CLASSICAL AND BAYESIAN INFERENTIAL PROCEDURES IN VARIOUS LIFE TESTING MODELS

THESIS

SUBMITTED TO

BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY
(A CENTRAL UNIVERSITY)
LUCKNOW

BABASAHEB
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UNIVERSITY

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प्रज्ञा शील करूणा

ESTABLISHED 1996

FOR THE AWARD OF DEGREE OF

Doctor of Philosophy

IN

APPLIED STATISTICS

Submitted by

Vaidehi Singh

Under the Supervision of

Dr. Surinder Kumar

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Enrolment No. - 440/14 2020

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Enrolment No. - 440/14 **2020**

DECLARATION

I, **Vaidehi Singh**, Enrolment No. **440/14**, hereby declare that the work which is being presented in the thesis entitled "**CLASSICAL AND BAYESIAN INFERENTIAL PROCEDURES IN VARIOUS LIFE TESTING MODELS**" in fulfillment of the requirements for the award of the degree of Doctor of Philosophy and submitted in the Department of Applied Statistics of the Babasaheb Bhimrao Ambedkar University(A Central University), Lucknow is an authentic record of my own work carried out during a period from September, 2016 to August, 2020 under the supervision of Dr. Surinder Kumar, Professor, Department of Applied Statistics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma to this or any other University.

This is also declared that the thesis is essentially free from all kind of plagiarism.

Date:

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CERTIFICATE

This is to certify that the thesis titled "**CLASSICAL AND BAYESIAN INFERENTIAL PROCEDURES IN VARIOUS LIFE TESTING MODELS**" submitted by **Ms. Vaidehi Singh** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other University.

The thesis submitted to Babasaheb Bhimrao Ambedkar University Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations -1999 as amended in 2008/2010/2013* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

Date: Supervisor

Head of the Department

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List of Publications

1. Kumar, S., Singh, V. and Vaish, M. (2018): Robustness Study of the Sequential testing procedures for the New Weibull-Pareto Distribution, *J. Stat. Appl. Pro.*, Vol. 7(1), 59-73.

2. Kumar, S., Singh, V. and Gautam, P. (2018): Sequential testing procedure and their robustness study for Erlang distribution, *International Research Journal of Natural and Applied Sciences*, Vol. 5(3), 222-244.

3. Kumar, S. and Singh, V. (2018): Robustness Study if the SPRT for the Generalised Inverse Weibull (GIW) Distribution when the parameters are misspecified, *Pramana Research Journal*, Vol. 8(11), 242-263.

4. Kumar, S., Singh, V. and Gautam, P. (2019): Stress-Strength relationship between Exponentiated Weibull distribution and Power Function distribution, *International Journal of Research and Analytical Reviews*, Vol. 6(1), 201-204.

5. Kumar, S., Singh, V. and Gautam, P. (2020): Study of disaster probability when Strength follows Power Function distribution and Stress follows Odd Generalised Exponential Gompertz Distribution (OGE-G), *J. Stat. Appl. Pro.*, Vol. 9(3), 1-10. (Accepted)

6. Kumar, S. and Singh, V. (2020): Bayesian estimation for the scale parameter of a family of lifetime distributions under different priors, *Thailand Statistician.* (Accepted)

7. Kumar, S. and Singh, V. (2020): Sequential analysis and Robustness study for the parameters of Positive exponential family of distributions with known coefficient of variation. (Submitted)

8. Kumar, S. and Singh, V. (2020): Point estimator, Confidence interval for $P(Y > X)$ through Transformation method and the probability of disaster for a Positive Exponential Family of Distributions (PEFD). (Submitted)

9. Kumar, S. and Singh, V. (2020): Bayesian estimation methods for two parameter Generalized Rayleigh distribution under different loss functions and Type II censoring. (Submitted)

Contents

Chapter 1

Introduction

In (1940), Mahalanobis [77] stated the significance of sampling procedure in steps and developed various sampling designs for estimating the expanse of jute crop in the whole state of Bengal. This seminal evolution in large-scale sample survey is of national importance as regarded by many, including Abraham Wald, the forerunner of sequential analysis. The theory and methodology of sequential tests was developed by Wald and his collaborators to lessen the number of sampling inspections without negotiating the reliability of the terminal decisions. Sequential analysis began its march with deep-rooted motivations in response to demands for testing the efficiency of anti-aircraft gunnery and other weapons during the First World War. Fewer sampling inspections with accurate outcome were essential to gain advantage in the front line. This procedure solves a wide range of practical problems from queuing, inventory, life tests, reliability, quality control and designs of experiments. The collection of books and research articles are mentioned which provides immense knowledge about this field are: Wald (1947)[113], Bechhofer et al. (1968)[13], Ghosh (1970)[54], Chow et al. (1971)[35], Chernoff (1972)[35], Schmitz (1972)[93], Wetherill (1975)[116], Gibbons et al. (1977)[56], Shiryaev (1978)[96], Gupta and Panchapakesan (1979)[60], Govindarajulu (1981)[57], Sen (1981)[94], Sen (1985)[95], Woodroofe (1982)[117], Siegmund (1985)[97], Gut (1988)[61], Mukhopadhyay and Solanky (1994)[81], Ghosh et al. (1997)[55] and Govindarajulu (2004)[58].

1.1 Sequential Analysis

Sequential procedures differ from other statistical procedures as here the sample size is not fixed in advance. The experimenter has the option of looking at a sequence of observations one(or a fixed number) at a time and decide whether to: stop sampling and take a decision; or to continue sampling and make a decision some time later. The order of the sequence of observations which the experimenter will take is specified in advance. Decision problems in which the experimenter may sequentially vary the treatments is of a higher order of difficulty and is called the sequential design problem. For example, if we wish to compare several drugs or treatments(as in sequential screening of cancer drugs), then it should be possible to drop some drugs out of the trials at an early stage, if the results from these are very poor when compared with the others. An essential feature of a sequential procedure is that the number of observations required to terminate the experiment is a random variable since it depends on the outcome of the observations. Sequential procedures are of interest because they are economical in the sense that we may reach to a decision earlier via a sequential procedure than via a fixed-sample size procedure. In sequential experiments we need to specify:

(a) the initial sample size

(b) a rule for termination of the experiment

(c) the additional number of observations to take if the experiment is to be continued

(d) a terminal decision rule.

1.1.1 Applications of Sequential Analysis

The role played by the sequential procedures in real life phenomenon are:

(a) The application of adaptive and sequential methods in clinical trials has significantly improved the flexibility, efficiency, therapeutic effect and validity of trials. For example, sequential testing is done to know the dose given to a patient with hypertension. The trials stop when the same dose has been tested consecutively for the certain number of cohorts.

(b) An investigator may wish to estimate to within 10 percent the mean weekly expenditure on tobacco per household. In order to determine the sample size he would need an estimate of the expenditure from household to household, and this might be obtainable only from the survey itself.

(c) A physician wishing to compare the effects of two drugs in the treatment of some disease may wish to stop the investigation if at some stage a convincing difference can already be demonstrated using the available data.

1.2 Sampling Inspection Plan

The plan by which one is able to determine whether the population should be rejected or accepted is termed as sampling inspection plan. This decision is based on the number of defectives found in a random sample. The rejection takes place if the number of defectives exceeds a predefined value or level.

The double sampling plan introduced by Dodge and Romig (1929)[39] is the earliest sequential procedure, in which a lot contains 'n' items and the rejection(acceptance) depends on the number of defectives(d) in the sample i.e. $d \geq c(d < c)$, respectively. The drawback of the scheme is that the number of defective might have been more than 'c', that too earlier than the sample size 'n'. To overcome this drawback, an alternate scheme was proposed in which one item is taken at a time. The lot is rejected(accepted) as the number of defectives in the sample is $\geq c(\geq n - c + 1)$. The size of the sample is atmost 'n' and atleast 'c'. This particular scheme is known as curtailed inspection.

1.3 Sequential Probability Ratio Test (SPRT)

In 1947, Wald[113] developed the concept of SPRT for testing simple null hypothesis H_0 vs. simple alternative hypothesis H_1 where $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1 (> \theta_0)$. Let $f(x, \theta)$ denote the pdf (pmf) of a random variable X where θ is a scale parameter. Further, let the successive independent observations on X are $X_1, X_2, ..., X_m$ where 'm' is any positive integer. The probability that $(X_1, X_2, ..., X_m)$ is observed (Likelihood function) is p_m, where $p_m = \prod_{i=1}^{m} f(x_i, \theta)$. Let $p_{1m} = \prod_{i=1}^{m} f(x_i, \theta_1)$ and $p_{0m} = \prod_{i=1}^{m} f(x_i, \theta_0)$ stands for the probability that $(X_1, X_2, ..., X_m)$ is observed under H_0 and H_1, respectively. We consider the likelihood ratio:

$$\lambda_m = \frac{p_{1m}}{p_{0m}} = \frac{\prod_{i=1}^{m} f(x_i, \theta_1)}{\prod_{i=1}^{m} f(x_i, \theta_0)}; \quad m = 1, 2, ...$$

The SPRT for testing H_0 vs. H_1 is defined as:

Two constants A and B ($<$ A) are chosen. A and B are related to Type I and Type II errors, respectively. At each stage of experimentation say m^{th} stage the probability ratio is calculated. If this ratio satisfies the inequalities, then we reject H_0 if $\lambda_m < A; A > 1$ or we accept H_0 if $\lambda_m \geq B; B < 1$ or continue sampling by taking another observation, if $B < \lambda_m \leq A$. It is easier to operate with a logarithm and with the quantity,

$$Z_i = log \left[\frac{f(x_i; \theta_1)}{f(x_i; \theta_0)} \right]$$

Thus, if the m^{th} observation $(m > 1)$ has been taken in the sample, then we reject H_0 if, $\sum_{i=1}^{m} Z_i \geq logA$ and we accept H_0 if, $\sum_{i=1}^{m} Z_i \leq logB$ and proceed to examine the next $(m+1)^{th}$ observation if $logB \leq \sum_{i=1}^{m} Z_i \leq logA$, where A and B are the boundary points of strength of SPRT (α_0, β_0) defined as

$$A \leq \frac{1-\beta_0}{\alpha_0}; \quad B \geq \frac{\beta_0}{1-\alpha_0}$$

1.4 Properties of the SPRT

The properties of SPRT are:

(a) **Optimal Property:** In 1948, Wald and Wolfowitz proposed that among all tests (fixed-sample or sequential) for which $P(rejectH_0 \mid \theta_0) < \alpha$, $P(acceptH_0 \mid \theta_1) < \beta$ and for which $E(N \mid \theta_i) < \infty$; i $= 0, 1$ and the SPRT with error probabilities α and β minimizes both $E(N \mid \theta_0)$ and $E(N \mid \theta_1)$.

(b) **Monotonicity Property:** SPRT is said to have the monotonicity property, if at least one of the error probabilities decreases, when the upper stopping bound of the SPRT increases and the lower stopping bound decreases, unless the new test and the old test are equivalent. In this case, the error probabilities are unchanged. (Two tests are said to be equivalent if their sample paths differ on a set of probability zero under both the hypotheses).

(c) **Uniqueness Property:** There is at most one sequential probability ratio test for testing $H_0 : f(x) = f_0(x)$ vs. $H_1 : f(x) = f_1(x)$, that achieves a given α and β provided one of the following conditions hold:

(i) $f_1(X)/f_0(X)$ has a continuous distribution with positive probability on every interval in $(0, \infty)$.

(ii) The SPRT has stopping bounds which satisfy $0 < B < 1 < A$.

(iii) The SPRT has monotonicity property.

1.5 Performance Analysis of SPRT

The performance of the SPRT is evaluated through the following two concepts:

1.5.1 Operating Characteristic (OC) Function

Wald (1947)[113], formulated the creative method of obtaining the OC function of SPRT. OC function is the probability of accepting(rejecting) H_0 when $H_0 : \theta \in \omega (H_0 : \theta \notin \omega)$ and $\omega \in \Omega$ where Ω is the parametric space. It is denoted by $L(\theta)$ and describes how well the test procedure achieves the objective of making a correct decision. Since the acceptance(rejection) of H_0 any stage depends upon the common distribution of $f(x, \theta)$ of $X_1, X_2, ..., X_m (m = 1, 2, ...)$ which is determined by θ. The relationship of $L(\theta)$ to power and size can be easily established provided we assume that the sequential procedure terminate with probability 1. All possible cases are as follows:

- $L(\theta) =$ Probability of accepting H_0 when H_0 is true(false) i.e. $\theta \in \omega(\theta \notin \omega)$, respectively.
- $1 - L(\theta) =$ Probability of rejecting H_0 when H_0 is true(false) i.e. $\theta \in \omega(\theta \notin \omega)$, respectively.

Thus, $L(\theta) =$ Probability of correct decision; $\theta \in \omega$ and $1 - L(\theta) =$ Probability of correct decision; $\theta \notin \omega$. The OC function is defined as

$$L(\theta) = \frac{A^h - 1}{A^h - B^h}$$

$$= \frac{1 - A^h}{B^h - A^h}$$

where A and B are constants and $h \neq 0$ either $h < 0$ or $h > 0$. The OC curve is a plot that displays the probability of acceptance versus the percentage defective in a lot.

Approximate values of OC function						
h	-∞	-1	0	1	∞	h
θ	-	θ_1	-	θ_0	-	-
OC	0	β	$\frac{lnA}{lnA-lnB}$	$1-\alpha$	1	$\frac{A^h-1}{A^h-B^h}$

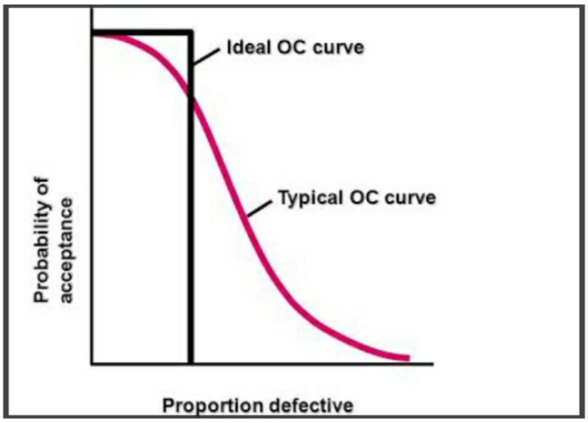


Figure 1.1: OC and Ideal OC curve

1.5.2 Average Sample Number (ASN) Function

A function which is used to describe the performance of a sequential test is the ASN function. This is the mean value of the sample number N requires to reach a terminal decision i.e. to accept H_0 or H_1 and therefore discontinue sampling. It explains how economic is the procedure in the terms of average amount of sampling required to reach a terminal decision. Obviously, a desirable feature of any sequential test is that ASN should be fairly small for all $\theta \in \Omega$ or atleast $\theta \in \omega$. The ASN function is defined as:

$$E_\theta(N) = \frac{L(\theta)logA - (1 - L(\theta))\,logB}{E_\theta(Z)}$$

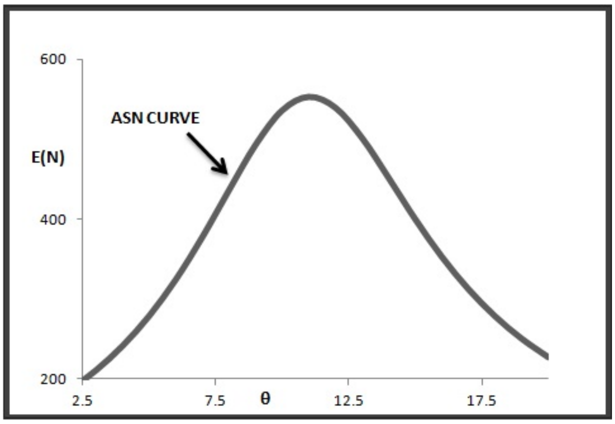


Figure 1.2: ASN curve

1.6 Concept of acceptance and rejection region

Let X_1, X_2, ... be i.i.d. random variables where $\theta \in R$. One wish to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1 (> \theta_0)$ having pre-assigned $0 < \alpha, \beta < 1$. Let $A = (1-\beta)/\alpha$ and $B = \beta/(1-\alpha)$. Now, $Z_i = ln\left[\frac{f(X_i|\theta_1)}{f(X_i|\theta_0)}\right]$, $i = 1, 2, ...$. The SPRT can be simplified as follows:
Let $Y(n) = \sum_{i=1}^{n} X_i$ and $N \equiv$ first integer $n(\geq)$ for which the inequality $Y(n) \leq c_1 + dn$ or $Y(n) \leq c_1 + dn$ holds with

$$c_1 = \frac{ln(B)}{\text{coeff. of } x_i},\ c_2 = \frac{ln(A)}{\text{coeff. of } x_i} \text{ and } d = \frac{\text{intercept form}}{\text{coeff. of } x_i}$$

At the stopped stage, if $Y(N) \leq c_1 + dN$, we accept H_0, but if $Y(N) \leq c_2 + dN$, we accept H_1.

1.7 Progress in the field of Sequential Analysis

Following Wald's (1947)[113] SPRT, Oakland (1950)[86] derived SPRT for testing between two simple hypotheses concerning the mean of the negative binomial distribution assuming dispersion parameter as known. Epstein and Sobel (1955)[46] developed SPRT for testing simple null hypothesis against simple alternative, for the scale parameter of an exponential distribution. Harter and Moore (1976)[66] conducted Monte Carlo study to investigate the robustness of exponential SPRT when the underlying distribution is a Weibull with shape parameter other than one. Montagne and Singpurwalla (1985)[80] generalized the results of Harter and Moore (1976)[66] from Weibull to a class of distribution shaving an increasing(decreasing) failure rate. They obtained inequalities for OC and ASN functions in order to demonstrate the robustness. Phatarfod (1971)[88] developed SPRT for testing composite hypothesis for the shape parameter of the gamma distribution. For some fixed sample size results concerning the robustness of testing and estimation procedures related to exponential distribution, one may refer to Barlow and Proschan (1967)[8] and Hager, Bain and Antle (1971)[62]. Chaturvedi, Kumar and Kumar (2000)[27] developed SPRT'S for the parameters of a family of continuous distributions. Pandit and Gudaganavar (2010)[87] developed SPRT for scale parameter of gamma and exponential distribution. Kharin (2011)[71] studied the robustness for Bayesian sequential testing procedures of composite hypothesis.

1.8 The Concept of Reliability

In today's scenario, there is an immense competition among the different kind of industrial set up. The focus lies in the efficiency and the durability of the manufactured products or goods. To improve or enhance the quality as well as the reliability of a system, it needs to be tested for a required or estimated period of time. With the help of extraordinary life testing models, one can obtain the significant or fruitful results so as to justify the study.

An experiment which is performed for a certain period of time by applying a set up in which a number of units as a product/device are subjected to certain predefined conditions and their performances are observed, popularly known as life testing experiments. It provides immense knowledge about the average life of a system or a product. During a life testing experiment, one or more failures may occur at anytime, since the lifetime of an item or a system is purely uncertain. Thus, the lifetime of a unit is considered as a random variable with a certain lifetime distribution. The motive of life testing is the concern regarding the failure.

The term 'reliability' refers to the potency of a system to perform its stated purpose efficiently for a specified period of time under the operational condition encountered. Any system will be absolutely reliable if some undesirable events, called failures, do not occur in the system's operation. Each and every system has its own set of such undesirable events. For example, a failure of a watch may be defined as a delay exceeding seconds over 24 hours period. For a mechanical system, a failure is a breakdown(a crack) of some of its parts or an increase in vibration above the permitted level, etc. One of the most dangerous failure of a nuclear reactor is a leak of radioactive material. For a missile, the failure could mean missing the target or exploding before hitting it.

In many fields of transport, industry, communication technology etc., the problem of increasing reliability of units becomes more important and urgent in connection with the complex mechanization and automation of industrial processes. The significance of this problem is explained by the fact that insufficient reliability of units engenders great loss in their servicing, partial stoppages of equipment, and these may be accidents with considerable damage to the equipment and even human injuries. In some cases, automatic devices are less profitable than non-automatic ones primarily because of their unreliability. The factors due to underestimation are associated with reliability cause expenditure in the course of the first few years of use to considerable exceed the original cost of the units. In the present scenario, the role of machines has a great impact on all spheres of human activity. The problem solved by the machines, mainly the control machines are becoming even more complicated. The increase in the complexity of the problem leads to the complexity in forming a machine to overcome it, and the ever-increasing complexity of system results in decrease in the reliability. On the other hand, the requirements on the reliable performance of these system become even more demanding. Reliability theory itself serves in the search for ways of resolving this by the two different approaches. Firstly, by increasing the quality and reliability of the individual elements of which the composite system is composed. Secondly, by developing special ways of constructing reliable complex systems from unreliable elements and also improvising methods of servicing such system during their use.

The basic feature of failure is that it has a random(stochastic) nature. A failure results in a joint action of many unpredictable random processes going on 'inside' the operating system as well as in the environment where the operations takes place. Therefore, with the help of statistical methodology, an adequate treatment of system reliability can be performed.

The reliability of a system can be characterized by distinguishing between the non-renewable(non-repairable) and renewable(repairable) systems. An appropriate example of non-renewable system is a missile or a rocket which is designed to carry out a single mission and cannot be reused. In such cases, the reliability characteristics are usually measured in terms of lifetime where lifetime is a random time from the beginning of the operation until the occurrence of a failure. Few examples in which lifetime data arise are, firstly, the electronic items are put under tests in a laboratory setting and are observed until they fail. Secondly, there are items which can be repaired where we are interested in the length between the two successive failures. Thirdly, in the medical studies while dealing with fatal diseases one takes into consideration the survival time of individuals with the disease, measured from the date of diagnosis or some other starting points.

The lifetime is not always measured in terms of calender time, it is also expressed in terms of minutes, hours, days or in number of cycles etc. In need to study the life time distributions where X is a random variable with $f(x)$ and $F(x)$ be the probability distribution function and distribution function, respectively. The reliability or survival probability of a new unit corresponding to a 'mission time' of duration 't' can be defined as follows

$$
\begin{aligned}
R(t) &= P(X > t) \\
&= 1 - P(X \leq t) \\
&= 1 - F(t)
\end{aligned}
$$

1.9 Conceptual Terminologies in Reliability

Let us consider a random variable X representing the 'lifetime' of a device or component, here $F(x)$ represents the distribution function and $f(x)$ corresponds probability density function (pdf).

1.9.1 Hazard Rate

The hazard rate(or conditional failure rate) is a metric which is usually used for identifying the appropriate probability distribution of a particular mechanism. The hazard rate $h(t)$ is defined as

$$
h(t) = \frac{f(t)}{1 - F(t)}
$$

$$
h(t) = \frac{f(t)}{R(t)} \tag{1.9.1}
$$

which is the function of time, has a probabilistic interpretation, namely, $h(t)dt$ represents the probability that a device of age 't' will fail in the interval $(t,\, t + \triangle t)$, i.e.

$$
h(t) = \lim_{\triangle t \to 0} \left[\frac{P(\text{a device of age t will fail in the interval}\,(t + \triangle t))}{\triangle t} \right]
$$

The hazard rate of a large population of statistically identical and independent items exhibits often a typical bathtub curve (Figure 1.3) with the following 3 phases:

a. Early failures: $h(t)$ decreases(in general) rapidly with time; failures in this phase are attributable to randomly distributed weaknesses in materials, components, or production processes.

b. Failures with constant(or nearly so) failure rate: $h(t)$ is approximately constant; failures in this period are Poisson distributed and often cataleptic.

c. Wearout failures: $h(t)$ increases with time; failures in this period are attributable to aging, wearout, fatigue, etc. (e.g. corrosion, electro-migration).

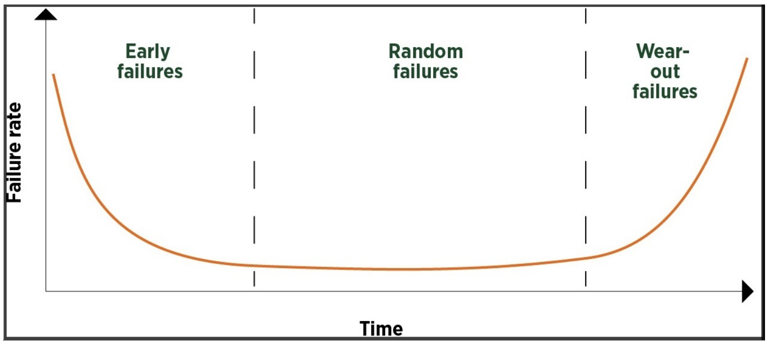


Figure 1.3: Bathtub curve

Now, we state the mathematical relationships between $h(t)$, $f(t)$ and $F(t)$. Since $t > 0$, obviously, $F(0) = 0$ and $F(\infty) = 1$. We have

$$\int_0^x f(s)ds = F(x)$$
$$\frac{d}{dx}F(x) = f(x)$$

Now, it follows from (1.9.1) that

$$h(x)dx = \frac{dF(x)}{1 - F(x)}$$

or,

$$\int_0^t h(x)dx = -ln[1 - F(x)]$$

Thus,

$$ln\left(\frac{1-F(t)}{1-F(0)}\right) = -\int_0^t h(x)dx$$

or,

$$1 - F(t) = exp\left(-\int_0^t h(x)dx\right) \tag{1.9.2}$$

Taking derivatives, one obtains from (1.9.2) that

$$f(t) = h(t)exp\left(-\int_0^t h(x)dx\right)$$

1.9.2 Cumulative Hazard Function

The cumulative hazard function is the integral of the hazard function. It can be interpreted as the probability of failure at time 't' given survival until time 't':

$$H(t) = \int_0^t h(x)dx$$

The functions $f(t)$, $F(t)$, $S(t)$, $h(t)$ and $H(t)$ are mathematically related by the following relations:

$$h(t) = -\frac{d}{dt}lnS(t)$$

$$S(t) = exp\left(-\int_0^t h(x)dx\right)$$

and

$$f(t) = h(t)exp\left(-\int_0^t h(x)dx\right)$$

Moreover, $S(\infty)$=0, $H(\infty) = \infty$, giving $h(t) \geq 0$ and $\int_0^\infty h(t)dt = \infty$

1.9.3 Complete and Censored Sample

If the test is allowed to run until all the 'n' components have failed and the lifetimes are recorded, the data set thereby obtained is said to be complete. Often, it is impractical or too expensive to wait until all the components have failed. Hence, censoring is applied to cease the test before all components have failed. In statistics, censoring is a condition in which the value of a measurement or observation is only partially known. For example, suppose a study is conducted to measure the impact of a drug on mortality rate. In such a study, it may be known that an individual's age at death is atleast 75 years. Such a situation can occur if the individual withdrew from the study at age 75 years or is currently alive at the age of 75 years. Censoring also occurs when a value occurs outside the range of a measuring instrument. For example, a bathroom scale might only measure upto 140 kgs. If a 160 kgs individual is weighed using the scale the observer would only know that the individual's weight is atleast 140 kgs.

Types of censoring:

(a). Type I censoring occurs if an experiment has a set number of subjects or items and stops the experiment at a predetermined time.

(b). Type II censoring occurs if an experiment has a set number of subjects or items and stops the experiment at a predetermined number are observed to have failed.

(c). Mixed or hybrid censoring is the combination of the Type I and Type II censoring. They have been adopted in competing risks set-up and in step-stress modeling. It can be further classified as Type I hybrid censoring(the termination time is pre-fixed) and the Type II hybrid censoring schemes(the termination time is a random variable).

(d). Random or non informative censoring is when each subject has a censoring time that is statistically independent of their failure time. The observed value is the minimum of the censoring and the failure times; subjects whose failure time is greater than their censoring time are right censored.

(e). Progressive Type II censoring is a censoring scheme that fulfills the drawbacks of the other censoring schemes as it allows the removal of units at various stages other than the terminal point of the experiment. In recent years, the progressive censoring scheme has received considerable attention in the statistical literature, see for instance the book by Balakrishnan and Aggrawalla (2000)[6] and an excellent review article by Balakrishnan (2007)[7].

1.9.4 Maintainability

Maintenance defines the set of activities performed on an item to retain it in or to restore it to a specified state. Maintenance is thus subdivided into preventive maintenance, carried out at predetermined intervals to reduce wearout failures, and corrective maintenance, carried out after failure recognition and intended to put the item into a state in which it can again perform the required function. Maintainability is a characteristic of an item, expressed by the probability that a preventive maintenance or a repair of the item will be performed within a stated time intentional for given procedures and resources(skill level of personnel, spare parts, test facilities, etc.). From a qualitative point of view, maintainability can be defined as the ability of an item to be retained in or restored to a specified state.

1.9.5 Availability

Availability is a broad term, expressing the ratio of delivered to expected service. It is often designated by A and used for the stationary and steady-state value of the point and average availability (PA = AA). Availability evaluations are often difficult, as logistic support and human factors should be considered in addition to reliability and maintainability. Ideal human and logistic support conditions are thus often assumed, yielding to the intrinsic(inherent) availability. Hereafter, availability is used as a synonym for intrinsic availability.

$$\text{Availability} = \frac{\text{Up-time}}{\text{Up-time} + \text{Down-time}}$$

where up-time is the actual period for which the equipment is required is available for use and down-time is the specified time for which the failed equipment is restored to operable condition.

1.9.6 Mean time to failure (MTTF)

We are often interested in knowing the mean time to failure of a component rather than the complete failure details. This parameter will be assumed to be same for all the components which are identical in the design and operate under identical conditions. If we have life-tests information on a population of N items with failure times t_1, t_2, ..., t_n, then the MTTF is defined as

$$MTTF = \frac{1}{N}\sum t_i$$

1.10 Stress-Strength Models

A stress-strength model compares the strength and stress on a system; it is used primarily in reliability engineering. In a stress-strength model, both stresses and strength are considered as separate random variables. Stress experienced by a component is often represented by the random variables designated X; strength of the component is represented by Y. A situation in which $X > Y$ is one in which the stresses are greater than the strengths and the component fails. If $Y > X$, the strengths are greater than the stresses. Then, we can define reliability as the probability a component will not fail: $P(X < Y)$. This $R = Pr(X < Y)$, is the basic stress-strength model, and refining it and applying it to real life analysis is the essence of stress-strength analysis.

1.10.1 Applications of Stress-Strength Models

The real life examples based on stress-strength models are:

(a). Reliability of Rocket Engines: When Y is the strength of a rocket chamber and X stands for the maximal chamber pressure which is generated when a solid propellent is ignited, $P(X < Y)$ is the probability that the engine will be fired successfully.

(b). Earthquake Resistance: The strength-stress model was used to study the risk an earthquake posed to a particular nuclear generator. With no concrete numbers to define the strength, the researcher took strength estimates from five experts and used the log-normal distribution as a model and a weighted least squares procedure to estimate the strength. A similar procedure was used for the stressor, and the conclusion $P(lnX < lnY) = 0.99978$ was reached a very reassuring number, if accurate.

(c). Medical Field: In a medical study, the reaction of leprosy patients to a medicine was modeled on a $P(X < Y)$ stress-strength model. Initial condition(infiltration status) was taken as X and Y as the change in health after 48 weeks of treatment. The null hypothesis, that initial infiltration values did not affect outcomes, was strongly rejected after an analysis of the data.

1.11 Classical Inferential Procedures in Reliability

The different methods used to estimate the reliability of a model are:

- Maximum Likelihood Estimation
- Unbiased Estimation
- Bayes Estimation
- Interval Estimation

1.12 Concept of Transformation Method

The point and interval estimation of the stress-strength reliability can be expressed by using the transformation method. Let us consider (X, Y) be a random vector with the probability density function $f(x, y|\theta)$. Suppose that there exist a random variable ε and η and a monotone function $u(.)$ with the inverse $v = u^{-1}$ such that

$$X = u(\varepsilon) \leftrightarrow \varepsilon = v(X), \qquad Y = u(\eta) \leftrightarrow \eta = v(Y).$$

Also, assume that the function u and v are strictly increasing, so that (ε, η) is the random vector with the pdf

$$g(\varepsilon, \eta|\tau) = f\left(u(\varepsilon), u(\eta)|\nu(\tau)\right) u^{'}(\varepsilon)u^{'}(\eta)$$

where the scalar or vector valued parameter τ is connected to θ by one to one transformation. Since, the function $u(.)$ is monotonically increasing, $P(\varepsilon < \eta) = P(u(\varepsilon) < u(\eta)) = P(X < Y)$. Hence, the model $R = P(X < Y)$ remains invariant in terms of τ and θ.

1.13 Advancement in the field of Reliability

Various authors have adopted classical inferential procedures in the field of life-testing and reliability theory. Cohen (1951)[37] and Harter (1969)[65] obtained local maximum likelihood estimators (LMLE'S). Harter and Moore (1966)[64] and later Calitz (1973)[20] noted that these LMLE'S appear to possess most of the desirable properties ordinarily associated with Maximum likelihood estimator (MLE). Epstein and Sobel (1953)[44] derived the MLE of the scale parameter of the one parameter exponential distribution in case of censoring from right. Epstein and Sobel (1954)[45] and Epstein (1960)[47] extended the foregoing analysis to the two-parameter exponential distribution with and without censoring. Mann et al. (1974)[78] summarized on industrial life testing point of view, estimation procedures for exponential, Weibull, log-normal and many other distributions, both for single sample and two sample problems, with censoring. Engelhardt and Bain (1974)[43] dealt with point estimation for two parameter Weibull distribution. Tyagi and Bhattacharya (1989a)[109] obtained the uniformly minimum variance unbiased estimators (UMVUE'S) of various parametric functions of Maxwell failure distribution. Chaturvedi and Rani (1998)[25] proposed generalized

Maxwell failure distribution and derived the UMVUE of the reliability function. Chaturvedi and Surinder (1999)[26] revisited the problem of obtaining UMVUE of the reliability function of one-parameter exponential distribution under Type I and II censorings. Tyagi et al. (1993)[111] derived the UMVUE of the reliability function for inverted Erlang failure distribution. Classical inferences for Weibull process have been developed by Muralidharan (2002)[83]. Classical inferential procedures for some families of lifetime distributions have been developed by Chaturvedi and Rani (1997)[23]. Chaturvedi and Pathak (2012)[32] presented the estimation of the reliability function for exponentiated Weibull distribution. Tripathi et al. (2016)[108] showed the study of estimating the shape parameter of a Pareto distribution under restrictions.

1.14 Bayesian Inference

Bayesian estimation is a framework for the formulation of statistical inference problems. In the prediction or estimation of a random process from a related observation signal, the Bayesian philosophy is based on combining the evidence contained in the signal with prior knowledge of the probability distribution of the process. Bayesian methodology includes the classical estimators such as maximum a posteriori (MAP), maximum-likelihood (ML), minimum mean square error (MMSE) and minimum mean absolute value of error (MAVE) as special cases. The estimation accuracy depends on the available information and on the efficiency of the estimator.

1.14.1 Bayes Theorem

Bayes theorem is an essential element of the Bayesian approach to statistical inference. It is also known as Bayes Rule or Bayes law and results in probability theory that relates conditional probabilities. An important application of Bayes theorem is that it gives a rule how to update or revise the strengths of evidence based beliefs in light of new evidence i.e. a posteriori. The theorem relates the conditional and marginal probabilities of stochastic events A and B such that

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}, P(B) > 0$$

1.14.2 Bayes Factor

The theory of Bayes factor is one way of performing hypothesis testing in the Bayesian framework. Suppose we have an observed event E and we wish to test a null hypothesis $H_0 : E = E_0$ vs. alternative hypothesis $H_1 : E = E_1$, where E_1 and E_0 are two events which are not necessarily mutually exclusive or even exhaustive of the event space. The Bayes Factor(BF) is defined as

$$BF = \frac{p_0/p_1}{\pi_0/\pi_1}$$

where p_0 and p_1 are the posterior probability of the null and alternative hypothesis, respectively and π_0 and π_1 are the prior probability of the null and alternative hypothesis, respectively. The magnitude of BF tells us how much effect the data has on the relative likelihood. The Bayes factor may be interpreted as the ratio of the likelihood based on the null hypothesis to the likelihood based on the alternative hypothesis. It is highly used in the disease testing procedure.

1.14.3 Bayesian models

A Bayesian model is a parametric model in the classical(or frequentist) sense, but with the addition of a prior probability distribution for the model parameter, which is treated as a random variable rather than an unknown constant. The basic components of a Bayesian model may be listed as:

- the data, denoted by y
- the parameter, denoted by θ

- the model distribution, $f(y \mid \theta)$, $F(y \mid \theta)$
- the prior distribution, $f(\theta)$, $F(\theta)$

1.15 Prior Distributions

The prior distribution is a way to summarize the available prior information. It may also be considered a tool which provides a unified inferential procedures having acceptable frequentist properties. There is no unique way of selecting a prior distribution but the resulting inference may be influenced by the prior information, i.e. the effect may be negligible, moderate or enormous. So, selecting a prior is still a tough task.

A prior $\pi(\theta)$ is said to be **non informative prior** if it contains no information about the parameter θ i.e. the distribution does not favour any value of θ over others. Some simple examples of such prior are $\pi(\theta) = 1$, $\pi(\theta) = 1/\theta$, $\pi(\theta) = \sqrt{I(\theta)}$. These may often lead to infinite mass and the density may be improper(i.e. it does not integrate to 1). Mostly, the principle of indifference is preferred for determining a non informative prior which assigns equal probabilities to all the possibilities. **Informative prior** is one which contains some specific information about the parameter. It is the pre-existing evidence which has already been taken into account is part of the prior and as more evidences accumulates, the posterior is determined largely by the evidence rather by the original assumption. In many circumstances, the sum or the integral of the prior values are not finite and do not provide sensible outcome for the posterior probabilities, such a prior is known as **Improper prior**. **Proper prior** is just the reciprocal of the improper prior as here one is able to get finite values for the posterior probabilities. Some frequently used priors are as follows:

- **Uniform Prior** is the highly accepted prior in a state of ignorance. We adopt a uniform prior when we have very little or no information regarding the parameter θ. This prior was highly used by Guass, Bernoulli and Laplace. Even, Bayes has also used it in his revolutionary work "An essay towards solving a problem of doctrine of chance".

