cdf is given by

$$F\left(x;a,b\right) = 1-\left(1-x^{a}\right)^{b}, \; 0 \le x < 1, \; a,b > 0$$

The Kumaraswamy distribution is widely used ditribution in hydrological problems and natural phenomenon. The Kumaraswamy distribution probability similar to the beta distribution. This distribution have several sub models.

V. Beta distribution

Beta distribution have two types namely beta distribution of first kind and beta distribution of second kind. We have discussed them separately as follows

a) Beta distribution of the first kind

A random variable X is said to have an Beta distribution of first kind if its pdf is given by

$$f\left(x\right)=\frac{1}{B\left(u,v\right)}x^{u-1}\left(1-x\right)^{v-1},\ 0\le x\le 1,u,v>0$$

Beta distribution arises as the distribution of an ordered variable from a uniform distribution $U\left(0,1\right)$. Suppose $X_{r:n}$ is an ordered sample from $U\left(0,1\right)$, then $X_{r:n}$ is distributed as $B\left(r,n-r+1\right)$. The standard uniform distribution $U\left(0,1\right)$.is the special case of beta distribution of first kind obtained by putting the exponents u and v equal to 1. If $v=1$ the distribution reduces to power function distribution.

b) Beta distribution of the Second kind

The continuous random variable X which is distributed according to probability law:

$$f\left(x\right)=\frac{1}{B\left(u,v\right)}\frac{x^{u-1}}{\left(1+x\right)^{u+v}},\ 0\le x<\infty,\ u,v>0$$

is known as a beta variate of the second kind with parameters u and v. Beta distribution of the second kind reduces to beta distribution of the first kind if we replace $1+x$ by $\frac{1}{y}$. The beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

VI Power Function distribution

A random variable X is said to have a power function distribution if its pdf and cdf are of the following form

$$f\left(x;\theta,\lambda\right)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1},\ 0\le x\le\lambda,\theta>0$$

$$F\left(x;\theta,\lambda\right)=\lambda^{-\theta}x^{\theta},\ 0\le x\le\lambda,\theta>0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y=\frac{1}{X}$ has a Pareto distribution.

VII. Pareto distribution

A random variable X is said to have the Pareto distribution if its probabihty density function $pdf\ f\left(x\right)$ and distribution function $cdf\ F\left(x\right)$ are of the form given below:

$$f\left(x\right)=\theta\lambda^{\theta}x^{-(\theta+1)},\ 0\le x\le\infty,\ \lambda,\theta>0$$

$$F\left(x\right)=1-\lambda^{\theta}x^{-\theta},\ 0\le x\le\infty,\ \lambda,\theta>0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

VIII. Kumaraswamy Power Function distribution

A random variable X is said to be Kumaraswamy-Power function distribution (Abdul-Moniem, 2017) with the following pdf

$$f\left(x\right)=\frac{ab\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1},\quad a,b,\theta>0,0\le x\le\lambda$$

and its cdf is given by

$$F\left(x\right)=1-\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b},\quad a,b,\theta>0,0\le x\le\lambda$$

the corresponding survival function $\left(SF\right)$ are

$$\overline{F}\left(x\right)=\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b}, \quad a,b,\theta>0, 0\le x\le\lambda$$

From pdf and $\overline{F}\left(x\right)$ we get,

$$\overline{F}\left(x\right)=\frac{1}{ab\theta}\left[\lambda^{a\theta}x^{1-a\theta}-x\right]f\left(x\right)$$

IX. Burr distribution

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below (Johnson et al, 1994):

$$F\left(x\right)=x, \ 0<x<1$$

$$F\left(x\right)=\left(1+e^{-x}\right)^{-k}, \ -\infty<x<\infty$$

$$F\left(x\right)=\left(1+x^{-c}\right)^{-k}, \ 0\le x<\infty$$

$$F\left(x\right)=\left[1+\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{-k}, \ 0\le x\le c$$

$$F\left(x\right)=\left(1+ce^{-\tan x}\right)^{-k}, \ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$$

$$F\left(x\right)=\left[1+ce^{-k\sinh x}\right]^{-k}, \ -\infty<x<\infty$$

$$F\left(x\right)=2^{-k}\left(1+\tanh x\right)^{k} \quad -\infty<x<\infty$$

$$F\left(x\right)=\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)^{k}, \ -\infty<x<\infty$$

$$F(x)=1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2},\ \ -\infty<x<\infty$$

$$F(x)=\left(1-e^{-x^{2}}\right)^{k},\ \ 0\le x<\infty$$

$$F(x)=\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k},\ \ 0\le x\le 1$$

$$F(x)=1-\left(1+x^{c}\right)^{-k},\ \ 0\le x<\infty$$

where k and c are positive parameters.

Special attention is given to type XII, whose *pdf* is given as

$$f(x)=kcx^{c-1}\left(1+x^{c}\right)^{-(k+1)},\ \ 0\le x<\infty,\ \ k,\ c>0$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c=1,$ it is called Lomax distribution whereas at $k=1,$ it is known as Log-logistic distribution.

X. Power Lomax distribution

It is three parameters continuous probability distribution used for reduction times of bladder cancer data. A random variable is said to have the Power Lomax distribution Rady, *et al.* (2016) if its *pdf* is of the form

$$f(x)=\alpha\beta\lambda^{-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)},\ \alpha,\beta,\lambda>0,\ \ \mathrm{x}\ge 0$$

and the *df* is given by

$$F(x)=1+\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)},\alpha,\beta,\lambda>0,\ \ \mathrm{x}\ge 0$$

XI. Sushila distribution

A rv X is said to have a Sushila distribution (Shanker *et al.* (2013)) if its df as given by

$$F(x)=1-\frac{\lambda(\sigma+1)+\sigma x}{\lambda(\sigma+1)}e^{-\frac{\sigma}{\lambda}x};\ x>0,\ \sigma>0,\ \lambda>0$$

And the corresponding pdf is given by

$$f(x)=\frac{\sigma^2}{\lambda(1+\sigma)}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma}{\lambda}x};\ x>0,\ \sigma>0,\ \lambda>0$$

The pdf of Sushila distribution was introduced by Shanker *et al.* (2013). At $\lambda=1$, it reduces to Lindley distribution (Lindley, 1958) having pdf as given by

$$f(x)=\frac{\sigma^2}{1+\sigma}(1+x)e^{-\sigma x};\ x>0,\ \sigma>0$$

(Ghitany *et al.*, 2008) have explored some interesting properties of this distribution and showed that Lindley distribution gives better lifetime model than the exponential distribution in applications. Sankaran (1970) introduced the discrete Poisson-Lindley distribution after mixing Poisson and Lindley distribution. Zakarzadeh and Dolati (2009) introduced the generalization of Lindley distribution having three parameters.

It is observed that Lindley distribution is a particular case of Sushila distribution. The pdf of Sushila distribution can be shown as a mixture of exponential $\left(\frac{\sigma}{\lambda}\right)$ and gamma $\left(2,\frac{\sigma}{\lambda}\right)$ distribution as follows:

The relation between pdf and survival function $\overline{F}(x)$ we have

$$\left(\lambda\left(1+\sigma\right)+\sigma x\right) fx=\sigma^{2}\left(1+\frac{x}{\lambda}\right)\left(1-F\left(x\right)\right)$$

XII. Lindley distribution

A rv X is said to have Lindley distribution (Lindley, 1958) if its pdf is given by

$$f\left(x\right)=\frac{\theta^{2}}{\left(1+\theta\right)}\left(1+x\right)e^{-\theta x};\ x>0,\ \theta>0$$

Corresponding cdf is given by

$$F\left(x\right)=1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x};\ \ x>0,\ \theta>0$$

Lindley (1958) proposed Lindley distribution in the context of Bayesian statistics, as a counter example of fudicial statistics. Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distribution with shape parameter 2. Therefore, many properties of the mixture distribution can be translated for the Lindley distribution.

1.11. Thesis Plan

The research work presented in this thesis have divided into seven Chapters.

Chapters 1, is introductory nature. In this we have discussed various concepts of ordered random variables such as order statistics, record values generalized order statistics and their sub modes.

Chapter 2, is based on $k-th$ upper record values. We have established the recurrence relations for single and product moments from Muth distribution. In last Section, of this Chapter a characterization result is obtained from single moment of $k-th$ upper record values from Muth distribution.

Chapter 3, the recurrence relations of generalized order statistics have been obtained for Kumaraswamy power function distribution. Characterization of Kumaraswamy power function distribution also obtained which is based on recurrence relation of single moments of generalized order statistics.

Chapter 4, we have developed recurrence relations for single as well as product moments from generalized order statistics for Power Lomax distribution. Characterization results are also established for Power Lomax distribution from recurrence relation of single moments for generalized order statistics.

In Chapter 5, we have developed recurrence relations for single and product moments from generalized order statistics for Sushila distribution. Their special case are also discussed for order statistics and record values. Characterization results are also established for Sushila distribution from recurrence relation of single moments for generalized order statistics.

In Chapter 6, the problem of Stress-Strength reliability model is studied for Erlang-truncated exponential distributed stress. We have studied the problem by establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows an Erlang-truncated exponential distribution and Strength follows a Power function distribution. Further, these results are explained with an example and are utilized to get optimum cost of any item when the cost function is linear in terms of parameters

In this Chapter 7, the problem of Stress-Strength reliability model is studied for Sushila distributed stress. We have studied the problem by establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows a Sushila distribution and Strength follows a Power function distribution

Chapter 2

Recurrence Relation for Single and Product Moments of K-th Record Values from Muth Distribution and its Characterization

CHAPTER 2

RECURRENCE RELATION FOR SINGLE AND PRODUCT MOMENTS OF *K*-th RECORD VALUES FROM MUTH DISTRIBUTION AND ITS CHARACTERIZATION

2.1. Introduction

Chandler (1952) introduced the record values and discussed statistical properties of record values. He structured the theory of record values as a model for successive extremes in a sequence of iid random variables. Later, Feller (1966) discussed application of record values in gambling problems. Record values are useful in many real life situations such as sports and athletic events, oil and mining surveys, climatology, medicine, traffic, industrial stress testing, bioscience and among others. Many authors have discussed recurrence relations and characterization for $k-th$ upper record values. Here we have mentioned some remarkable contributions in the theory of records such as Dziubdziela and Kopociński (1976), Gaver (1976), Glick (1978), Grudzień (1982), Deheuvels (1984b), Galambos (1987), Resnick (1987), Nevzorov (1987), Nagaraja (1988), Arnold *et al.* (1992), Ahsanullah (1995), Balakrishnan and Ahsanullah (1995), Grudzien, and Szynal, (1985), Grudzień and Szynal (1997), Pawlas and Szynal (1998), Pawlas and Szynal (1999), Pawlas and Szynal (2000), Raqab and Ahsanullah (2000), Bieniek and Szynal (2002), Bieniek and Szynal (2007), Khan and Zia (2009), Khan *et al.* (2010), Khan *et al.* (2015), Ahsanullah and Nevzorov (2015), Khan *et al.* (2016), Khan and Khan (2016a, 2016b). This Chapter comprises of four sections. In Section 2.2, we have established the recurrence relation based on single moment of $k-th$ upper record values from Muth distribution. In Section 2.3, we have derived the recurrence relation by using product moment of $k-th$ upper record values from Muth distribution. In Section 2.4 we have characterized Muth distribution based on recurrence relation of single moment of $k-th$ upper record values.

Let $\{X_n; n \geq 1\}$ be a sequence of iid random variable with *cdf* $F(x)$ and the *pdf* $f(x)$. Let $X_1, X_2, ..., X_n$ be a sample of size n from any population. For any $k \in \mathbb{N}$, we define the sequence $\{U_n^{(k)}; n \geq 1\}$ of $k-th$ upper record times of $\{X_n; n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\left\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_{n+k-1}^{(k)}} \right\}$$

For $k=1$ and $n=1,2,\ldots$ we can write $U_n^{(1)} = U_n$. $\{U_n; n \geq 1\}$ is the sequence of record times of $\{X_n; n \geq 1\}$.

The *pdf* of $k-th$ upper record value $X_n^{(k)}; n \geq 1$ is given by Dziubdziela and Kopocinski (1976) as follows

$$f_{X_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1} f(x). \tag{2.1.1}$$

The joint *pdf* of $X_m^{(k)}$ and $X_n^{(k)}; 1 \leq m < n,\ n \geq 2$ is given by Grudzien (1982) as follows

$$f_{X_m^{(k)}, X_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}\left[-\ln \bar{F}(x)\right]^{m-1} \frac{f(x)}{\bar{F}(x)}$$

$$\times\left[\ln \bar{F}(x) - \ln \bar{F}(y)\right]^{n-m-1}\left[\bar{F}(y)\right]^{k-1} f(y); \qquad -\infty < x < y < \infty \tag{2.1.2}$$

Where $\bar{F}(x) = 1 - F(x)$ denotes the survival function.

A random variable X is said to have a Muth distribution with shape parameter α if its *pdf* is given by

$$f(x;\alpha) = \left(e^{\alpha x} - \alpha\right)\exp\left\{\alpha x - \frac{1}{\alpha}\left(e^{\alpha x} - 1\right)\right\}; \qquad x > 0, \alpha \in (0,1] \tag{2.1.3}$$

and corresponding *cdf* is given by

$$F(x;\,\alpha)=1-\exp\left[\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right],\qquad x>0,\alpha\in(0,1] \tag{2.1.4}$$

Therefore from (2.1.3) and (2.1.4), it is evident that the relation between pdf and cdf is given as

$$f(x;\,\alpha)=\left(e^{\alpha x}-\alpha\right)\left(1-F(x)\right) \tag{2.1.5}$$

The relation in (2.1.5) will be utilized to derive the recurrence relation for moments of $k-th$ upper record from Muth distribution. Muth distribution was introduced by Muth (1977) as an alternative reliability model. He also discussed its application in reliability theory. For next three decades, this distribution was unnoticed. Leemis and MacQueston (2008) summarized the interrelationship among several important univariate continuous distributions. Leemis and MacQueston (2008) has shown that Muth distribution reduces to standard exponential distribution if $\alpha\to 0$. Recently, Jodrá *et al.* (2015) derived the various statistical properties of Muth distribution. Jodrá *et al.* (2015) also obtained the maximum likelihood estimator, least square estimator, weighted least square estimator and moment estimator for shape parameter α and compared the efficiency of these estimator by simulation study.