- **Jeffery's prior** is defined in terms of the Fisher information:

$$\pi_J(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ is $k \times k$ Fisher's information matrix given by

$$I(\theta) = -E\left[\frac{\partial^2 log f(x \mid \theta)}{\partial\theta_i \partial\theta_j}\right]$$

Jeffreys priors work well for single parameter models, but not for models with multidimensional parameters. It is nearly uniformly distributed for the location parameter.

- **Conjugate Prior** is a natural parameter family of distributions such that the posterior distribution also belong to the same family. These priors make the computations much easier. Sufficient statistic plays an important role in Bayesian inference in order to construct them. Let $X \sim f(x \mid \theta)$ and $\pi(\theta)$ be the prior distribution on Θ. Then, π is said to be a conjugate family, if the corresponding posterior distribution $h(\theta \mid x)$ belongs to the same family as $\pi(\theta)$. Few examples of conjugate prior are gamma, beta and inverted gamma prior etc.

1.16 Posterior distribution

Bayesian inference requires determination of the posterior probability distribution of θ. This task is equivalent to finding the posterior pdf of θ, which may be done using the equation

$$f(\theta \mid y) = \frac{f(\theta) f(y \mid \theta)}{f(y)}$$

where $f(y)$ is the unconditional pdf of y given as

$$f(y) = \int f(y \mid \theta) dF(\theta)$$
$$= \begin{cases} \int f(\theta) f(y \mid \theta) d(\theta); & \theta \text{ is continuous} \\ \sum_\theta f(\theta) f(y \mid \theta); & \theta \text{ is discrete} \end{cases}$$

Bayesian inference is based on the properties of posterior distribution. The relationship between the posterior density $f(\theta \mid y)$, the prior density $f(\theta)$ and the likelihood $L(\theta)$ is given as

$$f(\theta \mid y) \propto L(\theta) \times f(\theta)$$

1.17 Loss functions

Let $\mathcal{A}$ be an arbitrary space of actions. A non negative function 'l' that maps $\Theta \times \mathcal{A}$ into $\mathcal{R}$ is called a loss function. The value $l(\theta, a)$ is the loss to the statistician if he takes action 'a' when 'θ' is the true parameter value. If we use the decision function $\delta(X)$ and loss function 'l' and 'θ' is the true parameter value, the loss is the random variable $l(\theta, \delta(X))$. Some well known loss functions are as follows:

- **Squared Error Loss Function (SELF)** is a symmetric loss function which gives equal importance to the loss incurred due to over and under estimation. But in situations where the loss is asymmetric, we need to have some kind of asymmetric loss functions.

- **Weighted Squared Error Loss Function (WSELF)** is basically obtained by the product of SELF and a function of the parameter 'θ'.

- **Quadratic Loss Function (QLF)** is known as modified SELF in which we multiply SELF with a constant other than 1.

- **Precautionary Loss Function (PLF)** is one of asymmetric loss function which is proposed by Norstrom (1996)[85]. The loss function approach infinitely near the origin to prevent underestimation and giving conservative estimators, especially when low failure rates are being estimated.

- **LINEX loss Function** was suggested by Varian (1975)[112] and was used by Zellner (1986)[118]. It is an asymmetric loss function used for the estimation of a scaler parameter and prediction of a scalar random variable.

- **Generalised Entropy Loss Function (GELF)** was purported by Calabria and Pulcini (1996)[19], as a valid alternative to the modified LINEX loss. Putting a=1, in the GELF we get the entopy loss function(ELF).

- **K-Loss Function (KLF)** was proposed by Wasan (1970)[115], to fit for a measure of inaccuracy for an estimator of a scale parameter of a distribution on $R^+ = (0, \infty)$.

Bayes estimator and Posterior risk under different loss functions		
Loss Function	**Bayes Estimator**	**Posterior Risk**
$SELF = (\theta - d)^2$	$E(\theta \mid x)$	$Var(\theta \mid x)$
$WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1} \mid x))^{-1}$	$E(\theta \mid x) - (E(\theta^{-1} \mid x))^{-1}$
$QLF = \left(1 - \frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1}\mid x)}{\theta^{-2}\mid x}$	$1 - \frac{E(\theta^{-1}\mid x)^2}{\theta^{-2}\mid x}$
$PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2 \mid x)}$	$2\left[\sqrt{E(\theta^2 \mid x)} - E(\theta \mid x)\right]$
$LINEX = exp(c(d-\theta)) - c(d-\theta) - 1$	$-\frac{1}{c} logE(e^{-c\theta})$	$E(\theta \mid x)$
$LLF = (log\theta - logd)^2$	$exp(E(log\theta \mid x))$	$Var(log\theta \mid x)$
$KLF = \left(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}}\right)^2$	$\sqrt{\frac{E(\theta\mid x)}{E(\theta^{-1}\mid x)}}$	$2\left[E(\theta \mid x)E(\theta^{-1} \mid x) - 1\right]$
$ELF = \left[\frac{d}{\theta} - log\frac{d}{\theta} - 1\right]$	$\left[E(\theta^{-1} \mid x)\right]^{-1}$	$E(log\theta \mid x) - logE(\theta^{-1} \mid x)$
$GELF = \left(\frac{d}{\theta}\right)^a - alog\left(\frac{d}{\theta}\right) - 1$	$\left[E\left((\theta)^{-a}\right)\right]^{-1/a}$	$E\left[\left(\frac{d}{\theta}\right)^a - alog\left(\frac{d}{\theta}\right) - 1\right]$

1.18 Types of risk

1.18.1 Risk

The risk is a function which quantifies the quality of the estimator $\hat{\theta}$ at θ. It is expressed as $R_\theta(\hat{\theta}) = E_\theta\left[l(\theta, \hat{\theta})\right]$, where $l(\theta, \hat{\theta})$ denotes the loss function. In other words, the risk function of a decision rule is defined as $R(\theta, \delta) = E\left[L(\theta, \delta(Y))\right]$, where the expectation is taken with respect to $f(y \mid \theta)$.

1.18.2 Bayes risk

The Bayes approach is an average-case analysis which considers the average risk of an estimator over all $\theta \in \Theta$. Concretely, we set a probability distribution called prior (π) on Θ. Then, the average risk (w.r.t. π) is defined as

$$\begin{aligned} R_\pi(\hat{\theta}) &= E_{\theta\sim\pi} R_\theta(\hat{\theta}) \\ &= E_{\theta,X}\, l(\theta, \hat{\theta}) \end{aligned}$$

The Bayes risk for a prior (π) is the minimum that the average risk can achieve, i.e.

$$R_\pi^* = \inf_\theta R_\pi(\hat{\theta})$$

In terms of decision rule, Bayes risk with respect to prior distribution $g(\theta)$ can be expressed as $r(g, \delta) = E(R(\theta, \delta))$ where the expectation is taken with respect to $g(\theta)$. A decision rule which minimises the bayes risk and is termed as Bayes rule.

1.18.3 Posterior Risk

The posterior risk is defined as the expected loss, where the expectation is taken with respect to the posterior distribution of 'θ'.

1.19 Evolution in the field of Bayesian Analysis

The Bayesian ideas in reliability were introduced by Bhattacharya (1967)[15] and considered the Bayesian estimation for one parameter exponential distribution under uniform and beta priors. Similar estimators have been obtained by Bhattacharya and Kumar (1986)[16], Bhattacharya and Tyagi (1988)[17], Harris and Singpurwalla (1968)[63], Soland (1969)[99] and Canavos and Tsokos (1971)[21]. For a detailed review on Bayesian reliability estimation, one may refer to Martz and Waller (1982)[79], Tyagi and Bhattacharya (1989b)[110], Alvandi (1990)[2], Basu and Ebrahimi (1991)[12], Calabria and Pulcini (1994)[18], Chaturvedi and Rani (1998)[25], Chaturvedi and Singh (2006)[29], Chaturvedi and Tomer (2002)[28], Chaturvedi et al. (2007)[30] and Chaturvedi and Pathak (2013)[33].

1.20 Significant Aspects

1.20.1 Methods of Random Number Generation

(a). The inverse transform: There is a simple and sometimes useful transformation known as the probability integral transform that allows us to transform any random variable into a uniform random variable and more importantly, vice versa. For example, if X has density $f(x)$ and cdf $F(x)$, then we have the relation $F(x) = \int_{-\infty}^{x} f(t)dt$ and if we set $U = F(X)$, then U is a random variable distributed from uniform *U(0, 1)*.

(b). Accept-reject method: There are many distributions for which the inverse transform method and even general transformations will fail to generate the required random variables.

For such cases, we must turn to indirect methods; i.e. methods in which we generate a candidate random variable and only accept it subject to passing a test. As we will see, this class of methods is extremely powerful and will allow us to simulate from virtually any distribution. These so-called Accept-Reject methods only require us to know the functional form of the density $f(x)$ of interest(called the target density) up to a multiplicative constant. We use a simpler density $g(x)$ to simulate, called the instrumental or candidate density, to generate the random variable for which the simulation is actually done. The only constraints we impose on this candidate density $g(x)$ are

(i). $f(x)$ and $g(x)$ have compatible supports (i.e., $g(x) > 0$ when $f(x) > 0$).

(ii). There is a constant M with $f(x)/g(x) \leq M$ for all x.

In this case, X can be simulated as follows: First, we generate $Y \sim g$ and independently we generate $U \sim U_{[0,1]}$. If

$$U \leq \frac{1}{M} \frac{f(Y)}{g(Y)}$$

then, we set X = Y. If the inequality is not satisfied, we then discard Y and U and start again.

1.20.2 Markov Chain Monte Carlo (MCMC) Techniques

MCMC methods are a class of algorithms for sampling from a probability distribution based on constructing a Markov chain that has the desired distribution of its equilibrium distribution. The state of the chain after a number of steps is then used as a sample of the desired distribution. The quality of the sample improves as a function of the number of steps. In Bayesian statistics, they are also used for generating samples that gradually populate the rare failure region in rare event sampling. **Gibbs sampling** or a Gibbs sampler is a MCMC algorithm for obtaining a sequence of observations which are approximated from a specified multivariate probability distribution, when direct sampling is difficult. It generates a Markov chain of samples, each of which is correlated with nearby samples. **Metropolis-Hastings algorithm** is a MCMC method for obtaining a sequence of random samples from a probability distribution for which direct sampling is difficult. They are generally used for sampling from multi-dimensional distributions, especially, when the number of dimensions is high.

1.21 Content of the Thesis

In this particular chapter, i.e. Chapter 1 provides introductory review of the research topics covered in the thesis and also state the significance as well as their applications.

In Chapter 2, sequential testing procedures are developed for testing the hypotheses regarding the parameters of the New Weibull-Pareto Distribution (NWPD). Theoretical expression for the Operating Characteristics (OC) and Average Sample Number (ASN) functions are derived for the scale parameters of the distribution. The robustness of the SPRT'S in respect of OC and ASN functions is studied, when the distribution under study has undergone a change. The results are presented through Tables and Graphs, so that one can see the numerical evaluated departures in OC and ASN functions.

In Chapter 3, the sequential testing procedures are developed for testing the hypotheses regarding the shape and rate parameters of the Positive Exponential Family of Distribution(PEFD). The robustness of the SPRT'S in respect of OC and ASN functions are studied, when the distribution under study has undergone a change. The acceptance and rejection regions for H_0 against H_1 are derived in case of rate parameter. The expressions of OC and ASN functions for the robustness of the SPRT in case of rate parameter, when the coefficient of variation is known are also derived and studied. Finally, the results are presented through Tables and Graphs, so that one can see the numerical evaluated departures in OC and ASN functions.

In Chapter 4, the problem of Sequential Probability Ratio Test (SPRT) is considered for Generalised Inverse Weibull Distribution (GIWD). The GIWD has hazard function which has a unimodal shape. Hence, the GIWD could be an appropriate model for fitting the data which has the convex and then the concave shape. Robustness of the SPRT is studied for the parameters involved in this model, under the conditions when these parameters have undergone a change.

In Chapter 5, the probability of disaster is studied when the strength of the items follows power function distribution and the stress of the manufactured items/devices follows OGE-G distribution. In order to study the probability of disaster, a relationship between the parameters of OGE-G and power function distribution is established through the reliability measure $P = Pr(Y > X)$. Finally, through the relationship among the parameters involved in the model is used to get the optimum cost function when the cost function is linear in terms of parameters.

In Chapter 6, the estimation of $R(t)$, $R = P(Y > X)$ and $\alpha = P(X > Y)$ for the Positive Exponential Family of Distribution (PEFD) is considered. The UMVUE'S, MLE'S and Confidence Interval are derived. These estimators are derived through the method of Transformation. The $\alpha = P(X > \gamma)$, which is termed as probability of disaster is also derived when random stress X follows PEFD and finite strength follows Power function distribution.

In Chapter 7, the Bayes estimators for the scale parameter of a family of lifetime distributions are considered under the assumptions of non-informative and conjugate priors. The uniform and inverted gamma priors are observed to obtain the posterior distribution for the scale parameter of this family of lifetime distributions under different loss functions. Finally, the performance is compared by the values obtained through MCMC simulation techniques.

In Chapter 8, firstly, the Bayes estimators for the positive and negative powers of the parameters are obtained. Then, through using these Bayes estimators, the estimates of the reliability function and stress-strength reliability under Type II censoring for Generalized Two Parameter Rayleigh Distribution are obtained. Estimation procedure is done for different priors under different loss functions.

Chapter 2

Robustness Study of the Sequential testing procedures for the New Weibull-Pareto Distribution(NWPD)

2.1 Introduction

Wald (1947)[113], is the first who developed the concept of sequential testing of statistical hypotheses for testing between two simple hypotheses. The concept of sequential testing is heavily dominated by the Sequential Probability Ratio Test (SPRT). He derived the theoretical expressions for the Operating Characteristics (OC) and Average Sample Number (ASN) functions, to study the performance of the SPRT'S.

The SPRT has been applied by various authors, to deal with testing problems, for references, Oakland (1950)[86] developed SPRT for testing the simple vs. simple hypothesis concerning the mean of the negative binomial distribution, Epstein and Sobel (1955)[46] dealt the testing of simple hypothesis problem regarding the mean of one parameter exponential distribution through SPRT, Johnson (1966)[68] applied SPRT for testing the hypothesis for the scale parameter of the Weibull distribution when the shape parameter is known, Phatarford (1971)[88] dealt the problem of testing the composite hypothesis for the shape parameter of the gamma distribution through SPRT, when the scale parameter is unknown, Bain and Engelhardt (1982)[5] applied SPRT for testing the hypothesis for the shape parameter of a non-homogeneous Poisson process and Chaturvedi et al. (2000)[27] developed SPRT for testing simple and composite hypothesis regarding the parameters of a class of distributions representing various life-testing models. Bacanli and Demirhan (2008)[4] developed a group sequential test when response variable has an inverse Gaussian distribution with known parameter.

The robustness of the SPRT in respect of OC and ASN functions has been studied by several authors, when the distribution under consideration has undergone a change, while dealing with various probabilistic models. For references, Harter and Moore (1976)[66] gives sampling plans for reliability tests under the assumption of a constant failure rate and by using Monte Carlo techniques the robustness of the exponential SPRT is studied, when the underlying distribution is a Weibull distribution, Montagne and Singpurwalla (1985)[80] investigated the robustness of the sequential life-testing procedure with respect to the risks and the expected sample sizes for the exponential distribution when the life length is not exponential, Hubbard and Allen (1991)[67] applied SPRT on the mean of the negative binomial distribution when the dispersion parameter is known and the robustness of the test to the misspecification of dispersion parameter is studied. Chaturvedi et al. (1998)[24] considered a family of life-testing models and studied the robustness of the SPRT'S for various parameters involved in the model and also generalised the results of Montagne and Singpurwalla (1985)[80].

2.2 Set-up of the problem

In this chapter, we consider the NWPD proposed by Nasiru and Lugnterah (2015)[84] with probability density function (pdf) given by

$$f(x;\beta,\partial,\theta)=\frac{\beta\partial}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left\{-\partial\left(\frac{x}{\theta}\right)^{\beta}\right\};\quad x>0 \tag{2.2.1}$$

and cumulative distribution function (cdf)

$$F(x;\beta,\partial,\theta)=1-\exp\left\{-\partial\left(\frac{x}{\theta}\right)^{\beta}\right\};\quad x>0 \tag{2.2.2}$$

where β is a shape and ∂, θ are the scale parameters, respectively. Weibull and Exponential distributions are the specific cases of (2.2.1) for $\partial=1$ and for $\partial=1$, $\beta=1$, respectively.

In Sections 2.3, 2.4, 2.5 and 2.6, respectively, we developed SPRT'S for testing the simple null hypotheses for the parameters ∂ and θ involved in the model (2.2.1). The robustness of the SPRT'S in respect of OC and ASN functions is studied [see Remarks 2.3, 2.4, 2.5 and 2.6]. In Section 2.7, the acceptance and rejection regions for H_0 vs. H_1 in case of θ are derived and plotted in Figure 2.9. Finally, in Section 2.8, the results and findings are presented through Tables and Figures.

2.3 SPRT for testing the hypothesis regarding '∂' for known values of θ and β

The SPRT for testing the simple null hypothesis $H_0:\partial=\partial_0$ against the simple alternative $H_1:\partial=\partial_1(\partial_1>\partial_0)$ is defined as

$$Z_i=\ln\left[\frac{f(x_i;\beta,\partial_1,\theta)}{f(x_i;\beta,\partial_0,\theta)}\right] \tag{2.3.1}$$

$$Z_i = \ln\left(\frac{\partial_1}{\partial_0}\right) - (\partial_1 - \partial_0)\left(\frac{x_i}{\theta}\right)^{\beta} \tag{2.3.2}$$

or,

$$e^{Z_i} = \left(\frac{\partial_1}{\partial_0}\right)\exp\left\{-(\partial_1 - \partial_0)\left(\frac{x_i}{\theta}\right)^{\beta}\right\} \tag{2.3.3}$$

Now, we choose two numbers A and B such that $0 < B < 1 < A$. At the n^{th} stage, accept H_0 if $\sum_{i=1}^{n} Z_i \leq \ln B$, reject H_0 if $\sum_{i=1}^{n} Z_i \geq \ln A$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. If $\alpha \in (0,1)$ and $\beta \in (0,1)$ are Type I and Type II errors, respectively, then according to Wald (1947)[113], A and B are approximately given by

$$A \approx \frac{1-\beta}{\alpha} \text{ and } B \approx \frac{\beta}{1-\alpha} \tag{2.3.4}$$

The OC function $L(\theta)$ is given by

$$L(\theta) = \frac{A^h - 1}{A^h - B^h} \tag{2.3.5}$$

where 'h' is the non-zero solution of

$$E[e^{Z_i}]^h = 1 \tag{2.3.6}$$

or,

$$\int_0^{\infty}\left[\frac{f(x_i;\beta,\partial_1,\theta)}{f(x_i;\beta,\partial_0,\theta)}\right]^h f(x_i;\beta,\partial,\theta)dx = 1 \tag{2.3.7}$$

From (2.2.1) and (2.3.3), we obtain

$$E[e^{Z_i}]^h = \frac{\partial\left(\frac{\partial_1}{\partial_0}\right)^h}{h(\partial_1 - \partial_0) + \partial} \tag{2.3.8}$$

On substituting (2.3.8) in (2.3.6), we get

$$\partial = \frac{h(\partial_1 - \partial_0)}{\left(\frac{\partial_1}{\partial_0}\right)^h - 1} \tag{2.3.9}$$

The expression (2.3.9) is not very useful for finding the values of OC and ASN functions, hence, we will further evaluate (2.3.9) in the following manner to obtain the desired results.

$$h \ln \left(\frac{\partial_1}{\partial_0} \right) = \ln \left[1 + h \left(\frac{\partial_1 - \partial_0}{\partial} \right) \right] \tag{2.3.10}$$

Using the expansion of $\ln(1+x)$; $-1 < x < 1$ in (2.3.10), retaining the terms up to third degree in 'h' and on simplifying, we obtain the real roots of 'h' from (2.3.11)

$$\left\{ \frac{1}{3} \left(\frac{\partial_1 - \partial_0}{\partial} \right)^3 \right\} h^2 - \left\{ \frac{1}{2} \left(\frac{\partial_1 - \partial_0}{\partial} \right)^2 \right\} h + \left\{ \left(\frac{\partial_1 - \partial_0}{\partial} \right) - \ln \left(\frac{\partial_1}{\partial_0} \right) \right\} = 0 \tag{2.3.11}$$

The ASN function is approximately given by

$$E(N|\partial) = \frac{L(\partial) \ln B + [1 - L(\partial)] \ln A}{E(Z)} \tag{2.3.12}$$

provided that $E(Z) \neq 0$, where

$$E(Z) = \ln \left(\frac{\partial_1}{\partial_0} \right) - \left(\frac{\partial_1 - \partial_0}{\partial} \right) \tag{2.3.13}$$

From (2.3.12) ASN function under H_0 and H_1 are given by

$$E_0(N) = \frac{(1-\alpha) \ln B + \alpha \ln A}{\ln \left(\frac{\partial_1}{\partial_0} \right) - \left(\frac{\partial_1 - \partial_0}{\partial} \right)} \tag{2.3.14}$$

and

$$E_1(N) = \frac{\beta \ln B + (1-\beta) \ln A}{\ln \left(\frac{\partial_1}{\partial_0} \right) - \left(\frac{\partial_1 - \partial_0}{\partial} \right)} \tag{2.3.15}$$

Remarks 2.3: Let us consider the problem of testing the simple null hypothesis $H_0 : \partial = 13$ against the simple alternative hypothesis $H_1 : \partial = 15$, for $\alpha = \beta = 0.05$. The numerical values of OC and ASN functions are shown in Table 2.1 and their curves are plotted in Figure 2.1 and 2.2, respectively. Table and Figures show that the approximation gives satisfactorily results.

2.4 Robustness of the SPRT for '∂' when θ has undergone a change

Let us suppose that the parameter θ has undergone a change to θ^* and then the probability distribution in (2.2.1) becomes $f(x;\beta,\partial,\theta^*)$. In order to study the robustness of SPRT developed in Section 2.3 with respect to OC and ASN functions, the values of 'h' are obtained by solving the following equation

$$\int_0^\infty \left[\frac{f(x_i;\beta,\partial_1,\theta)}{f(x_i;\beta,\partial_0,\theta)}\right]^h f(x_i;\beta,\partial,\theta^*)dx = 1 \tag{2.4.1}$$

$$\left(\frac{\partial_1}{\partial_0}\right)^h \frac{\beta\partial}{\theta^*}\int_0^\infty \left(\frac{x}{\theta^\beta}\right)^{\beta-1} \exp\left[-\left\{\frac{h(\partial_1-\partial_0)}{\theta^\beta}+\frac{\partial}{\theta^{*\beta}}\right\}x^\beta\right]dx = 1$$

$$\frac{\left(\frac{\partial_1}{\partial_0}\right)^h \partial}{\left\{\frac{h(\partial_1-\partial_0)}{\theta^\beta}+\frac{\partial}{\theta^{*\beta}}\right\}\theta^{*\beta}} = 1$$

$$\frac{\left(\frac{\partial_1}{\partial_0}\right)^h \partial}{\left(\frac{\theta^*}{\theta}\right)^\beta h(\partial_1-\partial_0)+\partial} = 1$$

Finally, we get

$$\partial = \frac{(p)^\beta h(\partial_1-\partial_0)}{\left(\frac{\partial_1}{\partial_0}\right)^h - 1} \tag{2.4.2}$$

where $p = \frac{\theta^*}{\theta}$.

The expression (2.4.2) is not of much use for calculating the numerical values of OC and ASN functions. In order to handle the situation, we rewrite (2.4.2) as

$$h\ln\left(\frac{\partial_1}{\partial_0}\right) = \ln\left[1+h\left(\frac{\partial_1-\partial_0}{\partial}\right)\left(\frac{\theta^*}{\theta}\right)^\beta\right] \tag{2.4.3}$$

Using the expansion of $\ln(1+x)$; $-1 < x < 1$ in (2.4.3) and retaining the terms up to third degree in 'h' and on simplifying, we obtain the following quadratic equation in 'h'

$$\left\{\frac{p^{3\beta}}{3}\left(\frac{\partial_1 - \partial_0}{\partial}\right)^3\right\} h^2 - \left\{\frac{p^{2\beta}}{2}\left(\frac{\partial_1 - \partial_0}{\partial}\right)^2\right\} h + \left\{p^{\beta}\left(\frac{\partial_1 - \partial_0}{\partial}\right) - \ln\left(\frac{\partial_1}{\partial_0}\right)\right\} = 0 \quad (2.4.4)$$

where $p = \frac{\theta^*}{\theta}$

The Robustness of the SPRT with respect to ASN is studied by replacing the denominator of (2.3.12) by

$$E_{\theta^*}(Z) = \int_0^\infty z f(x; \beta, \partial, \theta^*) dx$$

or,

$$\begin{aligned} E_{\theta^*}(Z) &= E\left[\ln\left(\frac{\partial_1}{\partial_0}\right) - (\partial_1 - \partial_0)\frac{x^\beta}{\theta^\beta}\right] \\ &= \ln\left(\frac{\partial_1}{\partial_0}\right) - \frac{(\partial_1 - \partial_0)}{\theta^\beta} E(x^\beta) \\ &= \ln\left(\frac{\partial_1}{\partial_0}\right) - \frac{(\partial_1 - \partial_0)}{\theta^\beta}\frac{\theta^{*\beta}}{\partial} \\ &= \ln\left(\frac{\partial_1}{\partial_0}\right) - \frac{(\partial_1 - \partial_0)}{\partial}\left(\frac{\theta^*}{\theta}\right)^\beta \\ &= \ln\left(\frac{\partial_1}{\partial_0}\right) - \frac{(\partial_1 - \partial_0)}{\partial}(p)^\beta \end{aligned}$$

where $p = \frac{\theta^*}{\theta}$.

Remarks 2.4: Let us consider the example of testing null hypothesis $H_0 : \partial = 13$ vs. $H_1 : \partial = 15$, for $\alpha = \beta = 0.05$. The numerical values of OC and ASN functions are obtained for $p = 1$, $p > 1$ and $p < 1$, in order to study robustness of the SPRT and are presented in Table 2.2 and 2.3, respectively. The OC and ASN curves are plotted in Figure 2.3 and 2.4, respectively. It follows from Figure 2.3 that the OC function curve shifts to left(right) for $p < 1 (p > 1)$ of the curve corresponding to $p = 1$ and the similar pattern is followed by the ASN function curve in Figure 2.4. It is evident from both the curves that the SPRT is highly sensitive for a minor change in θ.

2.5 SPRT for testing the hypothesis regarding 'θ' when ∂ is known

The SPRT for testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$ is defined as

$$Z_i = \ln \left[\frac{f(x_i; \beta, \partial, \theta_1)}{f(x_i; \beta, \partial, \theta_0)} \right] \tag{2.5.1}$$

or,

$$Z_i = \ln \left(\frac{\theta_0}{\theta_1} \right)^{\beta} - \partial \left(\frac{1}{{\theta_1}^{\beta}} - \frac{1}{{\theta_0}^{\beta}} \right) {x_i}^{\beta} \tag{2.5.2}$$

or,

$$e^{Z_i} = \left(\frac{\theta_0}{\theta_1} \right)^{\beta} \exp \left\{ -\partial \left(\frac{1}{{\theta_1}^{\beta}} - \frac{1}{{\theta_0}^{\beta}} \right) {x_i}^{\beta} \right\} \tag{2.5.3}$$

From (2.2.1) and (2.5.3), we get

$$E[e^{Z_i}]^h = \frac{\left(\frac{\theta_0}{\theta_1} \right)^{\beta h}}{1 + h\theta^{\beta} \left(\frac{1}{{\theta_1}^{\beta}} - \frac{1}{{\theta_0}^{\beta}} \right)} \tag{2.5.4}$$

We get from (2.5.4) that

$$\theta = \left[\frac{\left(\frac{\theta_0}{\theta_1} \right)^{\beta h} - 1}{h \left(\frac{1}{{\theta_1}^{\beta}} - \frac{1}{{\theta_0}^{\beta}} \right)} \right]^{1/\beta} \tag{2.5.5}$$

The expression (2.5.5) is not very useful in calculating the numerical values of OC and ASN functions. Again, we may rewrite (2.5.3) as

$$\beta h \ln \left(\frac{\theta_0}{\theta_1} \right) = \ln \left[1 + h\theta^{\beta} \left(\frac{1}{{\theta_1}^{\beta}} - \frac{1}{{\theta_0}^{\beta}} \right) \right] \tag{2.5.6}$$

Using the expansion for $\ln(1+x)$; $-1 < x < 1$ in (2.5.6), retaining the terms up to third degree in 'h' and on simplifying, we obtain the following quadratic equation in 'h'

$$\left\{\frac{\theta^{3\beta}}{3}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right)^3\right\}h^2-\left\{\frac{\theta^{2\beta}}{2}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right)^2\right\}h+\left\{\theta^{\beta}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right)-\beta\ln\left(\frac{\theta_0}{\theta_1}\right)\right\}=0 \tag{2.5.7}$$

The ASN function is approximately given by

$$E(N|\theta)=\frac{L(\theta)\ln B+[1-L(\theta)]\ln A}{E(Z)} \tag{2.5.8}$$

provided that $E(Z)\neq 0$, where

$$E(Z)=\ln\left(\frac{\theta_0}{\theta_1}\right)^{\beta}-\theta^{\beta}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right) \tag{2.5.9}$$

From (2.5.8) ASN function under H_0 and H_1 are given by

$$E_0(N)=\frac{(1-\alpha)\ln B+\alpha\ln A}{\ln\left(\frac{\theta_0}{\theta_1}\right)^{\beta}-\theta^{\beta}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right)} \tag{2.5.10}$$

and

$$E_1(N)=\frac{\beta\ln B+(1-\beta)\ln A}{\ln\left(\frac{\theta_0}{\theta_1}\right)^{\beta}-\theta^{\beta}\left(\frac{1}{\theta_1{}^{\beta}}-\frac{1}{\theta_0{}^{\beta}}\right)} \tag{2.5.11}$$

Remarks 2.5: Let us consider the problem of testing the simple null hypothesis $H_0 : \theta = 12$ against the simple alternative hypothesis $H_1 : \theta = 15$, for $\alpha = \beta = 0.05$. The numerical values of OC and ASN functions are shown in Table 2.4 and their curves are plotted in Figure 2.5 and 2.6, respectively. It is evident from the Table 2.4 and Figures 2.5 and 2.6 that the approximation gives satisfactorily results.

2.6 Robustness of SPRT for 'θ' when ∂ has undergone a change

Let us suppose that the parameter ∂ has undergone a change then the probability distribution in (2.2.1) becomes $f(x;\beta,\partial^*,\theta)$. To study the robustness of the SPRT developed in Section 2.5 with respect to OC and ASN functions, the values of 'h' are obtained from the following equation

$$\int_0^\infty \left[\frac{f(x_i;\beta,\partial,\theta_1)}{f(x_i;\beta,\partial,\theta_0)}\right]^h f(x_i;\beta,\partial^*,\theta)dx = 1 \tag{2.6.1}$$

$$\left(\frac{\theta_0}{\theta_1}\right)^h \frac{\beta\partial^*}{\theta^\beta}\int_0^\infty x^{\beta-1}\exp\left[-\left\{\partial h\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)+\frac{\partial^*}{\theta^\beta}\right\}x^\beta\right]dx = 1$$

$$\frac{\left(\frac{\theta_0}{\theta_1}\right)^{\beta h}\frac{\partial}{\theta^\beta}}{\left\{\partial h\left(\frac{1}{\theta_1^\beta}-\frac{1}{\theta_0^\beta}\right)+\frac{\partial^*}{\theta^\beta}\right\}} = 1$$

$$\theta^\beta = \frac{\left\{\left(\frac{\theta_0}{\theta_1}\right)^{\beta h}-1\right\}\left(\frac{\partial^*}{\partial}\right)}{h\left(\frac{1}{\theta_1^\beta}-\frac{1}{\theta_0^\beta}\right)}$$

$$\theta = \left[\frac{\left\{\left(\frac{\theta_0}{\theta_1}\right)^{\beta h}-1\right\}\left(\frac{\partial^*}{\partial}\right)}{h\left(\frac{1}{\theta_1^\beta}-\frac{1}{\theta_0^\beta}\right)}\right]^{1/\beta} \tag{2.6.2}$$

The expression (2.6.2) is not very useful in calculating the numerical values of OC and ASN functions. Rewrite (2.6.2) to obtain the real roots of 'h'.

$$\beta h\ln\left(\frac{\theta_0}{\theta_1}\right) = \ln\left[1+h\theta^\beta\left(\frac{\partial}{\partial^*}\right)\left(\frac{1}{\theta_1^\beta}-\frac{1}{\theta_0^\beta}\right)\right] \tag{2.6.3}$$

Using the expansion of $\ln(1+x)$; $-1 < x < 1$ in (2.6.3), retaining the terms up to third degree in 'h' and on simplifying, we obtain the real roots for 'h' from the following equation

$$\left\{\frac{\phi^3\theta^{3\beta}}{3}\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)^3\right\}h^2-\left\{\frac{\phi^2\theta^{2\beta}}{2}\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)^2\right\}h+\left\{\phi\theta^\beta\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)-\beta\ln\left(\frac{\theta_0}{\theta_1}\right)\right\}=0 \tag{2.6.4}$$

where $\phi = \frac{\partial}{\partial^*}$

The Robustness of the SPRT with respect to ASN can be studied by replacing the denominator of (2.5.8) by

$$E_{\partial^*}(Z) = \int_0^\infty zf(x;\beta,\partial^*,\theta)dx$$

or,

$$\begin{aligned}
E_{\partial^*}(Z) &= E\left[\ln\left(\frac{\theta_0}{\theta_1}\right)^\beta - \partial\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)x^\beta\right] \\
&= \ln\left(\frac{\theta_0}{\theta_1}\right)^\beta - \partial\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)E[x^\beta] \\
&= \ln\left(\frac{\theta_0}{\theta_1}\right)^\beta - \partial\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)\frac{\theta^\beta}{\partial^*} \\
&= \ln\left(\frac{\theta_0}{\theta_1}\right)^\beta - \frac{\partial}{\partial^*}\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)\theta^\beta
\end{aligned}$$

$$E_{\partial^*}(Z) = \ln\left(\frac{\theta_0}{\theta_1}\right)^\beta - \phi\left(\frac{1}{{\theta_1}^\beta}-\frac{1}{{\theta_0}^\beta}\right)\theta^\beta \tag{2.6.5}$$

where $\phi = \frac{\partial}{\partial^*}$.

Remarks 2.6: Let us consider the problem of testing null hypothesis $H_0 : \theta = 12$ vs. $H_1 : \theta = 15$ for $\alpha = \beta = 0.05$. To study the robustness of the SPRT, the numerical values of OC and ASN functions are obtained for $\phi = 1$, $\phi > 1$ and $\phi < 1$ and are given in Table 2.5 and 2.6, respectively. The OC and ASN curves are plotted in Figure 2.7 and 2.8, respectively. It follows from Figure 2.7 that the OC function curve shifts to right(left) for $\phi < 1(\phi > 1)$ of the curve corresponding to $\phi = 1$ and the similar pattern is followed by the ASN function curve in Figure 2.8. The curves show that the SPRT is highly sensitive for changes in '∂'.

2.7 Acceptance and Rejection boundaries for NWPD

The nature of SPRT in case of NWPD is described as, let $X_1, X_2, X_3, \ldots$ be (iid) random variables from NWPD where $\theta > 0$. The problem of testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$ is considered. Let A and B be approximately given by (2.3.4) where $\alpha > 0, \beta < 1$ and Z_i is defined in (2.5.2).

Let us define, $Y(n) = \sum\limits_{i=1}^{n} X_i$ and N≡first integer $n(\geq 1)$, for which the inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with the constants

$$c_1 = \frac{\ln B}{\partial\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}, \; c_2 = \frac{\ln A}{\partial\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} \text{ and } d = \frac{\ln\left(\frac{\theta_0}{\theta_1}\right)}{\partial\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} \tag{2.7.1}$$

At the stopping stage, if $Y(N) \leq c_1 + dN$, we accept H_0 and if $Y(N) \geq c_2 + dN$, we reject H_0 for different values of N, where A and B are the fixed quantities. Figure 2.9 shows the acceptance and rejection region for H_0 under the case when $H_0 : \theta = 12$ vs. $H_1 : \theta = 15$, $\partial = 2$ and $\alpha = \beta = 0.05$. From (2.7.1), values of the constants are $c_1 = 0.33316$, $c_2 = -0.33316$ and $d = 6.694$, respectively. Thus, if $Y(N) \leq 0.33316 + 6.694N$, we accept H_0 and if $Y(N) \geq -0.3316 + 6.694N$, we accept H_1 and at the intermediate stage, we continue sampling.