2.2. Relations for single moment from Muth distribution

In this Section, we have derived the recurrence relation for single moment of $k-th$ upper record values from Muth distribution.

Theorem 2.1. For a positive integer $k\geq 1$, $n\geq 1$ and $r=0,1,2,...$

$$\mu_{n:k}^{r}=\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\frac{k}{r+p+1}\left[\mu_{n:k}^{r+p+1}-\mu_{n-1:k}^{r+p+1}\right]-\frac{\alpha k}{r+1}\left[\mu_{n:k}^{r+1}-\mu_{n-1:k}^{r+1}\right] \tag{2.2.1}$$

Proof. For $n\geq 1$ and $r=0,1,2,\dots$ we have from (2.1.1) and (2.1.5)

$$\mu_{n:k}^{r} = E\left[\left(X_{U(n)}^{(k)}\right)\right]^{r}$$

$$= \int_{0}^{\infty} x^{r} \frac{k^{n}}{(n-1)!}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1} f(x)\,dx$$

Now using (2.1.5), we have,

$$= \frac{k^{n}}{(n-1)!}\int_{0}^{\infty} x^{r}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1}\left(e^{\alpha x}-\alpha\right)\bar{F}(x)\,dx$$

$$= \frac{k^{n}}{(n-1)!}\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\int_{0}^{\infty} x^{r+p}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k} dx$$

$$-\frac{\alpha k^{n}}{(n-1)!}\int_{0}^{\infty} x^{r}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k} dx$$

Integrating the above equation by parts considering x^{r} for integration and rest part for differentiation, we have

$$\mu_{n:k}^{r} = \left[-\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\frac{k^{n}}{(n-1)!}\int_{0}^{\infty} k\frac{x^{r+p+1}}{r+p+1}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1}\left(-f(x)\right)dx\right.$$

$$\left.-\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\frac{k^{n}}{(n-1)!}\frac{n-1}{r+p+1}\int_{0}^{\infty} x^{r+p+1}\left[-\ln \bar{F}(x)\right]^{n-2}\left(\bar{F}(x)\right)^{k-1} f(x)\,dx\right]$$

$$-\left[\frac{\alpha k^{n}k}{(n-1)!}\int_{0}^{\infty}\frac{x^{r+1}}{r+1}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1}\left(-f(x)\right)dx\right.$$

$$\left.+\alpha\frac{k^{n}}{(n-1)!}\frac{n-1}{r+1}\int_{0}^{\infty} x^{r+1}\left[-\ln \bar{F}(x)\right]^{n-2}\frac{-f(x)}{-\bar{F}(x)}\left(\bar{F}(x)\right)^{k} dx\right]$$

$$= \sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\frac{k}{r+p+1}\left[\frac{k^{n}}{(n-1)!}\int_{0}^{\infty} x^{r+p+1}\left[-\ln \bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1} f(x)\,dx\right.$$

$$\left.-\frac{k^{n-1}}{(n-2)!}\int_{0}^{\infty} x^{r+p+1}\left[-\ln \bar{F}(x)\right]^{n-2}\left(\bar{F}(x)\right)^{k-1} f(x)\,dx\right]$$

$$-\frac{\alpha k}{r+1}\left[\frac{k^{n}}{(n-1)!}\int_{0}^{\infty}x^{r+1}\left[-\ln\bar{F}(x)\right]^{n-1}\left(\bar{F}(x)\right)^{k-1}f(x)dx\right.$$

$$\left.-\frac{k^{n-1}}{(n-2)!}\int_{0}^{\infty}x^{r+1}\left[-\ln\bar{F}(x)\right]^{n-2}\left(\bar{F}(x)\right)^{k-1}f(x)dx\right]$$

On simplifying the resulting expression, we derive the relation given in (2.2.1).

2.3. Relations for product moment from Muth distribution

In this Section, we have derived the expression for recurrence relation for product moment based on $k-th$ upper record values from Muth distribution.

Theorem 2.2. For $1\le m\le n-1$ and $r,s=0,1,2,\ldots$

$$\mu_{m,n:k}^{r,s}=\frac{k}{r+s+1}\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\left[\mu_{m,n:k}^{r,s+p+1}-\mu_{m,n-1:k}^{r,s+p+1}\right]-\frac{k\alpha}{s+1}\left[\mu_{m,n:k}^{r,s+1}-\mu_{m,n-1:k}^{r,s+1}\right] \tag{2.3.1}$$

and for $m\ge 1$ and $r,s=0,1,2,...$

$$\mu_{m,\mathrm{m}+1:k}^{r,s}=\frac{k}{r+s+1}\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\left[\mu_{m,\mathrm{m}+1:k}^{r,s+p+1}-\mu_{m:k}^{r,s+p+1}\right]-\frac{k\alpha}{s+1}\left[\mu_{m,n:k}^{r,s+1}-\mu_{m:k}^{r,s+1}\right] \tag{2.3.2}$$

Proof: For $1\le m\le n-1$ and $r,s=0,1,2,...$ we have from (2.1.2) and (2.1.5)

$$\mu_{m,n:k}^{r,s}=E\left[\left(X_{m}^{(k)}\right)^{r}\left(X_{n}^{(k)}\right)^{s}\right]$$

$$=\frac{k^{n}}{(m-1)!(n-m-1)!}\int_{0}^{\infty}x^{r}\left[-\ln\bar{F}(x)\right]^{m-1}\frac{f(x)}{\bar{F}(x)}I(x)dx \tag{2.3.3}$$

where

$$I(x)=\int_{x}^{\infty}y^{s}\left[\ln\bar{F}(x)-\ln\bar{F}(y)\right]^{n-m-1}\left[\bar{F}(y)\right]^{k-1}f(y)dy$$

On using (2.1.5), we get

$$I(x)=\int_x^\infty y^s\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}\left(e^{-\alpha y}-\alpha\right)\left[\overline{F}(y)\right]dy$$

Now, integrating the above integral by parts, we have

$$I_1=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{1}{s+p+1}\int_0^\lambda\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k}y^{s+p+1}\Big|_x^\infty$$

$$-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{1}{s+p+1}\int_x^\infty y^{s+p+1}\left\{\frac{d}{dy}\left(\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k}\right)\right\}dy$$

On simplification, we get

$$I_1=-\frac{n-m-1}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_x^\infty y^{s+p+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

$$+\frac{k}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_x^\infty y^{s+p+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

Similarly, we have, I_2

$$I_2=-\frac{\alpha(n-m-1)}{s+1}\int_x^\infty y^{s+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

$$+-\frac{k\alpha}{s+1}\int_x^\infty y^{s+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

Thus

$$I(x)=I_1+I_2$$

Hence, $I(x)$ will be

$$I(x)=-\frac{n-m-1}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_x^\infty y^{s+p+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

$$+\frac{k}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\int_{x}^{\infty}y^{s+p+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)\,dy$$

$$-\frac{\alpha(n-m-1)}{s+1}\int_{x}^{\infty}y^{s+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y)\,dy$$

$$+-\frac{k\alpha}{s+1}\int_{x}^{\infty}y^{s+1}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)\,dy$$

Substituting $I(x)$ in (2.3.3) and simplifying, we get the relation given in (2.3.1).

2.4. Characterization of Muth Distribution

In this Section, utilizing the result obtained in Section 2.2, we have characterized Muth distribution by using single moment of $k-th$ upper record values. In deriving the results, we try to find the distribution of random variables if certain statistical condition is fulfilled by random variables.

Theorem 2.3. For a positive integer $k\geq 1$ and let r be a non-negative integer, the necessary and sufficient condition for a random variable X to follow Muth distribution having *cdf* given in (2.1.4) is that

$$\mu_{n:k}^{r}=\sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\mu_{n:k}^{r+p+1}-\sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\mu_{n-1:k}^{r+p+1}-\frac{\alpha k}{r+1}\mu_{n:k}^{r+1}+\frac{\alpha k}{r+1}\mu_{n-1:k}^{r+1} \tag{2.4.1}$$

Proof. The necessary part follows immediately form (2.2.1), on the other hand, if the recurrence relation (2.4.1) is satisfied then on rearranging the terms in (2.4.1) and using (2.1.1), we have

$$\frac{k^{n}}{(n-1)!}\int_{0}^{\infty}x^{r}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}f(x)\,dx$$

$$=\sum_{p=0}^{\infty}\frac{\alpha^{p}}{p!}\frac{k}{r+p+1}\frac{k^{n}}{(n-1)!}\int_{0}^{\infty}x^{r+p+1}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}f(x)\,dx$$

$$-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k}{r+p+1}\frac{k^{n-1}}{(n-2)!}\int_0^{\infty}x^{r+p+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1}f(x)dx$$

$$-\frac{\alpha k}{r+1}\frac{k^n}{(n-1)!}\int_0^{\infty}x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}f(x)dx$$

$$+\frac{\alpha k}{r+1}\frac{k^{n-1}}{(n-2)!}\int_0^{\infty}x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1}f(x)dx \qquad (2.4.2)$$

Integrating the first and third integral by parts of R.H.S. of (2.4.2), we get

First integral become

$$=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k}{r+p+1}\frac{k^{n-1}}{(n-2)!}\int_0^{\infty}x^{r+p+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1}f(x)dx$$

$$+\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k^n}{(n-2)!}\int_0^{\infty}x^{r+p}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx$$

and third integral become

$$=-\frac{\alpha k}{r+1}\frac{k^{n-1}}{(n-2)!}\int_0^{\infty}x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1}f(x)dx$$

$$-\alpha\frac{k^n}{(n-1)!}\int_0^{\infty}x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx$$

Substituting the values of first and third integral into (2.4.2), we get

$$\frac{k^n}{(n-1)!}\int_0^{\infty}x^{r}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}f(x)dx$$

$$=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}k\frac{k^{n-1}}{(n-1)!}\int_0^{\infty}x^{r+p}\left[-\ln\left(1-F(x)\right)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx$$

$$-\alpha k\frac{k^{n-1}}{(n-1)!}\int_0^{\infty}x^{r}\left[-\ln\left(1-F(x)\right)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx$$

$$=\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}f(x)\,dx$$

$$-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k^n}{(n-1)!}\int_0^\infty x^{r+p}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx$$

$$+\alpha\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k}dx=0$$

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}\left[f(x)-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}x^p\overline{F}(x)+\alpha\overline{F}(x)\right]dx=0 \qquad (2.4.3)$$

Applying Müntz-Szász Theorem (Hwang and Lin, 1984), to (2.4.3), we get

$$f(x)-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}x^p\overline{F}(x)+\alpha\overline{F}(x)=0$$

or,

$$f(x)=\left(e^{\alpha x}-\alpha\right)\left(1-F(x)\right)$$

Thus, we get the relation given in (2.1.5).

Theorem 2.4. For r be a non-negative integer, the necessary and sufficient condition for a random variable X to be distributed with pdf given in (2.1.5) is that

$$\mu_{1:k}^{r}=\sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\mu_{1:k}^{r+p+1}-\frac{\alpha k}{r+1}\mu_{1:k}^{r+1} \qquad (2.4.4)$$

Proof. The necessary part follows from (2.1.5) on the other hand if (2.4.4) is satisfied then

$$\int_0^\infty x^r\left(1-F(x)\right)^{k-1}f(x)\,dx$$

$$=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k}{r+p+1}\int_0^\infty x^{r+p+1}\left(1-F(x)\right)^{k-1}f(x)\,dx$$

$$-\frac{\alpha k}{r+1}\int_0^\infty x^{r+1}\left(1-F\left(x\right)\right)^{k-1} f\left(x\right)dx$$

Integrating the R.H.S. of above integral by parts, we get

$$\int_0^\infty x^r\left(1-F\left(x\right)\right)^{k-1} f\left(x\right)dx=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_0^\infty x^{r+p}\left(1-F\left(x\right)\right)^k dx$$

$$-\alpha\int_0^\infty x^r\left(1-F\left(x\right)\right)^k f\left(x\right)dx$$

which reduces to,

$$\int_0^\infty x^r\left(1-F\left(x\right)\right)^{k-1}\left[f\left(x\right)-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}x^p\overline{F}\left(x\right)+\alpha\overline{F}\left(x\right)\right]dx=0 \qquad (2.4.5)$$

Now on generalization of Müntz-Szász Theorem (Hwang and Lin (1984)) to (2.4.5), we get,

$$f\left(x\right)=\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}x^p\left(1-F\left(x\right)\right)-\alpha\left(1-F\left(x\right)\right)$$

$$f\left(x\right)=\left(e^{\alpha x}-\alpha\right)\left(1-F\left(x\right)\right)$$

which shows that

$$F\left(x;\alpha\right)=1-\exp\left[\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right], \qquad x>0,\alpha\in\left(0,1\right]$$

2.5. Conclusion

This chapter demonstrate the recurrence relations between single and product moments of $k-th$ upper record values established in Section 2.2 and Section 2.3 respectively from Muth distribution. In Section 2.4 we have used recurrence relation of single moments to characterize the Muth distribution. Since recurrence relations reduce the amount of direct computation and hence reduce the time and labour. Therefore the relations established in the chapter useful in computing the moments of higher order from Muth distribution given through the recurrence relation for single moments of $k-th$ upper record values.

Chapter 3

Relations for Moments of Kumaraswamy Power Function Distribution Based on Generalized Order Statistics and a Characterization

CHAPTER 3

RELATIONS FOR MOMENTS OF KUMARASWAMY POWER FUNCTION DISTRIBUTION BASED ON GENERALIZED ORDER STATISTICS AND A CHARACTERIZATION

3.1. Introduction

Kamps (1995) introduced the concept of generalized order statistics as a unified theoretical set-up for ordered random variables which contain different varieties of ordered random variables. These models are ordinary order statistics, sequential order statistics, order statistics with non-integral sample size, progressive type II censored order statistics, record values and Pfeifer records. The use of such unifying concept has effectively applicable in reliability theory, life testing models, medicals and software analysis. The use of such unifying concepts has been steadily growing over the year. Several authors utilized the concept of generalized order statistics into their works due its structural similarities and common approaches. Various developments on generalized order statistics and related theories are available in the literature such as Kamps and Gather (1997), Keseling (1999), Ahsanullah (2000), Ahmad and Fawzy (2003), Al-Hussaini *et al.* (2005), Ahmad (2007), Khan *et al.* (2007), Kumar and Khan (2013), Khan *et al.* (2015a, 2015b), Khan and Khan (2016). This chapter organized into the four sections. In Section 3.2, we have deduced the recurrence relation for single moments from Kumaraswamy Power function distribution. In Section 3.3, we have deduced recurrence relations for product moments from Kumaraswamy power function distribution. In Section 3.4, we have characterized the Kumaraswamy power function distribution utilizing the recurrence relation of single moment obtaining from generalized order statistics. Also, we have discussed the particular cases for order statistics and $k-th$ upper record values.

The *pdf* of r^{th} *gos* is given by Kamps (1995)

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)\ ,\ -\infty\le x\le\infty \tag{3.1.1}$$

And the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(\mathrm{s},n,\tilde{m},k)$, $1\le r<s\le n$ is given by

$$f_{X(r,n,m,k),X(\mathrm{s},n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\overline{F}(x)\right]^{m} g_m^{r-1}\left[F(x)\right]$$

$$\times\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(x)\,f(y),-\infty\le x\le\infty \tag{3.1.2}$$

A random variable X is said to be Kumaraswamy-Power function distribution (Abdul Moniem, (2017)) with the following *pdf*

$$f(x)=\frac{ab\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1},\quad a,b,\theta>0, 0\le x\le\lambda \tag{3.1.3}$$

and its *cdf* is given by

$$F(x)=1-\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b},\quad a,b,\theta>0, 0\le x\le\lambda \tag{3.1.4}$$

the corresponding survival function (SF) are

$$\overline{F}(x)=\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b},\quad a,b,\theta>0, 0\le x\le\lambda \tag{3.1.5}$$

In view of (3.1.3) and (3.1.5), we get

$$\overline{F}(x)=\frac{1}{ab\theta}\left[\lambda^{a\theta}x^{1-a\theta}-x\right]f(x) \tag{3.1.6}$$

Relation (3.1.6) is used to obtain the recurrence relations for moments of generalized order statistics.

Power function distribution has large use in reliability theory due its flexibility. Meniconi and Barry (1996) shows that PFD is the best distribution for testing the reliability of electrical components. For more details, see Ahsanullah and Lutful-Kabir (1974), Ali *et al.* (2005), Sinha *et al.* (2008) and Chang (2007).

The Kumaraswamy Power function distribution was introduced by Abdul-Moniem (2017) which is the generalization of Kumaraswamy distribution. The Kumaraswamy power function distribution reduces to Kumaraswamy distribution for $\theta = 1$ and $\lambda = 1$. This two parameter distribution was introduced by Kumaraswamy (1980) on $(0,1)$. Its pdf has the following form

$$f(x) = abx^{a-1}\left(1-x^{a}\right)^{b-1},\ x \in (0,1), a, b, > 0 \tag{3.1.7}$$

And the corresponding df

$$F(x) = 1-\left(1-x^{a}\right)^{b},\ x \in (0,1), a, b, > 0 \tag{3.1.8}$$

It obvious that its pdf is unimodal, increasing, decreasing or constant for the specific choices of parameters a and b. Jones (2009) explored the background and genesis of this distribution. It is similar to beta distribution. This distribution was recognize for its use in hydrological problems and many natural phenomena. For additional information on this distribution and its application see Kumaraswamy (1980), Fletcher and Ponnambalam (1996), Sanchez *et al.* (2007), Seifi *et al.* (2000), Ganji *et al* (2006), and Courard-Hauri (2007). Several authors worked on Kumaraswamy distribution and its sub models on *gos* see Kumar (2011), Garg (2009), and Safi (2013).

3.2. Recurrence relations for single moment for Kumaraswamy power function distribution from *gos*

In this section the recurrence relation for single moments of *gos* from Kumaraswamy power function distribution has been deduced. Further, the recurrence relation for single moments of order statistics and record values are obtained as a special case of *gos*.

Theorem 3.2.1. Let X be a non-negative continuous random variable and follows Kumaraswamy power function distribution given in (3.1.3). Suppose that for any $j > 0$ and $1 \le r \le n$ and $E|X(r,n,m,k)|$ is finite, then

$$E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right]$$

$$= \frac{j\lambda^{a\theta}}{\gamma_r ab\theta} E\left[X^j(r,n,m,k)\right] - \frac{j}{\gamma_r ab\theta}\left[X^j(r,n,m,k)\right] \tag{3.2.1}$$

Proof. We have

$$E\left[X^j(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_0^\lambda x^j\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)\,dx$$

Integrating integral treating $\left[\overline{F}(x)\right]^{\gamma_r-1} f(x)$ as integrand and rest part for differentiation, we have

$$E\left[X^j(r,n,m,k)\right]$$

$$= \frac{C_{r-1}}{(r-1)!}\left[x^j g_m^{r-1}\left[F(x)\right]\frac{\left[\overline{F}(x)\right]^{\gamma_r}}{\gamma_r}\Bigg|_0^\lambda \right.$$

$$\left. -\int_0^\lambda \left\{ jx^{j-1} g_m^{r-1}\left[F(x)\right] + (r-1)\, g_m^{r-2}\left[F(x)\right] g_m'\left[F(x)\right] x^j f(x)\right\} \frac{-\left[\overline{F}(x)\right]^{\gamma_r}}{\gamma_r}\,dx \right]$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

$$+\frac{C_{r-1}(r-1)}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j}\left[\overline{F}(x)\right]^{\gamma_r}\left[\overline{F}(x)\right]^{m} g_m^{r-2}\left[F(x)\right]f(x)dx$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

$$+\frac{C_{r-1}(r-1)}{(r-1)\gamma_r(r-2)!}\int_0^{\lambda} x^{j}\left[\overline{F}(x)\right]^{\gamma_r+m} g_m^{r-2}\left[F(x)\right]f(x)dx$$

since $\frac{C_{r-1}}{\gamma_r}=C_{r-2}$, this implies that

$$E\left[X^j(r,n,m,k)\right]$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

$$+\frac{C_{r-1}}{\gamma_r(r-2)!}\int_0^{\lambda} x^{j}\left[\overline{F}(x)\right]^{\gamma_{r-1}-1} g_m^{r-2}\left[F(x)\right]f(x)dx$$

which reduces to

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

Now in view of (3.1.3), we get

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\lambda} x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]\left\{\frac{1}{ab\theta}\left[\lambda^{a\theta}x^{1-a\theta}-x\right]f(x)\right\}dx$$

$$=\frac{jC_{r-1}\lambda^{a\theta}}{\gamma_r(r-1)!ab\theta}\int_0^{\lambda}x^{j-a\theta}\left[\overline{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx$$

$$-\frac{jC_{r-1}}{\gamma_r(r-1)!ab\theta}\int_0^{\lambda}x^{j}\left[\overline{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx$$

$$=\frac{jC_{r-1}\lambda^{a\theta}}{\gamma_r(r-1)!ab\theta}E\left[X^{j-a\theta}(r-1,n.m.k)\right]-\frac{j}{\gamma_r ab\theta}E\left[X^{j}(r,n,m,k)\right]$$

or

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]$$

$$=\frac{j\lambda^{a\theta}}{\gamma_r ab\theta}E\left[X^{j-a\theta}(r,n,m,k)\right]-\frac{j}{\gamma_r ab\theta}E\left[X^{j}(r,n,m,k)\right]$$

Remark 3.1. Setting $m=0$ and $k=1$ in (3.2.1), we get the recurrence relation for single moments of order statistics of Kumaraswamy power function distribution as

$$E\left[X_{r:n}^{j}\right]-E\left[X_{r-1:n}^{j}\right]=\frac{j\lambda^{a\theta}}{ab\theta(n-r+1)}E\left[X_{r:n}^{j}\right]-\frac{j}{ab\theta(n-r+1)}\left[X_{r:n}^{j}\right]$$

Remark 3.2. Setting $m=-1$ and $k\geq 1$ in (3.2.1), we get the recurrence relation for single moments of upper $k-th$ record of Kumaraswamy power function distribution as

$$E\left[X^{j}(r,n,-1,k)\right]-E\left[X^{j}(r-1,n,-1,k)\right]$$

$$=\frac{j\lambda^{a\theta}}{ab\theta k}E\left[X^{j-a\theta}(r,n,-1,k)\right]-\frac{j}{ab\theta k}E\left[X^{j}(r,n,-1,k)\right]$$

Remark 3.3. Setting $\lambda=1$ and $\theta=1$ in (3.2.1) we get recurrence relation for single moments of *gos* from Kumaraswamy distribution.

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]$$

$$=\frac{j}{\gamma_r ab}E\left[X^{j-a}(r,n,m,k)\right]-\frac{j}{\gamma_r ab}\left[X^{j}(r,n,m,k)\right] \qquad \text{(Kumar, 2011)}$$

3.3. Product Moment of Kumaraswamy power function distribution from *gos*

In this Section, the recurrence relation for product moments of *gos* from Kumaraswamy power function distribution has been deduced. Further, the recurrence relation for single moments of order statistics and record values are obtained as a special case of *gos*.

Theorem 3.3.1. Let X(>0) be a continuous random variable which follows Kumaraswamy power function distribution given in (3.1.3). We assume that $\mathrm{E(X(r,n,m,k)X(s,n,m,k))}$ exists for any $i, j > 0$ and $1 \le \mathrm{r} < \mathrm{s} \le \mathrm{n}$, then the recurrence relation for the product moment is given by

$$E\left[X^i(r,n,m,k)\,X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)\,X^{j-1}(\mathrm{s}-1,n,m,k)\right]$$

$$=\frac{j\lambda^{a\theta}}{\gamma_s ab\theta}E\left[X^i(r,n,m,k)\,X^{j-a\theta}(\mathrm{s},n,m,k)\right]-\frac{j}{\gamma_s ab\theta}E\left[X^i(r,n,m,k)\,X^j(\mathrm{s},n,m,k)\right] \tag{3.3.1}$$

Proof. We have

$$E\left[X^i(r,n,m,k)\,X^j(r,n,m,k)\right]$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\lambda} x^j\left[\overline{F}(x)\right]^m f(x)\,g_m^{r-1}\left[F(x)\right]I(x)\,dx \tag{3.3.2}$$

where

$$I(x)=\int_x^{\lambda} y^j\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\,dy \tag{3.3.3}$$

On integrating (3.3.3) by parts, we get

$$I(x)=y^j\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\int_x^{\lambda}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y)\,dy$$

$$-\int_x^{\lambda}\left[\left\{\frac{d}{dy}y^j\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\right\}\int\left(\overline{F}(y)\right)^{\gamma_s-1} f(y)\,dy\right]dy$$

$$= y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left(\frac{-\left[\overline{F}(y)\right]^{\gamma_s}}{\gamma_s}\right)\Bigg|_x^{\lambda}$$

$$-\int_x^{\lambda}\left\{\frac{d}{dy}y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\right\}\left(\frac{-\left[\overline{F}(y)\right]^{\gamma_s}}{\gamma_s}\right)dy$$

$$=\int_x^{\lambda}y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\frac{\left(\overline{F}(y)\right)^{\gamma_s}}{\gamma_s}dy$$

$$=\frac{1}{\gamma_s}\int_0^{\lambda}\left[\overline{F}(x)\right]^{\gamma_s}\Big\{jy^{j-1}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}$$

$$+(s-r-1)y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}h_m^{'}\left[F(y)\right]f(y)\Big\}$$

$$=\frac{j}{\gamma_s}\int_x^{\lambda}y^{j-1}\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dy$$

$$+\frac{(s-r-1)}{\gamma_s}\int_x^{\lambda}y^j\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}\left[\overline{F}(x)\right]^m f(y)dy$$

$$=\frac{j}{\gamma_s}\int_x^{\lambda}y^{j-1}\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dy$$

$$+\frac{(s-r-1)}{\gamma_s}\int_x^{\lambda}y^j\left(\overline{F}(y)\right)^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}f(y)dy$$

Putting the value of $I(x)$ in (3.3.2), we get

$$E\left[X^i(r,n,m,k)X^j(r,n,m,k)\right]$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\lambda}x^j\left(\overline{F}(x)\right)^m f(x)g_m^{r-1}\left(F(x)\right)$$

$$\times\left\{\frac{j}{\gamma_s}\int_x^{\lambda} y^{j-1}\left(\overline{F}(y)\right)^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(y)\right)\right]^{s-r-1}dy\right.$$

$$\left.+\frac{s-r-1}{\gamma_s}\int_x^{\lambda} y^{j}\left(\overline{F}(y)\right)^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(y)\right)\right]^{s-r-2}f(y)dy\right\}dx$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\lambda}\int_x^{\lambda} x^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\lambda}\int_0^{\lambda} x^i\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]y^j\left[\overline{F}(y)\right]^{\gamma_s+m}$$

$$\times\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}f(y)dydx$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\lambda}\int_x^{\lambda} x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)(s-r-2)!}\int_0^{\lambda}\int_x^{\lambda} x^i y^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}f(y)dydx$$

$$E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(\mathrm{s}-1,n,m,k)\right]$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\lambda}\int_x^{\lambda} x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)(s-r-2)!}\int_0^{\lambda}\int_x^{\lambda} x^i y^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2} f(y)dydx$$

$$E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(\mathrm{s}-1,n,m,k)\right]$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\lambda}\int_x^{\lambda}x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

After simplification it gives (3.3.1).