2.8 Tables and Figures

Table 2.1: OC and ASN Function for ($H_0 : \partial = 13, H_1 : \partial = 15, \alpha = \beta = 0.05$)

∂	$L(\partial)$	$E(N)$	∂	$L(\partial)$	$E(N)$
12.4	0.99	159.810	14.2	0.34	413.371
12.6	0.99	183.473	14.4	0.22	384.705
12.8	0.97	212.476	14.6	0.14	346.856
13.0	0.95	247.619	14.8	0.08	307.783
13.2	0.91	288.857	15.0	0.05	272.012
13.4	0.85	334.095	15.2	0.03	241.208
13.6	0.75	377.757	15.4	0.02	215.434
13.8	0.63	410.695	15.6	0.01	194.095
14.0	0.48	423.680	15.8	0.01	176.433

Table 2.2: OC Function for different values of p ($H_0 : \partial = 13, H_1 : \partial = 15, \alpha = \beta = 0.05$)

∂	$p = .96$	$p = .98$	$p = 1$	$p = 1.02$	$p = 1.04$
12.0	0.99	1.00	1.00	1.00	1.00
12.2	0.98	0.99	1.00	1.00	1.00
12.4	0.96	0.98	0.99	1.00	1.00
12.6	0.93	0.97	0.99	0.99	1.00
12.8	0.87	0.94	0.97	0.99	1.00
13.0	0.79	0.90	0.95	0.98	0.99
13.2	0.66	0.82	0.91	0.96	0.98
13.4	0.51	0.71	0.85	0.93	0.97
13.6	0.36	0.57	0.75	0.87	0.94
13.8	0.24	0.42	0.63	0.79	0.89
14.0	0.15	0.29	0.48	0.68	0.82
14.2	0.09	0.18	0.34	0.54	0.72
14.4	0.05	0.11	0.22	0.40	0.59
14.6	0.03	0.06	0.14	0.27	0.45
14.8	0.02	0.04	0.08	0.17	0.32
15.0	0.01	0.02	0.05	0.11	0.21
15.2	0.00	0.01	0.03	0.06	0.13
15.4	0.00	0.01	0.02	0.04	0.08
15.6	0.00	0.00	0.01	0.02	0.05
15.8	0.00	0.00	0.01	0.01	0.03
16.0	0.00	0.00	0.00	0.01	0.02
16.2	0.00	0.00	0.00	0.00	0.01

Table 2.3: ASN Function for different values of p
($H_0 : \partial = 13, H_1 : \partial = 15, \alpha = \beta = 0.05$)

∂	$p = .96$	$p = .98$	$p = 1$	$p = 1.02$	$p = 1.04$
12.0	171.036	144.504	124.632	109.377	97.377
12.2	198.459	165.165	140.498	121.858	107.428
12.4	232.202	190.607	159.810	136.813	119.284
12.6	272.750	221.868	183.473	154.911	133.412
12.8	318.832	259.639	212.476	176.973	150.415
13.0	365.674	303.378	247.619	203.929	171.036
13.2	404.026	349.827	288.857	236.622	196.137
13.4	422.916	391.650	334.095	275.333	226.567
13.6	416.583	418.47	377.757	318.832	262.812
13.8	388.984	422.174	410.695	363.045	304.262
14.0	350.145	402.825	423.680	400.329	348.066
14.2	309.365	368.300	413.371	421.371	388.110
14.4	272.012	328.311	384.705	420.285	415.719
14.6	240.036	289.695	346.856	398.756	423.461
14.8	213.496	255.647	307.783	364.480	409.944
15.0	191.702	226.935	272.012	325.954	380.629
15.2	173.796	203.175	241.208	288.978	343.814
15.4	158.991	183.600	215.434	256.263	306.328
15.6	146.638	167.418	194.095	228.481	272.012
15.8	136.225	153.941	176.433	205.306	242.297
16.0	127.356	142.611	161.737	186.069	217.249
16.2	119.725	132.990	149.409	170.060	196.355
16.4	113.100	124.740	138.970	156.649	178.941
16.6	107.298	117.600	130.046	145.321	164.363

Table 2.4: OC and ASN Function for
($H_0 : \theta = 12, H_1 : \theta = 15, \alpha = \beta = 0.05$)

θ	$L(\theta)$	$E(N)$	θ	$L(\theta)$	$E(N)$
11.0	0.996	73.443	13.6	0.398	169.990
11.2	0.994	79.719	13.8	0.311	162.483
11.4	0.989	86.936	14.0	0.236	152.610
11.6	0.982	95.210	14.2	0.175	141.529
11.8	0.970	104.619	14.4	0.127	130.189
12.0	0.953	115.155	14.6	0.091	119.241
12.2	0.926	126.650	14.8	0.064	109.063
12.4	0.888	138.693	15.0	0.045	99.822
12.6	0.836	150.554	15.2	0.031	91.557
12.8	0.768	161.174	15.4	0.021	84.225
13.0	0.686	169.303	15.6	0.013	77.747
13.2	0.593	173.801	15.8	0.008	72.030
13.4	0.494	174.009			

Table 2.5: OC Function for different values of ϕ
($H_0 : \theta = 12, H_1 : \theta = 15, \alpha = \beta = 0.05$)

θ	$\phi = .96$	$\phi = .98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
11.0	0.999	0.998	0.996	0.993	0.988
11.2	0.998	0.997	0.994	0.989	0.980
11.4	0.997	0.994	0.989	0.981	0.966
11.6	0.995	0.990	0.982	0.968	0.945
11.8	0.991	0.984	0.970	0.949	0.914
12.0	0.985	0.973	0.953	0.919	0.869
12.2	0.976	0.957	0.926	0.878	0.808
12.4	0.962	0.933	0.888	0.821	0.731
12.6	0.941	0.899	0.836	0.748	0.639
12.8	0.911	0.852	0.768	0.661	0.538
13.0	0.869	0.790	0.686	0.563	0.436
13.2	0.814	0.714	0.593	0.463	0.341
13.4	0.744	0.625	0.494	0.367	0.258
13.6	0.661	0.530	0.398	0.282	0.191
13.8	0.569	0.434	0.311	0.211	0.138
14.0	0.475	0.344	0.236	0.154	0.098
14.2	0.384	0.266	0.175	0.111	0.068
14.4	0.301	0.200	0.127	0.078	0.047
14.6	0.231	0.148	0.091	0.055	0.032
14.8	0.173	0.107	0.064	0.037	0.021
15.0	0.127	0.077	0.045	0.025	0.013
15.2	0.092	0.054	0.031	0.017	0.008

Table 2.6: ASN Function for different values of ϕ
($H_0 : \theta = 12, H_1 : \theta = 15, \alpha = \beta = 0.05$)

θ	$\phi = .96$	$\phi = .98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
11.0	62.344	67.479	73.443	80.396	88.503
11.2	66.782	72.748	79.719	87.871	97.363
11.4	71.838	78.786	86.936	96.458	107.459
11.6	77.614	85.713	95.210	106.231	118.743
11.8	84.221	93.638	104.619	117.161	130.953
12.0	91.767	102.645	115.155	129.033	143.512
12.2	100.337	112.746	126.650	141.351	155.452
12.4	109.963	123.821	138.693	153.262	165.464
12.6	120.571	135.544	150.554	163.578	172.152
12.8	131.917	147.307	161.174	170.968	174.460
13.0	143.512	158.192	169.303	174.326	172.071
13.2	154.584	167.048	173.801	173.159	165.521
13.4	164.106	172.734	174.009	167.777	155.953
13.6	170.969	174.441	169.99	159.138	144.693
13.8	174.253	171.981	162.483	148.484	132.895
14.0	173.528	165.837	152.610	136.981	121.378
14.2	168.997	156.967	141.529	125.516	110.631
14.4	161.399	146.486	130.189	114.654	100.880
14.6	151.754	135.389	119.241	104.691	92.183
14.8	141.075	124.418	109.063	95.735	84.501
15.0	130.189	114.041	99.822	87.779	77.747
15.2	119.666	104.506	91.557	80.755	71.816

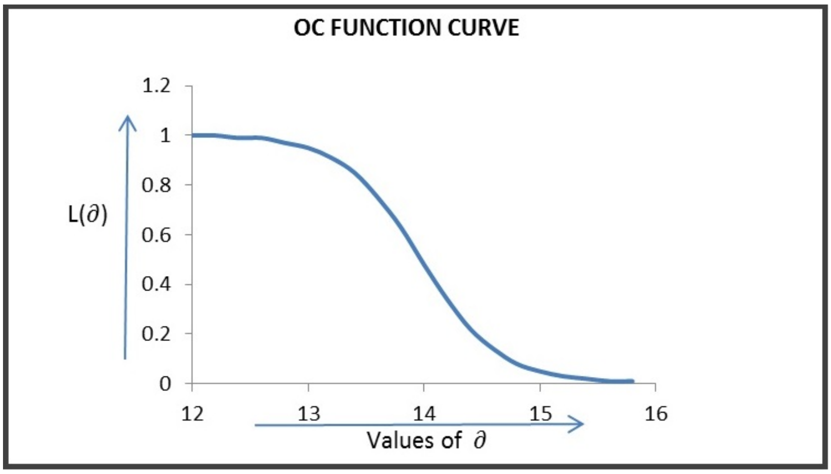


Figure 2.1

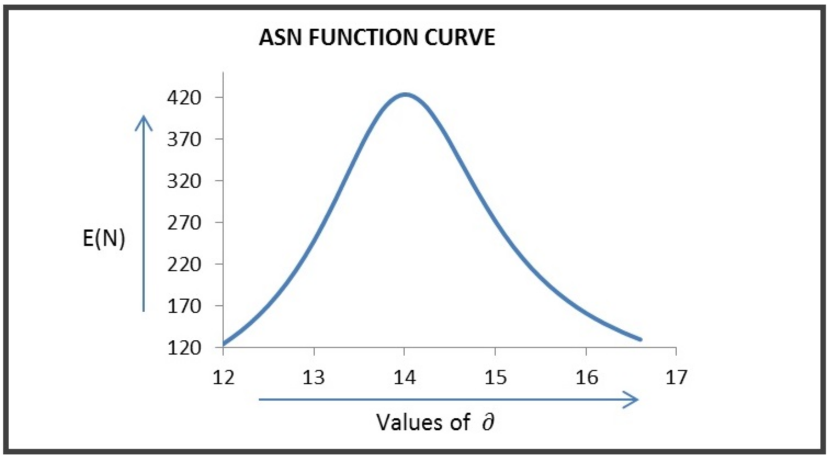


Figure 2.2

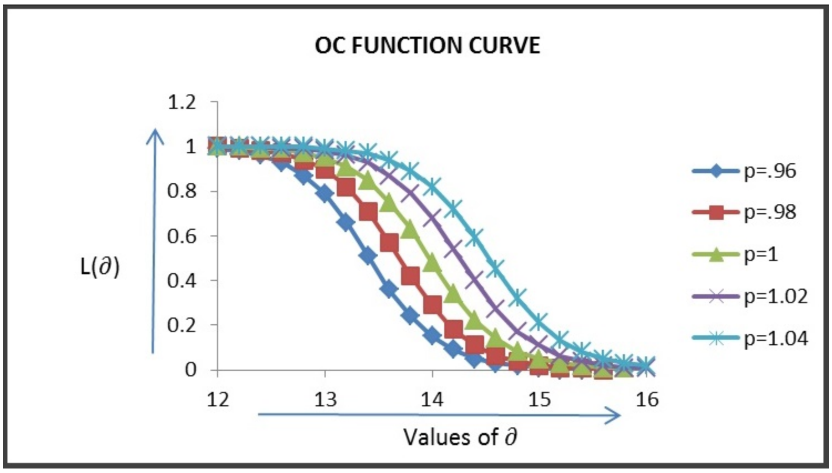


Figure 2.3

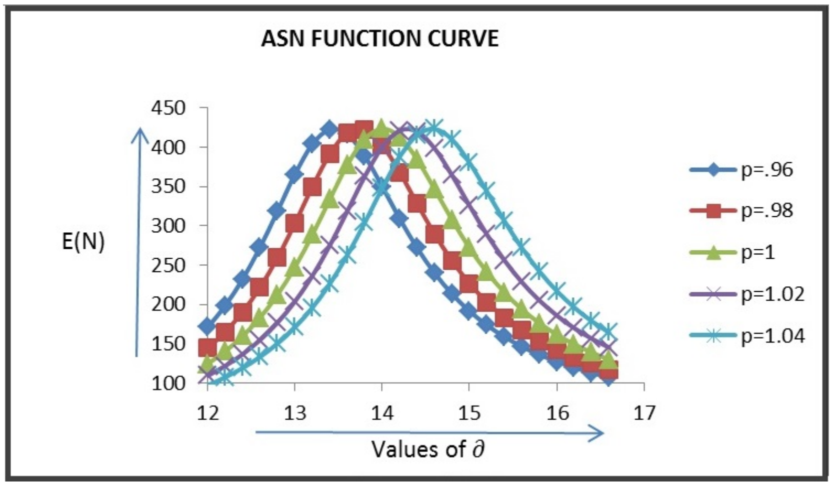


Figure 2.4

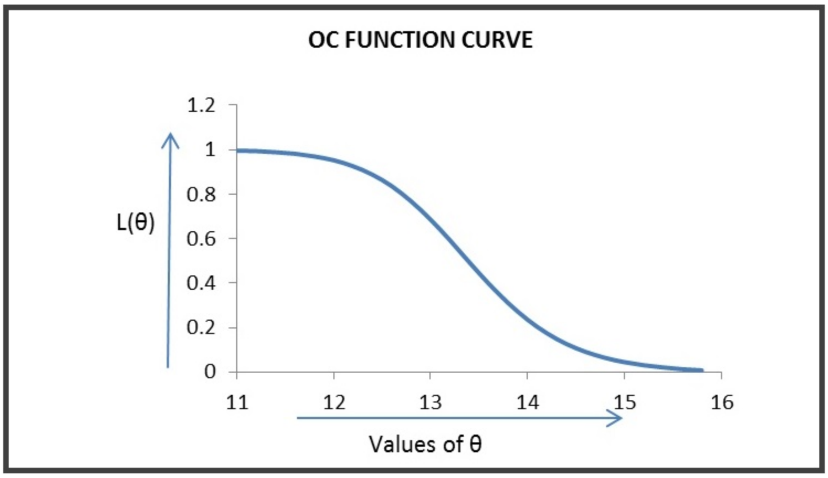


Figure 2.5

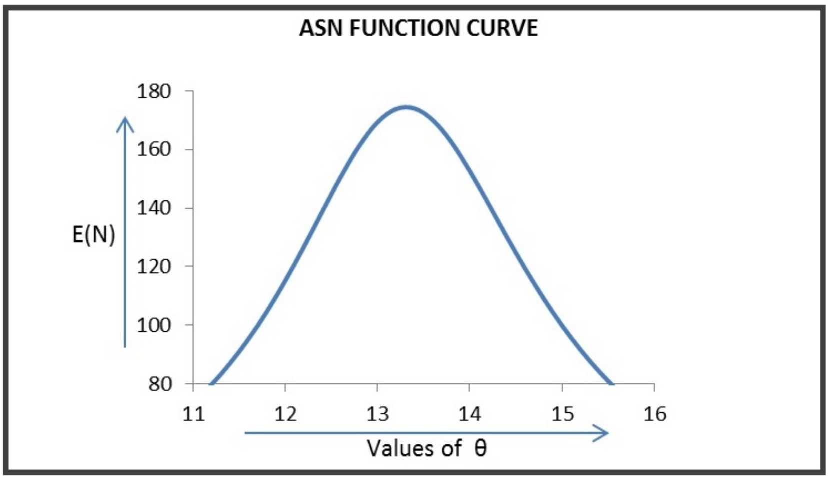


Figure 2.6

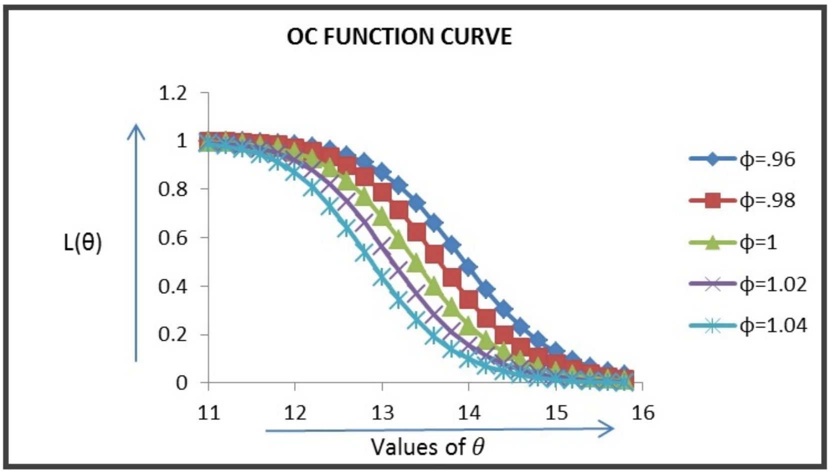


Figure 2.7

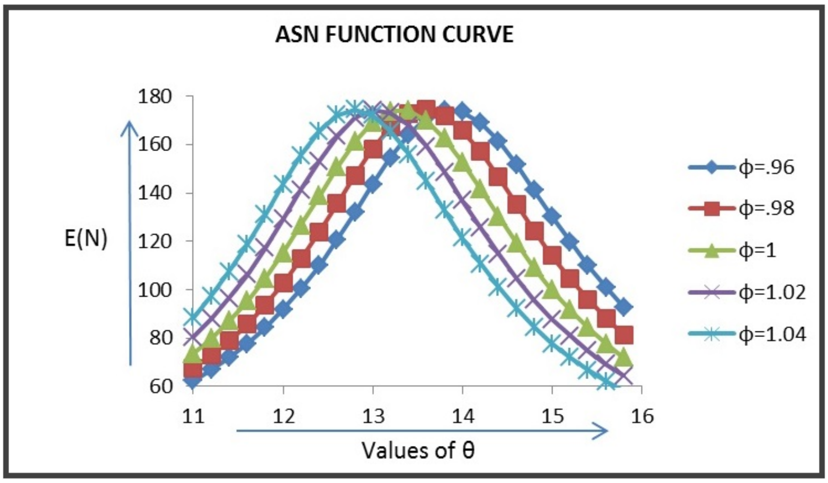


Figure 2.8

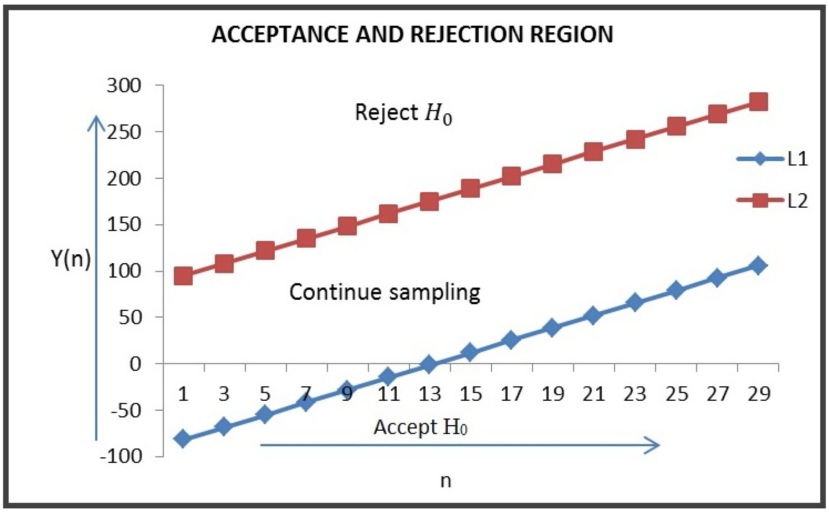


Figure 2.9

Chapter 3

Sequential analysis and Robustness study for the parameters of Positive exponential family of distributions with known coefficient of variation

3.1 Introduction

The origin of the concept of sequential analysis has been discussed in the previous chapter and the related references are also mentioned. In 1990, Joshi and Shah[69] developed the SPRT for testing the mean of an inverse Gaussian distribution, assuming the coefficient of variation (CV) to be known. They obtained only theoretical expressions for the OC and the ASN functions.

In this chapter, we are interested in obtaining the SPRT and its robustness for the mean of Positive Exponential Family of Distribution (PEFD) with known coefficient of variation (CV). The theoretical expressions as well as the numerical and graphical interpretation for the OC and ASN functions with their inferences are also studied for the considered model.

3.2 Positive exponential family of distribution

Liang (2008)[76] proposed a Positive Exponential Family of Distribution (PEFD). Let us consider a random variable (r.v.) X follows the PEFD presented by the probability density function (pdf)

$$f(x;\theta,\nu,\rho) = \frac{\rho x^{\rho\nu-1} e^{\left(\frac{-x^{\rho}}{\theta}\right)}}{\Gamma\nu\theta^{\nu}}; \quad x > 0, \theta, \nu, \rho > 0 \tag{3.2.1}$$

where, θ is assumed to be unknown and ρ, ν are known constants. When $\rho = \nu = 1$; we get one-parameter exponential distribution, for $\rho = 1$; we get gamma distribution, For ν=1; we get Weibull distribution, for $\nu > 0$, $\rho = 1$; we get Erlang distribution, for $\nu > 1/2$, $\rho = 2$; we get half-normal distribution, for $\nu > m/2$, $\rho = 2$; we get Chi-distribution, for $\nu = 1$, $\rho = 2$; we get Rayleigh distribution and for $\nu = p + 1$, $\rho = 2$; we get Generalized Rayleigh distribution.

In the model (3.2.1), for testing the simple null hypothesis against the simple alternative, the SPRT'S and the robustness of the SPRT'S in respect of OC and ASN functions are developed in the Section 3.3, 3.4, 3.5 and 3.6. In Section 3.7, the robustness of the SPRT for a mis-specified coefficient of variation is also studied. In Section 3.8, the acceptance and rejection regions for H_0 vs. H_1 for θ are derived and plotted in Figure 3.6. Finally, in Section 3.9 and Section 3.10, the results and findings as well as the Tables and Graphs are presented, respectively.

3.3 SPRT for testing the hypothesis regarding 'θ'

For a given sequence of observations X_1, X_2, X_3, ... from (3.2.1), the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ ($\theta_1 > \theta_0$) is taken into consideration. The SPRT for testing is defined as follows

$$Z_i = \ln\left(\frac{\theta_0}{\theta_1}\right)^{\nu} - \left[x_i^{\rho}\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right] \tag{3.3.1}$$

Wald (1947)[113] suggested the values of A and B $(0 < B < 1 < A)$ for the Type I error - $\alpha \in (0,1)$ and Type II error - $\beta \in (0,1)$ as defined in (2.3.4). The Operating Characteristic (OC) function $L(\theta)$ is defined in (2.3.5) where 'h' is the non-zero solution of (2.3.6).
From (3.2.1) and (3.3.1), we have

$$E\left[e^{Z_i}\right]^h = \frac{\left(\frac{\theta_0}{\theta_1}\right)^{h\nu}}{\left[1 + h\theta^{\nu}\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]} \tag{3.3.2}$$

taking logarithm and using the expression $\ln(1+x)$; $-1 < x < 1$ in (3.3.2). After retaining the terms up to third degree in 'h' and on simplifying, we obtain the following quadratic equation in 'h'

$$\frac{h^2}{3}\theta^3\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^3 - \frac{h}{2}\theta^2\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^2 + \theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) - \ln\left(\frac{\theta_0}{\theta_1}\right) = 0 \tag{3.3.3}$$

On solving (3.3.3), we get the numerical values of OC function. The ASN function is approximately obtained by using (2.5.8) provided that $E(Z) \neq 0$, where

$$E(Z) = \nu\left[\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right] \tag{3.3.4}$$

From (2.5.8), the ASN function under H_0 and H_1 is given by

$$E_0(N) = \frac{(1-\alpha)\ln B + \alpha \ln A}{\nu\left[\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]} \tag{3.3.5}$$

and

$$E_1(N) = \frac{\beta \ln B + (1-\beta)\ln A}{\nu\left[\ln\left(\frac{\theta_0}{\theta_1}\right) - \theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]} \tag{3.3.6}$$

3.4 SPRT for testing the hypothesis regarding 'ν'

To derive the OC and ASN function for the parameter ν under the simple null hypothesis $H_0 : \nu = \nu_0$ against the simple alternative hypothesis $H_1 : \nu = \nu_1 (\nu_1 > \nu_0)$. The value of Z_i is

$$Z_i = \frac{\Gamma\nu_0}{\Gamma\nu_1} \theta^{\nu_0 - \nu_1} x_i^{\rho(\nu_1 - \nu_0)} \tag{3.4.1}$$

Using (3.2.1) and (3.4.1) for the OC function, we have

$$\left(\frac{\Gamma\nu_0}{\Gamma\nu_1}\right)^h \frac{1}{\Gamma\nu} \Gamma[h(\nu_1 - \nu_0) + \nu] = 1 \tag{3.4.2}$$

taking the logarithm of both sides of (3.4.2), with $\ln(1 + x)$; $-1 < x < 1$ and using the approximation

$$\ln \Gamma x = \ln \sqrt{2\pi} - x + \left(x - \frac{1}{2}\right) \ln x \tag{3.4.3}$$

we have

$$\frac{h^2}{6}\left(\frac{\nu_1 - \nu_0}{\nu}\right)^3 (\nu + 1) - \frac{h}{4}\left(\frac{\nu_1 - \nu_0}{\nu}\right)^2 (2\nu + 1) - \left(\nu_0 - \frac{1}{2}\right) \ln \nu_0 + \left(\nu_1 - \frac{1}{2}\right) \ln \nu_1$$
$$- \left(1 + \ln \nu - \frac{1}{2\nu}\right)(\nu_1 - \nu_0) = 0 \tag{3.4.4}$$

which is quadratic equation in 'h'. The numerical values of OC function is now obtain from equation (3.4.4). We get the values of ASN by replacing the denominator of (2.5.8) by

$$E(Z_i|\nu) = \ln \Gamma\nu_0 - \ln \Gamma\nu_1 + (\nu_1 - \nu_0) \ln \lambda + (\nu_1 - \nu_0) E(\ln x_i)$$

Using the result of Gradshteyn and Ryzhik(1965, p.576, 4.352(1))[59] that

$$\psi(x) = \ln x - \frac{1}{2x} \tag{3.4.5}$$

$$E(Z_i|\nu) = \left(\nu_0 - \frac{1}{2}\right)\ln\nu_0 - \left(\nu_1 - \frac{1}{2}\right)\ln\nu_1 + \left(1 + \ln\nu - \frac{1}{2\nu}\right)(\nu_1 - \nu_0) \qquad (3.4.6)$$

3.5 Robustness of the SPRT for 'θ' when 'ν' has undergone a change.

Let the parameter ν has undergone a change to ν^* and then probability distribution in (3.2.1) becomes $f(x_i;\theta,\nu^*,\rho)$. For the robustness of SPRT developed in Section 3.3 with respect to OC function, the values of 'h' are obtained by the equation (3.2.1) and (3.3.1) as

$$\left(\frac{\theta_0}{\theta_1}\right)^{h\nu}\left[1 + h\theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right]^{-\nu^*} = 1 \qquad (3.5.1)$$

taking logarithm on both sides of equation (3.5.1) and using the expansion of $\ln(1+x)$; $-1 < x < 1$, we have

$$\frac{Qh^2}{3}\theta^3\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^3 - \frac{Qh}{2}\theta^2\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^2 + Q\theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) - \ln\left(\frac{\theta_0}{\theta_1}\right) = 0 \qquad (3.5.2)$$

which is quadratic equation in 'h'. On solving (3.5.2), we get the real roots of 'h'. The robustness of the SPRT with respect to ASN is studied by replacing the denominator of (2.5.8) by

$$E(Z|\theta) = \ln\left(\frac{\theta_0}{\theta_1}\right) - Q\theta\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \qquad (3.5.3)$$

where $Q = \frac{\nu^*}{\nu}$

3.6 Robustness of the SPRT for 'ν' when 'θ' has undergone a change.

Let us suppose that the parameter θ has undergone a change to θ^* and then probability distribution in (3.2.1) becomes $f(x_i;\theta^*,\nu,\rho)$. For OC function, value of 'h' are obtained by the equation

$$\left(\frac{\Gamma\nu_0}{\Gamma\nu_1}\right)^h \frac{1}{\Gamma\nu}\theta^{h(\nu_1-\nu_0)}\frac{\rho}{(\theta^*)^\nu}\int_0^\infty x^{h(\nu_1-\nu_0)+\nu-1}e^{\left(-\frac{x^\rho}{\theta^*}\right)}dx = 1$$

$$\left(\frac{\Gamma\nu_0}{\Gamma\nu_1}\right)^h \frac{1}{\Gamma\nu}\phi^{h(\nu_1-\nu_0)}\Gamma[h(\nu_1-\nu_0)+\nu] = 1 \tag{3.6.1}$$

where $\phi = \frac{\theta^*}{\theta}$.

Taking logarithm on both sides of equation(3.6.1) and using the approximation(3.4.3), we get the roots of 'h' by the following equation

$$\frac{h^2}{6}\left(\frac{\nu_1-\nu_0}{\nu}\right)^3(\nu+1) - \frac{h}{4}\left(\frac{\nu_1-\nu_0}{\nu}\right)^2(2\nu+1) - \left(\nu_0-\frac{1}{2}\right)\ln\nu_0 + \left(\nu_1-\frac{1}{2}\right)\ln\nu_1$$
$$-(\nu_1-\nu_0)\ln\phi - \left(1+\ln\nu-\frac{1}{2\nu}\right)(\nu_1-\nu_0) = 0 \tag{3.6.2}$$

The robustness of the SPRT with respect to ASN is studied by replacing the denominator of (2.5.8) by

$$E(Z_i|\nu) = \ln\Gamma\nu_0 - \ln\Gamma\nu_1 + (\nu_1-\nu_0)\ln\lambda + (\nu_1-\nu_0)E(\ln x_i)$$

Using the result of (3.4.5), we get

$$E(Z_i|\nu) = \left(\nu_0-\frac{1}{2}\right)\ln\nu_0 + \left(\nu_1-\frac{1}{2}\right)\ln\nu_1 + \left(1+\ln\phi+\ln\nu-\frac{1}{2\nu}\right)(\nu_1-\nu_0) \tag{3.6.3}$$

3.7 Robustness of the SPRT for 'θ' with known coefficient of variation(CV)

For PEF-distribution, the mean and variance are $(\theta\nu)$ and $(\theta^2\nu)$ for $\rho = 1$, respectively. The coefficient of variation (CV) is $\frac{1}{\sqrt{\nu}}$. Let us suppose that coefficient of variation changes from c to $c*$, so that, the (pdf) of (3.2.1) shifts to $f(x_i;\theta,c^*,\rho)$.

From (3.3.1) and (3.2.1), OC and ASN functions are

$$\left(\frac{\theta_0}{\theta_1}\right)^{\frac{h}{c^2}}\left[1+h\theta\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\right]^{\left(\frac{-1}{c^*}\right)^2}=1 \tag{3.7.1}$$

taking logarithm on both sides of equation (3.7.1) and using the expansion of $\ln(1+x)$; $-1 < x < 1$. In order to obtain the roots of the given equation, we get the following quadratic equation

$$\frac{\psi h^2}{3}\theta^3\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)^3-\frac{\psi h}{2}\theta^2\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)^2+\psi\theta\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)-\ln\left(\frac{\theta_0}{\theta_1}\right)=0 \tag{3.7.2}$$

On solving (3.7.2), we get the real roots of 'h'. The robustness of the SPRT with respect to ASN function is studied by replacing the denominator of (2.5.8) by

$$E(Z|\lambda)=\ln\left(\frac{\theta_0}{\theta_1}\right)-\psi\theta\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right) \tag{3.7.3}$$

where $\psi=\left(\frac{c}{c^*}\right)^2$

3.8 Acceptance and Rejection Region for PEFD

To test the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$ having pre-assigned $0 < \alpha, \beta < 1$. Z_i is defined as

$$Z_i=\nu\ln\left(\frac{\theta_0}{\theta_1}\right)+x_i\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)$$

Let us define, $Y(n)=\sum_{i=1}^{n}X_i$ and N $\equiv$ first integer $n(\geq 1)$, for which the inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with the constants

$$c_1=\frac{\ln B}{\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)}, c_2=\frac{\ln A}{\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)} \text{ and } d=\frac{\nu\ln\left(\frac{\theta_0}{\theta_1}\right)}{\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)} \tag{3.8.1}$$

3.9 Results and Findings

I. For Section 3.3 and 3.4, the numerical values of the OC and ASN functions for the parameters θ and ν are presented in Table 3.1 and 3.2, respectively. The OC and ASN curves are plotted in Figure 3.1 and Figure 3.2, respectively. The Tables and Figures shows that the approximation gives the satisfactory results.

II. To study the robustness of θ, the numerical values of OC and ASN functions are derived for Section 3.5 and are presented in Table 3.3 and 3.4. Finally, the values of OC and ASN curves are plotted in Figure 3.3 for various values of 'Q'. The OC curve shifts to the right(left) and ASN curve shifts to the right upward (left downward) for $Q < 1(Q > 1)$. From both the curves, it is evident that the SPRT is highly sensitive for any change in ν.

III. Similarly for Section 3.6, the robustness is studied for ν and the values and curves of OC and ASN functions are presented and shown for various values of ϕ in Table 3.5 and 3.6 and Figure 3.4, respectively. The OC curve shifts to the right(left) and ASN curve shifts to the right upward (left downward) for $\phi < 1(\phi > 1)$. From both the curves, it is evident that the SPRT is highly sensitive for any change in θ.

IV. For Section 3.7, the values and curves of OC and ASN functions for various values of ψ are presented and shown in Table 3.7 and 3.8 and Figure 3.5, respectively. The OC curve shifts to the right (left) and ASN curve shifts to the right upward (left downward) for $\psi < 1(\psi > 1)$. From both the curves, it is evident that the SPRT is highly sensitive for any change in ψ.

V. In case of Section 3.8, Figure 3.6 shows the acceptance and rejection regions for H_0 under the case when $H_0 : \theta_0 = 13$ vs $H_1 : \theta_1 = 15$ for $\alpha = \beta = 0.05$ and $\nu = 2$. The values of constants $c_1 = -287.0828$, $c_2 = 287.0828$ and $d = -27.90466$, respectively. Thus, if $Y(N) \leq -27.90466N + 287.0828$, we accept H_0 and if $Y(N) \geq -27.90466N - 287.0828$, we accept H_1. At the intermediate stages, we continue sampling.