Remark 3.4. Setting $m=0$ and $k=1$ in (3.3.1), we obtain the recurrence relation for product moments of order statistics of Kumaraswamy power function distribution as

$$E\left[X_{r:n}^iX_{\mathrm{s}:n}^j\right]-E\left[X_{r:n}^iX_{\mathrm{s}-1:n}^{j-1}\right]$$

$$=\frac{j\lambda^{a\theta}}{ab\theta(n-s+1)}E\left[X_{r:n}^iX_{\mathrm{s}:n}^{j-a\theta}\right]-\frac{j}{ab\theta(n-s+1)}E\left[X_{r:n}^iX_{\mathrm{s}:n}^j\right]$$

Remark 3.5. Setting $m=-1$ and $k\geq1$ in (3.3.1), we obtain the recurrence relation for product moments of upper $k-th$ record of Kumaraswamy power function distribution as

$$E\left[X^i(r,n,-1,k)X^j(\mathrm{s},n,-1,k)\right]-E\left[X^i(r,n,-1,k)X^j(\mathrm{s}-1,n,-1,k)\right]$$

$$=\frac{j\lambda^{a\theta}}{ab\theta k}E\left[X^i(r,n,-1,k)X^{j-a\theta}(\mathrm{s},n,-1,k)\right]-\frac{j}{ab\theta k}E\left[X^i(r,n,-1,k)X^j(s,n,-1,k)\right]$$

Remark 3.6. Setting $\lambda=1$ and $\theta=1$ in (3.3.1) we get recurrence relation for product moments of *gos* from Kumaraswamy distribution.

$$E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(\mathrm{s}-1,n,m,k)\right]$$

$$=\frac{j}{\gamma_s ab}E\left[X^i(r,n,m,k)X^{j-a}(\mathrm{s},n,m,k)\right]-\frac{j}{\gamma_s ab}E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]$$

(Kumar, 2011)

3.4. Characterization of Kumaraswamy Power Function Distribution

In this Section, we have deduced the characterization result from Kumaraswamy power function distribution using the recurrence relation of single moments based on *gos*.

Theorem 3.4.1. The necessary and sufficient condition for a random variable X to be distributed with pdf given in (3.1.5), for $m \geq -1$ is that

$$E\left[X^{j}\left(r,n,m,k\right)\right]-E\left[X^{j}\left(r-1,n,m,k\right)\right]$$

$$=\frac{j\lambda^{a\theta}}{\gamma_{r}ab\theta}E\left[X^{j-a\theta}\left(r,n,m,k\right)\right]-\frac{j}{\gamma_{r}ab\theta}E\left[X^{j}\left(r,n,m,k\right)\right] \tag{3.4.1}$$

Proof. From (3.4.1), we have

$$j\lambda^{a\theta}\left[X^{j-a\theta}\left(r,n,m,k\right)\right]-j\left[X^{j-a\theta}\left(r-1,n,m,k\right)\right]$$

$$=\gamma_{r}ab\theta E\left[X^{j}\left(r,n,m,k\right)\right]-\gamma_{r}ab\theta E\left[X^{j}\left(r,n,m,k\right)\right]$$

$$\gamma_{r}ab\theta\left[\frac{c_{r-1}}{(r-1)!}\int_{0}^{\lambda}x^{j}\left[\overline{F}\left(x\right)\right]^{\gamma_{r}-1}g_{m}^{r-1}\left[F\left(x\right)\right]f\left(x\right)dx\right.$$

$$\left.-\frac{(r-1)c_{r-1}}{\gamma_{r}(r-1)!}\int_{0}^{\lambda}x^{j}\left[\overline{F}\left(x\right)\right]^{\gamma_{r}+m}g_{m}^{r-2}\left[F\left(x\right)\right]f\left(x\right)dx\right]$$

$$=\gamma_{r}ab\theta\frac{c_{r-1}}{(r-1)!}\int_{0}^{\lambda}x^{j}\left[\overline{F}\left(x\right)\right]^{\gamma_{r}}g_{m}^{r-2}\left[F\left(x\right)\right]f\left(x\right)\left[\frac{g_{m}\left[F\left(x\right)\right]}{\overline{F}\left(x\right)}-\frac{(r-1)\left[\overline{F}\left(x\right)\right]^{m}}{\gamma_{r}}\right]dx$$

Let

$$h\left(x\right)=-\frac{\left[\overline{F}\left(x\right)\right]^{\gamma_{r}}g_{m}^{r-1}\left[F\left(x\right)\right]}{\gamma_{r}}$$

Differentiating both sides with respect to x , we get

$$h'(x)=\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-2}\left[F(x)\right] f(x)\left[\frac{g_m\left[F(x)\right]}{\overline{F}(x)}-\frac{(r-1)\left[\overline{F}(x)\right]^{\gamma_r}}{\gamma_r}\right]$$

Now

$$j\lambda^{a\theta}E\left[X^{j-a\theta}(r,n,m,k)\right]-jE\left[X^{j-a\theta}(r-1,n,m,k)\right]=\gamma_r ab\theta\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^j h'(x)\,dx$$

Integrating by parts and using the value of $h(x)$, we get

$$j\lambda^{a\theta}E\left[X^{j-a\theta}(r,n,\mathrm{m},\mathrm{k})\right]-jE\left[X^{j}(r-1,n,m,k)\right]$$

$$=j\gamma_r ab\theta\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] f(x)\,dx$$

This implies that

$$\lambda^{a\theta}\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^{j-a\theta}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)\,dx$$

$$-\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^{j}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)\,dx$$

$$-\gamma_r ab\theta\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] f(x)\,dx$$

Which reduces to

$$\frac{c_{r-1}}{(r-1)!}\int_0^{\lambda}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)\left[\lambda^{a\theta}x^{1-a\theta}-x-ab\theta\frac{\overline{F}(x)}{f(x)}\right]dx=0$$

Now applying Müntz-Szász Theorem (Hwang and Lin 1984)

$$\lambda^{a\theta}x^{1-a\theta}-x-ab\theta\frac{\overline{F}(x)}{f(x)}=0$$

$$ab\theta\frac{\overline{F}(x)}{f(x)}=\lambda^{a\theta}x^{1-a\theta}-x$$

$$ab\theta\left(\overline{F}(x)\right)=\left[\lambda^{a\theta}x^{1-a\theta}-x\right]f(x)$$

$$\overline{F}(x)=\frac{1}{ab\theta}\left[\lambda^{a\theta}x^{1-a\theta}-x\right]f(x)$$

Which prove that $\overline{F}(x)$ has the form in (3.1.6).

3.5. Conclusions

In this Chapter, we have established the recurrence relations for single and product moments of generalized order statistics from Kumaraswamy power function distribution. Particular cases for recurrence relations of single and product moments are also discussed for order statistics and upper $k-th$ record values. We have also obtained characterization of the Kumaraswamy power function distribution using the recurrence relation for single moments.

Chapter 4

On Generalized Order Statistics from Power Lomax Distribution and its Characterization

CHAPTER 4

ON GENERALIZED ORDER STATISTICS FROM POWER LOMAX DISTRIBUTION AND ITS CHARACTERIZATION

4.1. Introduction

This Chapter established the recurrence relations for single and product moments of *gos* from the Power Lomax distribution. Results for order statistics and $k-th$ upper record values are also established. Further, characterization of Power Lomax distribution based on recurrence relation for single moments of *gos* is also explored.

The whole Chapter is divided into four Sections. In Section 4.2, we have obtained the recurrence relation for single moments of *gos* for Power Lomax distribution. In Section 4.3, we deduced the recurrence relation for product moments of *gos* for Power Lomax distribution. In Section 4.4, the characterization results based on the recurrence relation for single moments of gos for Power Lomax distribution are developed.

Recurrence relations are quite useful in computing the moments. The result given in present paper can be used to compute the moments of ordered random variables if the present population follows Power Lomax distribution. There is sufficient literature available regarding the recurrence relations of distributions based on *gos* and the characterization result based on these recurrence relations. For the review of literature, one may refer to Ahmad and Fawzy (2003), Al-Husainni *et.al.*(2005), Kumar and Khan (2013), Kumar (2015), Pawlas and Szynal (2001), Khan and Khan (2012) for the textbook reference, the readers are referred to Ahsanullah and Nevzorov (2001), Arnold *et al.*(1992).

A random variable X follows Power Lomax distribution Rady, *et al.* (2016) if its *pdf* is of the form

$$f(x)=\alpha\beta\lambda^{-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)},\ \alpha,\beta,\lambda>0,\ \text{x}\geq 0 \tag{4.1.1}$$

and the *df* is given by

$$F(x)=1+\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)},\alpha,\beta,\lambda>0,\ \text{x}\geq 0 \tag{4.1.2}$$

Now in view of (4.1.1) and (4.1.2),

$$\bar{F}(x)=\frac{1}{\alpha\beta}\left[x+\lambda x^{1-\beta}\right]f(x) \tag{4.1.3}$$

The relation (4.1.6) will be utilized to establish recurrence relation for moments of *gos.*

4.2. Recurrence relation for single moments of gos from Power Lomax distribution

In this Section, the recurrence relation for single moments of gos from Power Lomax distribution has been explored. Further, the recurrence relation for single moments of order *stastistics* and record values are obtained as a particular case of gos.

Theorem 4.2.1: Let X be a non negative continous random variable and follows Power Lomax distribution given in (4.1.4). Suppose that for any $j>0$ and $1\leq r\leq n$ and $E\left|X(r,n,m,k)\right|$ is finite, then

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]$$

$$=\frac{j}{\alpha\beta\gamma}E\left[X^{j}(r,n,m,k)\right]+\frac{\lambda j}{\alpha\beta\gamma}E\left[X^{j-\beta}(r,n,m,k)\right] \tag{4.2.1}$$

Proof. We have

$$E\left[X^{j}(r,n,m,k)\right]$$

$$=\frac{C_{r-1}}{(r-1)!}\int_{0}^{\infty}x^{j}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx \tag{4.2.2}$$

Integrating by parts taking $\left[\bar{F}\left(x\right)\right]^{\gamma_r-1} f\left(x\right)$ as the part to be integrated and rest part for differentiation

$$E\left[X^j(r,n,m,k)\right]=\frac{jC_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

$$+\frac{C_{r-2}}{(r-2)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1} g_m^{r-2}\left[F(x)\right]f(x)dx$$

which implies that

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx$$

Now in view of equation (4.1.3), we have

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]f(x)dx$$

$$+\frac{\lambda jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^\infty x^{j-\beta}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]f(x)dx$$

Thus we get

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{j}{\alpha\beta\gamma_r}E\left[X^j(r,n,m,k)\right]-\frac{\lambda j}{\alpha\beta\gamma_r}E\left[X^{j-\beta}(r,n,m,k)\right]$$

and hence the result.

Remark 4.1. Putting $m=0$ and $k=1$ in (4.2.1), we obtain a recurrence relation for single moments of order statistics of the Power Lomax distribution as

$$E\left[X_{r:n}^j\right]-E\left[X_{r-1:n}^j\right]=\frac{j}{\alpha\beta\left(n-r+1\right)}E\left[X_{r:n}^j\right]+\frac{j}{\alpha\beta\gamma_r}E\left[X_{r:n}^{j-\beta}\right]$$

or

$$E\left[X_{r:n}^{j}\right]-E\left[X_{r-1:n}^{j}\right]=\frac{j}{\alpha\beta\left(n-r+1\right)}\left[E\left[X_{r:n}^{j}\right]+E\left[X_{r:n}^{j-\beta}\right]\right]$$

Remark 4.2. Putting $m=-1$ and $k\geq 1$ in (4.2.1), we obtain a recurrence relation for single moments of $k-th$ upper record values from Power Lomax distribution as

$$E\left[X_{U(r)}^{j}\right]^{k}-E\left[X_{U(r-1)}^{j}\right]^{k}=\frac{j}{\alpha\beta k}\left(E\left[X_{U(r)}^{j}\right]^{k}+\lambda E\left[X_{U(r)}^{j-\beta}\right]^{k}\right)$$

4.3. Recurrence relation for product moments of gos for Power Lomax distribution

Here the recurrence relation for product moments of gos for Power Lomax distribution has been explored. Further, the recurrence relation for product moments of order stastistics and record values are established as particular cases of gos.

Theorem 4.3.1 Let X be a non-negative continous random variable and follows Power Lomax distribution given in (4.1.2). Suppose that for any $i, j > 0$, $1 \leq \mathrm{r} < \mathrm{s} \leq \mathrm{n}$ and $\mathrm{E}\left|(\mathrm{X}(\mathrm{r,n,m,k})\mathrm{X}(\mathrm{s,n,m,k}))\right|$ is finite, then the recurrence relation for the product moment is

$$\begin{aligned}&E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k)X^{j-1}(s-1,n,m,k)\right]\\&\qquad=\frac{j}{\alpha\beta\gamma_{r}}E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right]\\&\qquad\qquad+\frac{\lambda j}{\alpha\beta\gamma_{r}}E\left[X^{i}(r,n,m,k)X^{j-\beta}(s,n,m,k)\right]\end{aligned} \tag{4.3.1}$$

Proof. In view of (1.7.2), we have

$$\begin{aligned}&E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right]\\&\qquad=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{0}^{\infty}x^{i}\left[\bar{F}(x)\right]^{m}g_{m}^{r-1}\left[F(x)\right]f(x)I(x)dx\end{aligned} \tag{4.3.2}$$

Where

$$I(x)=\int_x^\infty y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1}f(y)dy \tag{4.3.3}$$

Integrating $I(x)$ in (4.3.3) by parts treating $\left[\overline{F}(x)\right]^{\gamma_s-1} f(y)$ for integration and rest part for differentiation, we have

$$I(x)=y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\int_x^\infty\left[\overline{F}(y)\right]^{\gamma_s-1}f(y)dy$$

$$-\int_x^\infty\left[\left\{\frac{d}{dy}y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\right\}\int\left(\overline{F}(y)\right)^{\gamma_s-1}f(y)dy\right]dy$$

$$=y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\left(\frac{-\left[\overline{F}(y)\right]^{\gamma_s}}{\gamma_s}\right)\Bigg|_x^\infty$$

$$-\int_x^\infty\left\{\frac{d}{dy}y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\right\}\left(\frac{-\left[\overline{F}(y)\right]^{\gamma_s}}{\gamma_s}\right)dy$$

$$=\int_x^\infty y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\frac{\left(\overline{F}(y)\right)^{\gamma_s}}{\gamma_s}dy$$

$$=\frac{1}{\gamma_s}\int_0^\infty\left[\overline{F}(x)\right]^{\gamma_s}\left\{jy^{j-1}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}\right.$$

$$\left.+(s-r-1)y^j\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}h_m^{'}\left[F(y)\right]f(y)\right\}$$

$$=\frac{j}{\gamma_s}\int_x^\infty y^{j-1}\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dy$$

$$+\frac{(s-r-1)}{\gamma_s}\int_x^\infty y^j\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}\left[\overline{F}(x)\right]^m f(y)dy$$

$$=\frac{j}{\gamma_s}\int_x^\infty y^{j-1}\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dy$$

$$+\frac{(s-r-1)}{\gamma_s}\int_x^\infty y^{j}\left(\overline{F}(y)\right)^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}f(y)dy$$

Substituting the value of $I(x)$ in (4.3.2), we get

$$E\left[X^i(r,n,m,k)X^j(r,n,m,k)\right]$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^j\left(\overline{F}(x)\right)^m f(x)g_m^{r-1}\left(F(x)\right)$$

$$\times\left\{\frac{j}{\gamma_s}\int_x^\infty y^{j-1}\left(\overline{F}(y)\right)^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(y)\right)\right]^{s-r-1}dy\right.$$

$$\left.+\frac{s-r-1}{\gamma_s}\int_x^\infty y^j\left(\overline{F}(y)\right)^{\gamma_s+m}\left[h_m\left(F(y)\right)-h_m\left(F(y)\right)\right]^{s-r-2}f(y)dy\right\}dx$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^\infty x^i\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]y^j\left[\overline{F}(y)\right]^{\gamma_s+m}$$

$$\times\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-2}f(y)dydx$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1}dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)(s-r-2)!}\int_0^\infty\int_x^\infty x^i y^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s+m}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-2} f(y)dydx$$

$$E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(\mathrm{s}{-}1,n,m,k)\right]$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1} dydx$$

$$+\frac{(s-r-1)c_{s-1}}{\gamma_s(r-1)!(s-r-1)(s-r-2)!}\int_0^\infty\int_x^\infty x^i y^j\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s+m}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-2} f(y)dydx$$

$$E\left[X^i(r,n,m,k)X^j(\mathrm{s},n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(\mathrm{s}{-}1,n,m,k)\right]$$

$$=\frac{jc_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}\left[\overline{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$

$$\times\left[\overline{F}(y)\right]^{\gamma_s}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1} dydx$$

In view of (4.1.3), we get

$$E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(s-1,n,m,k)\right]$$

$$=\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^j y^{j-1}\left[\bar{F}(x)\right]^m g_m^{r-1}\left[F(x)\right]f(x)$$

$$\times\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}\left\{\frac{1}{\alpha\beta}(\mathrm{y}+\lambda y^{1-\beta})f(\mathrm{y})\right\}dydx$$

this implies that

$$E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k)X^{j-1}(s-1,n,m,k)\right]$$

$$=\frac{j}{\alpha\beta\gamma_r}E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]+\frac{\lambda j}{\alpha\beta\gamma_r}E\left[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)\right]$$

and hence the result follows.

Remark 4.3. Putting $m=0$ and $k=1$ in (4.3.1), we obtain a recurrence relation for product moments of order statistics of the Power Lomax distribution as

$$E\left[X_{r:n}^iX_{s:n}^j\right]-E\left[X_{r:n}^iX_{s-1:n}^j\right]=\frac{j}{\alpha\beta(n-s+1)}\left[E\left[X_{r:n}^iX_{s:n}^j\right]+E\left[X_{r:n}^iX_{s:n}^{j-\beta}\right]\right]$$

Remark 4.4. Putting $m=-1$ and $k\geq 1$ in (4.3.1), we obtain a recurrence relation for single moments of $k-th$ upper record values from Power Lomax distribution as

$$E\left[X^i(r,n,-1,k)X^j(s,n,-1,k)\right]-E\left[X^i(r,n,-1,k)X^{j-1}(s-1,n,-1,k)\right]$$

$$=\frac{j}{\alpha\beta\gamma_s}E\left[X^i(r,n,-1,k)X^j(s,n,-1,k)\right]$$

$$+\lambda E\left[X^i(r,n,-1,k)X^{j-\beta}(s,n,-1,k)\right]$$

4.4. Characterization of Power Lomax distribution

On applying a generalization of the *Müntz-Szász* Theorem (1984) the characterization result of Power Lomax distribution based on single moments of *gos* has been established.

Theorem 4.1. For $m\geq -1$, the necessary and sufficient condition for a random variable X to be distributed with pdf given in (1.6) is that

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]$$

$$=\frac{j}{\alpha\beta\gamma_r}E\left[X^j(r,n,m,k)\right]+\frac{\lambda j}{\alpha\beta\gamma_r}E\left[X^{j-\beta}(r,n,m,k)\right] \tag{4.4.1}$$

if and only if

$$\bar{F}(x) = \left(1 + \frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)}, \ \alpha, \beta, \lambda > 0, \ \mathrm{x} \geq 0$$

Proof. The necessary part follows immediately from (4.4.1). On the other hand if the recurrence relation (4.4.1) is satisfied, then

$$jE\left[X^{j}(r,n,m,k)\right] - \lambda E\left[X^{j-\beta}(r,n,m,k)\right]$$

$$= \alpha\beta\gamma_r \left[E\left[X^{j}(r,n,m,k)\right] - E\left[X^{j}(r-1,n,m,k)\right]\right]$$

$$= \alpha\beta\gamma_r \left[\frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^{j}\left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right] f(x)dx\right.$$

$$\left. + \frac{(r-1)C_{r-1}}{\gamma_r (r-1)!}\int_0^{\infty} x^{j}\left[\bar{F}(x)\right]^{\gamma_r + m} g_m^{r-2}\left[F(x)\right] f(x)dx\right]$$

$$= \alpha\beta\gamma_r \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^{j}\left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-2}\left[F(x)\right]$$

$$\times f(x)\left[\frac{g_m\left[F(x)\right]}{\left[\bar{F}(x)\right]} - \frac{(r-1)\left[\bar{F}(x)\right]^{m}}{\gamma_r}\right]dx \tag{4.4.2}$$

Let

$$\upsilon(x) = -\frac{\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]}{\gamma_r}$$

Differentiating both side of $\upsilon(x)$ we get

$$\upsilon'(x) = \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-2}\left[F(x)\right] f(x)\left[\frac{g_m\left[F(x)\right]}{\bar{F}(x)} - \frac{(r-1)\left[\bar{F}(x)\right]^{m}}{\gamma_r}\right]$$

Thus

$$jE\left[X^{j}(r,n,m,k)\right] - \lambda E\left[X^{j-\beta}(r,n,m,k)\right]$$

$$= \alpha\beta\gamma_r \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^{j}\upsilon'(x)dx \tag{4.4.3}$$

Integrating RHS of (4.4.3) by parts and using the value of $\upsilon(x)$ we get

$$jE\left[X^{j}(r,n,m,k)\right]-\lambda E\left[X^{j-\beta}(r,n,m,k)\right]$$

$$=\alpha\beta\gamma_r\frac{jC_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx$$

$$=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx$$

$$-\frac{\lambda C_{r-1}}{(r-1)!}\int_0^\infty x^{j-\beta}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx$$

$$=\alpha\beta\gamma_r\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]f(x)dx$$

which reduces

$$=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)\left[x+\lambda x^{1-\beta}-\alpha\beta\frac{\bar{F}(x)}{f(x)}\right]dx=0 \tag{4.4.4}$$

Applying a generalization of the *Müntz-Szász* Theorem (1984) to equation (4.4), which states that on a space L(a, b) of summable functions defined on the intervals (a, b) a sequence of functions $f_n(x)$ is complete on (a, b) if for any $g\in L(a,b)$ the equalities

$$\int_a^b f_n(x)g(x)dx=0;\quad \text{n}=1,2,....$$

imply that $g(x)=0$ almost everywhere on (a, b), then we get

$$\left[x+\lambda x^{1-\beta}-\alpha\beta\frac{\bar{F}(x)}{f(x)}\right]=0$$

which implies that

$$\bar{F}(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{-(\alpha+1)},\quad \alpha,\beta,\lambda>0,\ \ \text{x}\geq 0$$

4.5. Conclusion

In this Chapter, recurrence relations for single and product moments from generalized order statistics for Power Lomax distribution have been obtained. Also, particular cases for order statistics and $k-th$ upper record values are obtained. Further, characterization results for generalized order statistics were discussed by using the recurrence relation of single moments of generalized order statistics for Power Lomax distribution.

Chapter 5

Relations for Single and Product Moments of Generalized Order Statistics from Sushila Distribution

CHAPTER 5

RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF GENERALIZED ORDER STATISTICS FROM SUSHILA DISTRIBUTION

5.1. Introduction

The concept of generalized order statistics (gos) was given by Kamps (1995), which is given as below:

Let $X_1, X_2, \ldots, X_2$ be a sequence of $iid\ rv$ with absolutely continuous distribution function $cdf\ F(x)$ and the probability density function $pdf\ f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k \geq 1$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be the parameters such that $\gamma_r = k + n - r + M_r \geq 1$, for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called gos if their joint pdf is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n)$$

$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n) \tag{5.1.1}$$

on the cone $F^{-1}(0) < x_1 \leq \ldots \leq x_n < F^{-1}(1)$,

where $\bar{F}(x) = 1 - F(x)$.

Here we may consider two cases:

Case I. $\gamma_i = \gamma_j$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = m$.

Case II. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1$.

For case I, gos will be denoted as $X(r,n,m,k)$ and its pdf is given by Kamps (1995)

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \quad \alpha \leq x \leq \beta \tag{5.1.2}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y), \alpha \leq x < y \leq \beta \tag{5.1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(0)$, $x \in (0,1)$

For case II, the pdf of $X(r,n,\tilde{m},k)$ is (Kamps and Cramer (2001)

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1}, \tag{5.1.4}$$

and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,\tilde{m},k)X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)$$

$$\times \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \tag{5.1.5}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\,, \qquad \gamma_i=k+n-i+M_i$$

$$a_i(r)=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{(\gamma_j-\gamma_i)}\,, \quad 1\leq i\leq r\leq n$$

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{(\gamma_j-\gamma_i)}\,, \quad r+1\leq i\leq s\leq n\,.$$

It may be noted that $m_1=m_2=\ldots=m_{n-1}=m\neq -1$

$$a_i(r)=\frac{(-1)^{r-i}}{(r-1)!(m+1)^{r-1}}\binom{r-1}{r-i}$$

$$a_i^{(r)}(s)=\frac{(-1)^{s-i}}{(s-r-1)!(m+1)^{s-r-1}}\binom{s-r-1}{s-i}.$$

Consequently, the pdf of $X(r,n,\tilde{m},k)$ and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ reduce to the pdf of $X(r,n,m,k)$ and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, respectively.

Several models of ordered random variables such as order statistics and record value can be seen as special cases of gos. If $m=0$ and $k=1$, then $X(r,n,m,k)$ reduces to the r^{th} order statistic $X_{r:n}$ David and Nagaraja (2003). If $m=-1$ and $k=1$, then $X(r,n,m,k)$ is the r^{th} record values from an infinite sequence of $iid\ rv's$ Ahsanullah (1995). Other special cases are k^{th} record values $(m=-1,\ k\in\mathbb{N})$, Dziubdziela and Kopociński (1976), sequential order statistics $\left((\gamma_i=n-i+1)\beta_i;\beta_1,\beta_2,\ldots,\beta_n>0\right)$ and order statistics with non–integral sample size ($m=0,\ k=\alpha-n+1,\ \alpha=n,$ Stigler (1977), Rohatgi and Saleh (1988)).

Many authors utilized the concept of *gos* in their studies. References may be made to Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001), Ahmad and Fawzy (2003), Ahmad (2007), Khan *et al*. (2007), Athar *et al.* (2012), Saran *et al.*(2015), Khan and Khan (2016) among others. For textbook reference, one may referrred to Ahsanullah (1995), Ahsanullah and Nevzorov (2001), Kamps (1995) and Arnold *et al.* (1992). In this Chapter, we have obtained the recurrence relation of *gos* arising from the Sushila distribution.

A rv X is said to have a Sushila distribution (Shanker *et al.* (2013)) if its df as given by

$$F(x)=1-\frac{\lambda(\sigma+1)+\sigma x}{\lambda(\sigma+1)}e^{-\frac{\sigma}{\lambda}x};\ x>0,\ \sigma>0,\ \lambda>0 \tag{5.1.6}$$

And the corresponding pdf is given by

$$f(x)=\frac{\sigma^2}{\lambda(1+\sigma)}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma}{\lambda}x};\ x>0,\ \sigma>0,\ \lambda>0 \tag{5.1.7}$$

The Sushila distribution given in (5.1.7) was introduced by Shanker *et al.* (2013). At $\lambda=1$, it reduces to Lindley distribution (Lindley, 1958)) having pdf as given by

$$f(x)=\frac{\sigma^2}{1+\sigma}(1+x)e^{-\sigma x};\ x>0, \sigma>0, \tag{5.1.8}$$

(Ghitany *et al.*, 2008) have explored some interesting properties of this distribution and showed that Lindley distribution gives better lifetime model than the exponential distribution in applications. Sankaran (1970) introduced the discrete Poisson-Lindley distribution after mixing Poisson and Lindley distribution. Zakarzadeh and Dolati (2009) introduced the generalization of Lindley distribution having three parameters.

It is observed that Lindley distribution is a particular case of (5.1.7). The pdf (5.1.7) can be shown as a mixture of exponential $\left(\frac{\sigma}{\lambda}\right)$ and gamma $\left(2,\frac{\sigma}{\lambda}\right)$ distribution as follows:

$$f\left(x;\sigma,\lambda\right)=pf_1\left(x\right)+\left(1-p\right)f_2\left(x\right) \tag{5.1.9}$$

The relation between (4.1.6) and (4.1.7), we have

$$\left(\lambda\left(1+\sigma\right)+\sigma x\right)fx=\sigma^2\left(1+\frac{x}{\lambda}\right)\left(1-F\left(x\right)\right) \tag{5.1.10}$$

The relation (5.1.10) is used for obtaining the recurrence relations for moments of *gos* from Sushila distribution.

In this Chapter, we have established recurrence relations for single and product moments of generalized order statistics from Sushila distribution. This chapter comprise three Sections. In Section 2, we have established the recurrence relation based on single moment of generalized order statistics from Sushila distribution. In Section 3, we have obtained the recurrence relation based on product moment of generalized order statistics from Sushila distribution.