3.10 Tables and Figures

Table 3.1: OC and ASN Function under $\alpha = \beta = 0.05$ $(H_0 : \theta_0 = 13, H_1 : \theta_1 = 15)$

θ	$L(\theta)$	$E(N)$	θ	$L(\theta)$	$E(N)$
12.0	0.9986	146.6379	14.0	0.4649	421.7277
12.2	0.9969	162.8450	14.2	0.3268	401.3977
12.4	0.9936	182.6031	14.4	0.2148	365.6746
12.6	0.9903	206.8220	14.6	0.1344	324.0585
12.8	0.9936	236.4311	14.8	0.0813	283.5593
13.0	**0.9511**	272.0115	15.0	**0.0481**	247.6194
13.2	0.9745	312.9813	15.2	0.0280	217.1700
13.4	0.9101	356.2505	15.4	0.0161	191.8948
13.6	0.8427	394.9805	15.6	0.0091	171.0361
13.8	0.7423	419.4413	15.8	0.0051	153.7820

Table 3.2: OC and ASN Function under $\alpha = \beta = 0.05$ $(H_0 : \nu_0 = 13, H_1 : \nu_1 = 15)$

ν	$L(\nu)$	$E(N)$	ν	$L(\nu)$	$E(N)$
12.0	0.9978	9.2084	14.0	0.4909	29.2767
12.2	0.9958	10.2831	14.2	0.3489	28.5099
12.4	0.9921	11.5862	14.4	0.2299	26.4396
12.6	0.9852	13.1771	14.6	0.1429	23.7140
12.8	0.9727	15.1207	14.8	0.0852	20.9075
13.0	**0.9506**	17.4707	15.0	**0.0496**	18.3477
13.2	0.9128	20.2269	15.2	0.0284	16.1527
13.4	0.8514	26.1920	15.6	0.0091	12.8154
13.8	0.6350	28.4159	15.8	0.0051	11.5706

Table 3.3: Values of OC Function
($H_0 : \theta_0 = 13, H_1 : \theta_1 = 15, \alpha = \beta = 0.05$)

θ	$Q = 0.96$	$Q = 0.98$	$Q = 1$	$Q = 1.02$	$Q = 1.04$
12.0	0.9997	0.9994	0.9986	0.9969	0.9930
12.2	0.9994	0.9987	0.9970	0.9933	0.9851
12.4	0.9987	0.9972	0.9937	0.9860	0.9692
12.6	0.9974	0.9942	0.9871	0.9717	0.9389
12.8	0.9948	0.9884	0.9745	0.9448	0.8844
13.0	0.9898	0.9775	0.9512	0.8970	0.7952
13.2	0.9806	0.9577	0.9101	0.8188	0.6673
13.4	0.9640	0.9231	0.8427	0.7043	0.5125
13.6	0.9354	0.8662	0.7424	0.5604	0.3583
13.8	0.8884	0.7801	0.6113	0.4086	0.2306
14.0	0.8162	0.6636	0.4650	0.2745	0.1395
14.2	0.7155	0.5263	0.3269	0.1729	0.0811
14.4	0.5904	0.3874	0.2149	0.1042	0.0460
14.6	0.4552	0.2665	0.1345	0.0609	0.0256
14.8	0.3283	0.1738	0.0814	0.0350	0.0141
15.0	0.2238	0.1090	0.0482	0.0198	0.0076
15.2	0.1460	0.0667	0.0281	0.0111	0.0041
15.4	0.0925	0.0401	0.0162	0.0061	0.0021
15.6	0.0573	0.0238	0.0092	0.0033	0.0011

Table 3.4: Values of ASN Function
($H_0 : \theta_0 = 13, H_1 : \theta_1 = 15, \alpha = \beta = 0.05$)

θ	$Q = 0.96$	$Q = 0.98$	$Q = 1$	$Q = 1.02$	$Q = 1.04$
12.0	117.9642	130.7879	146.6379	166.6099	192.2676
12.2	127.9919	143.4212	162.8450	187.7760	220.2996
12.4	139.8035	158.5618	182.6031	213.9330	255.0499
12.6	153.8510	176.8701	206.8220	246.1303	297.0357
12.8	170.6990	199.1305	236.4311	284.9204	344.6670
13.0	191.0183	226.1628	272.0115	329.2600	392.0871
13.2	215.5267	258.5597	312.9813	374.7806	428.1692
13.4	244.8080	296.1020	356.2505	412.5785	440.5583
13.6	278.9059	336.7466	394.9805	431.4178	424.5372
13.8	316.5908	375.4905	419.4413	424.5431	387.0856
14.0	354.4084	404.3614	421.7277	395.0849	340.6018
14.2	386.2119	415.2478	401.3977	353.2120	294.8779
14.4	404.5178	404.8214	365.6746	308.9473	254.6703
14.6	404.1337	376.9041	324.0585	268.2821	221.1477
14.8	385.5071	339.5334	283.5593	233.4569	193.7957
15.0	354.2942	300.2335	247.6194	204.5853	171.5783
15.2	317.7116	263.6472	217.1700	180.9222	153.4506
15.4	281.3539	231.7132	191.8948	161.5276	138.5275
15.6	248.2856	204.7226	171.0361	145.5348	126.1096

Table 3.5: Values of OC Function ($H_0 : \nu_0 = 13, H_1 : \nu_1 = 15, \alpha = \beta = 0.05$)					
ν	$\phi = 0.96$	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
12.2	0.9994	0.9983	0.9958	0.9903	0.9787
12.4	0.9987	0.9967	0.9921	0.9820	0.9611
12.6	0.9975	0.9938	0.9852	0.9669	0.9305
12.8	0.9952	0.9882	0.9727	0.9405	0.8796
13.0	0.9910	0.9782	0.9506	0.8960	0.8003
13.2	0.9833	0.9604	0.9128	0.8251	0.6884
13.4	0.9693	0.9294	0.8514	0.7221	0.5501
13.6	0.9448	0.8779	0.7591	0.5896	0.4044
13.8	0.9033	0.7979	0.6350	0.4435	0.2743
14.0	0.8366	0.6852	0.4909	0.3072	0.1741
14.2	0.7382	0.5465	0.3489	0.1982	0.1054
14.4	0.6091	0.4008	0.2299	0.1213	0.0618
14.6	0.4635	0.2713	0.1429	0.0717	0.0356
14.8	0.3245	0.1720	0.0852	0.0414	0.0203
15.0	0.2112	0.1040	0.0496	0.0237	0.0115
15.2	0.1300	0.0610	0.0284	0.0134	0.0065
15.4	0.0771	0.0351	0.0161	0.0076	0.0036
15.6	0.0447	0.0200	0.0091	0.0043	0.0020

Table 3.6: Values of ASN Function ($H_0 : \nu_0 = 13, H_1 : \nu_1 = 15, \alpha = \beta = 0.05$)					
ν	$\phi = 0.96$	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$	$\phi = 1.04$
12.2	8.0434	9.0470	10.2831	11.8171	13.7175
12.4	8.8527	10.0687	11.5862	13.4827	15.8179
12.6	9.8159	11.3033	13.1771	15.5140	18.3199
12.8	10.9748	12.8073	15.1207	17.9528	21.1584
13.0	12.3819	14.6459	17.4707	20.7670	24.0814
13.2	14.1009	16.8816	20.2269	23.7618	26.5821
13.4	16.1989	19.5421	23.2560	26.4916	28.0034
13.6	18.7256	22.5548	26.1920	28.3015	27.8953
13.8	21.6623	25.6463	28.4159	28.6312	26.3483
14.0	24.8326	28.2725	29.2767	27.4095	23.9095
14.2	27.8024	29.7362	28.5099	25.1018	21.1947
14.4	29.8941	29.5709	26.4396	22.3555	18.6177
14.6	30.4711	27.8882	23.7140	19.6606	16.3631
14.8	29.3709	25.2770	20.9075	17.2626	14.4676
15.0	27.0272	22.3887	18.3477	15.2307	12.9001
15.2	24.1288	19.6563	16.1527	13.5457	11.6077
15.4	21.2315	17.2702	14.3239	12.1568	10.5380
15.6	18.6290	15.2662	12.8154	11.0089	9.6455

Table 3.7: Values of OC Function
$(H_0 : \theta_0 = 13, H_1 : \theta_1 = 15, \alpha = \beta = 0.05)$

θ	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$
12.2	0.9994	0.9970	0.9849
12.4	0.9987	0.9937	0.9687
12.6	0.9974	0.9871	0.9379
12.8	0.9947	0.9745	0.8827
13.0	0.9897	0.9512	0.7926
13.2	0.9803	0.9101	0.6637
13.4	0.9635	0.8427	0.5084
13.6	0.9344	0.7424	0.3545
13.8	0.8868	0.6113	0.2276
14.0	0.8138	0.4650	0.1375
14.2	0.7122	0.3269	0.0798
14.6	0.4511	0.1345	0.0251
14.8	0.3247	0.0814	0.0138
15.0	0.2208	0.0482	0.0075
15.4	0.0910	0.0162	0.0021
15.6	0.0564	0.0092	0.0010

Table 3.8: Values of ASN Function
$(H_0 : \theta_0 = 13, H_1 : \theta_1 = 15, \alpha = \beta = 0.05)$

θ	$\phi = 0.98$	$\phi = 1$	$\phi = 1.02$
12.2	128.2688	162.8450	221.0448
12.4	140.1371	182.6031	255.9894
12.6	154.2568	206.8220	298.1778
12.8	171.1966	236.4311	345.9449
13.0	191.6313	272.0115	393.2965
13.2	216.2803	312.9813	428.9734
13.4	245.7220	356.2505	440.6648
13.6	247.3173	171.0361	125.7705
13.8	279.9782	394.9805	423.9333
14.0	280.2862	191.8948	138.1281
14.2	316.6156	217.1700	152.9754
14.4	317.7702	419.4413	386.0318
14.6	353.3112	247.6194	171.0077
14.8	355.5624	421.7277	339.4071
15.0	384.8421	283.5593	193.1064
15.2	387.1238	401.3977	293.7435
15.4	403.9848	324.0585	220.3159
15.6	404.9570	365.6746	253.6803

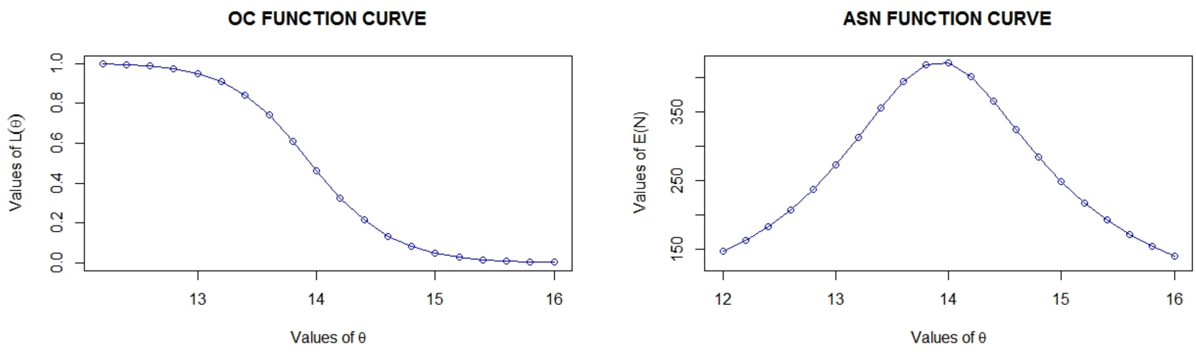


Figure 3.1

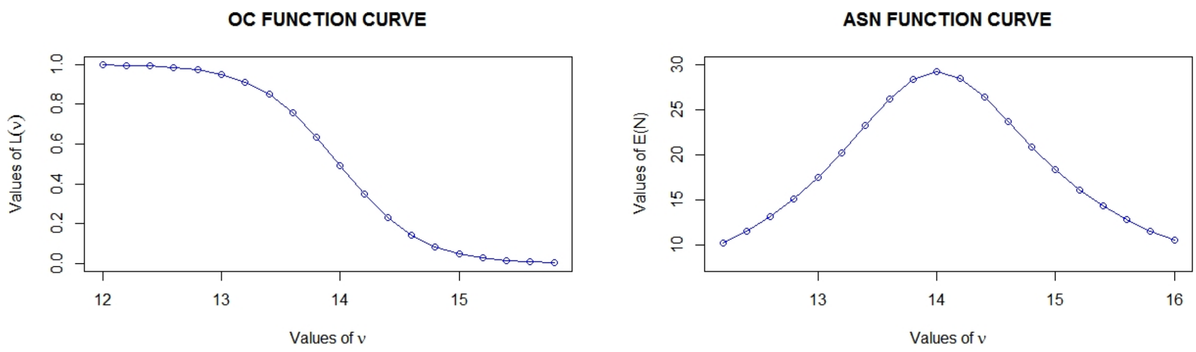


Figure 3.2

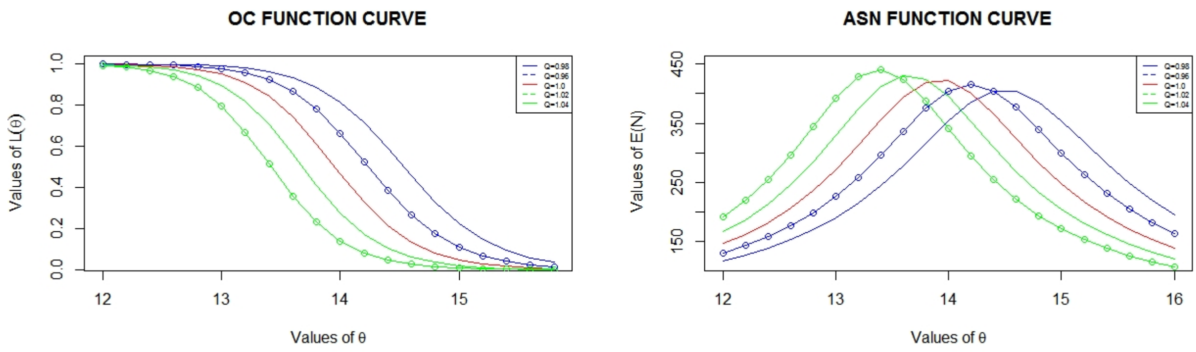


Figure 3.3

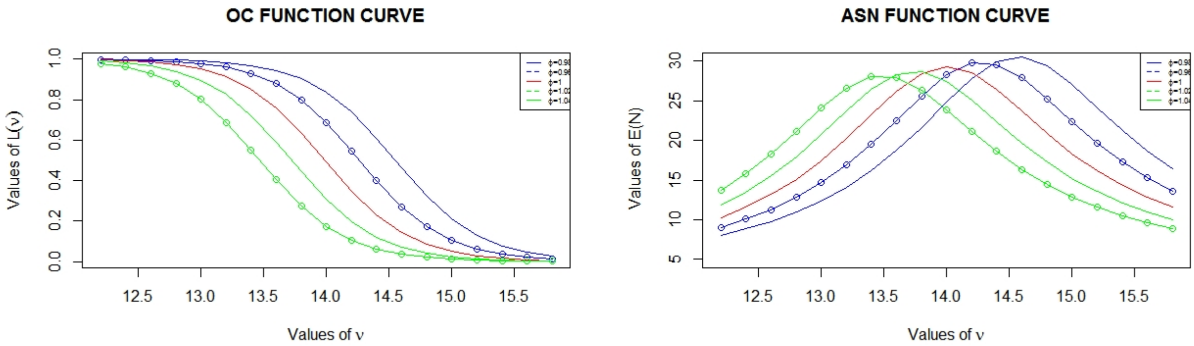


Figure 3.4

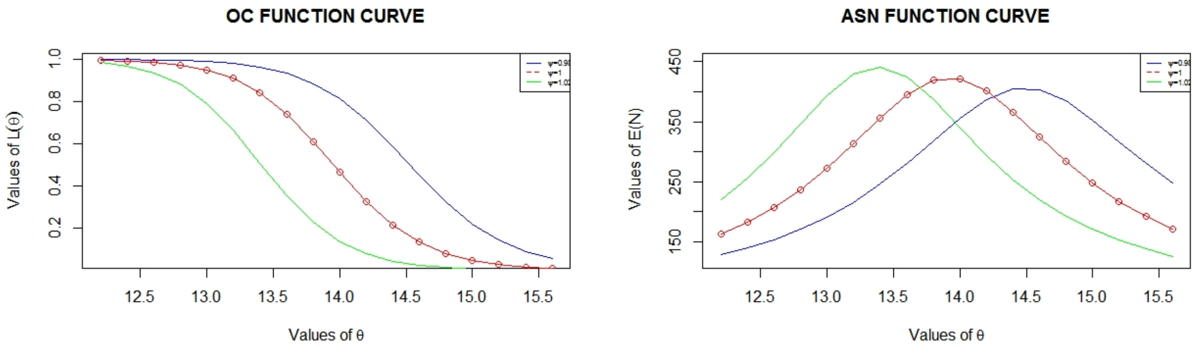


Figure 3.5

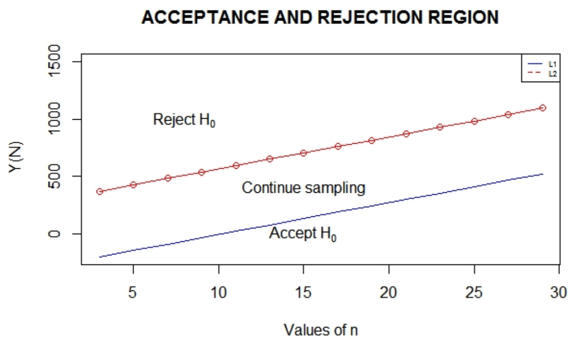


Figure 3.6

Chapter 4

Robustness Study of the SPRT for the Generalised Inverse Weibull(GIW) Distribution when the parameters are misspecified

4.1 Introduction

Wald's (1947)[113], idea of sequential testing procedure is motivated by the double sampling procedure of Dodge and Romig (1941)[40], where the decision to draw the second sample or not, depends upon the observations of the first sample. They presented this scheme in recognition of the fact that they required, on an average, lesser number of sample observations than the single sampling procedure. Sequential testing procedures have been applied by several authors to handle different testing problems related to various probabilistic models. For a brief review of the literature, one may refer to Barnard (1946)[9], Barnard (1947)[10], Foster et al. (1982)[52] and Fowler (1983)[53]. Robustness of the Sequential testing procedure in respect of OC and ASN functions, when the parameters under study are misspecified or undergone a change, is studied by several authors from time to time. For detailed review, one may refer to Surinder et al. (2018)[104].

Following the trend of performing the SPRT and its robustness, in this chapter, apart from making alterations in the scale parameters, the impact of the shape parameter is also taken into consideration. The study shows the the variation through the graphical representation when a slight change(i.e. increasing or decreasing) is made in the values of the shape parameter of the considered distribution.

4.2 Set-up of the problem

Consider the cdf F(t) of the inverse Weibull distribution proposed by Drapella (1993)[42]

$$F(t) = exp[-(\alpha/t)^{\beta}] \tag{4.2.1}$$

where t, α, $\beta > 0$. On upgrading F(t) to $Z(t)^{\gamma}$, the cdf given at (4.2.1) becomes

$$Z(t)^{\gamma} = exp[-\gamma(\alpha/t)^{\beta}]$$

where t, α, β, $\gamma > 0$ and the corresponding pdf is

$$z(t) = \gamma\beta\alpha^{\beta} exp[-\gamma(\alpha/t)^{\beta}], t > 0 \tag{4.2.2}$$

Equation (4.2.2) is the probability density function of generalised inverse Weibull distribution GIWD (α, β, γ), Felipe et al. (2011)[50].

4.3 SPRT for testing the hypothesis regarding α

The Sequential Probability Ratio Test of strength $(\hat{\alpha}, \hat{\beta})$ for testing the simple null hypothesis $H_0 : \alpha = \alpha_0$ against the simple alternative $H_1 : \alpha = \alpha_1 (> \alpha_0)$ is as follows:

Let $x_i(i = 1, 2, 3, ...)$ be the successive observations on X. Now computing Z_i which is defined as

$$Z_i = \log\left[\frac{f(x_i;\alpha_1,\beta,\gamma)}{f(x_i;\alpha_0,\beta,\gamma)}\right] \tag{4.3.1}$$

$$Z_i = \beta\log\left(\frac{\alpha_1}{\alpha_0}\right) - \gamma(\alpha_1^\beta - \alpha_0^\beta)\left(\frac{1}{x_i}\right)^\beta \tag{4.3.2}$$

From Chapter 2, considering equations (2.3.4) and (2.3.5), and using equation (2.3.6), we have

$$E\left[\frac{f(x;\alpha_1,\beta,\gamma)}{f(x;\alpha_0,\beta,\gamma)}\right]^h = 1 \tag{4.3.3}$$

$$\alpha^\beta\left[\left(\frac{\alpha_1}{\alpha_0}\right)^{\beta h} - 1\right] = h(\alpha_1^\beta - \alpha_0^\beta) \tag{4.3.4}$$

On solving the equation (4.3.4) by using the exponential function of the form $y = f(x) = a^x$ where $a > 1$, and retaining the terms upto 3rd degree in 'h', we get

$$\frac{\alpha^\beta h^2\beta^3}{3!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^3 + \frac{\alpha^\beta h\beta^2}{2!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^2 + \alpha^\beta\beta\log\frac{\alpha_1}{\alpha_0} - (\alpha_1^\beta - \alpha_0^\beta) = 0 \tag{4.3.5}$$

Equation (4.3.5) is a quadratic equation in 'h'. From this equation, we can compute 'h' for $\alpha > 0$. The values of OC function $L(\alpha)$ may be obtained by using equation (2.3.5).

The ASN function of the SPRT is given by

$$E[N|\alpha] = \frac{L(\alpha)\log B + [1 - L(\alpha)]\log A}{E(Z)} \tag{4.3.6}$$

where,

$$E(Z) = \log\left(\frac{\alpha_1}{\alpha_0}\right)^\beta - \left[\frac{(\alpha_1^\beta - \alpha_0^\beta)}{\alpha^\beta}\right] \tag{4.3.7}$$

Thus, finally the ASN function is given by

$$E[N|\alpha] = \frac{L(\alpha)\log B + [1 - L(\alpha)]\log A}{\log\left(\frac{\alpha_1}{\alpha_0}\right)^{\beta} - \left[\frac{(\alpha_1^{\beta} - \alpha_0^{\beta})}{\alpha^{\beta}}\right]} \tag{4.3.8}$$

4.4 Robustness of the SPRT for the parameter α

Let us suppose that the parameter γ misspecified and has undergone a change then the pdf (4.2.2) becomes $f(x; \alpha, \beta, \gamma^*)$. The robustness of the SPRT presented in Section 4.3 with respect to OC and ASN functions is studied by obtaining the values of 'h' from the following equation

$$E_{\gamma^*}[e^{Zh}] = \int_0^{\infty} \left[\frac{f(x; \alpha_1, \beta, \gamma)}{f(x; \alpha_0, \beta, \gamma)}\right]^h f(x; \alpha, \beta, \gamma^*)\mathrm{d}x \tag{4.4.1}$$

$$\gamma^* \alpha^{\beta}\left[\left(\frac{\alpha_1}{\alpha_0}\right)^{\beta h} - 1\right] = \gamma h(\alpha_1^{\beta} - \alpha_0^{\beta}) \tag{4.4.2}$$

Again, using the exponential function $y = f(x) = a^x$, $a > 1$ in equation (4.4.2) and retaining the terms upto 3rd degree in 'h', we get the following quadratic equation in 'h'

$$\frac{h^3\beta^3}{3!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^3 + \frac{\alpha^{\beta}h\beta^2}{2!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^2 + (\alpha_1^{\beta} - \alpha_0^{\beta})\left(\frac{\gamma}{\gamma^*}\right) - \alpha^{\beta}\beta\log\frac{\alpha_1}{\alpha_0} = 0 \tag{4.4.3}$$

$$\frac{h^3\beta^3}{3!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^3 + \frac{\alpha^{\beta}h\beta^2}{2!}\left[\log\frac{\alpha_1}{\alpha_0}\right]^2 + (\alpha_1^{\beta} - \alpha_0^{\beta})M - \alpha^{\beta}\beta\log\frac{\alpha_1}{\alpha_0} = 0 \tag{4.4.4}$$

where $M = \dfrac{\gamma}{\gamma^*}$.

Hence, the values of OC function $L(\alpha)$ is obtained from (2.3.5), on using the values of 'h' for $\alpha > 0$ computed from (4.4.4). The robustness of the SPRT for the parameter α is analysed by considering the cases (i) $\gamma > \gamma^*$, (ii) $\gamma = \gamma^*$ and (iii) $\gamma < \gamma*$. The Robustness of the SPRT with respect to ASN function is studied by replacing the denominator in equation (2.5.8) by

$$E_{\gamma^*}(Z) = \log\left(\frac{\alpha_1}{\alpha_0}\right)^{\beta} - \frac{(\alpha_1^{\beta} - \alpha_0^{\beta})}{\alpha^{\beta}}\left(\frac{\gamma}{\gamma^*}\right) \tag{4.4.5}$$

After computing the values of ASN function under the cases $\gamma > \gamma^*, \gamma = \gamma^*$ and $\gamma < \gamma^*$, respectively, the robustness is studied accordingly.

4.5 SPRT for testing the hypothesis regarding γ

The SPRT for testing the null hypothesis $H_0 : \gamma = \gamma_0$ against the simple alternative $H_1 : \gamma = \gamma_1 (> \gamma_0)$ is defined as

$$Z_i = \log\frac{f(x_i;\alpha,\beta,\gamma_1)}{f(x_i;\alpha,\beta,\gamma_0)} \tag{4.5.1}$$

$$Z_i = \log\left(\frac{\gamma_1}{\gamma_0}\right) - \left(\frac{\alpha}{x_i}\right)^{\beta}(\gamma_1 - \gamma_0) \tag{4.5.2}$$

The OC function of the SPRT for testing $H_0 : \gamma = \gamma_0$ against , $H_1 : \gamma = \gamma_1 (> \gamma_0)$ is obtained by using equation (2.3.5).

For each value of γ, the value of h is to be determined, such that $h \neq 0$. We have

$$E\left[\frac{f(x;\alpha,\beta,\gamma_1)}{f(x;\alpha,\beta,\gamma_0)}\right]^{h} = 1 \tag{4.5.3}$$

$$\gamma\left\{\left[\frac{\gamma_1}{\gamma_0}\right]^{h} - 1\right\} = h(\gamma_1 - \gamma_o) \tag{4.5.4}$$

On solving (4.5.4) by using the exponential function of the form $y = f(x) = a^x$ where $a > 1$, and retaining the terms upto 3rd degree in h, we get

$$\frac{h^3}{3!}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^3 + \frac{h}{2!}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^2 + \log\left[\frac{\gamma_1}{\gamma_0}\right] - \frac{(\gamma_1 - \gamma_0)}{\gamma} = 0 \tag{4.5.5}$$

Hence, we obtain the quadratic equation in h. The ASN function of the SPRT is given by:

$$E[N|\gamma] = \frac{L(\gamma)\log B + [1 - L(\gamma)]\log A}{E(Z)} \tag{4.5.6}$$

where,

$$E(Z) = \log\left[\frac{\gamma_1}{\gamma_0}\right] - \frac{\gamma_1 - \gamma_0}{\gamma} \tag{4.5.7}$$

By substituting the value of E(Z) in (4.5.5), we get the ASN as follows

$$E[N|\gamma] = \frac{L(\gamma)\log B + [1 - L(\gamma)]\log A}{\left[\log\left[\frac{\gamma_1}{\gamma_0}\right] - \frac{\gamma_1 - \gamma_0}{\gamma}\right]} \tag{4.5.8}$$

4.6 Robustness of the SPRT for parameter γ

Let us suppose that the parameter α has undergone a change, then (4.2.2) becomes $f(x;\alpha^*,\beta,\gamma)$. Considering the equations (2.3.6) and (2.3.7) we have,

$$(\gamma_1 - \gamma_0)\alpha^{\beta} h = \left\{\left[\frac{\gamma_1}{\gamma_0}\right]^h - 1\right\}\gamma\alpha^{*\beta} \tag{4.6.1}$$

Again, using the exponential function $y = f(x) = a^x$, $a > 1$ in (4.6.1) and retaining the terms upto 3rd degree in h, we get the following quadratic equation in h

$$\log\left[\frac{\gamma_1}{\gamma_0}\right] + \frac{h}{2}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^2 + \frac{h^2}{3!}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^3 = \left[\frac{\alpha}{\alpha^*}\right]^{\beta}\frac{(\gamma_1 - \gamma_0)}{\gamma}$$

$$\frac{h^2}{3!}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^3 + \frac{h}{2}\left\{\log\left[\frac{\gamma_1}{\gamma_0}\right]\right\}^2 + \log\left[\frac{\gamma_1}{\gamma_0}\right] - N^{\beta}\frac{(\gamma_1 - \gamma_0)}{\gamma} = 0 \tag{4.6.2}$$

where $N = \frac{\alpha}{\alpha^*}$

The robustness of SPRT with respect to ASN is studied by replacing the denominator of

(2.5.8) by

$$E_{\alpha^*}[Z] = \log\left(\frac{\gamma_1}{\gamma_0}\right) - \frac{(\gamma_1 - \gamma_0)}{\gamma}\left(\frac{\alpha}{\alpha^*}\right)^{\beta} \tag{4.6.3}$$

Now, the values of ASN function is computed from (2.5.8) through using (4.6.3). The robustness of the SPRT for the parameter γ is studied by taking the cases, where (i) $\alpha > \alpha^*$, (ii) $\alpha = \alpha^*$ and (iii) $\alpha < \alpha*$ and then analyse the ASN curve to check the robustness.

4.7 Acceptance and Rejection boundaries for GIWD

In order to test the simple hypotheses $H_0 : \alpha = \alpha_0$ against $H_1 : \alpha = \alpha_1$ having pre-assigned $0 < \alpha, \beta, < 1$. Considering (2.3.4), we defined Z_i as

$$Z_i = \beta \log\left(\frac{\alpha_1}{\alpha_0}\right) - \gamma(\alpha_1^{\beta} - \alpha_0^{\beta})\left(\frac{1}{x_i}\right)^{\beta} \tag{4.7.1}$$

Let us define, $Y(n) = \sum_{i=1}^{n} X_i$ and N≡first integer $n(\geq 1)$, for which the inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with the constants

$$c_1 = \frac{\ln B}{\gamma(\alpha_1 - \alpha_0)},\ c_2 = \frac{\ln A}{\gamma(\alpha_1 - \alpha_0)} \text{ and } d = \frac{\ln\left(\dfrac{\alpha_1}{\alpha_0}\right)}{\gamma(\alpha_1 - \alpha_0)} \tag{4.7.2}$$

4.8 Results and Conclusions

The theoretical expressions for the OC and ASN functions are obtained in Section 4.3 for the scale parameter α. The problem of testing simple null hypothesis versus simple alternative hypothesis is considered by fixing $\alpha_0 = 15$ and $\alpha_1 = 17$, $\alpha = \beta = 0.05$. For varying values of γ, the numerical values of OC and ASN functions are obtained. Table 4.1 and (Figure 4.1 and 4.2) depicts the values(curves) for the OC and ASN functions for the parameter α. For Section 4.4, the numerical values(curves) are obtained for $M = 1$, $M > 1$ and $M < 1$ (see Table 4.2 and 4.3) and Figures 4.3 and 4.4 shows that the SPRT is highly robust for a little misspecification in the parameter γ. The graphical representation of the OC and ASN function clarify that curves deviate towards right(left) for $M > 1$ ($M < 1$) from $M = 1$. Thus, in case of parameter α involved in the model (4.2.2) the SPRT is highly sensitive.

Following the same procedure for Section 4.6, the robust behaviour of the SPRT developed for γ is studied. The values(curves) of the OC and ASN functions for N=1, $N > 1$ and $N < 1$ are given in Table 4.9 and 4.10(Figure 4.11 and Figure 4.12). Here, we observe that the OC and ASN curves shift towards the right (left) for $N > 1(N < 1)$. It is evident from the observations that the SPRT is highly sensitive for even a minor change in the parameter α.

Alongwith, the study of scale parameters, the focus is also given to the shape parameter β, to study its effect, we considered different values of β i.e $\beta = 1, > 1$ and < 1 and the results are presented in Tables 4.4-4.7, 4.11-4.14; Figures 4.5-4.8, 4.13-4.16. From this study, it is concluded that due to little variation in the parameter β there is a drastic change in the shape of the curves.

In Section 4.7, the problem of constructing the acceptance and rejection regions for H_0 under the case when $H_0 : \alpha_0 = 15$ vs $H_1 : \alpha_1 = 17$ for $\alpha = \beta = 0.05$ and $\gamma = 0.5$ is considered and the findings are presented in Figure 4.17. The obtained values of constants are $c_1 = -2.9444$, $c_2 = 2.9444$ and $d = 0.1431$, respectively. Finally, it is concluded that if $Y(N) \leq 0.1252N + 2.9444$, accept H_0 and if $Y(N) \geq 0.1252N - 2.9444$, accept H_1. At the intermediate stages, continue sampling.

4.9 Tables and Figures

Table 4.1: OC and ASN Function for
$H_0 : \alpha_0 = 15,\ H_1 : \alpha_1 = 17, \alpha = \beta = 0.05$

α	$L(\alpha)$	$E(N)$	α	$L(\alpha)$	$E(N)$
13.5	0.9996	127.9958	16.5	0.1798	477.2827
14.0	0.9978	165.6792	17.0	0.0498	352.7286
14.5	0.9891	225.6010	17.5	0.0126	263.8483
15.0	0.9502	324.4736	18.0	0.0032	208.2099
15.5	0.8062	466.0721	18.5	0.0008	172.3676
16.0	0.4847	553.7270	19.0	0.0002	147.9020

Table 4.2: OC Function for $\beta = 1$
$(H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05)$

α	$M = 0.96$	$M = 0.98$	$M = 1$	$M = 1.02$	$M = 1.04$
13.5	0.9999	0.9998	0.9996	0.9990	0.9973
14.0	0.9996	0.9991	0.9978	0.9945	0.9859
14.5	0.9982	0.9956	0.9891	0.9729	0.9326
15.0	0.9915	0.9794	0.9502	0.8820	0.7409
15.5	0.9625	0.9127	0.8062	0.6180	0.3801
16.0	0.8549	0.7043	0.4847	0.2659	0.1195
16.5	0.5832	0.3592	0.1798	0.0770	0.0299
17.0	0.2549	0.1195	0.0498	0.0193	0.0071
17.5	0.0790	0.0325	0.0126	0.0047	0.0017
18.0	0.0215	0.0084	0.0032	0.0012	0.0004
18.5	0.0057	0.0022	0.0008	0.0003	0.0001
19.0	0.0015	0.0006	0.0002	0.0001	0.0000

Table 4.3: ASN Function for $\beta = 1$
$(H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05)$

α	$M = 0.96$	$M = 0.98$	$M = 1$	$M = 1.02$	$M = 1.04$
13.5	101.831	113.436	127.995	146.751	171.674
14.0	125.694	143.019	165.679	196.265	238.855
14.5	160.444	187.968	225.601	278.228	351.372
15.0	214.329	260.518	324.473	408.744	500.109
15.5	301.607	376.769	466.072	539.308	546.427
16.0	432.101	514.788	553.727	517.734	434.008
16.5	545.678	542.925	477.282	390.735	314.599
17.0	513.627	434.008	352.728	286.839	237.491
17.5	393.146	320.415	263.848	221.583	189.950
18.0	293.293	244.710	208.209	180.509	159.061
18.5	228.654	196.849	172.367	153.132	137.696
19.0	187.103	165.280	147.902	133.787	122.117

Table 4.4: OC Function for $\beta > 1$ ($H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05$)			
α	$M = 0.98$	$M = 1$	$M = 1.02$
13.5	0.9990	0.9996	0.9998
14.0	0.9947	0.9979	0.9992
14.5	0.9741	0.9896	0.9958
15.0	0.8868	0.9523	0.9803
15.5	0.6292	0.8134	0.9162
16.0	0.2754	0.4964	0.7137
16.5	0.0805	0.1869	0.3700
17.0	0.0203	0.0521	0.1245
17.5	0.0049	0.0133	0.0341
18.0	0.0012	0.0033	0.0088

Table 4.5: ASN Function for $\beta > 1$ ($H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05$)			
α	$M = 0.98$	$M = 1$	$M = 1.02$
13.5	146.4394	127.7334	113.2132
14.0	195.9239	165.3690	142.7532
14.5	278.2979	225.3610	187.6875
15.0	412.8893	325.3565	260.5346
15.5	588.1994	476.0245	379.3985
16.0	285.7719	350.2118	427.2487
16.5	221.0826	262.9952	318.6569
17.0	180.1645	207.7392	243.9831
17.5	152.8516	172.0219	196.3978
18.0	133.5448	147.6163	164.9343

Table 4.6: OC Function for $\beta < 1$ ($H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05$)			
α	$M = 0.98$	$M = 1$	$M = 1.02$
13.5	0.9989	0.9996	0.9998
14.0	0.9942	0.9977	0.9991
14.5	0.9716	0.9886	0.9954
15.0	0.8770	0.9479	0.9785
15.5	0.6066	0.7988	0.9089
16.0	0.2565	0.4729	0.6946
16.5	0.0736	0.1728	0.3486
17.0	0.0184	0.0476	0.1146
17.5	0.0045	0.0120	0.0311
18.0	0.0011	0.0030	0.0080
18.5	0.0003	0.0008	0.0021
19.0	0.0001	0.0002	0.0005

Table 4.7: ASN Function for $\beta < 1$ ($H_0 : \alpha_0 = 15, H_1 : \alpha_1 = 17, \alpha = \beta = 0.05$)			
α	$M = 0.98$	$M = 1$	$M = 1.02$
13.5	147.0640	128.2587	113.6602
14.0	196.6034	165.9886	143.2857
14.5	278.1236	225.8304	188.2458
15.0	404.3683	323.5154	260.4790
15.5	489.2650	455.7250	373.9965
16.0	538.9801	980.1456	490.9139
16.5	394.5228	488.5765	585.9467
17.0	287.8846	355.1826	440.6188
17.5	222.0800	264.6894	322.1396
18.0	180.8551	208.6788	245.4315
18.5	153.4134	172.7137	197.3016
19.0	134.0308	148.1883	165.6279

Table 4.8: OC and ASN Function for $\beta = 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)					
γ	$L(\gamma)$	$E(N)$	γ	$L(\gamma)$	$E(N)$
13.5	0.9995	127.9957	16.5	0.1797	477.2827
14.0	0.9978	165.6791	17.0	**0.0498**	352.7285
14.5	0.9891	225.6010	17.5	0.0126	263.8483
15.0	**0.9501**	324.4736	18.0	0.0031	208.2098
15.5	0.8062	466.0720	18.5	0.0007	172.3676
16.0	0.4846	553.7269	19.0	0.0002	147.9020

Table 4.9: OC Function for $\beta = 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)					
γ	$N = 1.04$	$N = 1.02$	$N = 1$	$N = 0.98$	$N = 0.96$
13.5	0.9999	0.9998	0.9995	0.9990	0.9973
14.0	0.9996	0.9991	0.9978	0.9945	0.9859
14.5	0.9981	0.9956	0.9891	0.9729	0.9326
15.0	0.9914	0.9794	0.9501	0.882	0.7409
15.5	0.9625	0.9127	0.8062	0.6180	0.3801
16.0	0.8549	0.7043	0.4846	0.2659	0.1195
16.5	0.5831	0.3592	0.1797	0.0770	0.0299
17.0	0.2548	0.1195	0.0498	0.0193	0.0071
17.5	0.0790	0.0325	0.0126	0.0047	0.0017
18.0	0.0214	0.0084	0.0031	0.0012	0.0004
18.5	0.0056	0.0022	0.0007	0.0003	0.0001
19.0	0.0015	0.0006	0.0002	0.0001	0.0

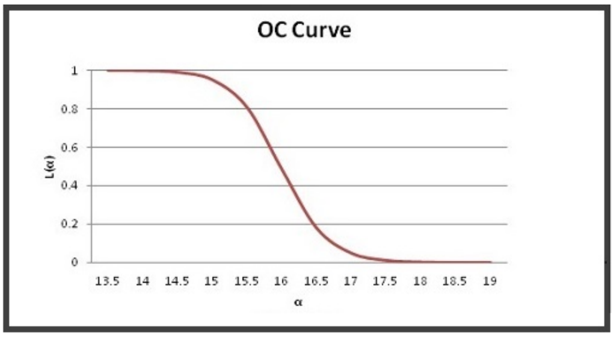
Table 4.10: ASN Function for $\beta = 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)					
γ	$N = 1.04$	$N = 1.02$	$N = 1$	$N = 0.98$	$N = 0.96$
13.5	101.830	113.436	127.995	146.751	171.674
14.0	125.694	143.019	165.679	196.265	238.855
14.5	160.444	187.968	225.601	278.228	351.372
15.0	214.329	260.518	324.473	408.744	500.109
15.5	301.607	376.769	466.072	539.308	546.427
16.0	432.101	514.788	553.726	517.734	434.009
16.5	545.678	542.925	477.282	390.735	314.599
17.0	513.627	434.008	352.728	286.839	237.491
17.5	393.146	320.415	263.848	221.583	189.950
18.0	293.293	244.710	208.209	180.509	159.061
18.5	228.654	196.849	172.367	153.132	137.696
19.0	187.103	165.280	147.902	133.787	122.117

Table 4.11: OC Function for $\beta > 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)			
γ	$N = 1.02$	$N = 1$	$N = 0.98$
13.5	0.9998	0.9996	0.9990
14.0	0.9991	0.9978	0.9945
14.5	0.9956	0.9891	0.9729
15.0	0.9794	0.9502	0.8819
15.5	0.9127	0.8062	0.6178
16.0	0.7045	0.4847	0.2657
16.5	0.3595	0.1798	0.0769
17.0	0.1196	0.0498	0.0193
17.5	0.0326	0.0126	0.0047
18.5	0.0022	0.0008	7 0.0003
19.0	0.0006	0.0002	0.0001

Table 4.12: ASN Function for $\beta > 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)			
γ	$N = 1.02$	$N = 1$	$N = 0.98$
13.5	113.423	127.995	146.772
14.0	142.999	165.679	196.301
14.5	187.935	225.601	278.290
15.0	260.463	324.473	408.835
15.5	376.684	466.072	539.355
16.0	514.719	553.727	517.666
16.5	542.965	477.282	390.651
17.0	434.095	352.728	286.782
17.5	320.481	263.848	221.547
18.0	244.752	208.209	180.485
18.5	196.877	172.367	153.115
19.0	165.300	147.902	133.774

Table 4.13: OC Function for $\beta < 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)			
γ	$N = 1.02$	$N = 1$	$N = 0.98$
13.5	0.9998	0.9996	0.9990
14.0	0.9991	0.9978	0.9945
14.5	0.9956	0.9891	0.9729
15.0	0.9794	0.9502	0.8821
15.5	0.9126	0.8062	0.6182
16.0	0.7041	0.4847	0.2661
16.5	0.3590	0.1798	0.0771
17.0	0.1194	0.0498	0.0193
17.5	0.0325	0.0126	0.0047
18.0	0.0084	0.0032	0.0012
18.5	0.0022	0.0008	0.0003
19.0	0.0006	0.0002	0.0001