5.2. Recurrence relations for single moments

Theorem 5.2.1. Let X be a non-negative continuous random variable and follows Sushila distribution given in (1.7). For Case II $\gamma_i\neq\gamma_j, i\neq j\in\left(1,2,\ldots,n-1\right)$, $k=1,2,\ldots$, $n\in\mathbb{N}$, $1\leq r\leq n$, $l=0,1,2,\ldots$

$$\lambda\left(1+\sigma\right)E\left[X^l\left(r,n,\tilde{m},k\right)\right]+\sigma E\left[X^{l+1}\left(r,n,\tilde{m},k\right)\right]$$

$$=\frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}(r,n,\tilde{m},k)\right]-E\left[X^{l+1}(r-1,n,\tilde{m},k)\right]\right]$$

$$+\frac{\sigma^2\gamma_r}{\lambda(l+2)}\left[E\left[X^{l+2}(r,n,\tilde{m},k)\right]-E\left[X^{l+2}(r-2,n,\tilde{m},k)\right]\right] \tag{5.2.1}$$

Proof. We have

$$\lambda(1+\sigma)E\left[X^{l}(r,n,\tilde{m},k)\right]+\sigma E\left[X^{l+1}(r,n,\tilde{m},k)\right]$$

$$=c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i-1}\right)\left(\lambda(1+\sigma)+\sigma x\right)fx\,dx$$

On using equation (5.1.10), we have

$$=c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i-1}\right)\sigma^2\left(1+\frac{x}{\lambda}\right)(1-F(x))\,dx$$

$$=\sigma^2 c_{r-1}\int_0^\infty x^l\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i-1}(1-F(x))\,dx$$

$$+\frac{\sigma^2}{\lambda}c_{r-1}\int_0^\infty x^{l+1}\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i-1}(1-F(x))\,dx \tag{5.2.2}$$

$$=\sigma^2 c_{r-1}\int_0^\infty x^l\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i}\,dx$$

$$+\frac{\sigma^2}{\lambda}c_{r-1}\int_0^\infty x^{l+1}\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i}\,dx$$

$$\lambda(1+\sigma)E\left[X^{l}(r,n,\tilde{m},k)\right]+\sigma E\left[X^{l+1}(r,n,\tilde{m},k)\right]=I+II$$

Now,

$$I=\sigma^2 c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^r a_i(r)(1-F(x))^{\gamma_i}\right)dx$$

Integrating by parts treating x^l as first function and rest of the part for as second function, we get

$$I=\sigma^2 c_{r-1}\left[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\cdot\frac{x^{l+1}}{l+1}\Bigg|_0^{\infty}-\int_0^{\infty} x^l\left(\frac{d}{dx}\left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right)\right)\left(\int x^l dx\right)dx\right]$$

$$=\frac{\sigma^2}{l+1}c_{r-1}\int_0^{\infty} x^{l+1}\left(\frac{d}{dx}\left(\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right)\right)dx$$

$$=\frac{\sigma^2}{l+1}c_{r-1}\int_0^{\infty} x^{l+1}\sum_{i=1}^{r} a_i(r)\gamma_i(1-F(x))^{\gamma_i-1}f(x)dx$$

on using (5.2.1) and $c_{r-1}=\gamma_r c_{r-2}$, we have

$$=\frac{\sigma^2}{l+1}c_{r-1}\int_0^{\infty} x^{x+1}\left[\gamma_r\left\{c_{r-1}\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}-c_{r-2}\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i}\right\}\right]fx\,dx$$

$$-\frac{\sigma^2\gamma_r}{l+1}\left[c_{r-1}\int_0^{\infty} x^{x+1}\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\,fx\,dx-c_{r-2}\int_0^{\infty}\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i}\,fx\,dx\right]$$

Therefore

$$I=\frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}(r,n,\tilde{m},k)\right]-E\left[X^{l+1}(r-1,n,\tilde{m},k)\right]\right]$$

Similarly, the second integral is given by

$$II=\frac{\sigma^2}{\lambda}c_{r-1}\int_0^{\infty} x^{l+1}\sum_{i=1}^{r} a_i(r)(i-F(x))^{\gamma_i}\,dx$$

$$=\frac{\sigma^2\gamma_r}{\lambda(l+2)}\left[E\left[X^{l+2}(r,n,\tilde{m},k)\right]-E\left[X^{l+2}(r-1,n,\tilde{m},k)\right]\right]$$

Substituting the value of the integral I and II in (5.2.3), we have

$$\lambda(1+\sigma)E\left[X^l(r,n,\tilde{m},k)\right]+\sigma E\left[X^{l+1}(r,n,\tilde{m},k)\right]$$

$$= \frac{\sigma^2 \gamma_r}{(l+1)} \Big[E\big[X^{l+1}(r,n,\tilde{m},k) \big] - E\big[X^{l+1}(r-1,n,\tilde{m},k) \big] \Big]$$

$$+ \frac{\sigma^2 \gamma_r}{\lambda(l+2)} \Big[E\big[X^{l+2}(r,n,\tilde{m},k) \big] - E\big[X^{l+2}(r-1,n,\tilde{m},k) \big] \Big]$$

and hence the result.

Corollary 5.1. For $m_1 = m_2 = \ldots = m_{n-1} = m$, the recurrence relation for single moments of *gos* from Sushila distribution is given by

$$\lambda(1+\sigma) E\big[X^{l}(r,n,m,k) \big] + \sigma E\big[X^{l+1}(r,n,m,k) \big]$$

$$= \frac{\sigma^2 \gamma_r}{(l+1)} \Big[E\big[X^{l+1}(r,n,m,k) \big] - E\big[X^{l+1}(r-1,n,m,k) \big] \Big]$$

$$+ \frac{\sigma^2 \gamma_r}{\lambda(l+2)} \Big[E\big[X^{l+2}(r,n,m,k) \big] - E\big[X^{l+2}(r-2,n,m,k) \big] \Big] \qquad (5.2.3)$$

Remark 5.1. At $\lambda = 1$ in (5.2.3), we get the recurrence relation for single moments of generalized order statistics from Lindley distribution as obtained by Saran *et al.* (2015).

Remark 5.2. Recurrence relation for single moments of order statistics (at $m = 0,\ k = 1$) from Sushila distribution is

$$\lambda(1+\sigma) E\big[X_{r:n}^{j} \big] + \sigma E\big[X_{r:n}^{j+1} \big]$$

$$= \frac{\sigma^2 (n-r+1)}{(l+1)} \Big[E\big[X_{r:n}^{l+1} \big] - E\big[X_{r-1:n}^{l+1} \big] \Big]$$

$$+ \frac{\sigma^2 (n-r+1)}{\lambda(l+2)} \Big[E\big[X_{r:n}^{l+2} \big] - E\big[X_{r-2:n}^{l+2} \big] \Big] \qquad (5.2.4)$$

Remark. 5.3. Recurrence relation for single moments of k^{th} upper record $(m=-1)$ from Sushila distribution is given by

$$\lambda(1+\sigma)E\left[X_{U(r)}^{(k)}\right]^{j}+\sigma E\left[X_{U(r)}^{(k)}\right]^{j+1}$$

$$=\frac{\sigma^{2}k}{(l+1)}\left[E\left[X_{U(r)}^{(k)}\right]^{l+1}-E\left[X_{U(r-1)}^{(k)}\right]\right]^{l+1}$$

$$+\frac{\sigma^{2}k}{\lambda(l+2)}\left[E\left[X_{U(r)}^{(k)}\right]^{l+2}-E\left[X_{U(r-2)}^{(k)}\right]^{l+2}\right]$$

5.3. Recurrence Relations for Product Moments

In this Section, the recurrence relation for product moments of *gos* from Sushila distribution has been obtained. Particular cases for recurrence relations of order statistics and $k-th$ upper record are also discussed.

Theorem 5.2. Let X be a non-negative continuous random variable and follows Sushila distribution given in (5.1.7).

Let case II be satisfied i.e. $\gamma_i \neq \gamma_j, i \neq j \in (1,2,\ldots,\ n-1)$. For Sushila distribution given in (5.1.7) and $u, v \in \mathbb{N}$,

$$\lambda(1+\sigma)E\left[X^{u}(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]+\sigma E\left[X^{u}(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$

$$=\frac{\sigma^{2}\gamma_{s}}{(v+1)}\left[E\left[X^{u}(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]-E\left[X^{u}(r,n,\tilde{m},k)X^{v+1}(s-1,n,\tilde{m},k)\right]\right]$$

(5.3.1)

Proof. We have

$$\lambda(1+\sigma)E\left[X^{u}(r,n,\tilde{m},k)X^{v}(s,n,\tilde{m},k)\right]+\sigma E\left[X^{u}(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$

$$=\int_{0}^{\infty}x^{u}\left\{\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right\}\frac{f(x)}{(1-F(x))}I(x)\,dx \tag{5.3.2}$$

where $I(x)$ is given by

$$I(x)=c_{s-1}\int_{x}^{\infty}y^{v}\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\frac{fy}{(1-F(y))}dy$$

substituting f(y) in above equation

$$=\sigma^{2}c_{s-1}\int_{x}^{\infty}y^{v}\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\left(1+\frac{y}{\lambda}\right)dy$$

$$I(x)=I_1(x)+I_2(x)$$

Where $I_1(x)$ is given by

$$I_1(x)=\frac{\sigma^{2}\gamma_s}{v+1}\left[\left\{c_{s-1}\int_{x}^{\infty}y^{v+1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(y)}\right)^{\gamma_i}\right.\right.$$

$$\left.\left.-c_{s-2}\int_{x}^{\infty}y^{v+1}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(y)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}dy \tag{5.3.5}$$

Similarly, $I_2(x)$ is given by

$$I_2(x)=\frac{\sigma^{2}\gamma_s}{\lambda(v+2)}\left[\left\{c_{s-1}\int_{x}^{\infty}y^{v+2}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right.\right.$$

$$\left.\left.-c_{s-2}\int_{x}^{\infty}y^{v+2}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}dy \tag{5.3.6}$$

Therefore

$$I(x) = I_1(x) + I_2(x)$$

substituting $I(x)$ in (5.3.2), we get

$$\lambda(1+\sigma)E\left[X^u(r,n,\tilde{m},k)X^v(s,n,\tilde{m},k)\right] + \sigma E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$

$$= \int_0^\infty x^u \left[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right] \frac{f(x)}{(1-F(y))} I(x)\,dx$$

$$= \int_0^\infty x^u \left\{\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right\} \frac{f(x)}{(1-F(y))} \left[\frac{\sigma^2\gamma_s}{v+1}\left[\left\{c_{s-1}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right.\right.\right.$$

$$\left.\left.-c_{s-2}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}\,dy$$

$$+\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[\left[\left\{c_{s-1}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\right.\right.\right.$$

$$\left.\left.\left.-c_{s-2}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s-1} a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}\right]dy$$

and hence the result given in (5.3.1).

Corollary 5.2. For $m_1 = m_2 = \ldots = m_{n-1} = m$, the recurrence relation for product moments of *gos* from Sushila distribution has the form

$$\lambda(1+\sigma)E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right] + \sigma E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right]$$
$$= \frac{\sigma^2\gamma_s}{(v+1)}\left[E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right] - E\left[X^u(r,n,m,k)X^{v+1}(s-1,n,m,k)\right]\right]$$

$$+\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[E\left[X^u(r,n,m,k)X^{v+2}(s,n,m,k)\right] - E\left[X^u(r,n,m,k)X^{v+2}(s-1,n,m,k)\right]\right] \qquad (3.6)$$

Remark 5.4. Putting $\lambda=1$ in (3.4), we get the recurrence relation for product moments of generalized order statistics from Lindley distribution obtained by Saran *et al.* (2015).

Remark 5.5. Recurrence relation for product moments of order statistics (at $m=0,\ k=1$) from Sushila distribution is

$$\lambda(1+\sigma)E\left[X_{r:n}^{u}X_{r:n}^{v+1}\right]+\sigma E\left[X_{r:n}^{u}X_{r:n}^{v+1}\right]$$

$$=\frac{\sigma^{2}(n-s+1)}{(v+1)}\left[E\left[X_{r:n}^{u}X_{\mathrm{s}:n}^{v+1}\right]-E\left[X_{r:n}^{u}X_{\mathrm{s}-1:n}^{v+1}\right]\right]$$

$$+\frac{\sigma^{2}(n-s+1)}{\lambda(v+2)}\left[E\left[X_{r:n}^{u}X_{\mathrm{s}:n}^{v+2}\right]-E\left[X_{r:n}^{u}X_{\mathrm{s}-1:n}^{v+2}X^{u}\left(r,n,\tilde{m},k\right)\right]\right] \tag{5.3.7}$$

Remark 5.6. Recurrence relation for product moments of k^{th} upper record $(m=-1)$ from Sushila distribution.

$$\lambda(1+\sigma)E\left[X_{U(r)}^{u}X_{U(s)}^{v+1}\right]+\sigma E\left[X_{U(r)}^{u}X_{U(s)}^{v+1}\right]$$

$$=\frac{\sigma^{2}(n-s+1)}{(v+1)}\left[E\left[X_{U(r)}^{u}X_{U(s)}^{v+1}\right]-E\left[X_{U(r)}^{u}X_{U(s-1)}^{v+1}\right]\right]$$

$$+\frac{\sigma^{2}(n-s+1)}{\lambda(v+2)}\left[E\left[X_{U(r)}^{u}X_{U(s)}^{v+2}\right]-E\left[X_{U(r)}^{u}X_{U(s-1)}^{v+2}\right]\right]$$

5.4. Conclusion

In this Chapter, we have demonstrated the recurrence relations for single and product moments of generalized order statistics from Sushila distribution. Particular cases for recurrence relations of single and product moments are also discussed for order statistics and $k-th$ upper record values. In last Section of this chapter we have obtained characterization result of the Sushila distribution using the recurrence relation for single moments of generalized order statistics.

Chapter 6

On Strength Reliability for Erlang-Truncated Exponential Distributed Stress

CHAPTER 6

ON STRENGTH RELIABILITY FOR ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTED STRESS

6.1. Introduction

Reliability, in general, is defined as the probability that a system, component or device will work adequately its intended function under the operational conditions. Improving reliability is an interesting topic of connotation for the researches. Reliability function '$R(t)$', a function of time 't', represents the probability that a device or item is still working at time 't' and defined as $R(t)=P(X>t)=1-F(t)$. Stress-strength reliability model is one of the most important statistical tools to measure the reliability, which is denoted as $P=P\mathrm{r}(X>Y)$. The stress-strength reliability model measures the performance of the item of strength Y subject to a stress X, where X and Y both are non-negative and continuous random variables. Church and Harris (1970) introduced the term stress-strength. For more references on the topic, one may have referred to Downton (1973), Kelly (1976), Chao (1982), Awad (1986). Some other references are Tong (1974), Sathe and Shah (1981) and Chaturvedi and Surinder (1999).