Table 4.14: ASN Function for$\beta < 1$ ($H_0 : \gamma_0 = 15, H_1 : \gamma_1 = 17, \alpha = \beta = 0.05$)			
γ	$N = 1.02$	$N = 1$	$N = 0.98$
13.5	113.4494	127.9958	146.7304
14.0	143.0392	165.6792	196.2303
14.5	188.0003	225.6010	278.1673
15.0	260.5734	324.4736	408.6527
15.5	376.855	466.0721	539.2613
16.0	514.8594	553.7270	517.8019
16.5	542.8852	477.2827	390.8183
17.0	433.9228	352.7286	286.8957
17.5	320.3505	263.8483	221.6191
18.0	244.6681	208.2099	180.5339
18.5	196.8219	172.3676	153.1492
19.0	165.2612	147.9020	133.8002



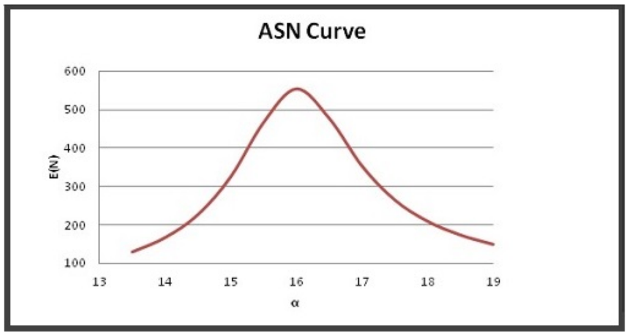
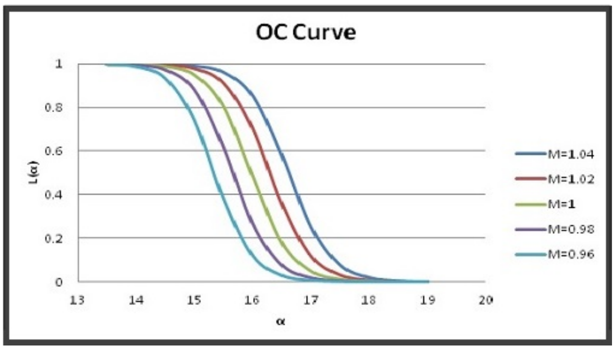


Figure 4.1 and Figure 4.2



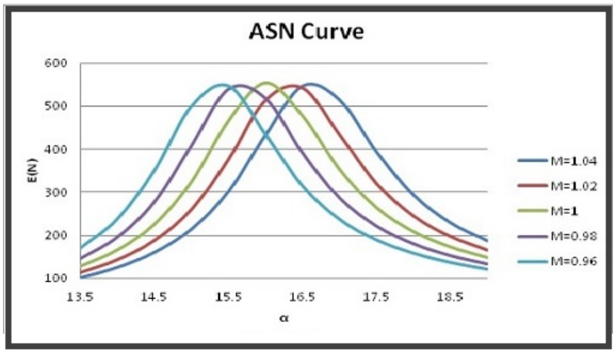
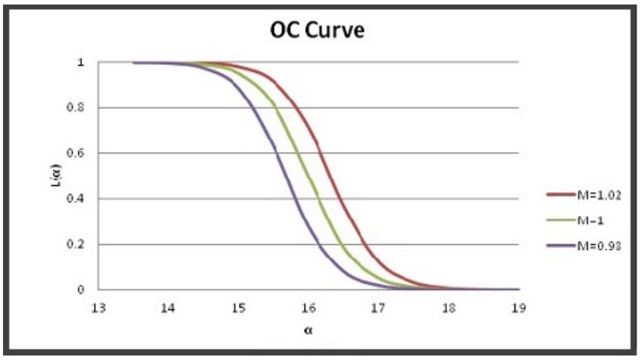


Figure 4.3 and Figure 4.4



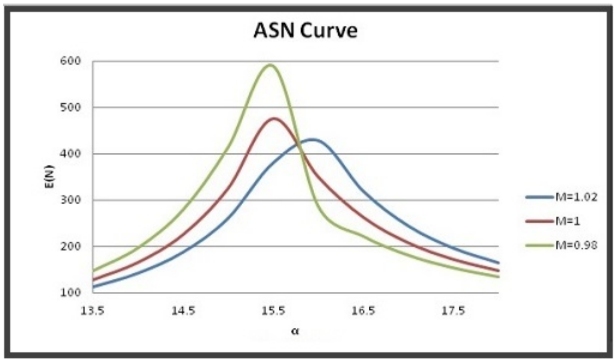
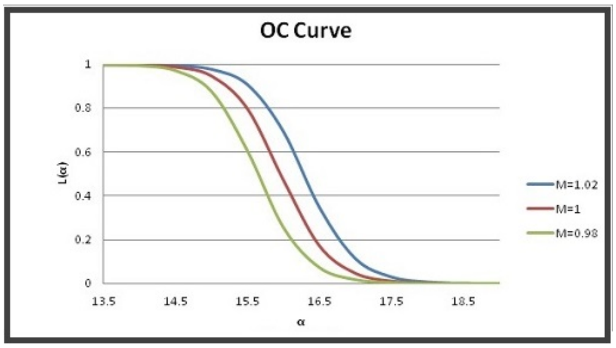


Figure 4.5 and Figure 4.6



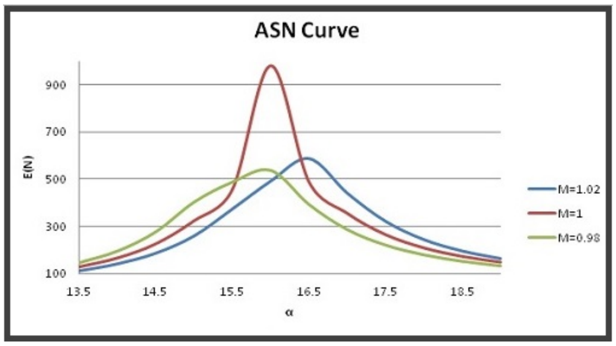
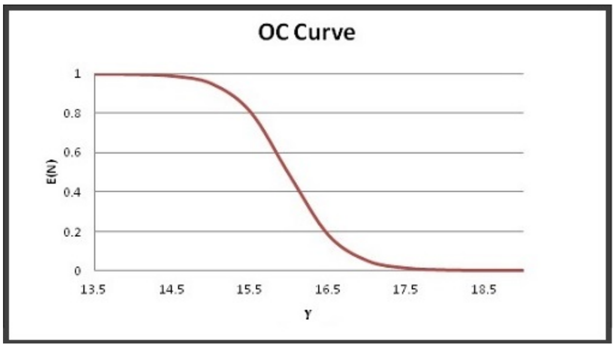


Figure 4.7 and Figure 4.8



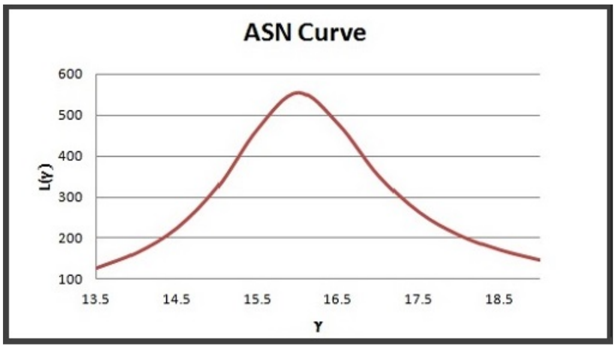
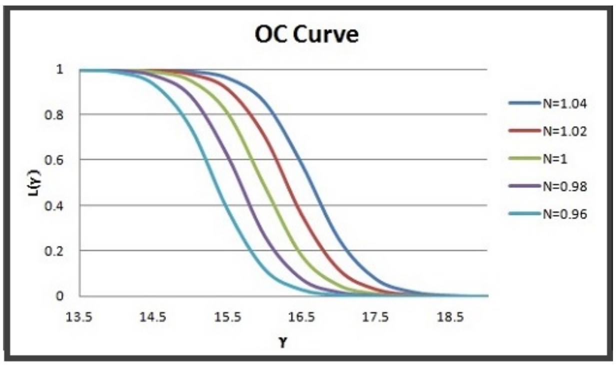


Figure 4.9 and Figure 4.10



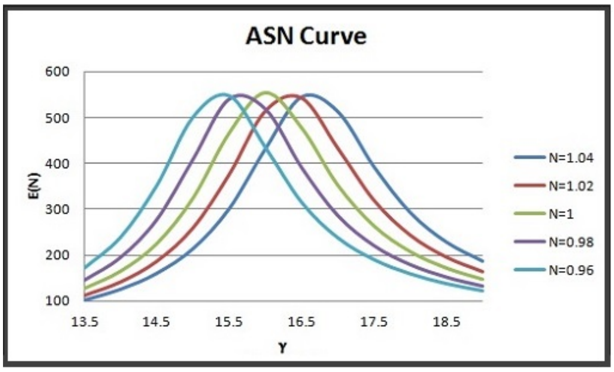
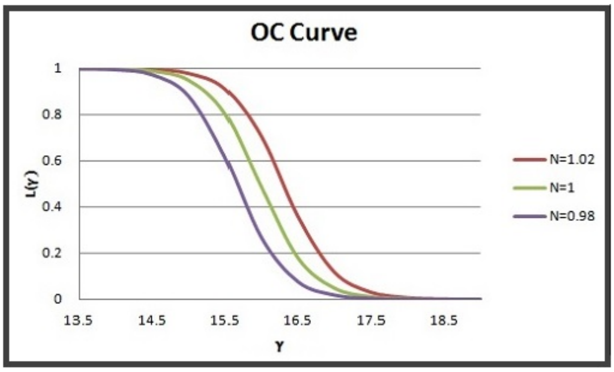


Figure 4.11 and Figure 4.12



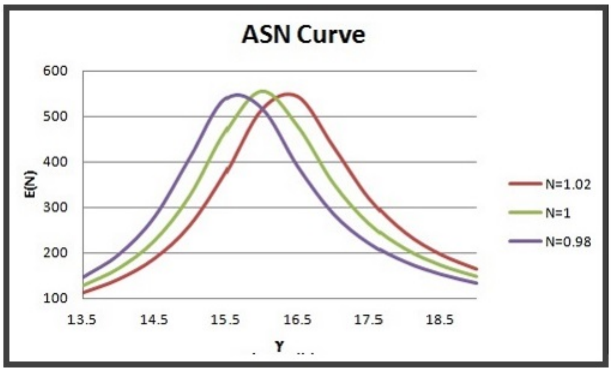
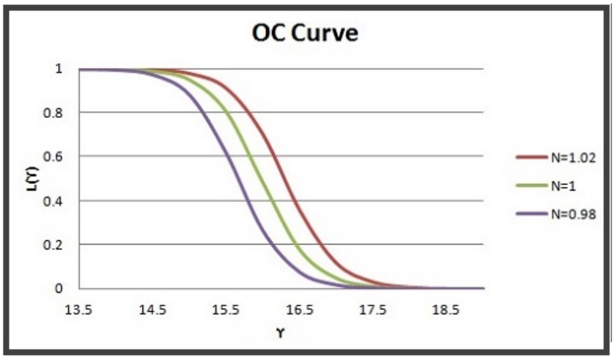


Figure 4.13 and Figure 4.14



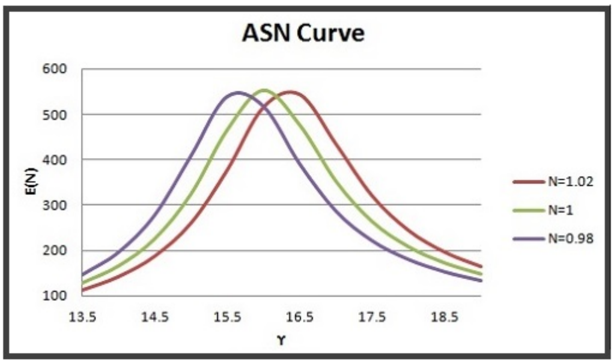


Figure 4.15 and Figure 4.16

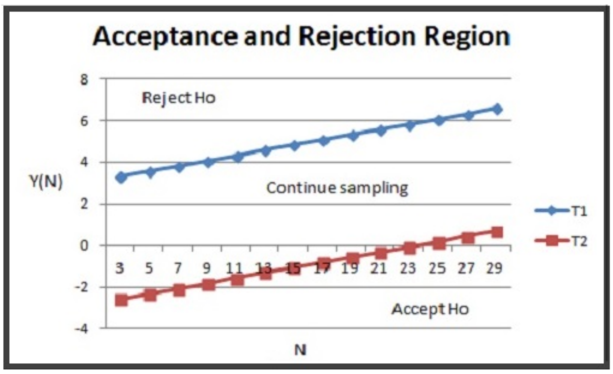


Figure 4.17

Chapter 5

Study of disaster probability when Strength follows Power Function Distribution and Stress follows Odd Generalised Exponential Gompertz (OGE-G) Distribution

5.1 Introduction

For any complex manufactured system, the reliability of its component or the whole system is always a topic of discussion for the manufacturers as well as the buyers. Overestimation and underestimation of stress and strength of the components, items or systems may engender to great losses in terms of system failures as well as human injuries. There are several statistical methods/models exist in the literature to study the reliability of a system. For example, $R(t) = P(X > t)$, where t is the given time, $R = Pr(X > Y)$, where X and Y represents the strength and stress of the model respectively, $P = Pr(X > \theta)$, where θ is the maximum range of the strength distribution etc. For a brief review, one may refer to Church and Harris (1970)[36], Enis and Geisser (1971)[48], Downton (1973)[41], Tong (1974)[107], Kelly (1976)[70], Sathe and Shah (1981)[92], Chao (1982)[22], Awad and Gharraf (1986)[3], Kundu and Gupta (2005)[73], Raqab and Kundu (2005)[90], Kundu and Raqab (2009)[74], Chaturvedi and Sharma (2010)[31], Rezaei et al. (2010)[91] and Surinder and Mayank (2017)[102].

In this chapter, the OGE-G lifetime model is considered, which has many advantages over the other well known life testing models such as Exponential, Generalised Exponential, Gompertz, Generalised Gompertz and Beta Gompertz distribution [see El-Damcese et al. (2015)[49]]. The probability density function (pdf) and cumulative density function (cdf) of the OGE-G distribution, which is considered here to represent the stress of the manufactured devices is defined as

$$f(x;\Theta) = \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} \tag{5.1.1}$$

and

$$F(x;\Theta) = \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta}; x > 0, \qquad \gamma, \lambda, \beta, c > 0 \tag{5.1.2}$$

where $\Theta = (c, \gamma, \lambda, \beta)$ and c, γ, λ are the scale parameters and β is the shape parameter, respectively.

Let us assume that the strength of the manufactured items/devices follows the power function distribution with probability density function (pdf)

$$g(y;\theta,\mu) = \frac{\mu}{\theta}\left(\frac{y}{\theta}\right)^{\mu-1}; 0 < y < \theta, \mu > 0 \tag{5.1.3}$$

This chapter has manifolds, in Section 5.2, the theoretical expressions for the probability of disaster is obtained. In Section 5.4 and Section 5.5, the stress-strength reliability for the model $R = Pr(Y > X)$ is obtained when the strength follows power distribution and when strength follows OGE-G distribution, respectively. In Section 5.3 and Section 5.6, the numerical study is done and the results are discussed. Finally, the whole study is illustrated with an example in Section 5.7.

5.2 Probability of disaster i.e. when $\alpha = P(X > \theta)$

Theorem 5.1: If the random variable X follows OGE-G distribution given at (5.1.1) and θ is the maximum range of a random variable Y which follows Power function distribution given at (5.1.3) respectively, then α is given by

$$\alpha = P(X > \theta)$$

$$= 1 - \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{p}-1)}-1\right\}}\right]^{\beta}$$

where, $p = c\theta$.

Proof: We know that

$$\alpha = P(X > \theta)$$

$$= \int_0^\infty \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} dx \tag{5.2.1}$$

On taking $1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}$= z, in equation (5.2.1), we get

$$\alpha = \int_{1-e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{c\theta}-1)}-1\right\}}}^{1} z^{\beta-1} dz$$

or,

$$\alpha = 1 - \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{p}-1)}-1\right\}}\right]^{\beta} \tag{5.2.2}$$

where, $p = c\theta$.

Hence, the theorem follows.

5.3 Numerical study for the Probability of Disaster α for different values of c and λ

From the expression (5.2.2), which is established for measuring the probability of disaster $\alpha = P(X > \theta)$, the numerical values are obtained for different combinations of p, c and λ and are presented in Table 5.1. It can be easily interpreted from Table 5.1 that the probability of disaster decreases with an increase in the value of p. The probability of disaster means the stress increases over the strength i.e. disaster will happen when $X > \theta$ [Alam and Roohi (2003)[1]]. Here, it is suggested that in order to overcome the problem of disaster (i.e. to attain the smallest value of $\alpha = P(X > \theta)$), the values of $p = c\theta$, where c is the scale parameter of OGE-G distribution and θ is the scale parameter of the power function distribution, should be considered in such a manner that the value of α tends to zero.

The values of p at different tolerance level for α are presented in Table 5.2. These values have an interpretation that as the tolerance level α decreases, the corresponding values of p increases. Further, these values are utilized to have an idea to obtain the minimum cost, which is shown in Section 5.7.

5.4 Stress-Strength Reliability when the random stress and strength follows OGE-G and Power function distribution

Theorem 5.2: Let $X \sim f(x;\Theta)$ and $Y \sim g(y;\theta,\mu)$, where X represents the stress and Y represents the strength, respectively, then $R = Pr(Y > X)$ is given by

$$R = \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{p}-1)}-1\right\}}\right]^{\beta} - \frac{1}{p^{\mu}}\int_{0}^{1-e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{p}-1)}-1\right\}}} \beta log\left[1+\frac{ct}{\lambda\gamma}\right] t^{\beta-1}dt \text{where}, p = c\theta. \tag{5.4.1}$$

where, $p = c\theta$.

Proof: We know that

$$R = \int_{0}^{\theta}\int_{x}^{\theta} f(x;\Theta)g(y;\theta,\mu)dydx \tag{5.4.2}$$

$$= \int_{0}^{\theta}\int_{x}^{\theta} \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} \frac{\mu}{\theta}\left(\frac{y}{\theta}\right)^{\mu-1} dydx \tag{5.4.3}$$

On substituting $y = \nu$x in (5.4.3), we get

$$= \int_{0}^{\theta}\int_{1}^{\frac{\theta}{x}} \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} \frac{\mu}{\theta}\left(\frac{vx}{\theta}\right)^{\mu-1} dvdx$$

$$= \int_{0}^{\frac{p}{c}}\int_{1}^{\frac{p}{cx}} \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} \mu\left(\frac{x}{\theta}\right)^{\mu} v^{\mu-1} dvdx$$

$$= \int_{0}^{\frac{p}{c}} \gamma\beta\lambda e^{cx} e^{\frac{\lambda}{c}(e^{cx}-1)} e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}} \left[1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}\right]^{\beta-1} \left(\frac{x}{\theta}\right)^{\mu} \left[\left(\frac{p}{cx}\right)^{\mu} - 1\right] dx \tag{5.4.4}$$

On substituting $1 - e^{-\gamma\left\{e^{\frac{\lambda}{c}(e^{cx}-1)}-1\right\}}$= t, in (5.4.4) and solving the above integrals, we get, the theorem follows.

Remarks 5.1: R is obtained for the fixed value of λ, c, γ, β and for different values of p and μ, from (5.4.1) and shown in Table 5.3. Interpretation of Table 5.3 is given in Section 5.6.

5.5 Stress-Strength Reliability when both the random stress and strength follows OGE-G distribution

Theorem 5.3: Let X and Y be two independent random variables from OGE-G distribution, where X and Y are the stress and the strength, respectively, which the item/component faces, then $R = Pr(Y > X)$ is given by

$$R = 1 - \int_0^\infty \gamma_1\beta_1\lambda_1 e^{c_1 x} e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)} e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)}-1\right\}} \left[1 - e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)}-1\right\}}\right]^{\beta_1-1} \left[1 - e^{-\gamma_2\left\{e^{\frac{\lambda_2}{c_2}(e^{c_2 x}-1)}-1\right\}}\right]^{\beta_2} dx \quad (5.5.1)$$

Proof: The pdfs follows the random stress X and random strength Y, respectively. The probability 'R' is obtained by solving the following expression

$$R = \int_0^\infty \int_x^\infty f(x;\Theta) f(y;\Theta) dy dx \quad (5.5.2)$$

$$= \int_0^\infty \gamma_1\beta_1\lambda_1 e^{c_1 x} e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)} e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)}-1\right\}} \left[1 - e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1 x}-1)}-1\right\}}\right]^{\beta_1-1} \left[\int_x^\infty \gamma_2\beta_2\lambda_2 e^{c_2 y} e^{\frac{\lambda_2}{c_2}(e^{c_2 y}-1)} e^{-\gamma_2\left\{e^{\frac{\lambda_2}{c_2}(e^{c_2 y}-1)}-1\right\}} \left[1 - e^{-\gamma_2\left\{e^{\frac{\lambda_2}{c_2}(e^{c_2 y}-1)}-1\right\}}\right]^{\beta_2-1} dy\right] dx \quad (5.5.3)$$

$$= \int_0^\infty \gamma_1\beta_1\lambda_1 e^{c_1x} e^{\frac{\lambda_1}{c_1}(e^{c_1x}-1)} e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1x}-1)}-1\right\}} \left[1-e^{-\gamma_1\left\{e^{\frac{\lambda_1}{c_1}(e^{c_1x}-1)}-1\right\}}\right]^{\beta_1-1}$$

$$\left[1-\left[1-e^{-\gamma_2\left\{e^{\frac{\lambda_2}{c_2}(e^{c_2x}-1)}-1\right\}}\right]^{\beta_2}\right] dx \qquad (5.5.4)$$

Hence, the theorem follows, on solving (5.5.4).

Remarks 5.2 The equation (5.5.1) cannot be evaluated further. To study the behaviour of the stress-strength model, the equation (5.5.1) is tackled with Mathematica software and the results are presented in Table 5.4 which concludes that as the value of β_2 increases, the probability 'R' converges to one, for decreasing values of β_1.

5.6 Discussion

Any manufactured items or components has maximum limit of its strength. For example, in case of an electric bulb its maximum voltage capacity is 220V, on the other hand, the accelerating capacity of an engine should not increase its maximum possible speed. Thus, it is desirable that the value of θ must have the maximum limit say, θ_0. For a fixed tolerance level α, suppose one wishes that θ_α is the required value of θ. In this particular case $\theta_\alpha < \theta_0$, one may obtain the desired value of μ say, μ_α , by using Table 5.3, so that the items or components are manufactured with the strength distribution parameters having parameters (μ_α, θ_α) and subsequently the required strength reliability may be achieved. In case, if $\theta_\alpha > \theta_0$ then one will have to either adjust α or needed some alterations in the manufactured items or components.

5.7 An illustrative example

Without loss of generality, let us suppose that the maximum possible value of p is 6.0. For $\alpha \le 0.01$, we must have $p \ge 5.25$. Since p cannot exceed 6.0, we have the option of fixing the item in such a way that $5.25 \le p \le 6.00$ i.e. $2.0 \le \theta \le 2.2$ and corresponding value of μ

leads to a maximum of $P(Y > X)$.

Let C_1 be the cost of adjusting one unit of θ and C_2 be the cost of adjusting one unit of μ. Minimize $C = C_1\theta + C_2\mu$ subject to $2.1 \leq \theta \leq 2.4$ and $P(Y > X) \geq 0.99$. The problem may be solved analytically as follows:

Using Table 5.3, for $p = 5.25$, 5.50, 5.75, 6.00 i.e. $\theta = 2.1$, 2.2, 2.3, 2.4 and find those values of μ for which $P(Y > X) \geq 0.99$. Evaluate the cost function for each pair of $(\theta,\ \mu)$. Clearly, the minimum of the cost lies at $2.1C_1 + 4C_2$ depending upon the numerical values of C_1 and C_2.

5.8 Tables

Table 5.1
Numerical Values for Probability of Disaster $\alpha = P(X > \theta)$ and p
for different values of c and λ at $\gamma = 0.05$ and $\beta = 0.05$

p	$\mathbf{c = 2.5, \lambda = 0.05}$	$c = 0.5, \lambda = 0.05$	$c = 2.5, \lambda = 0.5$	$c = 1.5, \lambda = 0.5$	$c = 2.5, \lambda = 1.5$
0.25	**0.335146**	0.279042	0.253111	0.233093	0.208776
0.50	**0.306994**	0.247991	0.220257	0.198406	0.170982
0.75	**0.287748**	0.226410	0.196959	0.173166	0.142125
1.00	**0.272036**	0.208422	0.177056	0.150930	0.115321
1.25	**0.258129**	0.192081	0.158420	0.129358	0.087933
1.50	**0.245237**	0.176427	0.139912	0.107088	0.587346
1.75	**0.232917**	0.160847	0.120702	0.083123	0.028786
2.00	**0.220888**	0.144852	0.100061	0.056925	0.005489
2.25	**0.208942**	0.127973	0.077341	0.029524	0.000015
2.75	**0.184632**	0.089554	0.026261	0.000072	0.0
3.25	**0.158660**	0.042331	0.000040	0.0	0.0
4.00	**0.112657**	0.000001	0.0	0.0	0.0
4.25	**0.094092**	0.0	0.0	0.0	0.0
5.50	**0.000100**	0.0	0.0	0.0	0.0

Table 5.2
Values of p at different tolerance level α, when $\gamma = 0.05$
$\beta = 0.05$, $c = 2.5$ and $\lambda = 0.05$

α	0.05	0.02	**0.01**	0.001	0.0001	0.00001
$p = c\theta$	4.74943	5.06156	**5.18660**	5.39246	5.49117	5.55367

Table 5.3
The Stress-Strength reliability of an item for $\lambda = 0.05, c = 2.5$,
$\gamma = 0.05, \beta = 0.05$ and varying the values of p and μ

p	$\mu = 2$	$\mu = 4$	$\mu = 6$	$\mu = 8$	$\mu = 10$
3.75	0.841800	0.868613	0.855261	0.870655	0.855448
4.00	0.857448	0.885475	0.887226	0.887336	0.887343
4.25	0.874561	0.904172	0.905811	0.905902	0.905907
4.50	0.893495	0.925137	0.926699	0.926776	0.926780
4.75	0.914364	0.948474	0.949986	0.950053	0.950056
5.00	0.936185	0.972840	0.974306	0.974364	0.974367
5.25	0.954621	**0.992503**	**0.993877**	**0.993927**	**0.993929**
5.50	0.962751	**0.998695**	**0.999883**	**0.999923**	**0.999924**
5.75	0.965973	**0.998971**	**0.999969**	**0.999984**	**0.999998**
6.00	0.968750	**0.999132**	**0.999976**	**0.999996**	**0.999999**

Table 5.4
Numerical values of the Stress-Strength model
$P(Y > X)$ at different values of underlying parameters

β_2	$\beta_1 = 1.50$	$\beta_1 = 1.25$	$\beta_1 = 0.75$	$\beta_1 = 0.50$	$\beta_1 = 0.25$
17.5	0.921053	0.933333	0.958904	0.972222	0.985915
20.0	0.930233	0.941176	0.963855	0.975612	0.987654
22.5	0.937500	0.947368	0.967742	0.978261	0.989011
25.0	0.943396	0.952381	0.970874	0.980392	**0.990099**
27.5	0.948276	0.956522	0.973451	0.982143	**0.990991**
30.0	0.952381	0.960000	0.975613	0.983607	**0.991736**
32.5	0.955882	0.962963	0.977444	0.984848	**0.992366**
35.0	0.958904	0.965517	0.979021	0.985915	**0.992908**
37.5	0.961538	0.967742	0.980392	0.986842	**0.993377**
40.0	0.963855	0.969697	0.981595	0.987654	**0.993789**
42.5	0.965909	0.971429	0.982659	0.988372	**0.994152**
45.0	0.967742	0.972973	0.983607	0.989011	**0.994475**
47.5	0.969388	0.974359	0.984456	0.989583	**0.994764**
50.0	0.970874	0.975615	0.985222	**0.990099**	**0.995025**
52.5	0.972222	0.976744	0.985915	**0.990566**	**0.995261**
55.0	0.973451	0.977778	0.986547	**0.990991**	**0.995475**
57.5	0.974576	0.978723	0.987124	**0.991379**	**0.995671**
60.0	0.975610	0.979592	0.987654	**0.991736**	**0.995851**
62.5	0.976563	0.980392	0.988142	**0.992063**	**0.996016**
65.0	0.977444	0.981132	0.988593	**0.992366**	**0.996169**
67.5	0.978261	0.981818	0.989011	**0.992647**	**0.996310**
70.0	0.979021	0.982456	0.989399	**0.992908**	**0.996441**
72.5	0.979730	0.983051	0.989761	**0.993151**	**0.996564**
75.0	0.980392	0.983607	**0.990099**	**0.993377**	**0.996678**
77.5	0.981013	0.984127	**0.990415**	**0.993597**	**0.996785**
80.0	0.981595	0.984615	**0.990712**	**0.993789**	**0.996885**

Table 5.5
Table for obtaining cost of manufacturing item

θ	μ	$C_1\theta + C_2\mu$	θ	μ	$C_1\theta + C_2\mu$
2.1	**4**	$\mathbf{2.1C_1 + 4C_2}$	2.3	4	$2.3C_1 + 4C_2$
2.1	6	$2.1C_1 + 6C_2$	2.3	6	$2.3C_1 + 6C_2$
2.1	8	$2.1C_1 + 8C_2$	2.3	8	$2.3C_1 + 8C_2$
2.1	10	$2.1C_1 + 10C_2$	2.3	10	$2.3C_1 + 10C_2$
2.2	4	$2.2C_1 + 4C_2$	2.4	4	$2.3C_1 + 4C_2$
2.2	6	$2.2C_1 + 6C_2$	2.4	6	$2.4C_1 + 6C_2$
2.2	8	$2.2C_1 + 8C_2$	2.4	8	$2.4C_1 + 8C_2$
2.2	10	$2.2C_1 + 10C_2$	2.4	10	$2.4C_1 + 10C_2$

Chapter 6

Point estimator, Confidence interval for $P(Y > X)$ through Transformation method and the probability of disaster for a Positive Exponential Family of Distributions (PEFD)

6.1 Introduction

Reliability measure $R(t) = P(X > t)$ which defines the failure free operation of components/items until time t and the measure $R = P(X > Y)$ commonly the reliability of components/items, where the random variable X and Y are the random stress and random strength. Another measure of reliability, $\alpha = P(X > \gamma)$ which represents the probability of disaster, where the variable X represents the stress and γ is the maximum strength of the components/items.

In the literature of reliability lot of work has been done since last few decades and their references have been discussed in Chapter 5. For a brief review of literature, the most popular article on the related study are Pugh (1963)[89], Basu (1964)[11], Chaturvedi and Surinder (1999)[26], Kotz et al. (2003)[72], Chaturvedi and Pathak (2012)[32], Surinder and Mayank (2014)[100], Surinder and Mukesh (2016)[101], Surinder et al. (2018)[103] . In the present chapter, we have considered a positive exponential family of distribution, which covers various lifetime distributions as their specific cases.

6.2 Set-up of the problem

Liang (2008)[76] proposed a positive exponential family of lifetime distributions, which covers gamma distribution as specific case. Let the random variable X has positive exponential family of distributions, then the pdf is given by

$$f(x;\Theta) = \frac{\rho x^{\rho\nu-1} e^{\left(\frac{-x^\rho}{\theta}\right)}}{\Gamma\nu\theta^\nu}; x > 0,\ \theta, \nu, \rho > 0 \tag{6.2.1}$$

where, θ is assumed to be unknown and ρ, ν are known constants. On assigning different values to ν and ρ, this family of distributions cover following pdfs

(1) For $\rho = \nu = 1$, we get one parameter exponential distribution.

(2) For $\rho = 1$, we get gamma distribution.

(3) For ν=1, we get Weibull distribution

(4) For $\nu > 0$, $\rho = 1$, we get Erlang distribution .

(5) For $\nu > 1/2$, $\rho = 2$, we get half-normal distribution.

(6) For $\nu > m/2$, $\rho = 2$, we get Chi-distribution.

(7) For $\nu = 1$, $\rho = 2$, we get Rayleigh distribution.

(8) For $\nu = p + 1$, $\rho = 2$, we get Generalized Rayleigh distribution.

Let the random variable Y considered as strength of items/components follows Power function distribution then the cdf and pdf are given by (6.2.2) and (6.2.3), respectively.

$$G(y;\mu,\gamma)=\left(\frac{y}{\gamma}\right)^{\mu} \tag{6.2.2}$$

and

$$g(y;\mu,\gamma)=\frac{\mu}{\gamma}\left(\frac{y}{\gamma}\right)^{\mu-1};0<y<\gamma,\ \mu>0 \tag{6.2.3}$$

6.3 MLE and UMVUE of $R = P(Y > X)$

The MLE and UMVUE of $R = P(Y > X)$ for PEFD by using the transformation method are evaluated in the following theorems.

Theorem 6.1: The MLE of $R = P(Y > X)$ is given by

$$\tilde{R}=\left(\frac{\nu_1\bar{\eta}}{\nu_2\bar{\xi}+\nu_1\bar{\eta}}\right)^{\nu_1}\frac{1}{B(\nu_1,\nu_2)}{}_2F_1\left(\nu_1,1-\nu_2;\nu_1;\frac{\nu_1\bar{\eta}}{\nu_2\bar{\xi}+\nu_1\bar{\eta}}\right) \tag{6.3.1}$$

where, $\bar{\xi}=\frac{1}{n_1}\sum_{i=1}^{n_1}X_i^{\rho_1}=\bar{T}_X$ and $\bar{\eta}=\frac{1}{n_2}\sum_{j=1}^{n_2}Y_j^{\rho_2}=\bar{T}_Y$

Proof: In order to transform the pdf (6.2.1), let us assume $x^{\rho_1}=\xi$, we get

$$f(\xi;\Theta)=\frac{\xi^{\nu_1-1}e^{-\left(\frac{\xi}{\lambda_1}\right)}}{\Gamma\nu_1{\lambda_1}^{\nu_1}};\ \xi,\lambda_1,\nu_1>0 \tag{6.3.2}$$

Similarly, for $\eta = y^{\rho_2}$

$$f(\eta;\Theta)=\frac{\xi^{\nu_1-2}e^{-\left(\frac{\eta}{\lambda_2}\right)}}{\Gamma\nu_2{\lambda_2}^{\nu_2}};\ \eta,\lambda_2,\nu_2>0 \tag{6.3.3}$$

where $\lambda_1=\theta_1$ and $\lambda_2=\theta_2$

Let ξ and η are two independent random variables with gamma pdfs given at (6.3.2) and (6.3.3). Thus,

$$\begin{aligned}R&=P(\eta>\xi)\\&=P(\frac{\eta}{\xi}>1)\end{aligned}$$

$$= P\left(\frac{\eta/\lambda_2}{\xi/\lambda_1} > \frac{\lambda_1}{\lambda_2}\right)$$
$$= P\left(\frac{\eta/\lambda_2}{\xi/\lambda_1} + 1 > \frac{\lambda_1}{\lambda_2} + 1\right)$$
$$R = P\left(\frac{\xi/\lambda_1}{\xi/\lambda_1 + \eta/\lambda_2} > \frac{\lambda_2}{\lambda_2 + \lambda_1}\right)$$

Since, we know that, if ξ and η be two independent random variables which follow gamma distribution with parameters (λ_1, ν_1) and (λ_2, ν_2) then $z = \frac{\xi/\lambda_1}{\xi/\lambda_1 + \eta/\lambda_2}$ is a beta (ν_1, ν_2) random variable with the pdf

$$f(z, \nu_1, \nu_2) = [B(\nu_1, \nu_2)]^{-1} z^{\nu_1 - 1} (1 - z)^{\nu_2 - 1}$$

or

$$R = I_{\frac{\lambda_2}{\lambda_2 + \lambda_1}}(\nu_1, \nu_2) \tag{6.3.4}$$

which is an incomplete beta function. Using the relation between an incomplete beta function and the hypergeometric series, we rewrite (6.3.4) as

$$R = \left(\frac{\lambda_2}{\lambda_2 + \lambda_1}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \left(\frac{\lambda_2}{\lambda_2 + \lambda_1}\right)\right) \tag{6.3.5}$$

The reliability $R = P(Y > X)$

$$R = \left(\tfrac{\theta_2}{\theta_2 + \theta_1}\right)^{\nu_1} \tfrac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \left(\tfrac{\theta_2}{\theta_2 + \theta_1}\right)\right)$$

Substituting the MLEs i.e. $\tilde{\lambda_1} = \frac{\bar{\xi}}{\nu_1}$ and $\tilde{\lambda_2} = \frac{\bar{\eta}}{\nu_2}$ in place of λ_1 and λ_2 in (6.3.5). The MLE of $R = P(\eta > \xi)$ is

$$\tilde{R} = \left(\tfrac{\nu_1 \bar{\eta}}{\nu_2 \bar{\xi} + \nu_1 \bar{\eta}}\right)^{\nu_1} \tfrac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \tfrac{\nu_1 \bar{\eta}}{\nu_2 \bar{\xi} + \nu_1 \bar{\eta}}\right)$$

where, $\bar{\xi} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1} = \bar{T}_X$ and $\bar{\eta} = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2} = \bar{T}_Y$

Hence, the theorem follows.

Corollary 6.1

1. MLE of $R = P(Y > X)$ for one parameter exponential distribution ($\rho = \nu = 1$)

$$\tilde{R} = \frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$

2. MLE of $R = P(Y > X)$ for gamma Distribution ($\rho = 1$)

$$\tilde{R} = \left(\frac{\nu_1 \bar{Y}}{\nu_2 \bar{X} + \nu_1 \bar{Y}}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \bar{Y}}{\nu_2 \bar{X} + \nu_1 \bar{Y}}\right)$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\rho_2}$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\rho_1}$

3. MLE of $R = P(Y > X)$ for Weibull Distribution ($\nu = 1$)

$$\tilde{R} = \frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\rho_2}$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\rho_1}$

4. MLE of $R = P(Y > X)$ for Erlang distribution ($\nu > 0, \rho = 1$)

$$\tilde{R} = \left(\frac{\nu_1 \bar{Y}}{\nu_2 \bar{X} + \nu_1 \bar{Y}}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \bar{Y}}{\nu_2 \bar{X} + \nu_1 \bar{Y}}\right)$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$

5. MLE of $R = P(Y > X)$ for half-normal distribution ($\nu = 1/2, \rho = 2$)

$$\tilde{R} = \left(\frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}\right)^{1/2} \frac{1}{\pi^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}\right)$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$

6. MLE of $R = P(Y > X)$ for Chi-distribution ($\nu > m/2, \rho = 2$)

$$\tilde{R} = \left(\frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}\right)^{m/2} \frac{1}{B\left(\frac{m}{2}, \frac{m}{2}\right)} {}_2F_1\left(\frac{m}{2}, 1 - \frac{m}{2}; \frac{m}{2}; \frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}\right)$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$

7. MLE of $R = P(Y > X)$ for Rayleigh distribution ($\nu = 1, \rho = 2$)

$$\tilde{R} = \frac{\bar{T_Y}}{\bar{T_X} + \bar{T_Y}}$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$

8. MLE of $R = P(Y > X)$ for Generalized Rayleigh distribution ($\nu = p + 1, \rho = 2$)

$$\tilde{R} = \left(\frac{\bar{T_Y}}{\bar{T_X}+\bar{T_Y}}\right)^{p+1} \frac{1}{B(p+1,p+1)} {}_2F_1\left(p+1,-p;p+1;\frac{\bar{T_Y}}{\bar{T_X}+\bar{T_Y}}\right)$$

where, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$

Theorem 6.2: The UMVUE of $P(Y > X)$ is given by

$$\hat{R} = \begin{cases} \frac{\rho_1\rho_2 B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]}\sum_{i=0}^{\infty}\frac{(-1)^i}{(n_2-1)\nu_2+i} \\ \binom{\nu_2-1}{i}\sum_{j=0}^{\infty}(-1)^j\binom{(n_1-1)\nu_1-1}{j}\left(\frac{T_Y}{T_X}\right)^{\nu_1+j}; \quad T_Y < T_X \\ \text{where}, 0 \le i \le \nu_1 - 1 < \infty \quad \text{and} \quad 0 \le j \le (n_1-1)\nu_1 - 1 < \infty \\ \frac{\rho_1\rho_2 B[(n_1-1)\nu_1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]}\sum_{i=0}^{\infty}\frac{(-1)^i}{(n_2-1)\nu_2+i} \\ \binom{\nu_2-1}{i}\sum_{j=0}^{\infty}(-1)^j\binom{(n_2-1)\nu_1+i}{j}\left(\frac{T_X}{T_Y}\right)^{j}; \quad T_Y > T_X \\ \text{where}, 0 \le i \le \nu_2 - 1 < \infty \quad \textit{and} \quad 0 \le j \le (n_2-1)\nu_2 + i < \infty \end{cases} \tag{6.3.6}$$

where, $T_X = \sum_{i=1}^{n_1} X_i^{\rho_1}$ and $T_Y = \sum_{i=1}^{n_2} Y_j^{\rho_2}$

Proof: Let ξ and η be two independent gamma distributions with pdfs (6.3.2) and (6.3.3). In order to obtain $P(\eta > \xi)$, we have to obtain the UMVUE of $f(\xi;\nu,\rho,\lambda)$ i.e. $\hat{f}(\xi;\nu,\rho,\lambda)$ and $f(\eta;\nu,\rho,\lambda)$ i.e. $\hat{f}(\eta;\nu,\rho,\lambda)$ which is given by

$$\hat{f}(\xi;\nu_1,\rho_1,\lambda_1) = \frac{\rho_1}{B[\nu_1,(n_1-1)\nu_1]}\left\{\frac{\xi^{\nu_1-1}}{(n_1\bar{\xi})^{\nu_1}}\right\}\left\{1-\frac{\xi}{n_1\bar{\xi}}\right\}^{(n_1-1)\nu_1-1}; \quad if \quad 0 < \xi < n_1\bar{\xi} \tag{6.3.7}$$

and

$$\hat{f}(\eta;\nu_2,\rho_2,\lambda_2) = \frac{\rho_2}{B[\nu_2,(n_2-1)\nu_2]}\left\{\frac{\eta^{\nu_2-1}}{(n_2\bar{\eta})^{\nu_2}}\right\}\left\{1-\frac{\eta}{n_2\bar{\eta}}\right\}^{(n_2-1)\nu_2-1}; \quad if \quad 0 < \eta < n_2\bar{\eta} \tag{6.3.8}$$

The Reliability is

$$\hat{R} = P(\eta > \xi) = \int_0^\infty \int_\xi^\infty \hat{f}(\eta;\nu_2,\rho_2,\lambda_2)\hat{f}(\xi;\nu_1,\rho_1,\lambda_1)d\eta d\xi$$

$$= \frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \int_0^{n_1\bar{\xi}} \int_{\xi}^{n_2\bar{\eta}} \frac{\eta^{\nu_2-1}}{(n_2\bar{\eta})^{\nu_2}} \left\{1-\frac{\eta}{n_2\bar{\eta}}\right\}^{(n_2-1)\nu_2-1}$$
$$\left\{\frac{\xi^{\nu_1-1}}{(n_1\bar{\xi})^{\nu_1}}\right\}\left\{1-\frac{\xi}{n_1\bar{\xi}}\right\}^{(n_1-1)\nu_1-1} d\eta d\xi$$

Let $1-\frac{\eta}{n_2\bar{\eta}} = w$

$$= \frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \int_0^{n_1\bar{\xi}} \int_0^{1-\frac{\xi}{n_2\bar{\eta}}} (1-w)^{\nu_2-1} w^{(n_2-1)\nu_2-1}$$
$$\left\{\frac{\xi^{\nu_1-1}}{(n_1\bar{\xi})^{\nu_1}}\right\}\left\{1-\frac{\xi}{n_1\bar{\xi}}\right\}^{(n_1-1)\nu_1-1} dwd\xi$$

$$= \frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}\binom{\nu_2-1}{i} \int_0^{min(n_1\bar{\xi},n_2\bar{\eta})} \left\{\frac{\xi^{\nu_1-1}}{(n_1\bar{\xi})^{\nu_1}}\right\}$$
$$\left\{1-\frac{\xi}{n_1\bar{\xi}}\right\}^{(n_1-1)\nu_1-1} dwd\xi$$

$$= \frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}\binom{\nu_2-1}{i} \int_0^{min(n_1\bar{\xi},n_2\bar{\eta})} \left\{\frac{\xi^{\nu_1-1}}{(n_1\bar{\xi})^{\nu_1}}\right\}$$
$$\left\{1-\frac{\xi}{n_1\bar{\xi}}\right\}^{(n_1-1)\nu_1-1}\left\{1-\frac{\xi}{n_2\bar{\eta}}\right\}^{(n_2-1)\nu_2+i} d\xi$$

Now, we consider the case when $n_1\bar{\xi} > n_2\bar{\eta}$ and

let $1-\frac{\xi}{n_2\bar{\eta}} = w$

$$=\frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}\binom{\nu_2-1}{i} \int_0^1 w^{(n_2-1)\nu_2+i}$$
$$\left\{1-\frac{n_2\bar{\nu_2}}{n_1\bar{\xi}}(1-z)\right\}^{(n_1-1)\nu_1-1}\left(\frac{n_2\bar{\eta}}{n_1\bar{\xi}}\right)^{\nu_1}(1-z)^{\nu_1-1} \quad dz$$

Using the Binomial expansion, we get

$$=\frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}\binom{\nu_2-1}{i} \int_0^1 \sum_{j=0}^{\infty}(-1)^j\binom{(n_1-1)\nu_1-1}{j}$$
$$\left(\frac{n_2\bar{\eta}}{n_1\bar{\xi}}\right)^{\nu_1+j}(1-z)^{\nu_1+j-1} \quad z^{(n_2-1)\nu_2+i} \quad dz$$

$$=\frac{\rho_1\rho_2 B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}\binom{\nu_2-1}{i} \sum_{j=0}^{\infty}(-1)^j\binom{(n_1-1)\nu_1-1}{j}$$
$$\left(\frac{n_2\bar{\eta}}{n_1\bar{\xi}}\right)^{\nu_1+j} ; \quad if \quad n_2\bar{\eta} < n_1\bar{\xi}$$

Similarly, we can take the case $n_2\bar{\eta} > n_1\bar{\xi}$, we get

$$= \frac{\rho_1\rho_2 B[(n_1-1)\nu_1, \nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)\nu_{+}i}{j} \left(\frac{n_2\bar{\xi}}{n_1\bar{\eta}}\right)^j$$

For obtaining the value of UMVUE substituting $n_1\bar{\xi} = \sum_{i=1}^{n_1} X_i{}^{\rho_1} = T_X$ and $n_2\bar{\eta} = \sum_{j=1}^{n_2} Y_j^{\rho_2} = T_Y$. Hence, the theorem follows.

Corollary 6.2

1. The UMVUE of $P(Y > X)$ for one parameter exponential ($\rho = \nu = 1$), Weibull ($\nu = 1$) and Rayleigh ($\nu = 1, \rho = 2$) distributions.

$$\hat{R} = \begin{cases} \frac{B[(n_2+i,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_1-2}{j} \left(\frac{T_Y}{T_X}\right)^{1+j} ; T_Y < T_X \\ \frac{B[(n_1-1,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_2-1+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} ; T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and $T_X = \sum_{i=1}^{n_1} X_i$

2. The UMVUE of $P(Y > X)$ for gamma distribution ($\rho = 1$) and Erlang distribution ($\nu > 0, \rho = 1$)

$$\hat{R} = \begin{cases} \frac{B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)\nu_1+i}{j} \left(\frac{T_Y}{T_X}\right)^{\nu_1+j} \\ \qquad ; T_Y < T_X \\ \frac{B[(n_1-1)\nu_1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)\nu_2+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} \\ \qquad ; T_Y > T_X \end{cases}$$

3. The UMVUE of $P(Y > X)$ for half-normal distribution $(\nu > \frac{1}{2}, \rho = 2)$

$$\hat{R} = \begin{cases} \frac{4B[(n_2-1)\frac{1}{2}+i+1,\frac{1}{2}+j]}{B[\frac{1}{2},(n_1-1)\frac{1}{2}]B[\frac{1}{2},(n_2-1)\frac{1}{2}]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\frac{1}{2}+i} \binom{\frac{1}{2}-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)\frac{1}{2}-1}{j} \left(\frac{T_Y}{T_X}\right)^{\frac{1}{2}+j} \\ \qquad ; T_Y < T_X \\ \frac{4B[(n_1-1)\frac{1}{2},\frac{1}{2}+j]}{B[\frac{1}{2},(n_1-1)\frac{1}{2}]B[\frac{1}{2},(n_2-1)\frac{1}{2}]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\frac{1}{2}+i} \binom{\frac{1}{2}-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)\frac{1}{2}+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} \\ \qquad ; T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

4. The UMVUE of $P(Y > X)$ for Chi-distribution $(\nu > \frac{m}{2}, \rho = 2)$

$$\hat{R} = \begin{cases} \frac{4B[(n_2-1)\frac{m}{2}+i+1,\frac{m}{2}+j]}{B[\frac{m}{2},(n_1-1)\frac{m}{2}]B[\frac{m}{2},(n_2-1)\frac{m}{2}]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\frac{m}{2}+i} \binom{\frac{m}{2}-1}{i} \sum_{j=0}^{\infty} (-1)^j \\ \binom{(n_1-1)\frac{m}{2}-1}{j} \left(\frac{T_Y}{T_X}\right)^{\frac{m}{2}+j} ; T_Y < T_X \\ \frac{4B[(n_1-1)\frac{m}{2},\frac{m}{2}+j]}{B[\frac{m}{2},(n_1-1)\frac{m}{2}]B[\frac{m}{2},(n_2-1)\frac{m}{2}]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\frac{m}{2}+i} \binom{\frac{m}{2}-1}{i} \sum_{j=0}^{\infty} (-1)^j \\ \binom{(n_2-1)\frac{m}{2}+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} ; T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

5. The UMVUE of $P(Y > X)$ for Generalized Rayleigh distribution $(\nu = p+1, \rho = 2)$

$$\hat{R} = \begin{cases} \frac{B[(n_2-1)(p+1)+i+1,(p+1)+j]}{B[(p+1),(n_1-1)(p+1)]B[(p+1),(n_2-1)(p+1)]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)(p+1)+i} \binom{(p+1)-1}{i} \\ \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)(p+1)+i}{j} \left(\frac{T_Y}{T_X}\right)^{(p+1)+j} ; T_Y < T_X \\ \frac{B[(n_1-1)(p+1),(p+1)+j]}{B[(p+1),(n_1-1)(p+1)]B[(p+1),(n_2-1)(p+1)]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)(p+1)+i} \binom{(p+1)-1}{i} \\ \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)(p+1)+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} ; T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

6.4 Confidence Interval of $R = P(Y > X)$

Theorem 6.3: The confidence interval for $R = P(Y > X)$ is

$$P\left(I_{\frac{(\nu_1\bar{T_Y}/\nu_2\bar{T_X})F_{1-\sigma_2}}{(\nu_1\bar{T_Y}/\nu_2\bar{T_X})F_{1-\sigma_2}+1}}(\nu_1,\nu_2) < R < I_{\frac{(\nu_1\bar{T_Y}/\nu_2\bar{T_X})F_{\sigma_1}}{(\nu_1\bar{T_Y}/\nu_2\bar{T_X})F_{\sigma_1}+1}}(\nu_1,\nu_2)\right) = 1-\sigma \tag{6.4.1}$$

where, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i{}^{\rho_1}$ and $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j{}^{\rho_2}$

Proof: It follows from the above theorems that ξ and η be two independent random variables with Gamma (ν_1, λ_1) and Gamma (ν_2, λ_2) respectively, then

$$\lambda = \frac{\lambda_2}{\lambda_1};\ \tilde{\lambda} = \frac{\nu_1\bar{\eta}}{\nu_2\bar{\xi}}$$

where, $\lambda_1 = \frac{\bar{\xi}}{\nu_1}$ and $\lambda_2 = \frac{\bar{\eta}}{\nu_2}$. As we know that,

$$\frac{2n_1\bar{\xi}}{\lambda_1} \sim Gamma(n_1\nu_1, 2) \equiv \chi^2_{2n_1\nu_1}$$

Similarly,

$$\frac{2n_2\bar{\eta}}{\lambda_1} \sim Gamma(n_2\nu_2, 2) \equiv \chi^2_{2n_2\nu_2}$$

where, χ^2_α is the pdf of Chi-squared distribution with α degree of freedom. Hence,

$$\begin{aligned}\frac{\tilde{\lambda}}{\lambda} &= \frac{2n_2\bar{\eta}/n_2\nu_2\lambda_2}{2n_1\bar{\xi}/n_1\nu_1\lambda_1} \\ &= \frac{\chi^2_{2n_2\nu_2}/2n_2\nu_2}{\chi^2_{2n_1\nu_1}/2n_1\nu_1} \sim F(2n_1\nu_1, 2n_2\nu_2)\end{aligned} \tag{6.4.2}$$

where, $F(\varepsilon_1, \varepsilon_2)$ denotes Snedecors F-distribution with ε_1 and ε_2 degree of freedom.

$$\frac{\tilde{\lambda}}{\lambda} \sim F(2n_1\nu_1, 2n_2nu_2)$$

For any δ denoted by $F_\delta = F_\delta(2n_1\nu_1, 2n_2\nu_2)$, then the relation to F_δ and $1-\delta$ quantile of $F_\delta(2n_1\nu_1, 2n_2\nu_2)$ distribution is

$$F_\delta(2n_1\nu_1, 2n_2\nu_2) = [F_{1-\delta}(2n_1\nu_1, 2n_2\nu_2)]^{-1}$$

Let σ_1 and σ_2 be non-negative numbers such that $\sigma_1 + \sigma_2 = \sigma$. Then

$$P(\tilde{\lambda}F_{1-\sigma_2} < \lambda < \tilde{\lambda}F_{\sigma_1}) = 1 - \sigma \tag{6.4.3}$$

Since, $R = I_{\frac{\lambda}{\lambda+1}}(\nu_1, \nu_2)$ and $I_Z(a,b)$ is an increasing function of z for any a, b. So, $I_{\frac{\lambda}{\lambda+1}}(\nu_1, \nu_2)$ as the function of λ . Hence, (6.4.3) becomes

$$P\left(I_{\frac{\tilde{\lambda}F_{1-\sigma_2}}{\tilde{\lambda}F_{1-\sigma_2}+1}}(\nu_1, \nu_2) < R < I_{\frac{\tilde{\lambda}F_{\sigma_1}}{\tilde{\lambda}F_{\sigma_1}+1}}(\nu_1, \nu_2)\right) = 1 - \sigma \tag{6.4.4}$$

After substituting $\tilde{\lambda} = \frac{\nu_1\bar{\eta}}{\nu_2\bar{\xi}}$ and then using $\lambda_1 = \theta_1$, $\lambda_2 = \theta_2$ we get $R = I_{\frac{\lambda_2}{\lambda_1+\lambda_2}}(\nu_1, \nu_2) = I_{\frac{\theta_2}{\theta_1+\theta_2}}(\nu_1, \nu_2)$ and the confidence interval for R is

$$P\left(I_{\frac{(\nu_1\bar{\eta}/\nu_2\bar{\xi})F_{1-\sigma_2}}{(\nu_1\bar{\eta}/\nu_2\bar{\xi})F_{1-\sigma_2}+1}}(\nu_1, \nu_2) < R < I_{\frac{(\nu_1\bar{\eta}/\nu_2\bar{\xi})F_{\sigma_1}}{(\nu_1\bar{\eta}/\nu_2\bar{\xi})F_{\sigma_1}+1}}(\nu_1, \nu_2)\right) = 1 - \sigma$$

where, $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\rho_1} = \bar{\xi}$ and $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\rho_2} = \bar{\eta}$

Hence the theorem follows.

Corollary 6.3

1. Confidence interval for one parameter exponential distribution ($\rho = \nu = 1$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}} < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}\right) = 1 - \sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(1, 1)$

2. Confidence interval for gamma Distribution ($\rho = 1$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}(\nu_1,\nu_2) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}(\nu_1,\nu_2)\right) = 1-\sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(\nu_1,\nu_2)$

3. Confidence interval for Weibull Distribution ($\nu = 1$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}(1,1) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}(1,1)\right) = 1-\sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(1,1)$

4. Confidence interval for Erlang Distribution ($\nu > 0, \rho = 1$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}(\nu_1,\nu_2) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}(\nu_1,\nu_2)\right) = 1-\sigma$$

where $\bar{\lambda} = \frac{\nu_1\bar{\eta}}{\nu_2\bar{\xi}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(\nu_1,\nu_2)$

5. Confidence interval for half-normal Distribution ($\nu > 1/2, \rho = 2$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}\left(\frac{1}{2},\frac{1}{2}\right) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}\left(\frac{1}{2},\frac{1}{2}\right)\right) = 1-\sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}\left(\frac{1}{2},\frac{1}{2}\right)$

6. Confidence interval for Chi-distribution ($\nu > m/2, \rho = 2$)

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}\left(\frac{m}{2},\frac{m}{2}\right) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}\left(\frac{m}{2},\frac{m}{2}\right)\right) = 1-\sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}\left(\frac{m}{2}, \frac{m}{2}\right)$

7. Confidence interval for Rayleigh distribution $(\nu = 1, \rho = 2)$

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}(1,1) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}(1,1)\right) = 1 - \sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(1,1)$

8. Confidence interval for Generalized Rayleigh distribution $(\nu = p+1, \rho = 2)$

$$P\left(I_{\frac{\bar{\lambda}F_{1-\sigma_2}}{\bar{\lambda}F_{1-\sigma_2}+1}}(p+1,p+1) < R < I_{\frac{\bar{\lambda}F_{\sigma_1}}{\bar{\lambda}F_{\sigma_1}+1}}(p+1,p+1)\right) = 1 - \sigma$$

where $\bar{\lambda} = \frac{\bar{Y}}{\bar{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(p+1,p+1)$.

6.5 Probability of Disaster $P(Y > \gamma)$

Theorem 6.4: If the stress and finite strength are denoted by the random variables X and Y which follows PEFD and Power function distribution, that are shown in (6.2.1) and (6.2.3), respectively. Then, probability of disaster α is given by

$$\alpha = \frac{1}{\Gamma\nu}\int_k^{\infty} u^{\nu-1}e^{-u}du \tag{6.5.1}$$

where, $k = \frac{\gamma^\rho}{\theta}$

Proof: We know that

$$\begin{aligned}\alpha &= P(Y > \gamma)\\ &= \int_{\gamma}^{\infty} \frac{\rho x^{\rho\nu-1} e^{\frac{-x^{\rho}}{\theta}}}{\Gamma\nu\theta^{\nu}} dx\\ &= \frac{\rho}{\Gamma\nu\theta^{\nu}} \int_{\gamma}^{\infty} x^{\rho\nu-1} e^{\frac{-x^{\rho}}{\theta}} dx\end{aligned}$$

Let $\frac{x^{\rho}}{\theta} = u$

$$\begin{aligned}\alpha &= \frac{1}{\Gamma\nu} \int_{\frac{\gamma^{\rho}}{\theta}}^{\infty} u^{\nu-1} e^{-u} du\\ &= \Gamma(\nu, k)\end{aligned}$$

which is the upper incomplete gamma function, where, $k = \frac{\gamma^{\rho}}{\theta}$.

6.6 Numerical Analysis

From Section 6.5, the probability of disaster $P(Y > \gamma)$ can be measured. The numerical values are obtained for different values of ν which is presented in Table 6.1. It can be easily interpreted from Table 6.1 that the probability of disaster decreases with an increase in the value of k. In order to overcome the problem of disaster (i.e. to attain the smallest value of $\alpha = P(X > \gamma)$, the values of $k = \frac{\gamma^{\rho}}{\theta}$, where ρ and θ is the parameter of PEF-distribution and γ is the scale parameter of the power function distribution, should be considered in such a manner that the value of α tends to zero.

Alternatively, we may also obtain the numerical values of k for fixed values ν from equation (6.5.1). These values are used to obtain the optimum cost for manufacturing of item at desired tolerance level.

6.7 Stress-Strength Reliability

Theorem 6.5: The stress-strength model 'R', where X follows PEFD and Y follows power function distribution is

$$R = \frac{1}{\Gamma\nu}\int_0^k u^{\nu-1}e^{-u}du - \frac{1}{\Gamma\nu k^{\mu/\alpha}}\int_0^k u^{\left(\frac{\mu}{\alpha}+\nu-1\right)}e^{-u}du \tag{6.7.1}$$

Proof: From (6.2.1) and (6.2.3), we have

$$\begin{aligned} R &= \int_0^{\gamma} \frac{\rho x^{\rho\nu-1}e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu}\left[\int_x^{\gamma}\frac{\mu}{\gamma}\left(\frac{y}{\gamma}\right)^{\mu-1}dy\right]dx \\ &= \int_0^{\gamma} \frac{\rho x^{\rho\nu-1}e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu}\left[\frac{1}{\gamma^\mu}\left\{\gamma^\mu - x^\mu\right\}\right]dx \\ &= \int_0^{\gamma} \frac{\rho x^{\rho\nu-1}e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu}\left[\frac{1}{\gamma^\mu}\left\{\gamma^\mu - x^\mu\right\}\right]dx \\ &= \int_0^{\gamma} \frac{\rho x^{\rho\nu-1}e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu}dx - \int_0^{\gamma}\frac{\rho x^{\rho\nu+\mu-1}e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu\gamma^\mu}dx \end{aligned}$$

Taking $\frac{x^\rho}{\theta} = u$ and solving the above integrals we finally get (6.7.1) and hence, the theorem follows.

6.8 The Stress-Strength Reliability R when both X and Y follows PEFD

Theorem 6.6: Let $X, Y \sim$ PEFD, where X and Y represents the stress and the strength of the items/components. The reliability 'R' is

$$R = \frac{\rho_1}{{\theta_1}^{\nu_1}\Gamma\nu_1}\int_{x=0}^{\infty} x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)e^{(-x^{\rho_2}/\theta_2)}}\sum_{k=0}^{\nu_2-1}\frac{1}{k!}\left(\frac{x^{\rho_2}}{\theta_2}\right)^k dx \tag{6.8.1}$$

Proof: Random variable X follows the PEF-distribution with pdf

$$f(x,\Theta) = \frac{\rho_1 x^{\rho_1\nu_1-1} e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1 \theta_1^{\nu_1}} \tag{6.8.2}$$

Random variable Y follows the PEF-distribution with pdf

$$f(y,\Theta) = \frac{\rho_2 y^{\rho_2\nu_2-1} e^{(-y^{\rho_2}/\theta_2)}}{\Gamma\nu_2 \theta_2^{\nu_2}} \tag{6.8.3}$$

The Reliability 'R' is

$$R = \int_{x=0}^{\infty} \left\{ \frac{\rho_1 x^{\rho_1\nu_1-1} e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1 \theta_1^{\nu_1}} \right\} \left[\int_{y=x}^{\infty} \frac{\rho_2 y^{\rho_2\nu_2-1} e^{(-y^{\rho_2}/\theta_2)}}{\Gamma\nu_2 \theta_2^{\nu_2}} dy \right] dx$$

Let $\frac{y^{\rho_2}}{\theta_2} = t$

$$R = \int_{0}^{\infty} \left\{ \frac{\rho_1 x^{\rho_1\nu_1-1} e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1 \theta_1^{\nu_1}} \right\} \left[\int_{\frac{x^{\rho_2}}{\theta_2}}^{\infty} \frac{t^{\nu_2-1}}{\Gamma\nu_2} e^{-t} dt \right]$$

Hence, the theorem follows, after solving this upper incomplete gamma function.

6.9 Discussion

When an item/device is manufactured and if the strength of an item follows Power function distribution, it is expected that the maximum feasible values of γ may have an upper limit say γ_0 . For example, the maximum accelerating speed of a turbine must not be increased its permissible capacity. At a fixed tolerance level α, suppose γ_α is the desired value of γ. In case $\gamma_\alpha < \gamma_0$, we may obtain the required value of μ say μ_α, by using Table 6.3, so that the item is manufactured with the strength distribution having parameters $(\mu_\alpha, \gamma_\alpha)$ and consequently, the desired strength reliability can be achieved. However, if $\gamma_\alpha > \gamma_0$, we will have to either adjust α or look for an alternate item.

6.10 Study of the cost with an example

Let us assume that the maximum feasible value of k is 12. When $\alpha \leq 0.01$ the value of m must be greater or equal to 5.1 i.e. $m \geq 5.1$. As the value of m cannot exceed 12 , then one needs to fix the item / device in a way such that $5.1 \leq k \leq 12$ i.e. $2.7 \leq \gamma \leq 4.1$ and thus, the corresponding values of μ leads to a maximum of $P(Y > X)$. The cost factor of adjusting the parameters may be taken into consideration here as the cost of varying γ and μ may be different. Theoretically, the costs may be an increasing or decreasing function of γ and μ depending upon the nature of the parameters. Usually, Cost (Y) is an increasing function Y, if Y is the mean strength. In our study, $E(Y) = \mu\gamma/(\mu + 1)$, which implies that the mean strength increases by increasing either of the two parameters. Hence, we may assume the two costs to be an increasing function of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem may be further evaluated as follows:

Let E_1 and E_2 be the costs of adjusting one unit of γ and μ, respectively.
Minimize $E = \gamma E_1 + \mu E_2$ subject to $2.7 \leq \gamma \leq 4.1$ and $P(Y > X) \geq 0.99$.
Analytically, the problem may be simplified as follows:
On using Table 6.3 for k = 5.1, 6.2, 7.6, 9.3 and 11.4 i.e. γ = 2.76, 3.04, 3.37, 3.73 and 4.1, respectively and obtain those values of μ for which $P(Y > X) \geq 0.99$, the cost function for each pair of (γ, μ) is evaluated. Table 6.4 depicts that the minimum cost lies at $3.37E_1 + 20E_2$ depending upon the numerical values of E_1 and E_2.

6.11 Tables

Table 6.1 Numerical Values for the Probability of disaster $\alpha = P(X > \theta)$ and k for different values of ν						
k	$\nu = 0.001$	$\nu = 0.01$	$\nu = 0.5$	$\nu = 1.05$	$\nu = 1.5$	$\nu = 2$
2.3	0.000016	0.000331	0.031972	0.109131	0.203542	**0.330854**
2.8	0.0	0.000172	0.017961	0.066687	0.132778	**0.231078**
3.4	0.0	0.000081	0.009116	0.036880	0.078555	**0.146842**
4.1	0.0	0.000034	0.004189	0.018454	0.042054	**0.084521**
5.1	0.0	0.000012	0.001404	0.006851	0.016940	**0.037190**
6.2	0.0	0.0	0.000429	0.002300	0.006131	**0.014612**
7.6	0.0	0.0	0.000087	0.000572	0.000165	**0.004304**
9.3	0.0	0.0	0.000016	0.000105	0.000331	**0.000942**
11.4	0.0	0.0	0.0	0.000013	0.000044	**0.000138**
13.9	0.0	0.0	0.0	0.0	0.0	**0.000014**

Table 6.2 Values of m at different tolerance level α for $\nu = 2$						
α	0.05	0.02	0.01	0.001	.0001	0.00001
k	2.996020	3.912310	4.605460	6.908040	9.210630	11.513200

Table 6.3 The Strength reliability of an item for $\nu = 2, \rho = 2$, and varying values of k and μ and varying values of k and μ						
k	$\mu = 2$	$\mu = 5$	$\mu = 10$	$\mu = 15$	$\mu = 20$	$\mu = 30$
3.4	0.464769	0.666495	0.761480	0.794106	0.809972	0.825255
4.1	0.536852	0.749440	0.840064	0.868720	0.882015	0.894349
5.1	0.616331	0.831032	0.910101	0.932106	0.941591	0.949892
6.2	0.680103	0.887158	0.951925	0.967406	0.973505	0.978478
7.6	0.737474	0.928900	0.977654	0.987122	0.990404	0.992822
9.3	0.785057	0.956229	0.990580	0.995696	0.997186	0.998139
11.4	0.824575	0.973529	0.996360	0.998797	0.999357	0.999650
13.9	0.856116	0.983855	0.998620	0.999695	0.999878	0.999951
17.0	0.882353	0.990239	0.999493	0.999930	0.999981	0.999995

Table 6.4 Table for Optimum cost of manufactured items					
γ	μ	$\gamma E_1 + \mu E_2$	γ	μ	$\gamma E_1 + \mu E_2$
3.37	20	$3.37E_1 + 20E_2$	3.70	30	$3.7E_1 + 30E_2$
3.37	30	$3.37E_1 + 30E_2$	4.10	10	$4.1E_1 + 10E_2$
3.70	10	$3.7E_1 + 10E_2$	4.10	15	$4.1E_1 + 15E_2$
3.70	15	$3.7E_1 + 15E_2$	4.10	20	$4.1E_1 + 20E_2$
3.70	20	$3.7E_1 + 20E_2$	4.10	30	$4.1E_1 + 30E_2$

Chapter 7

Bayesian estimation for the scale parameter of a family of lifetime distributions under different priors

7.1 Introduction

In the present chapter, we consider a family of lifetime distributions proposed by Chaturvedi and Rani (1997)[23] which is defined as

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}e^{\left(-x^c/\theta^b\right)}}{\theta^{ab}\Gamma a}; \; x > 0 \tag{7.1.1}$$

where $\theta^{b/c}$ is the scale parameter and a, b, c are the shape parameters. The model given at (7.1.1) covers various lifetime distributions as specific cases, for example, one-parameter exponential distribution, gamma distribution, generalized gamma distribution, Erlang distribution, Weibull distribution, half-normal distribution, Rayleigh distribution, Chi-distribution and Maxwells failure distribution may obtain through assigning the different values of a, b and c.

In the Bayesian decision theory, loss function plays a very significant role in obtaining a good estimator. The performance of a Bayes estimator mainly depends upon the assumed prior distribution and loss function. In the decision theory, most widely used symmetric unbounded loss function is called the Squared Error Loss Function (SELF) proposed by Legendre (1805)[75]. This loss function is suggested for the situations when overestimation and underestimation are given equal importance. Another more general type of loss function is the Quadratic Loss Function (QLF) [see Taguchi (1986)[106]]. In those situations where overestimation is more serious than underestimation it is suggested to consider the asymmetric loss function in place of SELF [see Ferguson (1967)[51], Varian (1975)[112], Berger (1980)[14], Zellner (1986)[118]]. Norstrom (1996)[85] introduced an asymmetric loss function called Precautionary Loss Function (PLF). This loss function is very simple to use and a good alternative for other asymmetric loss functions.

The Squared Error Loss Function(SELF) is defined as

$$L(\theta, \theta_{SELF}) = (\theta - \theta_{SELF})^2$$

The Quadratic Loss Function(QLF) is defined as

$$L(\theta, \theta_{QLF}) = \left(\frac{\theta - \theta_{QLF}}{\theta}\right)^2$$

The Precautionary Loss Function(PLF) is defined as

$$L(\theta, \theta_{PLF}) = \frac{(\theta_{PLF} - \theta)^2}{\theta}$$

The objective of this chapter is to find the Bayes estimators and the Posterior risks for the scale parameter of the family of lifetime distributions under different loss functions and priors. Further, on utilizing theses Bayes estimators and posterior risks, for all the specific cases are obtained for the purpose of their comparisons. The chapter is organized as: in Section 7.2, we obtain the posterior distributions in case of uniform and inverted gamma priors. Using these posterior distributions, the Bayes estimates are evaluated under the different loss functions i.e. SELF, QLF and PLF in Section 7.3. In Section 7.4, the posterior risk is deduced with the help of the Bayes estimators under different loss functions as mentioned in Section 7.3 for both the priors. In Section 7.5, a simulation study is done based on the deduced results and is demonstrated through Tables and Graphs in Section 7.7. Section 7.6, contains the conclusion and a brief summary of the results.

7.2 Posterior distributions under the assumption of different priors

Posterior distributions for scale parameter of the family of lifetime distributions given at (7.1.1) are obtained, assuming the non-informative and informative priors.

Theorem 7.1: Posterior distribution for the scale parameter $\theta^{b/c}$ using the uniform prior, which is defined as $P(\theta) = k$ (where k is constant) is given by

$$P(\theta|x) = \frac{b\theta^{-nab}e^{-\sum {x_i}^c/\theta^b}(\sum x_i)^{na-\frac{1}{b}}}{\Gamma\big(na - \frac{1}{b}\big)} \tag{7.2.1}$$

where $x = (x_1, x_2, ..., x_n)$.

Proof: The likelihood function of the family of lifetime distribution is

$$L(\theta|x) = \tfrac{c^n e^{(-\sum {x_i}^c/\theta^b)}}{\theta^{nab}(\Gamma a)^n} \textstyle\prod_{i=1}^n x_i^{ac-1}$$

$$L(\theta|x) \propto \theta^{-nab}e^{(-\sum {x_i}^c/\theta^b)} \tag{7.2.2}$$

Posterior distribution is obtained as

$$P(\theta|x) = \frac{p(\theta)L(\theta|x)}{\int_0^\infty p(\theta)L(\theta|x)} \tag{7.2.3}$$

$$P(\theta|x) = \tfrac{\theta^{-nab}e^{-\sum x_i^c/\theta^b}}{\int_0^\infty \theta^{-nab}e^{-\sum x_i^c/\theta^b}d\theta}$$

Thus, we get

$$P(\theta|x) = \frac{b\theta^{-nab}e^{-\sum x_i^c/\theta^b}(\sum x_i^c)^{na-\frac{1}{b}}}{\Gamma\left(na-\frac{1}{b}\right)}$$

Hence, the theorem follows.

Theorem 7.2: Posterior distribution of scale parameter $\theta^{b/c}$ using Inverted Gamma prior which is defined as $g(\theta) = \frac{b\tau^{v/b}\theta^{-(v+1)}e^{-\tau/\theta^b}}{\Gamma(v/b)}$, is given by

$$P(\theta|x) = \frac{b\left(\tau + \sum x_i^c\right)^{na+\frac{v}{b}}\theta^{-(nab+v+1)}e^{-(\tau+\sum x_i^c)/\theta^b}}{\Gamma\left(na+\frac{v}{b}\right)} \tag{7.2.4}$$

where $x = (x_1, x_2, ..., x_n)$.

Proof: Posterior distribution of scale parameter using Inverted Gamma prior is

$$P(\theta|x) = \frac{g(\theta)L(\theta|x)}{\int_0^\infty g(\theta)L(\theta|x)}$$

$$P(\theta|x) = \frac{\theta^{-(nab+v+1)}e^{-(\tau+\sum x_i^c)/\theta^b}}{\int_0^\infty \theta^{-(nab+v+1)}e^{-(\tau+\sum x_i^c)/\theta^b}d\theta}$$

Thus, we get

$$P(\theta|x) = \frac{b\left(\tau+\sum x_i^c\right)^{na+\frac{v}{b}}\theta^{-(nab+v+1)}e^{-(\tau+\sum x_i^c)/\theta^b}}{\Gamma\left(na+\frac{v}{b}\right)}$$

Hence, the theorem follows.

7.3 Bayesian estimation under three different Loss Functions

Theorem 7.3: Bayes estimator for the scale parameter under SELF using Uniform Prior is given

$$\left\{\theta^{b/c}\right\}_{SELF} = \frac{\left(\sum x_i{}^c\right)^{1/c}\Gamma\left(na-\frac{1}{b}-\frac{1}{c}\right)}{\Gamma\left(na-\frac{1}{b}\right)} \tag{7.3.1}$$

where, $x = (x_1, x_2, ..., x_n)$

Proof: The Bayes estimate under SELF can be obtained by using the expression given as follows

$$\left\{\theta^{b/c}\right\}_{SELF} = E\left(\theta^{b/c}|x\right) \tag{7.3.2}$$

$$E\left(\theta^{b/c}|x\right) = \int_0^\infty \theta^{b/c} P(\theta|x) d\theta \tag{7.3.3}$$

Substituting the value of $P(\theta|x)$ from (7.2.1) in (7.3.3), we get

$$E\left(\theta^{b/c}|x\right) = \int_0^\infty \frac{b\theta^{b/c}\theta^{-nab}e^{-(\sum x_i^c)/\theta^b}\left(\sum x_i^c\right)^{na-\frac{1}{b}}}{\Gamma\left(na-\frac{1}{b}\right)} d\theta$$

$$\left\{\theta^{b/c}\right\}_{SELF} = \frac{(\sum x_i{}^c)^{1/c}\Gamma\left(na-\frac{1}{b}-\frac{1}{c}\right)}{\Gamma\left(na-\frac{1}{b}\right)}$$

Hence, the theorem follows.