El-Alosey (2007) proposed Erlang-truncated exponential distribution, denoted by $ETE(\beta,\lambda)$. Random variable (rv) X is said to be distributed as ETE distribution if its probability density function (pdf) is given by

$$f(x;\beta,\lambda)=\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)};\quad x\geq 0,\beta>0,\lambda>0, \tag{6.1.1}$$

where β is the shape parameter and λ is the scale parameter.

In the present study, Erlang-truncated exponential distribution is taken under consideration to study the strength-reliability of item for Erlang-truncated exponential distributed stress.

6.2. Strength reliability for finite strength

A finite random time power function distribution has been chosen to represent the strength, denoted by $'Y'$, which justify the fact that the life time of items or devices may confined to only in finite range. On the other hand an infinite range of stress $'X'$ may be justifiable because of the fact that large stress may tend towards infinity. Here, strength of item, denoted by $'Y'$ follows power function distribution and its pdf is

$$g(y;a,\theta)=\left(\frac{a}{\theta}\right)\left(\frac{y}{\theta}\right)^{a-1}; \quad 0<y<\theta, a>0 \tag{6.2.1}$$

where $'\theta'$ is the scale parameter and $'a'$ is the shape parameter. The maximum value of strength is θ. Thus, the total unreliability of the devices/items is obtained by $P(X>Y)$, where $\theta>Y$, which is termed as probability of disaster by Alam and Roohi (2003). In the above setup, underlying problem may handle mathematically as follows

$$\alpha=P(X>\theta)=\int_{\theta}^{\infty} f(x;\beta,\lambda)\,dx$$

$$=\int_{\theta}^{\infty} \beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\,dx$$

On substituting $\beta x\left(1-e^{-\lambda}\right)=t$, we get

$$\alpha=P(X>\theta)=e^{-m\left(1-e^{-\lambda}\right)} \tag{6.2.2}$$

where $m=\beta\theta$

Table 6.1. Numerical values for Probability of disaster $\alpha = P\left(X > \theta\right)$ for different values of m and λ

m	$P\left(X > \theta\right)$						
	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3$	$\lambda = 3.5$
0.1	0.961417	0.938744	0.925254	0.917166	0.912295	0.909354	0.907574
0.2	0.924323	0.881241	0.856095	0.841194	0.832283	0.826924	0.82369
0.3	0.88866	0.82726	0.792105	0.771515	0.759288	0.751966	0.74756
0.4	0.854373	0.776586	0.732899	0.707607	0.692695	0.683803	0.678466
1	0.674712	0.531464	0.459843	0.421193	0.399351	0.386659	0.379158
1.5	0.554214	0.387445	0.311828	0.273351	0.252367	0.240432	0.233469
2	0.455236	0.282454	0.211456	0.177403	0.159481	0.149505	0.143761
2.5	0.373935	0.205913	0.143392	0.115134	0.100783	0.092965	0.088522
3	0.307153	0.150114	0.097236	0.074721	0.063689	0.057807	0.054508
3.5	0.252298	0.109435	0.065938	0.048493	0.040248	0.035946	0.033564
4	0.20724	0.07978	0.044714	0.031472	0.025434	0.022352	0.020667

4.5	0.170229	0.058161	0.030321	0.020425	0.016073	0.013899	0.012726
5	0.139827	0.0424	0.020561	0.013256	0.010157	0.008642	0.007836
5.5	0.114855	0.03091	0.013943	0.008603	0.006419	0.005374	0.004825
6	0.094343	0.022534	0.009455	0.005583	0.004056	0.003342	0.002971
6.6	0.074504	0.015421	0.005932	0.003323	0.002339	0.00189	0.00166
7	0.063654	0.011976	0.004348	0.002352	0.00162	0.001292	0.001127
7.7	0.048329	0.007694	0.002524	0.001284	0.000852	0.000664	0.000571

Table 6.2. Values of m for tolerance levels α and for fixed value of $\lambda = 0.5$.

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m	2.541494	3.30656	4.317922	5.082988	7.624482	10.16598	12.70747

Remarks.

1. **Table 6.1** depicts the probability of disaster for Erlang-truncated exponential distributed stress. It is interested to note that the probability of disaster decreases when the values of m and λ increase.

2. **Table 6.2** depicts the values of m for different values of α for fixed $\lambda = 0.5$. It is obvious that values of m increases as α decreases. In other words, if we want to have a small tolerance level, the ultimate strength capacity must be increased.

6.3. Stress and Strength Reliability

For the stress-strength model the probability $P = \Pr(Y > X)$, when the random variable X and Y follows the pdf (6.1.1) and (6.2.1), respectively is given by the following theorem.

Theorem 6.1. $P = \Pr(Y > X)$ is given by

$$P = P(Y > X) = 1 - e^{-m\left(1-e^{-\lambda}\right)} - \frac{1}{\left(m\left(1-e^{-\lambda}\right)\right)^{a}}\gamma\left(a+1, m\left(1-e^{-\lambda}\right)\right) \tag{6.3.1}$$

where $m = \beta\theta$

Proof.

$$P(Y > X) = \int_{0}^{\theta}\int_{y=x}^{\theta} f(x;\beta,\lambda)\, g(y;a,\theta)\, dy\, dx \tag{6.3.2}$$

Since $m = \beta\theta \Rightarrow \theta = \dfrac{m}{\beta}$

$$\int_{0}^{\frac{m}{\beta}}\int_{y=x}^{\frac{m}{\beta}} f(x;\beta,\lambda)\, g(y;a,\theta)\, dy\, dx$$

Substitute $y = vx$ in (6.3.2), we have

$$= \int_{0}^{\frac{m}{\beta}}\int_{y=vx}^{\frac{m}{\beta x}} f(x;\beta,\lambda)\, g(y;a,\theta)\, xdvdx$$

$$= \int_{0}^{\frac{m}{\beta}}\int_{y=vx}^{\frac{m}{\beta x}} \beta\left(1-e^{-\lambda}\right) e^{-\beta x\left(1-e^{-\lambda}\right)} \left(\frac{a}{\theta}\right)\left(\frac{vx}{\theta}\right)^{a-1} xdvdx$$

$$=\int_{0}^{\frac{m}{\beta}}\int_{y=vx}^{\frac{m}{\beta x}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\left(\frac{a}{\theta}\right)\frac{v^{a-1}x^{a-1}}{\theta^{a-1}}xdvdx$$

$$=\int_{0}^{\frac{m}{\beta}}\int_{y=1}^{\frac{m}{\beta x}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\left(\frac{ax^{a}}{\theta^{a}}\right)v^{a-1}dvdx$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\frac{ax^{a}}{\theta^{a}}\left(\int_{y=1}^{\frac{m}{\beta x}}v^{a-1}dv\right)dx$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\frac{ax^{a}}{\theta^{a}}\left[\left(\frac{v^{a}}{a}\right)_{1}^{\frac{m}{\beta x}}\right]dx$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\frac{x^{a}}{\theta^{a}}\left[\left(\frac{m}{\beta x}\right)^{a}-1\right]$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\frac{x^{a}}{\theta^{a}}\left[\left(\frac{m}{\beta x}\right)^{a}-1\right]dx$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}\frac{x^{a}}{\theta^{a}}\left(\frac{m}{\beta x}\right)^{a}dx-\frac{1}{\theta^{a}}\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}x^{a}$$

$$=\int_{0}^{\frac{m}{\beta}}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)_{a}}dx-\frac{1}{\theta^{a}}\int_{0}^{\frac{m}{\beta}}x^{a}\beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}dx \tag{6.3.3}$$

Now, substituting $\beta x\left(1-e^{-\lambda}\right)=u$ in (6.3.3), we get

$$=\int_{0}^{m\left(1-e^{-\lambda}\right)}\beta\left(1-e^{-\lambda}\right)e^{-u}\frac{du}{\beta\left(1-e^{-\lambda}\right)e^{-u}}-\frac{1}{\theta^{a}}\int_{0}^{m\left(1-e^{-\lambda}\right)}u^{a+1-1}e^{-u}du$$

Incomplete gamma function is $\gamma\left(a,z\right)=\int_{0}^{z}t^{a-1}e^{-t}dt$

$$P\left(Y>X\right)=1-e^{-m\left(1-e^{-\lambda}\right)}-\frac{1}{\left(m\left(1-e^{-\lambda}\right)\right)^{a}}\gamma\left(a+1,m\left(1-e^{-\lambda}\right)\right) \tag{6.3.4}$$

This completes the proof.

Table 6.3. Strength-reliability of an item for selected values of m and a for $\lambda=0.5$

$m\rightarrow$ $a\downarrow$	8	9	10	11	12	13	14
0.5	0.8638	0.9126	0.9447	0.9655	0.9787	0.987	0.992
1	0.8649	0.9109	0.9419	0.9626	0.9761	0.9849	0.9905
1.5	0.8818	0.9213	0.9479	0.9657	0.9776	0.9855	0.9907
2	0.9024	0.9351	0.9568	0.9713	0.981	0.9875	0.9918
2.5	0.9202	0.9472	0.9649	0.9766	0.9845	0.9897	0.9931
3	0.9335	0.9562	0.9709	0.9806	0.9871	0.9914	0.9942
3.5	0.9425	0.9622	0.9749	0.9833	0.9888	0.9925	0.995
4	0.9483	0.9659	0.9773	0.9849	0.9899	0.9932	0.9954
4.5	0.9519	0.9681	0.9788	0.9858	0.9905	0.9936	0.9957
5	0.954	0.9694	0.9795	0.9863	0.9908	0.9938	0.9958
5.5	0.9553	0.9701	0.98	0.9865	0.9909	0.9939	0.9959
6	0.9561	0.9705	0.9802	0.9867	0.991	0.9939	0.9959

6.4. Discussion

While manufacturing an item, if the strength of an item follows Power function distribution, it is likely that the possible values of θ may have an upper limit say θ_0. For example, the acceleration capability of a machine must be subject to the maximum possible speed. For a fixed tolerance level a, suppose θ_a is the desired value of θ. In case $\theta_a < \theta_0$, we may obtain the required values of $'a'$ and $'a_a'$, by using Table 3, so that the item is manufactured with the strength distribution having parameters (a_a, θ_a) and consequently the desired strength reliability is achieved. However, if $\theta_a > \theta_0$, we have to either adjust a or look for alternate item.

6.5. An Illustrative Example

Without loss of generality, it β can be assumed to be unity, so that $m = \theta$. Now suppose the maximum possible value of θ is 14. For $\alpha \leq 0.01$, we must have $m = \theta \geq 5$. Since θ cannot exceed 14, we have the opportunity of finding the maximum value of $P(Y > X)$ for an item in such a way that $5 \leq \theta \leq 14$ and corresponding value of $'a'$. The cost of adjustment in the value of the parameter may also be taken into consideration here as the cost of varying θ and $'a'$ can be different. Theoretically, the cost factor might be increasing function or decreasing function in θ and $'a'$ according to the nature of the parameter. In our case $E(Y) = \frac{a\theta}{(a+1)}$ implies that the mean strength increase by increasing either of two parameters. Hence, it can be assumed that the cost is an increasing function of the respective parameters. If we assume that the costs are directly proportional to the required parametric values, the formulation of the problem may be as follows:

Let C_1 be the cost of adjusting in a unit of parameter $'a'$ and C_2 be the cost of adjusting in a unit of parameter θ. Then our objective function may be in the following form minimize $C = C_1 a + C_2 \theta$ subject to $5 \leq \theta \leq 14$ and $P(Y > X) \geq 0.99$.

The problem can be solved analytically as follows:

Look into **Table 6.3** for $\theta = 12, 13$ and 14 and find those values of $'a'$ for which $P(Y > X) \geq 0.99$. Evaluate the cost function for each pair of (a, θ):

Table 6.4. Table for obtaining the optimum cost of manufacturing item.

a	θ	$C = C_1 a + C_2 \theta$
4.5	12	$C = 4.5C_1 + 12C_2$
5	12	$C = 5C_1 + 12C_2$
5.5	12	$C = 5.5C_1 + 12C_2$
6	12	$C = 6C_1 + 12C_2$
6.5	12	$C = 6.5C_1 + 12C_2$
3	13	$C = 3C_1 + 13C_2$
3.5	13	$C = 3.5C_1 + 13C_2$
4	13	$C = 4C_1 + 13C_2$
4.5	13	$C = 4.5C_1 + 13C_2$
5	13	$C = 5C_1 + 13C_2$
5.5	13	$C = 5.5C_1 + 13C_2$
6	13	$C = 6C_1 + 13C_2$
6.5	13	$C = 6.5C_1 + 13C_2$
0.5	14	$C = 0.5C_1 + 14C_2$
1	14	$C = C_1 + 14C_2$
1.5	14	$C = 1.5C_1 + 14C_2$

2	14	$C = 2C_1 + 14C_2$
2.5	14	$C = 2.5C_1 + 14C_2$
3	14	$C = 3C_1 + 14C_2$
3.5	14	$C = 3.5C_1 + 14C_2$
4	14	$C = 4C_1 + 14C_2$
4.5	14	$C = 4.5C_1 + 14C_2$
5	14	$C = 5C_1 + 14C_2$
5.5	14	$C = 5.5C_1 + 14C_2$
6	14	$C = 6C_1 + 14C_2$
6.5	14	$C = 6.5C_1 + 14C_2$

Clearly, the minimum of the cost lies at $C = 0.5C_1 + 14C_2$ depending upon the numerical values of C_1 and C_2.

6.6. Conclusion

Stress-strength reliability models are considered to estimate the reliability of the manufacturing devices facing Erlang-truncated exponential distribution. The power function distributed strength and the Erlang-truncated exponential distributed stress of the manufacturing items are considered. A stress-strength relationship is established among the parameters. Also calculated the optimum cost for the cost function which is for the linear cost function in terms of parameters, on the basis of these results.