Theorem 7.4: Bayes estimator for the scale parameter under SELF using the Inverted Gamma prior is given as

$$\left\{\theta^{b/c}\right\}_{SELF} = \frac{(\tau + \sum x_i{}^c)^{1/c}\Gamma\left(na + \frac{\nu}{b} - \frac{1}{c}\right)}{\Gamma\left(na + \frac{\nu}{b}\right)} \tag{7.3.4}$$

where, $x = (x_1, x_2, ..., x_n)$

Proof: Bayes estimator for the scale parameter under SELF can be evaluated through the expression by substituting the value of $P(\theta|x)$ from (7.2.4) in (7.3.3), we get

$$\left\{\theta^{b/c}\right\}_{SELF} = \int_0^\infty \frac{b\theta^{b/c}(\tau+\sum x_i)^{na+\frac{v}{b}}\theta^{-(nab+v+1)}e^{-(\tau+\sum x_i)/\theta^b}d\theta}{\Gamma\left(na+\frac{v}{b}\right)}$$

$$\left\{\theta^{b/c}\right\}_{SELF} = \frac{(\tau+\sum x_i{}^c)^{1/c}\Gamma\left(na+\frac{\nu}{b}-\frac{1}{c}\right)}{\Gamma\left(na+\frac{\nu}{b}\right)}$$

Hence, the theorem follows.

Theorem 7.5: Bayes estimator of the scale parameter under QLF using Uniform prior is given as

$$\left\{\theta^{b/c}\right\}_{QLF} = \frac{\left(\sum x_i{}^c\right)^{1/c}\Gamma\left(na - \frac{1}{b} + \frac{1}{c}\right)}{\Gamma\left(na - \frac{1}{b} + \frac{2}{c}\right)} \tag{7.3.5}$$

where, $x = (x_1, x_2, ..., x_n)$

Proof: The Bayes estimate of the scale parameter under QLF is given by

$$\left\{\theta^{b/c}\right\}_{QLF} = \frac{E\left(\theta^{-b/c}\right)}{E\left(\theta^{-2b/c}\right)} \tag{7.3.6}$$

$$E\left(\theta^{-b/c}\right) = \int_0^\infty \theta^{-b/c} P(\theta|x) d\theta \tag{7.3.7}$$

$$E\left(\theta^{-2b/c}\right) = \int_0^\infty \theta^{-2b/c} P(\theta|x) d\theta \tag{7.3.8}$$

Substituting the value of $P(\theta|x)$ from (7.2.1) in (7.3.7) and (7.3.8), we get

$$E\left(\theta^{-b/c}\right) = \frac{\left(\sum {x_i}^c\right)^{-1/c} \Gamma\left(na - \frac{1}{b} + \frac{1}{c}\right)}{\Gamma\left(na - \frac{1}{b}\right)} \tag{7.3.9}$$

$$E\left(\theta^{-2b/c}\right) = \frac{\left(\sum {x_i}^c\right)^{-2/c} \Gamma\left(na - \frac{1}{b} + \frac{2}{c}\right)}{\Gamma\left(na - \frac{1}{b}\right)} \tag{7.3.10}$$

Substituting the values from (7.3.9) and (7.3.10) in (7.3.6), we get

$$\left\{\theta^{b/c}\right\}_{QLF} = \frac{\left(\sum {x_i}^c\right)^{1/c} \Gamma\left(na - \frac{1}{b} + \frac{1}{c}\right)}{\Gamma\left(na - \frac{1}{b} + \frac{2}{c}\right)}$$

Hence, the theorem follows.

Theorem 7.6: Bayes estimator of the scale parameter under QLF using Inverted Gamma prior is given as

$$\left\{\theta^{b/c}\right\}_{QLF} = \frac{\left(\tau + \sum {x_i}^c\right)^{1/c} \Gamma\left(na + \frac{v}{b} + \frac{1}{c}\right)}{\Gamma\left(na + \frac{v}{b} + \frac{2}{c}\right)} \tag{7.3.11}$$

where, $x = (x_1, x_2, ..., x_n)$

Proof: To obtain the Bayes estimate of the scale parameter under QLF the expression in (7.3.6) is used and we have

$$E\left(\theta^{-b/c}\right) = \frac{\left(\tau + \sum {x_i}^c\right)^{-1/c} \Gamma\left(na + \frac{v}{b} + \frac{1}{c}\right)}{\Gamma\left(na + \frac{v}{b}\right)} \tag{7.3.12}$$

$$E\left(\theta^{-2b/c}\right)=\frac{\left(\tau+\sum {x_i}^c\right)^{-2/c}\Gamma\left(na+\frac{v}{b}+\frac{2}{c}\right)}{\Gamma\left(na+\frac{v}{b}\right)} \tag{7.3.13}$$

Substituting the values from (7.3.12) and (7.3.13) in (7.3.6), we get

$$\left\{\theta^{b/c}\right\}_{QLF}=\frac{\left(\tau+\sum {x_i}^c\right)^{1/c}\Gamma\left(na+\frac{v}{b}+\frac{1}{c}\right)}{\Gamma\left(na+\frac{v}{b}+\frac{2}{c}\right)}$$

Hence, the theorem follows.

Theorem 7.7: Bayes estimator for the scale parameter under PLF using Uniform prior is given by

$$\left\{\theta^{b/c}\right\}_{PLF}=\left\{\frac{\Gamma\left(na+\frac{1}{b}-\frac{2}{c}\right)}{\Gamma\left(na+\frac{1}{b}\right)}\right\}^{1/2}\left(\sum {x_i}^c\right)^{1/c} \tag{7.3.14}$$

where, $x=(x_1,x_2,...,x_n)$

Proof: Bayes estimator under PLF using Uniform prior is computed as

$$\left\{\theta^{b/c}\right\}_{PLF}=\left\{E\left(\theta^{2b/c}|x\right)\right\}^{1/2} \tag{7.3.15}$$

$$E\left(\theta^{2b/c}|x\right)=\int_0^\infty \theta^{2b/c}P(\theta|x)d\theta \tag{7.3.16}$$

Substituting the value of $P(\theta|x)$ from (7.2.1) in (7.3.16), we get

$$E\left(\theta^{2b/c}|x\right)=\left\{\frac{\Gamma\left(na+\frac{1}{b}-\frac{2}{c}\right)}{\Gamma\left(na+\frac{1}{b}\right)}\right\}\left(\sum {x_i}^c\right)^{2/c} \tag{7.3.17}$$

Substituting the value from (7.3.17) in (7.3.15), we get

$$\left\{\theta^{b/c}\right\}_{PLF}=\left\{\frac{\Gamma\left(na+\frac{1}{b}-\frac{2}{c}\right)}{\Gamma\left(na+\frac{1}{b}\right)}\right\}^{1/2}\left(\sum {x_i}^c\right)^{1/c}$$

Hence, the theorem follows.

Theorem 7.8: Bayes estimator under PLF using Inverted Gamma prior is given by

$$\left\{\theta^{b/c}\right\}_{PLF}=\left\{\frac{\Gamma\left(na+\frac{v}{b}-\frac{2}{c}\right)}{\Gamma\left(na+\frac{v}{b}\right)}\right\}^{1/2}\left(\tau+\sum {x_i}^c\right)^{1/c} \tag{7.3.18}$$

where, $x = (x_1, x_2, ..., x_n)$

Proof: Bayes estimator under PLF using Inverted Gamma prior is computed from (7.3.15), we have

$$E\left(\theta^{2b/c}|x\right) = \left\{\frac{\Gamma\left(na + \frac{v}{b} - \frac{2}{c}\right)}{\Gamma\left(na + \frac{v}{b}\right)}\right\}\left(\tau + \sum x_i{}^c\right)^{2/c} \tag{7.3.19}$$

Substituting the value from (7.3.19) in (7.3.15), we get

$$\left\{\theta^{b/c}\right\}_{PLF} = \left\{\frac{\Gamma\left(na+\frac{v}{b}-\frac{2}{c}\right)}{\Gamma\left(na+\frac{v}{b}\right)}\right\}^{1/2}\left(\tau + \sum x_i{}^c\right)^{1/c}$$

Hence, the theorem follows.

7.4 Posterior risks under different loss functions

Considering both the priors for the scale parameter of the family of lifetime distributions and after obtaining the Bayes estimators under different loss functions, the posterior risks are obtained under the assumptions of SELF, QLF and PLF.

7.4.1 Posterior risk of Bayes estimator under different loss functions using Uniform prior are:

I. Under SELF, the expression for the posterior risk of the Bayes estimator is

$$P\left[\left\{\theta^{b/c}\right\}_{SELF}\right] = E\left(\theta^{2b/c}|x\right) - \left\{E\left(\theta^{b/c}|x\right)\right\}^2 \tag{7.4.1}$$

Substituting the values from (7.3.1) and (7.3.17) in (7.4.1), we get

$$P\left[\left\{\theta^{b/c}\right\}_{SELF}\right] = \left\{\Gamma\left(na - \frac{1}{b} - \frac{2}{c}\right) - \frac{\left(\Gamma\left(na - \frac{1}{b} - \frac{1}{c}\right)\right)^2}{\Gamma\left(na - \frac{1}{b}\right)}\right\}\frac{\left(\sum x_i{}^c\right)^{2/c}}{\Gamma\left(na - \frac{1}{b}\right)} \tag{7.4.2}$$

II. Under QLF, the expression for the posterior risk of the Bayes estimator is

$$P\left[\left\{\theta^{b/c}\right\}_{QLF}\right] = 1 - \frac{\left\{E\left(\theta^{-b/c}|x\right)\right\}^2}{E\left(\theta^{-2b/c}|x\right)} \tag{7.4.3}$$

On utilizing the values from (7.3.19) and (7.3.10) in (7.4.3), we get

$$P\left[\left\{\theta^{b/c}\right\}_{QLF}\right] = 1 - \left\{\frac{\left(\Gamma\left(na - \frac{1}{b} + \frac{1}{c}\right)\right)^2}{\Gamma\left(na - \frac{1}{b}\right)\Gamma\left(na - \frac{1}{b} + \frac{2}{c}\right)}\right\} \tag{7.4.4}$$

III. Under PLF, the expression for the posterior risk of the Bayes estimator is

$$P\left[\left\{\theta^{b/c}\right\}_{PLF}\right] = 2\left\{\left\{\theta^{b/c}\right\}_{PLF} - E\left(\theta^{b/c}|x\right)\right\} \tag{7.4.5}$$

Substituting the values from (7.3.1) and (7.3.14) in (7.4.5), we get

$$P\left[\left\{\theta^{b/c}\right\}_{PLF}\right] = 2\left[\left\{\frac{\Gamma\left(na - \frac{1}{b} - \frac{2}{c}\right)}{\Gamma\left(na - \frac{1}{b}\right)}\right\}^{1/2}\left(\sum {x_i}^c\right)^{1/c} - \frac{\Gamma\left(na - \frac{1}{b} - \frac{1}{c}\right)}{\Gamma\left(na - \frac{1}{b}\right)}\left(\sum {x_i}^c\right)^{1/c}\right] \tag{7.4.6}$$

where, $x = (x_1, x_2, ..., x_n)$

7.4.2 Posterior risk of Bayes estimator under different loss functions using Inverted Gamma prior are:

I. Under SELF, utilising the results from (7.3.4) and (7.3.19) in (7.4.1), we get

$$P\left[\left\{\theta^{b/c}\right\}_{SELF}\right] = \left\{\Gamma\left(na + \frac{v}{b} - \frac{2}{c}\right) - \frac{\left(\Gamma\left(na + \frac{v}{b} - \frac{1}{c}\right)\right)^2}{\Gamma\left(na + \frac{v}{b}\right)}\right\}\frac{\left(\tau + \sum {x_i}^c\right)^{2/c}}{\Gamma\left(na + \frac{v}{b}\right)} \tag{7.4.7}$$

II. Under QLF, utilising the results from (7.3.12) and (7.3.13) in (7.4.3), we get

$$P\left[\left\{\theta^{b/c}\right\}_{QLF}\right] = 1 - \left\{\frac{\left(\Gamma\left(na + \frac{v}{b} + \frac{1}{c}\right)\right)^2}{\Gamma\left(na + \frac{v}{b}\right)\Gamma\left(na + \frac{v}{b} + \frac{2}{c}\right)}\right\} \tag{7.4.8}$$

III. Under PLF, utilising the results from (7.3.4) and (7.3.18) in (7.4.5), we get

$$P\left[\left\{\theta^{b/c}\right\}_{PLF}\right] = 2\left[\left\{\frac{\Gamma\left(na + \frac{v}{b} - \frac{2}{c}\right)}{\Gamma\left(na + \frac{v}{b}\right)}\right\}^{1/2}\left(\tau + \sum {x_i}^c\right)^{1/c} - \frac{\Gamma\left(na + \frac{v}{b} - \frac{1}{c}\right)}{\Gamma\left(na + \frac{v}{b}\right)}\left(\tau + \sum {x_i}^c\right)^{1/c}\right] \tag{7.4.9}$$

7.5 Simulation Study

To generate the random numbers for the model (7.1.1), we fail to use the method of inverse transformation since the cdf is not in a closed form, therefore another method called the accept-reject method is used. In this method, we only need to know the functional form of the probability density function $f(x)$, which is known as the target density. For this target density $f(x)$, a simpler density $g(x)$ for simulation called the instrumental density or candidate density is developed. Two constraints are imposed on the candidate density $g(x)$, firstly, $f(x)$ and $g(x)$ have compatible support(i.e. $g(x) > 0$, when $f(x) > 0$); secondly, there must exist a constant M with $f(x)/g(x) \geq M, \forall x$, called the normalising constant. R software is used for the purpose of computational work and simulation is done through MCMC techniques. The behaviour of the instrumental density and the target density is shown in Figure 7.1.

We have generated 5000 random numbers from the model (7.1.1), further, random

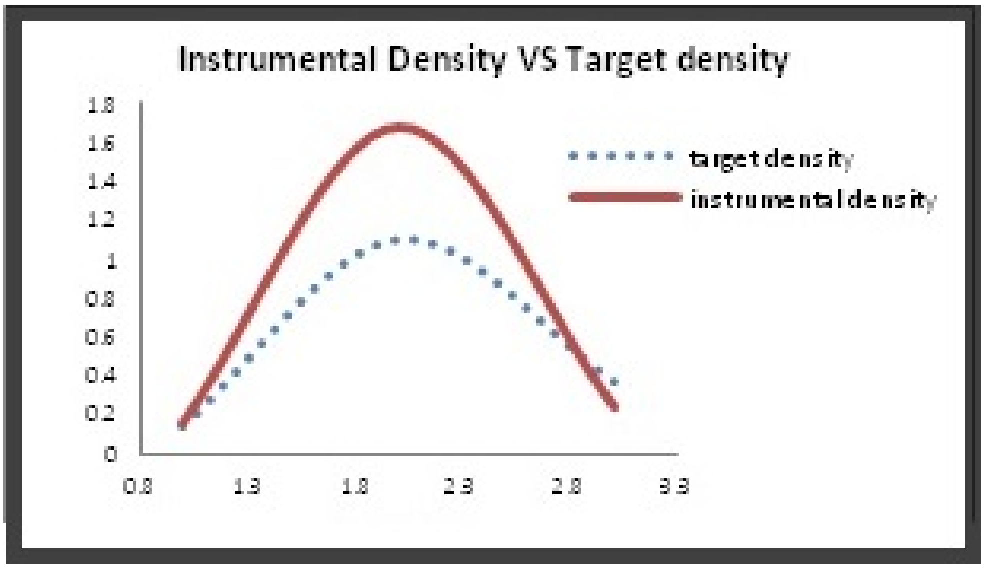


Figure 7.1: Plots of Instrumental density and Target density

numbers for all the specific distributions are obtained for different values of parameter θ and a, b and c. The Bayes Estimates and Posterior Risks are obtained through simulation process and are presented in Tables 7.1-7.10 and Figures 7.2-7.10.

7.6 Conclusion

In case of uniform prior, the distributions considered in the family of lifetime distributions i.e. one-parameter exponential distribution, gamma, generalised gamma, Erlang, Weibull, Rayleigh, Chi and Maxwell distribution, here an increase in the values of θ brings about an increase in the posterior risk for SELF and PLF. In case of inverted gamma prior, for $\theta = 0.5$ and $\theta = 1$, the value of the posterior risk increases and decreases afterwards i.e. for $\theta = 1$ to $\theta = 2$ for SELF and PLF.

In half-normal distribution, for the uniform prior, the values of posterior risk increases for $\theta = 0.5$ to $\theta = 1.5$ and it decreases afterwards i.e. $\theta = 1.5$ to $\theta = 2$ for SELF and PLF. In case of inverted gamma prior, for $\theta = 0.5$ to $\theta = 2$, the posterior risk decreases for SELF and PLF.

7.7 Tables and Graphs

Table 7.1: Bayes estimator (Posterior risk) for one-parameter exponential distribution (a=b=c=1)				
		Uniform Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.616334 (0.059599)	0.493067 (0.100000)	0.658889 (0.085109)
	20	0.549284 (0.018589)	0.494355 (0.050000)	0.565208 (0.031849)
	35	0.525709 (0.008877)	0.495668 (0.028571)	0.533859 (0.016302)
	40	0.519791 (0.007485)	0.493802 (0.025000)	0.526769 (0.013955)
	50	0.515860 (0.005773)	0.495226 (0.020000)	0.521319 (0.010918)
1.0	10	1.263462 (0.251220)	1.010770 (0.100000)	1.350698 (0.174471)
	20	1.126900 (0.078401)	1.014210 (0.050000)	1.159570 (0.065341)
	35	1.075154 (0.037189)	1.013716 (0.028571)	1.091824 (0.033340)
	40	1.069618 (0.031685)	1.016137 (0.025000)	1.083976 (0.028716)
	50	1.056978 (0.024239)	1.014699 (0.020000)	1.068163 (0.022370)
1.5	10	1.909606 (0.571681)	1.527685 (0.100000)	2.041455 (0.263697)
	20	1.698386 (0.178077)	1.528547 (0.050000)	1.747624 (0.098478)
	35	1.618531 (0.084189)	1.526043 (0.028571)	1.643626 (0.050189)
	40	1.604996 (0.071344)	1.524746 (0.025000)	1.626541 (0.043089)
	50	1.590025 (0.054837)	1.526424 (0.020000)	1.606851 (0.033652)
2.0	10	2.508897 (0.985467)	2.007117 (0.100000)	2.682123 (0.346453)
	20	2.235502 (0.308219)	2.011951 (0.050000)	2.300312 (0.129621)
	35	2.130332 (0.145832)	2.008599 (0.028571)	2.163362 (0.066061)
	40	2.114622 (0.123849)	2.008891 (0.025000)	2.143007 (0.056771)
	50	2.093400 (0.095750)	2.009664 (0.020000)	2.115553 (0.044306)
		Inverted Gamma Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.559675 (0.039803)	0.462339 (0.086957)	0.591681 (0.064014)
	20	0.522054 (0.015394)	0.473491 (0.046512)	0.535978 (0.027847)
	35	0.506421 (0.007860)	0.478672 (0.027397)	0.513924 (0.015006)
	40	0.502808 (0.006719)	0.478576 (0.024096)	0.509296 (0.012976)
	50	0.499151 (0.005233)	0.479767 (0.019417)	0.504271 (0.010239)
1.0	10	1.092519 (0.153092)	0.902515 (0.086957)	1.154998 (0.124959)
	20	1.038407 (0.060973)	0.941811 (0.046512)	1.066103 (0.055391)
	35	1.014651 (0.031582)	0.959053 (0.027397)	1.029684 (0.030065)
	40	1.014259 (0.027364)	0.965379 (0.024096)	1.027347 (0.026175)
	50	1.004832 (0.021217)	0.965809 (0.019417)	1.015138 (0.020617)
1.5	10	0.756853 (0.073398)	0.625226 (0.086956)	0.800136 (0.086566)
	20	0.708624 (0.028428)	0.642706 (0.046512)	0.727524 (0.037799)
	35	0.692715 (0.014717)	0.654758 (0.027397)	0.702978 (0.020526)
	40	0.690960 (0.012708)	0.657661 (0.024096)	0.699876 (0.017832)
	50	0.686315 (0.009905)	0.659662 (0.019417)	0.693354 (0.014079)
2.0	10	0.577460 (0.042439)	0.477033 (0.086956)	0.610484 (0.066047)
	20	0.537899 (0.016341)	0.487862 (0.046512)	0.552246 (0.028693)
	35	0.518565 (0.008245)	0.490150 (0.027397)	0.526248 (0.015366)
	40	0.517110 (0.007109)	0.492189 (0.024096)	0.523783 (0.013345)
	50	0.513094 (0.005533)	0.493168 (0.019417)	0.513095 (0.010525)

Table 7.2: Bayes estimator (Posterior risk) for gamma distribution (b=c=1)				
			Uniform Prior	
θ	n	SELF	QLF	PLF
0.5	10	0.619987 (0.060192)	0.495989 (0.100000)	0.662794 (0.085614)
	20	0.550574 (0.018701)	0.495516 (0.050000)	0.566535 (0.031924)
	35	0.523382 (0.008798)	0.493474 (0.028571)	0.531497 (0.016229)
	40	0.520069 (0.007488)	0.494066 (0.025000)	0.527051 (0.013962)
	50	0.516158 (0.005778)	0.495512 (0.020000)	0.521620 (0.010924)
1.0	10	1.258947 (0.248857)	1.007157 (0.100000)	1.345871 (0.173848)
	20	1.123378 (0.077946)	1.011040 (0.050000)	1.155946 (0.065136)
	35	1.074761 (0.037124)	1.013346 (0.028571)	1.091425 (0.033327)
	40	1.063472 (0.031331)	1.010298 (0.025000)	1.077747 (0.028551)
	50	1.055541 (0.024181)	1.013319 (0.020000)	1.066711 (0.022341)
1.5	10	1.899472 (0.566154)	1.519578 (0.100000)	2.030621 (0.262298)
	20	1.699482 (0.178351)	1.529534 (0.050000)	1.748752 (0.098541)
	35	1.612568 (0.083570)	1.520421 (0.028571)	1.637571 (0.050005)
	40	1.599385 (0.070872)	1.519416 (0.025000)	1.620854 (0.042938)
	50	1.589286 (0.054817)	1.525714 (0.020000)	1.606104 (0.033636)
2.0	10	2.528018 (0.989834)	2.022415 (0.100000)	2.702565 (0.349094)
	20	2.233305 (0.307941)	2.009974 (0.050000)	2.298051 (0.129494)
	35	2.128148 (0.145475)	2.006539 (0.028571)	2.161144 (0.065993)
	40	2.113259 (0.123618)	2.007596 (0.025000)	2.141626 (0.056734)
	50	2.092234 (0.095005)	2.008544 (0.020000)	2.114374 (0.044281)
			Inverted Gamma Prior	
θ	n	SELF	QLF	PLF
0.5	10	0.565307 (0.040675)	0.466993 (0.086956)	0.597636 (0.064658)
	20	0.523231 (0.015455)	0.474558 (0.046512)	0.537186 (0.027910)
	35	0.507180 (0.007883)	0.479389 (0.027397)	0.514694 (0.015028)
	40	0.503008 (0.006724)	0.478767 (0.024097)	0.509499 (0.012981)
	50	0.499413 (0.005245)	0.480018 (0.019418)	0.019417 (0.010244)
1.0	10	1.097167 (0.154226)	0.906355 (0.086956)	1.159912 (0.125490)
	20	1.039202 (0.061116)	0.942531 (0.046512)	1.066919 (0.055433)
	35	1.019085 (0.031831)	0.963245 (0.027397)	1.034184 (0.030196)
	40	1.013597 (0.027325)	0.964748 (0.024096)	1.026676 (0.026158)
	50	0.971392 (0.021464)	0.971392 (0.019417)	1.021006 (0.020731)
1.5	10	0.760731 (0.074034)	0.628430 (0.086956)	0.804236 (0.087009)
	20	0.713280 (0.028788)	0.646928 (0.046511)	0.732304 (0.038048)
	35	0.692619 (0.014710)	0.654667 (0.027397)	0.702881 (0.020523)
	40	0.690799 (0.012692)	0.657507 (0.024096)	0.699713 (0.017828)
	50	0.685946 (0.009895)	0.659307 (0.019417)	0.692982 (0.014071)
2.0	10	0.577604 (0.042347)	0.477151 (0.086957)	0.610636 (0.066064)
	20	0.536252 (0.016244)	0.486368 (0.046512)	0.550555 (0.028605)
	35	0.519546 (0.008277)	0.491078 (0.027397)	0.527244 (0.015394)
	40	0.516446 (0.007095)	0.491557 (0.024097)	0.523110 (0.013328)
	50	0.511550 (0.005497)	0.491684 (0.019417)	0.516797 (0.010493)

Table 7.3: Bayes estimator (Posterior risk) for generalised gamma distribution (b=c=1)

θ	n	SELF	QLF	PLF
Uniform Prior				
0.5	10	0.618316 (0.060048)	0.494652 (0.100000)	0.661007 (0.085383)
	20	0.550183 (0.018653)	0.495164 (0.050000)	0.566133 (0.031901)
	35	0.524468 (0.008829)	0.494498 (0.028571)	0.532599 (0.016263)
	40	0.520957 (0.007515)	0.494909 (0.025000)	0.527949 (0.013986)
	50	0.515889 (0.005772)	0.495254 (0.020000)	0.521349 (0.010919)
1.0	10	1.264785 (0.251754)	1.011828 (0.100000)	1.352112 (0.174654)
	20	1.122264 (0.077861)	1.010038 (0.050000)	1.154801 (0.065073)
	35	1.071760 (0.036958)	1.010516 (0.028571)	1.088377 (0.033234)
	40	1.067494 (0.031574)	1.014120 (0.025000)	1.081824 (0.028658)
	50	1.054250 (0.024118)	1.012080 (0.020000)	1.065407 (0.022313)
1.5	10	1.904788 (0.569712)	1.52383 (0.100000)	2.036304 (0.263032)
	20	1.691720 (0.176556)	1.522548 (0.050000)	1.740766 (0.098091)
	35	1.618215 (0.084095)	1.525745 (0.028571)	1.643305 (0.050181)
	40	1.601838(0.071091)	1.521746 (0.025000)	1.623340 (0.043004)
	50	1.587035 (0.054649)	1.523554 (0.020000)	1.603830 (0.033589)
2.0	10	2.518618 (0.995297)	2.014894 (0.100000)	2.692516 (0.347796)
	20	2.237152 (0.309052)	2.013437 (0.050000)	2.302011 (0.129717)
	35	2.129976 (0.145719)	2.008263 (0.028571)	2.163001 (0.066049)
	40	2.121500 (0.124607)	2.015425 (0.025000)	2.149977 (0.056956)
	50	2.091698 (0.094919)	2.008030 (0.020000)	2.113833 (0.044269)
Inverted Gamma Prior				
θ	n	SELF	QLF	PLF
0.5	10	0.561705 (0.040110)	0.464017 (0.086957)	0.593828 (0.064246)
	20	0.523895 (0.015495)	0.475161 (0.046512)	0.537868 (0.027946)
	35	0.504856 (0.007816)	0.477193 (0.027397)	0.512336 (0.014959)
	40	0.502558 (0.006711)	0.478338 (0.024096)	0.509043 (0.012969)
	50	0.499059 (0.005231)	0.479675 (0.019417)	0.504175 (0.010237)
1.0	10	1.095272 (0.153718)	0.904789 (0.086956)	1.157908 (0.125273)
	20	1.043062 (0.061566)	0.946033 (0.046511)	1.070881 (0.055639)
	35	1.020020 (0.031897)	0.964129 (0.027397)	1.035133 (0.030224)
	40	1.013221 (0.027299)	0.964391 (0.024096)	1.026296 (0.026148)
	50	1.007588 (0.021339)	0.968458 (0.019417)	1.017922 (0.020669)
1.5	10	0.756921 (0.073290)	0.625282 (0.086956)	0.800207 (0.086574)
	20	0.714881 (0.028918)	0.648380 (0.046511)	0.733947 (0.038134)
	35	0.693802 (0.014768)	0.655786 (0.027397)	0.704082 (0.020558)
	40	0.689429 (0.012648)	0.656199 (0.024096)	0.698321 (0.017792)
	50	0.686690 (0.009913)	0.660022 (0.019417)	0.693733 (0.014086)
2.0	10	0.576835 (0.042367)	0.476516 (0.086956)	0.609824 (0.065976)
	20	0.536937 (0.016294)	0.486989 (0.046512)	0.551258 (0.028641)
	35	0.521229 (0.008327)	0.492669 (0.027397)	0.528952 (0.015444)
	40	0.517312 (0.007114)	0.492381 (0.024096)	0.523986 (0.013351)
	50	0.513402 (0.005538)	0.493464 (0.019417)	0.518667 (0.010532)

Table 7.4: Bayes estimator (Posterior risk) for erlang distribution ($a > 0, b = c = 1$)				
		Uniform Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.618993 (0.060134)	0.495195 (0.100000)	0.661732 (0.085477)
	20	0.548955 (0.018594)	0.494059 (0.050000)	0.564868 (0.031830)
	35	0.524356 (0.008834)	0.494393 (0.028571)	0.532486 (0.016260)
	40	0.520586 (0.007494)	0.494557 (0.025000)	0.527574 (0.013976)
	50	0.515081 (0.005756)	0.494483 (0.020000)	0.520537 (0.010901)
1.0	10	1.265817 (0.251529)	1.012654 (0.100000)	1.353215 (0.174797)
	20	1.123568 (0.077909)	1.011211 (0.050000)	1.156142 (0.065148)
	35	1.074347 (0.037087)	1.012955 (0.028571)	1.091004 (0.033315)
	40	1.065961 (0.031465)	1.012663 (0.025000)	1.080270 (0.028618)
	50	1.057226 (0.024266)	1.014937 (0.020000)	1.068414 (0.022376)
1.5	10	1.913619 (0.573794)	1.530895 (0.100000)	2.045745 (0.264252)
	20	1.697824 (0.177912)	1.528042 (0.050000)	1.747046 (0.098445)
	35	1.618863 (0.084234)	1.526357 (0.028571)	1.643963 (0.050200)
	40	1.602879 (0.071170)	1.522735 (0.025000)	1.624395 (0.043032)
	50	1.592441 (0.055025)	1.528743 (0.020000)	1.609292 (0.033703)
2.0	10	2.507497 (0.985658)	2.005998 (0.100000)	2.680627 (0.346260)
	20	2.243270 (0.310479)	2.018943 (0.050000)	2.308306 (0.130072)
	35	2.132715 (0.146059)	2.010846 (0.028571)	2.165783 (0.066135)
	40	2.116199 (0.123993)	2.010389 (0.025000)	2.144605 (0.056813)
	50	2.096589 (0.095379)	2.012725 (0.020000)	2.118775 (0.044373)
		Inverted Gamma Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.561165 (0.040065)	0.463571 (0.086956)	0.593257 (0.064184)
	20	0.523645 (0.015506)	0.474934 (0.046512)	0.537611 (0.027932)
	35	0.505983 (0.007844)	0.478258 (0.027397)	0.513479 (0.014993)
	40	0.503124 (0.006729)	0.478877 (0.024096)	0.509616 (0.012985)
	50	0.498664 (0.005222)	0.479298 (0.019417)	0.503778 (0.010229)
1.0	10	1.089719 (0.152235)	0.900203 (0.086957)	1.152038 (0.124638)
	20	1.039060 (0.061074)	0.942403 (0.046512)	1.066774 (0.055426)
	35	1.014910 (0.031581)	0.959298 (0.027397)	1.029946 (0.030073)
	40	1.013759 (0.027345)	0.964901 (0.024096)	1.026840 (0.026162)
	50	1.008705 (0.021385)	0.969532 (0.019417)	1.019051 (0.020692)
1.5	10	0.754716 (0.072926)	0.623461 (0.086956)	0.797877 (0.086322)
	20	0.713766 (0.028839)	0.647369 (0.046512)	0.732803 (0.038074)
	35	0.692037 (0.014687)	0.654117 (0.027397)	0.702290 (0.020506)
	40	0.689773 (0.012655)	0.656531 (0.024096)	0.698673 (0.017801)
	50	0.686413 (0.009906)	0.659756 (0.019417)	0.693453 (0.014081)
2.0	10	0.577418 (0.042485)	0.476997 (0.086956)	0.610439 (0.066043)
	20	0.536998 (0.016281)	0.487045 (0.046512)	0.551320 (0.028645)
	35	0.522087 (0.008351)	0.493479 (0.027397)	0.529822 (0.015470)
	40	0.518337 (0.007143)	0.493357 (0.024096)	0.525026 (0.013377)
	50	0.513238 (0.005536)	0.493306 (0.019417)	0.518502 (0.010528)

Table 7.5: Bayes estimator (Posterior risk) for Weibull distribution (a=1, b=c)				
Uniform Prior				
θ	n	SELF	QLF	PLF
0.5	10	0.626289 (0.061418)	0.501031 (0.100000)	0.669531 (0.085947)
	20	0.559029 (0.019286)	0.503126 (0.050000)	0.575236 (0.031852)
	35	0.533863 (0.009165)	0.503357 (0.028571)	0.542141 (0.016257)
	40	0.531427 (0.007821)	0.504855 (0.025000)	0.538560 (0.013976)
	50	0.524947 (0.005979)	0.503949 (0.020000)	0.530502 (0.010909)
1.0	10	1.270088 (0.253769)	1.016070 (0.100000)	1.357781 (0.175386)
	20	1.129065 (0.078688)	1.016159 (0.050000)	1.161798 (0.065466)
	35	1.071738 (0.036919)	1.010496 (0.028571)	1.088355 (0.033234)
	40	1.064445 (0.031392)	1.011223 (0.025000)	1.078733 (0.028577)
	50	1.054733 (0.024143)	1.012544 (0.020000)	1.065895 (0.022323)
1.5	10	1.900357 (0.565738)	1.520285 (0.100000)	2.031567 (0.262420)
	20	1.698752 (0.178223)	1.528877 (0.050000)	1.748001 (0.098498)
	35	1.613077 (0.083599)	1.520901 (0.028571)	1.638087 (0.050021)
	40	1.611101 (0.071899)	1.530546 (0.025000)	1.632727 (0.043253)
	50	1.585802 (0.054568)	1.522369 (0.020000)	1.602583 (0.033563)
2.0	10	2.511191 (0.988524)	2.008953 (0.100000)	2.684576 (0.346770)
	20	2.227257 (0.306221)	2.004531 (0.050000)	2.291828 (0.129143)
	35	2.135355 (0.146505)	2.013335 (0.028571)	2.168463 (0.066216)
	40	2.113051 (0.123653)	2.007398 (0.025000)	2.141415 (0.056728)
	50	2.091700 (0.094916)	2.008032 (0.020000)	2.113835 (0.044270)
Inverted Gamma Prior				
θ	n	SELF	QLF	PLF
0.5	10	0.577418 (0.042485)	0.476997 (0.086956)	0.610439 (0.066043)
	20	0.536997 (0.016281)	0.487045 (0.046512)	0.551320 (0.028645)
	35	0.522087 (0.008351)	0.493479 (0.027397)	0.529822 (0.015470)
	40	0.518337 (0.007143)	0.493357 (0.024096)	0.525026 (0.013377)
	50	0.513238 (0.005536)	0.493306 (0.019417)	0.518502 (0.010528)
1.0	10	0.577418 (0.042485)	0.476997 (0.086957)	0.610439 (0.066043)
	20	0.536998 (0.016281)	0.487045 (0.046512)	0.551320 (0.028645)
	35	0.522087 (0.008351)	0.493479 (0.027397)	0.529822 (0.015470)
	40	0.518337 (0.007143)	0.493357 (0.024096)	0.525026 (0.013377)
	50	0.513238 (0.005536)	0.493306 (0.019417)	0.518502 (0.010528)
1.5	10	0.755752 (0.073145)	0.624317 (0.086956)	0.798972 (0.086440)
	20	0.715454 (0.028941)	0.648901 (0.046512)	0.734537 (0.038164)
	35	0.693336 (0.014743)	0.655346 (0.027397)	0.703608 (0.020544)
	40	0.690731 (0.012689)	0.657442 (0.024096)	0.024096 (0.017826)
	50	0.687263 (0.009932)	0.660574 (0.019417)	0.694313 (0.014098)
2.0	10	0.579361 (0.042703)	0.478603 (0.086956)	0.612494 (0.066265)
	20	0.537892 (0.016351)	0.487852 (0.046512)	0.552238 (0.028693)
	35	0.521209 (0.008327)	0.492650 (0.027397)	0.528932 (0.015444)
	40	0.517716 (0.007125)	0.492765 (0.024096)	0.524396 (0.013361)
	50	0.513656 (0.005545)	0.493709 (0.019417)	0.518925 (0.010537)