Chapter 7

A Study of Strength-Reliability For Sushila Distributed Stress

CHAPTER 7

A STUDY OF STRENGTH-RELIABILITY FOR SUSHILA DISTRIBUTED STRESS

7.1. Introduction

Reliability of any system become more significant as industries are introducing more and more complex mechanization and automation in the industrial process to meet the increasing demand of society. The science of reliability is concerned with evaluating the risks and their consequences. One of the most important and commonly used statistical models for evaluating the risk and their consequences is the stress-strength testing model. The probability model $P = P\left(X > Y\right)$ represents the performance of an item of Y strength and subjected stress on it. Here X and Y are non-negative, independent and continuously distributed random variables. The term stress-strength was originally introduced by Church and Harries (1970). A lot of works have been done in this field by various researchers. For a literature review, Downton (1973), Tong (1974), Kelly (1976), Sathe and Vande (1981), Chao (1982), Awad (1986), Chaturvedi and Surinder (1999), Alam and Roohi (2003) can referred.

Shankar *et al.* (2013), proposed Shushila distribution, which is the mixture distribution of exponential and gamma distributions. Lindley distribution (Lindley, (1958)) is the particular case of Shushila distribution. In this paper Shushila distribution has been considered for the stress of an item. We obtain strength-reliability of an item for Shushila distributed stress.

7.2. Strength reliability for finite strength

An infinite stress distribution is justifiable in the sense that huge stress may tends to infinity but the strength of various devices/equipments depends upon its subcomponents which may not be recorded as infinite lifetime. Here, the maximum value of the unreliability of items is obtained by $P\left(X>\theta\right)$. Alam and Roohi (2003) defined it as the probability of disaster.

It is assumed that the random variable X represent the stress that faces, follows the Sushila distribution with probability density function *pdf*

$$f\left(x;\lambda,\sigma\right)=\frac{\sigma^2}{\lambda\left(\sigma+1\right)}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma x}{\lambda}},\ x>0,\lambda>0,\sigma>0 \qquad (7.2.1)$$

we obtain the strength reliability when the finite strength follows power function distribution having *pdf*

$$g\left(y\right)=\left(\frac{a}{\theta}\right)\left(\frac{y}{\theta}\right)^{a-1},\ 0<y<\theta,\ a>0 \qquad (7.2.2)$$

where θ is the scale parameter and a is the shape parameter.

Theorem 7.1. If the random variables X and Y follows the Sushila distribution given in (7.2.1) and power function distribution given in (7.2.2) respectively, then $\alpha=P\left(X>\theta\right)$ is given by

$$\alpha\equiv P(X>\theta)=\frac{\sigma}{(\sigma+1)}\left[1+m+\frac{1}{\sigma}\right]e^{-m\sigma} \qquad (7.2.3)$$

where $m=\frac{\theta}{\lambda}$

Proof. We know that

$$\alpha\equiv P(X>\theta)=\frac{\sigma^2}{\lambda(\sigma+1)}\int_\theta^\infty(1+\frac{x}{\lambda})e^{-\frac{\sigma x}{\lambda}}dx$$

$$=\frac{\sigma}{(\sigma+1)}\left[e^{-\frac{\sigma\theta}{\lambda}}+\frac{\theta}{\lambda}e^{-\frac{\sigma\theta}{\lambda}}+\frac{1}{\sigma}e^{-\frac{\sigma\theta}{\lambda}}\right]$$

Or,

$$\alpha = \frac{\sigma}{(\sigma+1)}\left[1+m+\frac{1}{\sigma}\right]e^{-m\sigma} \tag{7.2.4}$$

where $m = \frac{\theta}{\lambda}$

Hence, the theorem follows.

Table 7.1. Numerical values for Probability of disaster $\alpha = P\left(X > \theta\right)$ for different values of m and σ

m	$P\left(X > \theta\right)$				
	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$	$\sigma = 2.5$
0.5	0.9086	0.7582	0.6141	0.4905	0.3888
1.0	0.8087	0.5518	0.3570	0.2256	0.1407
1.5	0.7085	0.3905	0.2003	0.0996	0.0487
2.0	0.6131	0.2707	0.1095	0.0427	0.0164
2.5	0.5253	0.1847	0.0588	0.0180	0.0054
3.0	0.4463	0.1245	0.0311	0.0074	0.0017
3.5	0.3765	0.0830	0.0163	0.0030	0.0005
4.0	0.3158	0.0549	0.0084	0.0012	0.0001
4.5	0.2635	0.0361	0.0043	0.00049	0.000054
5.0	0.2189	0.0236	0.0022	0.000197	0.000017

Alternatively, we may also obtain the values of m for fixed values of σ at different tolerance level α from the equation.

$$\alpha = \frac{\sigma}{(\sigma+1)}\left[1+m+\frac{1}{\sigma}\right]e^{-m\sigma}$$

$$\alpha e^{m\sigma} = \frac{\sigma}{(\sigma+1)}\left[1+m+\frac{1}{\sigma}\right]$$

for $\sigma = 0.5$,we get the expression for

$$f(m) = \alpha e^{0.5m} - \frac{m}{3} - 1 \tag{7.2.5}$$

Here, the equation (7.2.5) is a nonlinear equation and hence solved by Newton-Raphson method for different real values of m using Mathematica Software.

Table. 7.2. Values of m for tolerance levels α and for $\sigma = 0.5$

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m	7.01639	8.71618	10.889	12.494	17.6763	22.7178	27.6756

Remarks.

1. **Table 7.1.** Depicts the probability of disaster for Sushila distributed stress. It is interested to note that the probability of disaster decreases for increasing values of m and also decreases when the values of α increases.
2. **Table 7.2.** Shows the values of m for various values of α and for fixed $\sigma = 0.5$. It is obvious that values of m increases as α decreases. In other words, if we want to have a small tolerance level, the ultimate strength capacity must be increased.

7.3. Stress and Strength-Reliability

For the stress strength model, the probability $P=\Pr\left(Y>X\right)$, when the random variables X and Y follow pdfs (7.2.1) and (7.2.2) respectively, is given according to the following theorem.

Theorem 7.2. $P=\Pr\left(Y>X\right)$ is given by

$$\begin{aligned} P\left(Y>X\right)=&\frac{\sigma}{\left(1+\sigma\right)}\left(1-e^{-m\sigma}\right)+\frac{1}{\left(1+\sigma\right)}\left(1-e^{-m\sigma}-m\sigma e^{-m\sigma}\right) \\ &-\frac{\sigma}{\left(1+\sigma\right)}\cdot\frac{1}{\left(m\sigma\right)^{a}}\gamma\left(a+1,m\sigma\right)-\frac{1}{\left(1+\sigma\right)}\cdot\frac{1}{\left(m\sigma\right)^{a}}\gamma\left(a+2,m\sigma\right) \end{aligned} \tag{7.3.1}$$

where $m=\dfrac{\theta}{\lambda}$ and $\gamma\left(a,m\right)=\int_0^m u^{a-1}e^{-u}du,$ is incomplete gamma function.

Proof.

$$P(Y>X)=\int_0^{\theta}\int_{y=x}^{\theta} f(x;\lambda,\sigma)g(y;a,\theta)dydx \tag{7.3.2}$$

Substituting $y=vx$ in equation (7.3.2) we get

$$P(Y>X)=\int_0^{\theta}\int_{y=vx}^{\frac{m\lambda}{x}} xf(x;\lambda,\sigma)g(vx;a,\theta)dvdx$$

$$=\int_0^{\theta}\int_{v=1}^{\frac{m\lambda}{x}}\frac{\sigma^2}{\lambda\left(1+\sigma\right)}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma x}{\lambda}}\left(\frac{a}{\theta}\right)\left(\frac{vx}{\theta}\right)^{a-1}dvdx$$

$$=\frac{\sigma^2}{\lambda\left(1+\sigma\right)}\int_0^{\theta}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma}{\lambda}x}\left(1-\frac{x^a}{\theta^a}\right)dx$$

Setting $m=\dfrac{\theta}{\lambda}$,

$$=\frac{\sigma^2}{\lambda\left(1+\sigma\right)}\int_0^{m\lambda}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma}{\lambda}x}\left(1-\frac{x^a}{m^a\lambda^a}\right)dx$$

$$= \frac{\sigma^2}{\lambda(1+\sigma)}\int_0^{m\lambda}\left(1+\frac{x}{\lambda}\right)e^{-\frac{\sigma}{\lambda}x}dx - \int_0^{m\lambda}\frac{x^a}{m^a\lambda^a}\left(1+\frac{x}{\lambda}\right)dx$$

$$= \frac{\sigma^2}{\lambda(1+\sigma)}\int_0^{m\lambda}e^{-\frac{\sigma}{\lambda}x}dx + \frac{\sigma^2}{\lambda(1+\sigma)}\int_0^{m\lambda}\frac{x}{\lambda}e^{-\frac{\sigma}{\lambda}x}dx - \frac{\sigma^2}{\lambda(1+\sigma)}\cdot\frac{1}{m^a\lambda^a}\int_0^{m\lambda}x^a e^{-\frac{\sigma}{\lambda}x}dx$$
$$-\frac{\sigma^2}{\lambda(1+\sigma)}\cdot\frac{1}{m^a\lambda^a}\int_0^{m\lambda}x^{a+1}e^{-\frac{\sigma}{\lambda}x}dx$$

$$P(Y > X) = I + II - III - IV$$

Now Integral is given by

$$I = \frac{\sigma^2}{\lambda(1+\sigma)}\int_0^{m\lambda}e^{-\frac{\sigma}{\lambda}x}dx$$

$$= \frac{\sigma}{(\sigma+1)}\left[1-e^{-m\sigma}\right]$$

$$II = \frac{\sigma^2}{\lambda(1+\sigma)}\int_0^{m\lambda}\frac{x}{\lambda}e^{-\frac{\sigma}{\lambda}x}dx$$

$$= \frac{1}{(1+\sigma)}\left(1-e^{-m\sigma}-m\lambda e^{-m\sigma}\right)$$

$$III = \frac{\sigma^2}{\lambda(1+\sigma)}\cdot\frac{1}{m^a\lambda^a}\int_0^{m\lambda}x^a e^{-\frac{\sigma}{\lambda}x}dx$$

$$= \frac{\sigma^2}{\lambda(1+\sigma)}\cdot\frac{1}{m^a\lambda^a}\cdot\frac{\lambda^{a+1}}{\sigma^{a+1}}\int_0^{m\lambda}t^{a+1-1}e^{-t}dt$$

$$= \frac{\sigma}{(1+\sigma)}\cdot\frac{1}{(m\sigma)^a}\gamma(a+1, m\sigma)$$

where $\gamma(a,m) = \int_0^m u^{a-1}e^{-u}du$, is incomplete gamma function.

$$IV = \frac{\sigma^2}{\lambda(1+\sigma)} \cdot \frac{1}{m^a \lambda^a} \int_0^{m\lambda} x^{a+1} e^{-\frac{\sigma}{\lambda}x} dx$$

$$= \frac{1}{(1+\sigma)} \cdot \frac{1}{m^a \lambda^a} \cdot \left(\frac{\lambda}{\sigma}\right)^a \cdot \gamma(a+2, m\sigma)$$

Thus, we get

$$P(Y > X) = I + II - III - IV$$

Therefore,

$$\begin{aligned} P(Y > X) = &\frac{1}{(1+\sigma)}\left\{\sigma\left(1-e^{-m\sigma}\right)+\left(1-e^{-m\sigma}-m\sigma e^{-m\sigma}\right)\right\} \\ &-\frac{1}{(1+\sigma)}\frac{1}{(m\sigma)^a}\left\{\sigma\gamma(a+1, m\sigma)-\gamma(a+2, m\sigma)\right\} \end{aligned} \tag{7.3.3}$$

Hence, the theorem follows.

Table 7.3. Strength-Reliability of an item for selected values of m and a for $\sigma = 1.5$

$m \rightarrow$ $a \downarrow$	2	3	4	5	6
1.0	0.6528	0.9051	0.9773	0.9952	0.9990
1.5	0.6613	0.9072	0.9768	0.9948	0.9989
2.0	0.6806	0.9146	0.9781	0.9948	0.9989
2.5	0.7033	0.924	0.9804	0.9952	0.9989
3.0	0.7258	0.9334	0.9829	0.9957	0.9990
3.5	0.7466	0.9415	0.9852	0.9963	0.9991

4.0	0.7653	0.9483	0.9870	0.9967	0.9992
4.5	0.7817	0.9536	0.9884	0.9971	0.9993
5.0	0.7960	0.9577	0.9895	0.9973	0.9993
5.5	0.8086	0.9607	0.9902	0.9975	0.9994
6.0	0.8195	0.9630	0.9907	0.9976	0.9994

Remarks.

3. **Table 7.3.** Shows that the strength reliability of item is increasing when we increase the values of m accordingly. If we increase the parametric values of power function i.e. ‘a’, the strength reliability of the system also increases.

7.4. Conclusion

In Chapter 7, the problem of Stress-Strength reliability model is studied for Sushila distributed stress. We have established the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items in order to study the problem under consideration. It is also considered that the Stress is distributed as Sushila distribution and Strength distributed as Power function distribution.

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List of Publications

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[1] M. J. S. Khan, **Ajay Kumar** and Arti Sharma (2018): Recurrence Relation for Single & Product Moments of Kth Record Values from Muth Distribution and Its Characterization. Asia Pacific Journal of Research, 1(LVV), 196-201.

[2] M. J. S. Khan, Surinder Kumar and **Ajay Kumar** (2018): Relations for Moments of Kumaraswami Power Function Distribution Based on Ordered Random Variables and a Characterization. International Journal of Advanced Research in Science, Engineering and Technology, 5(2), 5250-5257.

[3] **Ajay Kumar** (2018): On generalized order statistics from Power Lomax distribution and its characterization. **(Communicated)**.

[4] **Ajay Kumar** and M. A. R. Khan (2018): Relations for single and product moments of generalized order statistics from Sushila distribution. International Journal of Statistika and Mathematika, 26(1), 06-13.

[5] Surinder Kumar and **Ajay Kumar** (2016): A study of strength-reliability for shushila distributed stress. International Research Journal of Mathematics, Engineering and IT, 3(11), 22-28.

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