Table 7.6: Bayes estimator (Posterior risk) for half normal distribution (a=0.5, b=c=2)				
		Uniform Prior		
θ	n	SELF	QLF	PLF
0.5	10	1.073432 (0.094389)	0.954162 (0.053934)	1.112330 (0.089864)
	20	1.022171 (0.032949)	0.968373 (0.025961)	1.037305 (0.048385)
	35	1.002159 (0.016344)	0.972684 (0.014596)	1.010018 (0.032816)
	40	1.000531 (0.014014)	0.974876 (0.012737)	1.007313 (0.030240)
	50	0.994868 (0.010834)	0.974564 (0.010152)	1.000173 (0.026329)
1.0	10	2.200836 (0.397029)	1.956299 (0.053934)	2.280587 (0.377996)
	20	2.114273 (0.141233)	2.002996 (0.025961)	2.145577 (0.207394)
	35	2.064155 (0.069378)	2.003444 (0.014596)	2.080341 (0.139298)
	40	2.065371 (0.059807)	2.012413 (0.012737)	2.079372 (0.129055)
	50	2.050667 (0.046079)	2.008817 (0.010151)	2.061603 (0.111985)
1.5	10	3.323679 (0.903468)	2.954381 (0.053933)	3.444118 (0.860157)
	20	3.162555 (0.315722)	2.996104 (0.025961)	3.209379 (0.463623)
	35	3.116065 (0.158107)	3.024416 (0.014596)	3.140501 (0.317450)
	40	3.093174 (0.133948)	3.013862 (0.012738)	3.114142 (0.289040)
	50	3.078275 (0.103819)	3.015453 (0.010151)	3.094691 (0.252309)
2.0	10	4.385883 (0.073584)	3.898563 (0.053933)	4.544813 (0.062984)
	20	4.164003 (0.030513)	3.944845 (0.025961)	4.225655 (0.027987)
	35	4.096360 (0.015996)	3.975878 (0.014596)	4.128482 (0.015217)
	40	4.075651 (0.013768)	3.971147 (0.012738)	4.103279 (0.013190)
	50	4.064293 (0.010836)	3.981348 (0.010151)	4.085967 (0.010456)
		Inverted Gamma Prior		
θ	n	SELF	QLF	PLF
0.5	10	1.024350 (0.069039)	0.926793 (0.046435)	1.054855 (0.061009)
	20	0.992198 (0.028350)	0.943798 (0.024086)	1.005690 (0.026985)
	35	0.976269 (0.014783)	0.948769 (0.013984)	0.983581 (0.014624)
	40	0.976240 (0.012798)	0.952135 (0.012269)	0.982599 (0.012718)
	50	0.974807 (0.010063)	0.955503 (0.009851)	0.979844 (0.010075)
1.0	10	1.998831 (0.266664)	1.808466 (0.046435)	2.058355 (0.119049)
	20	1.981259 (0.113307)	1.884612 (0.024086)	2.008201 (0.053885)
	35	1.969730 (0.060194)	1.914245 (0.013984)	1.984482 (0.029505)
	40	1.971454 (0.052176)	1.922776 (0.012268)	1.984296 (0.025683)
	50	1.973194 (0.041236)	1.934121 (0.009852)	1.983390 (0.020393)
1.5	10	1.384587 (0.127602)	1.252722 (0.046435)	1.425820 (0.082465)
	20	1.361242 (0.053567)	1.294840 (0.024085)	1.379753 (0.037022)
	35	1.346935 (0.028175)	1.308994 (0.013984)	1.357023 (0.020175)
	40	1.351357 (0.024556)	1.317990 (0.012268)	1.360160 (0.017605)
	50	1.351143 (0.019352)	1.324388 (0.009851)	1.358125 (0.013964)
2.0	10	1.051952 (0.072826)	0.951765 (0.046435)	1.083279 (0.062654)
	20	1.018381 (0.029846)	0.968704 (0.024085)	1.032230 (0.027697)
	35	1.004142 (0.015635)	0.975856 (0.013984)	1.011663 (0.015041)
	40	1.007066 (0.013621)	0.982200 (0.012269)	1.013626 (0.013119)
	50	1.003789 (0.010673)	0.983912 (0.009852)	1.008977 (0.010374)

Table 7.7: Bayes estimator (Posterior risk) for Rayleigh distribution (a=1, b=1, c=2)				
Uniform Prior				
θ	n	SELF	QLF	PLF
0.5	10	0.721679 (0.018394)	0.681586 (0.027381)	0.733036 (0.026217)
	20	0.707218 (0.007393)	0.688607 (0.013070)	0.712146 (0.015739)
	35	0.699225 (0.003841)	0.688942 (0.007325)	0.701878 (0.011055)
	40	0.700078 (0.003332)	0.691102 (0.006389)	0.702384 (0.010288)
	50	0.697341 (0.002598)	0.690225 (0.005088)	0.699159 (0.009014)
1.0	10	1.494524 (0.078921)	1.411495 (0.027381)	1.518044 (0.112481)
	20	1.463518 (0.031692)	1.425004 (0.013070)	1.473715 (0.067462)
	35	1.446449 (0.016462)	1.425177 (0.007325)	1.451938 (0.047373)
	40	1.441123 (0.014120)	1.422647 (0.006389)	1.445871 (0.043594)
	50	1.443232 (0.011143)	1.428505 (0.005088)	1.446996 (0.038651)
1.5	10	2.238633 (0.176700)	2.114265 (0.027381)	2.273863 (0.251842)
	20	2.191208 (0.071031)	2.133544 (0.013070)	2.206476 (0.151202)
	35	2.169087 (0.036993)	2.137189 (0.007325)	2.177318 (0.106455)
	40	2.168755 (0.031968)	2.140951 (0.006389)	2.175901 (0.098697)
	50	2.164156 (0.025040)	2.142073 (0.005088)	2.169799 (0.086854)
2.0	10	2.972134 (0.311048)	2.807015 (0.027382)	3.018908 (0.443323)
	20	2.886929 (0.122903)	2.810957 (0.013070)	2.907044 (0.261622)
	35	2.858461 (0.064157)	2.816425 (0.007325)	2.869309 (0.184622)
	40	2.848495 (0.055123)	2.811976 (0.006389)	2.857880 (0.170181)
	50	2.846554 (0.043292)	2.817507 (0.005089)	2.853976 (0.150165)
Inverted Gamma Prior				
θ	n	SELF	QLF	PLF
0.5	10	0.696578 (0.014050)	0.663408 (0.023519)	0.705800 (0.018443)
	20	0.684935 (0.006358)	0.668229 (0.012119)	0.689339 (0.008808)
	35	0.682998 (0.003496)	0.673378 (0.007017)	0.685474 (0.004958)
	40	0.683962 (0.003052)	0.675518 (0.006153)	0.686131 (0.004335)
	50	0.684149 (0.002422)	0.677375 (0.004938)	0.685878 (0.003459)
1.0	10	1.361886 (0.054515)	1.297035 (0.023519)	1.379916 (0.036059)
	20	1.374044 (0.025662)	1.340531 (0.012119)	1.382879 (0.017670)
	35	1.380158 (0.014286)	1.360719 (0.007017)	1.385167 (0.010018)
	40	1.382307 (0.012472)	1.365242 (0.006153)	1.386689 (0.008762)
	50	1.384185 (0.009912)	1.370480 (0.004938)	1.387685 (0.006999)
1.5	10	0.940943 (0.025952)	0.896136 (0.023519)	0.953405 (0.024913)
	20	0.941939 (0.012086)	0.918965 (0.012119)	0.947991 (0.012113)
	35	0.942955 (0.006680)	0.929674 (0.007017)	0.946378 (0.006845)
	40	0.947493 (0.005866)	0.935796 (0.006153)	0.950496 (0.006006)
	50	0.946113 (0.004636)	0.936745 (0.004938)	0.948505 (0.004784)
2.0	10	0.715841 (0.014894)	0.681753 (0.023519)	0.725317 (0.018953)
	20	0.706655 (0.006775)	0.689419 (0.012119)	0.711198 (0.009087)
	35	0.706401 (0.003745)	0.696452 (0.007017)	0.708965 (0.005127)
	40	0.706102 (0.003252)	0.697384 (0.006153)	0.708339 (0.004475)
	50	0.703006 (0.002558)	0.696045 (0.004938)	0.704783 (0.003554)

Table 7.8: Bayes estimator (Posterior risk) for chi-distribution (a=0.5, b=1, c=2)				
			Uniform Prior	
θ	n	SELF	QLF	PLF
0.5	10	1.150475 (0.127288)	1.006665 (0.060437)	1.199199 (0.112520)
	20	1.049248 (0.036951)	0.990956 (0.027382)	1.065760 (0.052666)
	35	1.015573 (0.017334)	0.984798 (0.015035)	1.023795 (0.034261)
	40	1.017079 (0.014875)	0.990313 (0.013070)	1.024165 (0.031664)
	50	1.006147 (0.011321)	0.985185 (0.010361)	1.011629 (0.027221)
1.0	10	2.375528 (0.543602)	2.078587 (0.060437)	2.476136 (0.480536)
	20	2.167816 (0.157839)	2.047382 (0.027382)	2.201932 (0.224961)
	35	2.106973 (0.074612)	2.043126 (0.015035)	2.124031 (0.147466)
	40	2.094754 (0.063211)	2.039629 (0.013070)	2.109349 (0.134558)
	50	2.075192 (0.048202)	2.031959 (0.010361)	2.086501 (0.115899)
1.5	10	3.587984 (1.237248)	3.139486 (0.060436)	3.739942 (1.093707)
	20	3.261786 (0.357243)	3.080576 (0.027381)	3.313118 (0.509163)
	35	3.150174 (0.166696)	3.054714 (0.015035)	3.175677 (0.329465)
	40	3.134777 (0.141433)	3.052283 (0.013070)	3.156619 (0.301067)
	50	3.118812 (0.108882)	3.053836 (0.010361)	3.135806 (0.261800)
2.0	10	4.725959 (2.143402)	4.135214 (0.060437)	4.926113 (1.894733)
	20	4.294922 (0.618901)	4.056315 (0.027381)	4.362513 (0.882092)
	35	4.157921 (0.290307)	4.031923 (0.015035)	4.191582 (0.573773)
	40	4.133677 (0.245742)	4.024896 (0.013071)	4.162479 (0.523107)
	50	4.096936 (0.187645)	4.011583 (0.010362)	4.119261 (0.451180)
			Inverted Gamma Prior	
θ	n	SELF	QLF	PLF
0.5	10	0.995774 (0.061534)	0.905249 (0.044377)	1.023765 (0.055981)
	20	0.978673 (0.026840)	0.932069 (0.023519)	0.991629 (0.025913)
	35	0.970899 (0.014405)	0.943930 (0.013791)	0.978063 (0.014328)
	40	0.970313 (0.012485)	0.946647 (0.012119)	0.976553 (0.012478)
	50	0.968854 (0.009841)	0.949857 (0.009755)	0.973810 (0.009911)
1.0	10	1.943938 (0.237183)	1.767216 (0.044377)	1.998580 (0.109285)
	20	1.957754 (0.107572)	1.864527 (0.023515)	1.983672 (0.051836)
	35	1.956865 (0.058506)	1.902508 (0.013791)	1.971305 (0.028879)
	40	1.957759 (0.050825)	1.910009 (0.012119)	1.970348 (0.025177)
	50	1.962409 (0.040354)	1.923931 (0.009755)	1.972447 (0.020074)
1.5	10	1.345106 (0.113378)	1.222824 (0.044377)	1.382916 (0.075619)
	20	1.335297 (0.050126)	1.271711 (0.023519)	1.352975 (0.035355)
	35	1.342544 (0.027571)	1.305251 (0.013791)	1.352450 (0.019813)
	40	1.343104 (0.023942)	1.310345 (0.012119)	1.351740 (0.017272)
	50	1.338646 (0.018791)	1.312398 (0.009755)	1.345493 (0.013693)
2.0	10	1.029682 (0.065831)	0.936074 (0.044377)	1.058625 (0.057887)
	20	1.006171 (0.028363)	0.958258 (0.023519)	1.019491 (0.026640)
	35	1.000004 (0.015283)	0.972226 (0.013791)	1.007383 (0.014758)
	40	0.999284 (0.013245)	0.974911 (0.012119)	1.005710 (0.012851)
	50	0.996819 (0.010416)	0.977274 (0.009755)	1.001918 (0.010196)

Table 7.9: Bayes estimator (Posterior risk) for Maxwells Failure distribution (a=1.5, b=1, c=2)				
		Uniform Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.571435 (0.007049)	0.551026 (0.017693)	0.576954 (0.012769)
	20	0.568476 (0.003070)	0.558675 (0.008583)	0.571020 (0.008141)
	35	0.567385 (0.001649)	0.561876 (0.004842)	0.568791 (0.005868)
	40	0.567322 (0.001430)	0.562510 (0.004228)	0.568546 (0.005454)
	50	0.566442 (0.001126)	0.562515 (0.003372)	0.567413 (0.004816)
1.0	10	1.175675 (0.029861)	1.133687 (0.017695)	1.187031 (0.054092)
	20	1.171068 (0.013015)	1.150878 (0.008583)	1.176308 (0.034514)
	35	1.173134 (0.007071)	1.161744 (0.004842)	1.176041 (0.025156)
	40	1.169028 (0.006083)	1.159121 (0.004228)	1.171551 (0.023187)
	50	1.172273 (0.004830)	1.164352 (0.003373)	1.174282 (0.020652)
1.5	10	1.769124 (0.067526)	1.705941 (0.017694)	1.786212 (0.122319)
	20	1.761770 (0.029504)	1.731395 (0.008583)	1.769653 (0.078237)
	35	1.751380 (0.015737)	1.734376 (0.004842)	1.755720 (0.055986)
	40	1.758235 (0.013762)	1.743335 (0.004228)	1.762028 (0.052463)
	50	1.756130 (0.010830)	1.744265 (0.003372)	1.759140 (0.046306)
2.0	10	2.343734 (0.118174)	2.260029 (0.017694)	2.366373 (0.214066)
	20	2.316804 (0.050913)	2.276860 (0.008583)	2.327170 (0.135006)
	35	2.311144 (0.027385)	2.288705 (0.004843)	2.316871 (0.097420)
	40	2.313503 (0.023771)	2.293897 (0.004228)	2.318494 (0.090610)
	50	2.306405 (0.018671)	2.290821 (0.003372)	2.310357 (0.079830)
		Inverted Gamma Prior		
θ	n	SELF	QLF	PLF
0.5	10	0.564553 (0.006031)	0.546341 (0.015990)	0.569440 (0.009773)
	20	0.561251 (0.002815)	0.552050 (0.008162)	0.563634 (0.004766)
	35	0.557401 (0.001542)	0.552142 (0.004705)	0.558742 (0.002682)
	40	0.555047 (0.001332)	0.550460 (0.004123)	0.556214 (0.002334)
	50	0.556666 (0.001064)	0.552980 (0.003305)	0.557601 (0.001869)
1.0	10	1.102664 (0.023307)	1.067095 (0.015996)	1.112209 (0.019089)
	20	1.120704 (0.011269)	1.102332 (0.008165)	1.125463 (0.009517)
	35	1.126075 (0.006304)	1.115452 (0.004706)	1.128786 (0.005420)
	40	1.128644 (0.005512)	1.119317 (0.004123)	1.131018 (0.004747)
	50	1.128929 (0.004377)	1.121453 (0.003306)	1.130825 (0.003791)
1.5	10	0.765775 (0.011220)	0.741073 (0.015997)	0.772404 (0.013257)
	20	0.768474 (0.005302)	0.755876 (0.008162)	0.771737 (0.006521)
	35	0.768564 (0.002938)	0.761314 (0.004705)	0.770414 (0.003699)
	40	0.770561 (0.002572)	0.764193 (0.004123)	0.772182 (0.003241)
	50	0.772728 (0.002053)	0.767610 (0.003305)	0.774025 (0.002595)
2.0	10	0.582261 (0.006416)	0.563478 (0.015997)	0.587301 (0.010080)
	20	0.576829 (0.002979)	0.567373 (0.008163)	0.579279 (0.004899)
	35	0.575355 (0.001645)	0.569927 (0.004706)	0.576740 (0.002769)
	40	0.574294 (0.001426)	0.569548 (0.004124)	0.575502 (0.002415)
	50	0.574748 (0.001134)	0.570942 (0.003306)	0.575714 (0.001930)

Table 7.10(a): Minimum and maximum values of the posterior risk for different loss function in case of Uniform prior

	$\theta = 0.5$		$\theta = 1.0$		$\theta = 1.5$		$\theta = 2.0$	
Distribution	min	max	min	max	min	max	min	max
One-parameter exponential distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Gamma distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Generalised gamma distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Erlang distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Weibull distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Half normal Distribution	SELF	QLF	QLF	SELF	QLF	SELF	QLF	SELF
Rayleigh distribution	SELF	QLF	QLF	PLF	QLF	PLF	QLF	PLF
Chidistribution	QLF	SELF	QLF	SELF	QLF	SELF	QLF	SELF
Maxwell Failure distribution	SELF	QLF	QLF	PLF	QLF	PLF	QLF	PLF

Table 7.10(b): Minimum and maximum values of the posterior risk for different loss function in case of inverted gamma prior

	$\theta = 0.5$		$\theta = 1.0$		$\theta = 1.5$		$\theta = 2.0$	
Distribution	min	max	min	max	min	max	min	max
One-parameter exponential distribution	SELF	QLF	QLF	SELF	SELF	QLF	SELF	QLF
Gamma distribution	SELF	QLF	QLF	SELF	SELF	PLF	SELF	QLF
Generalized gamma distribution	SELF	QLF	QLF	SELF	SELF	QLF	SELF	QLF
Erlang distribution	SELF	QLF	QLF	SELF	SELF	QLF	SELF	QLF
Weibull distribution	SELF	QLF	SELF	QLF	SELF	QLF	SELF	QLF
Half normal distribution	QLF	SELF	QLF	SELF	QLF	SELF	QLF	SELF
Rayleigh distribution	SELF	QLF	QLF	SELF	QLF	SELF	SELF	QLF
Chidistribution	QLF	SELF	QLF	SELF	QLF	SELF	QLF	SELF
Maxwells Failure distribution	SELF	QLF	QLF	SELF	SELF	QLF	SELF	QLF

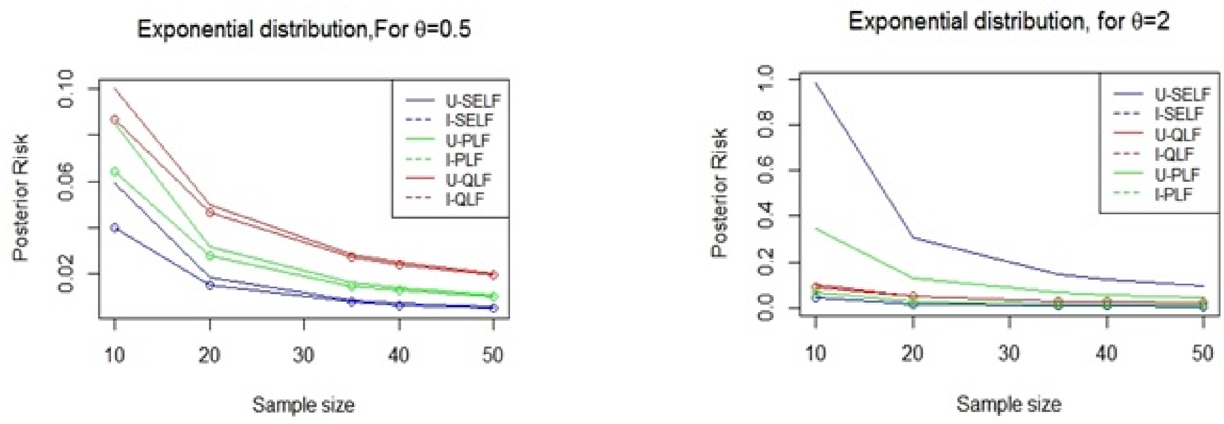


Figure 7.2

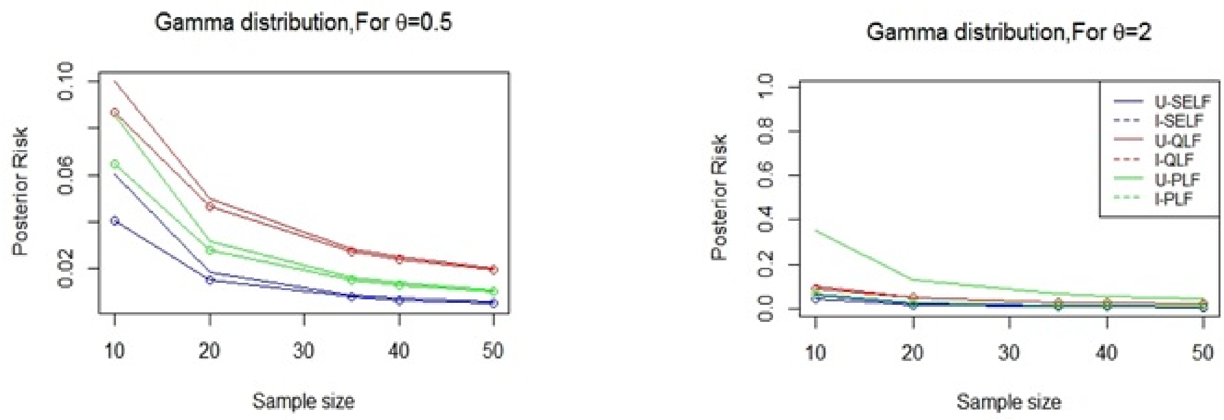


Figure 7.3

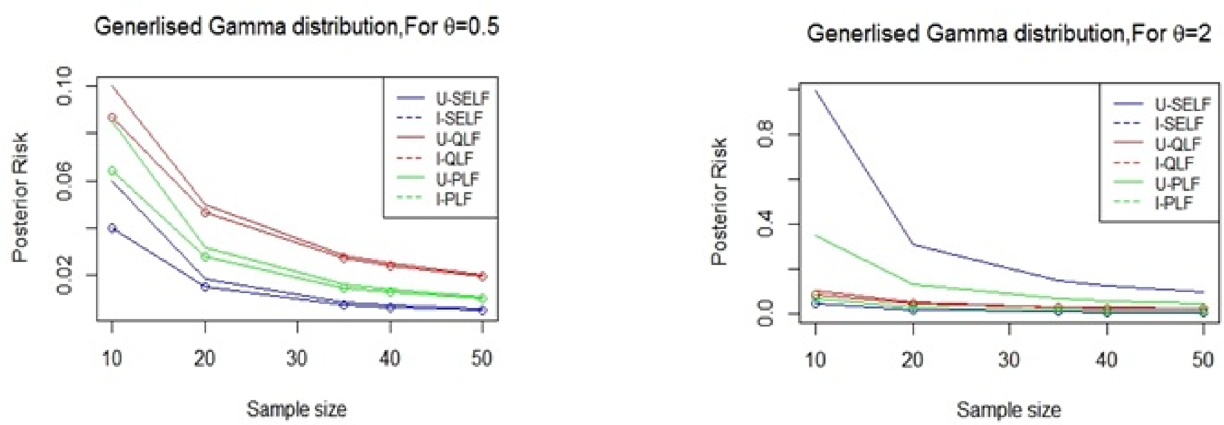


Figure 7.4

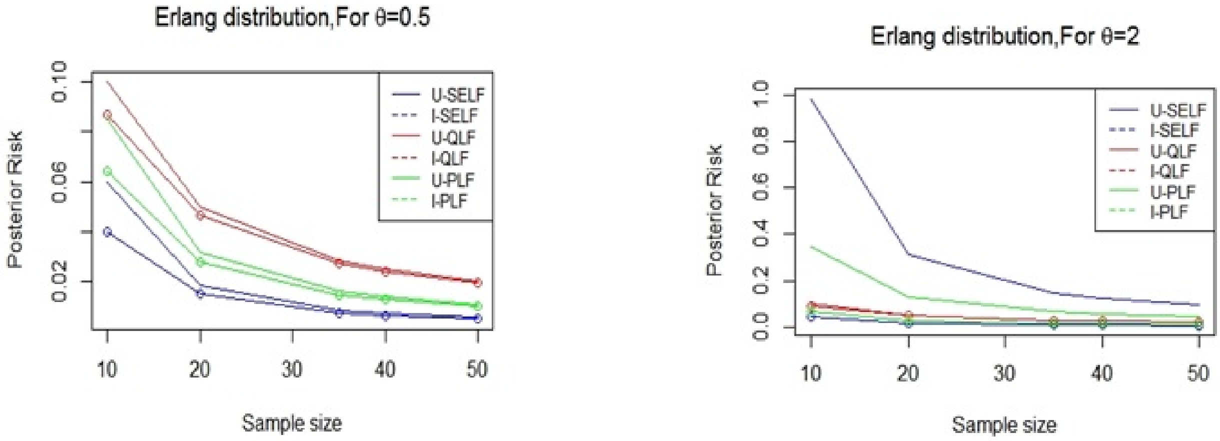


Figure 7.5

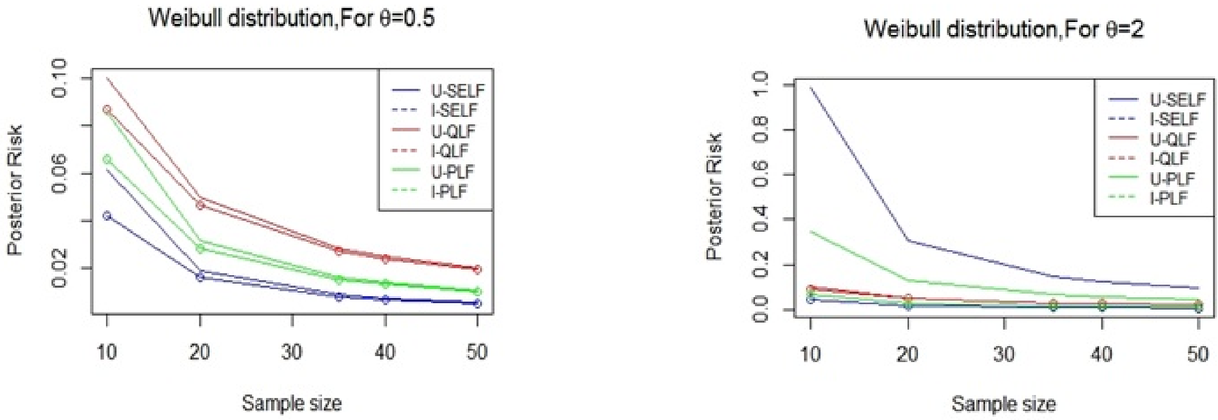


Figure 7.6

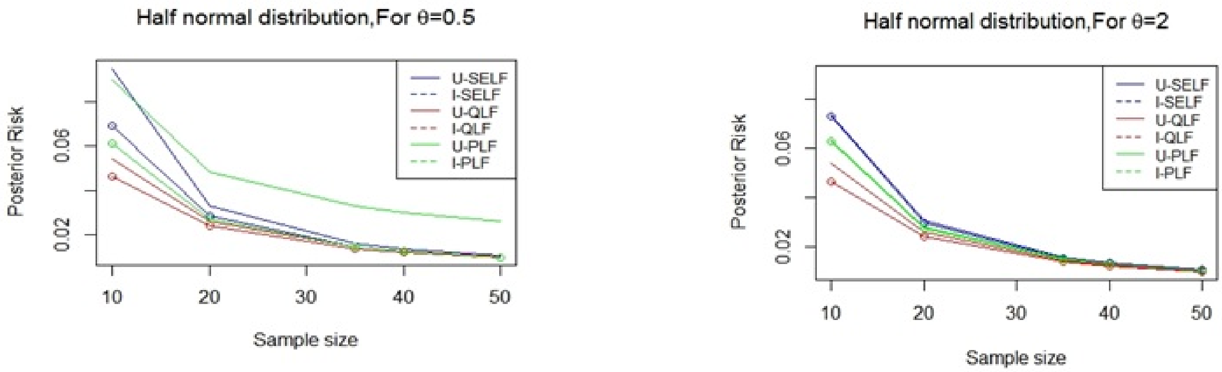


Figure 7.7

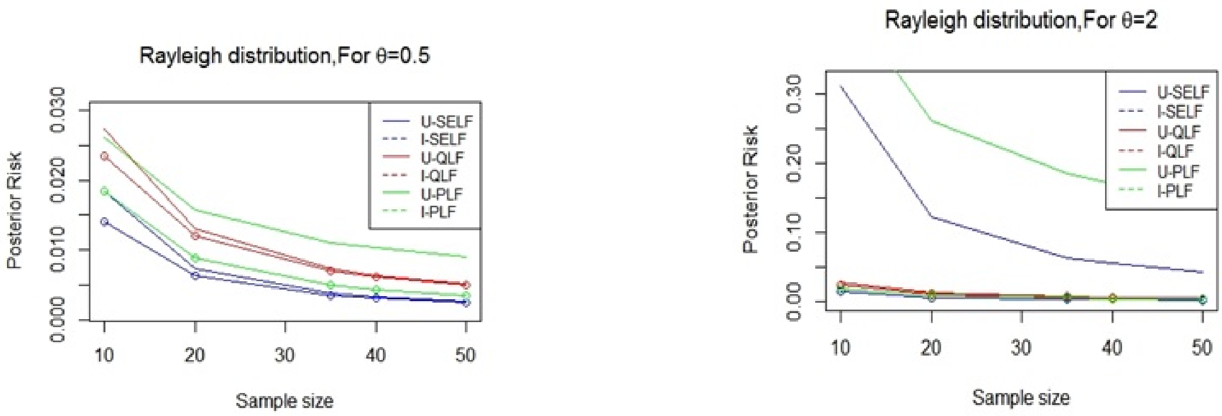


Figure 7.8

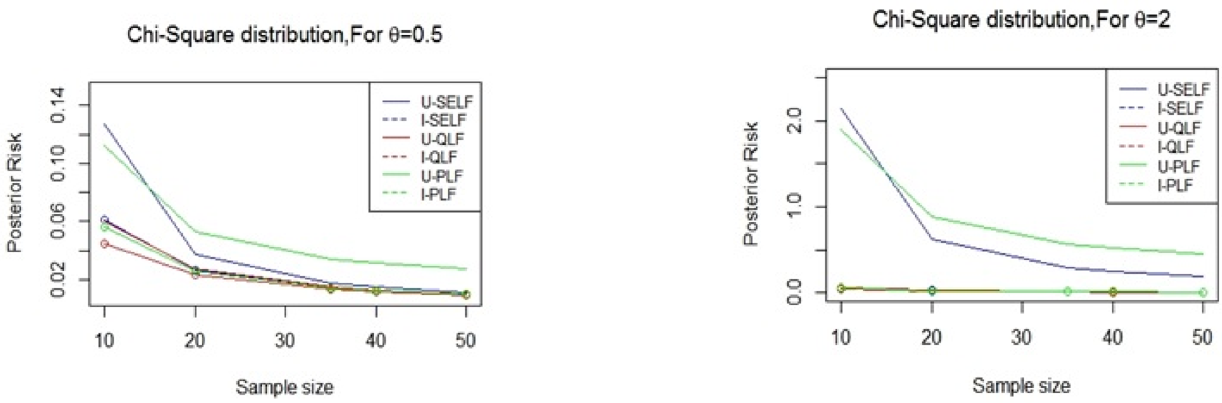


Figure 7.9

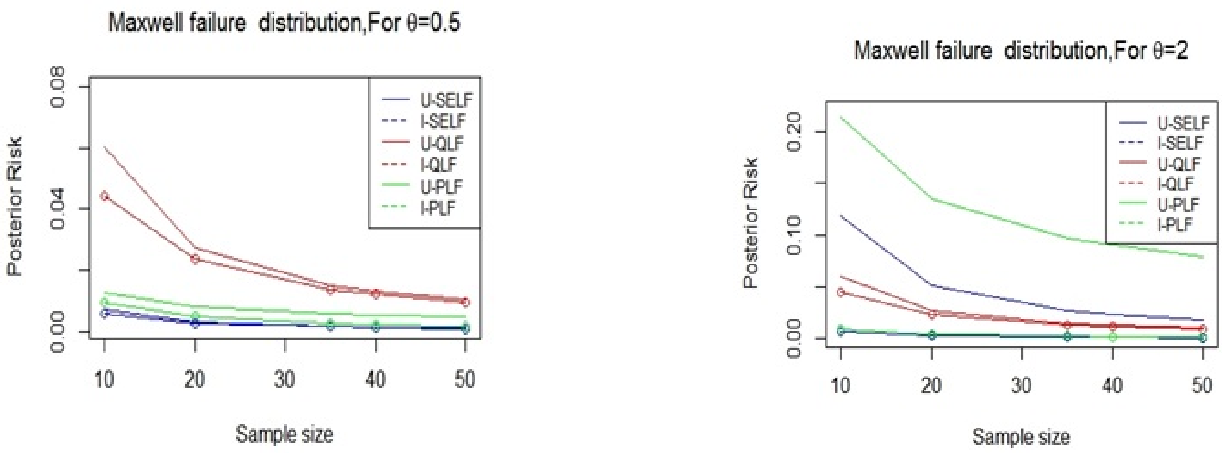


Figure 7.10

Chapter 8

Bayesian Estimation Methods for two parameter Generalised Rayleigh Distribution under different Loss functions and Type II censoring

8.1 Introduction

In this chapter, we have considered the two parameter Generalized Rayleigh distribution proposed by Surles and Padgett (2001)[105]. The cumulative density function (cdf), probability density function (pdf) and the reliability function of this distribution are given by (8.1.1), (8.1.2) and (8.1.3), respectively.

$$F(x;\alpha,\tau) = \left(1 - e^{-\tau x^2}\right)^{\alpha}; \qquad x > 0, \tag{8.1.1}$$

$$f(x;\alpha,\tau) = 2\alpha\tau x e^{-\tau x^2}\left(1 - e^{-\tau x^2}\right)^{\alpha-1}; \qquad x > 0 \tag{8.1.2}$$

$$R(t) = 1 - \left(1 - e^{-\tau x^2}\right)^{\alpha}; \qquad x > 0. \tag{8.1.3}$$

where $\alpha > 0$ is the shape parameter and $\tau > 0$ is the scale parameter. Here, we have considered uniform prior, quasi prior and natural conjugate prior under the different loss functions like Squared Error Loss Function (SELF), Quadratic Loss Function (QLF) and Generalised Entropy Loss Function (GELF) to obtain the posterior distribution. A comparative study has been performed in order to study the impact of the different priors under different the loss functions where the major role is played by the Bayes estimators of the positive and negative powers of the parameter involved in the model.

8.2 Set-up of the Problem

Let a random variable X follows generalized two parameter Rayleigh distribution whose pdf is given in (8.1.2). In the whole chapter, we assume that α is unknown but τ is known. Suppose n items are put on test and the test is terminated after r ordered items are noticed. Let $0 \geq X_{(1)} \geq X_{(2)} ... \geq X_{(r)}$, $0 \geq r \geq n$, be the lifetimes of the first r ordered observations. Here $(n-r)$ items survived until $X_{(r)}$. The likelihood function is given by

$$L(\alpha \mid \underline{x}) = \alpha^r exp(-\alpha s_r) \tag{8.2.1}$$

$$s_r = -\left[\sum_{i=1}^{r} log\left\{1 - e^{-\tau x_{(i)}^2}\right\} + (n-r) log\left\{1 - e^{-\tau x_{(r)}^2}\right\}\right] \tag{8.2.2}$$

The prior $\pi(\alpha)$ and posterior density $h(\alpha \mid s_r)$ in each case i.e. for the uniform, quasi and natural conjugate prior, respectively, are given as follows

$$\pi_U(\alpha) = m \tag{8.2.3}$$

$$h_U(\alpha \mid s_r) = \frac{s_r^{r+1}}{\Gamma(r+1)} \alpha^r exp\left\{-\alpha s_r\right\} \tag{8.2.4}$$

$$\pi_Q(\alpha) = \frac{1}{\alpha^d} \tag{8.2.5}$$

$$h_Q(\alpha \mid s_r) = \frac{s_r^{r-d+1}}{\Gamma(r-d+1)} \alpha^{r-d} exp\left\{-\alpha s_r\right\} \tag{8.2.6}$$

$$\pi_C(\alpha) = \frac{d^v}{\Gamma\nu} \alpha^{\nu-1} exp(-d\alpha); \qquad \alpha, d, \nu > 0 \tag{8.2.7}$$

$$h_C(\alpha \mid s_r) = \frac{(s_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \alpha^{r+\nu-1} exp\left\{-\alpha(s_r+\mu)\right\} \tag{8.2.8}$$

Let us make the transformation $U = -log\left\{1 - e^{-\tau x^2}\right\}$. After the transformation, we see that U follows exponential distribution with pdf

$$f(u;\alpha) = \alpha e^{-\alpha u}; \qquad u > 0$$

According to the additive property of the exponential distribution, the pdf of S_r is

$$h(s_r, \alpha) = \frac{\alpha^r}{\Gamma r} s_r^{r-1} exp(-\alpha s_r); \qquad s_r > 0 \tag{8.2.9}$$

The risk is defined as

$$R_B(\hat{\theta}_B) = E_{S_r|\theta}\left\{L(\hat{\theta}_B, \theta)\right\}$$

and the posterior risk is given by

$$R_P B(\hat{\theta}_B) = E_{\theta|S_r}\left\{L(\hat{\theta}_B, \theta)\right\}$$

where $\hat{\theta}_B$ is the Bayes estimate of the parameter θ.

8.3 Bayes estimators $\hat{\alpha}^p$ and $\hat{\alpha}^{-p}$ of α^p and α^{-p}

The Bayes estimator is defined as the posterior mean. In each case the Bayes Estimators $\hat{\alpha}^p_{SB}$, $\hat{\alpha}^p_{QB}$ and $\hat{\alpha}^p_{GB}$ of SELF, QLF and GELF, respectively, are defined as

$$\hat{\alpha}^p_{SB} = E(\alpha^p \mid x) \tag{8.3.1}$$

$$\hat{\alpha}_{QB}^{p} = \frac{E(\alpha^{-p} \mid x)}{E(\alpha^{-2p} \mid x)} \tag{8.3.2}$$

$$\hat{\alpha}_{GB}^{p} = \left[E_{\alpha|S_r}\left((\alpha^{p})^{-a}\right)\right]^{-1/a} \tag{8.3.3}$$

8.3.1 The Bayes estimators under Uniform prior

The Bayes estimators under uniform prior for different loss functions are discussed in the following theorems.

Theorem 8.1: For a positive integer p, $\hat{\alpha}_{USB}^{p}$ and $\hat{\alpha}_{USB}^{-p}$ under SELF are:

$$\hat{\alpha}_{USB}^{p} = \frac{\Gamma(r+1+p)}{\Gamma(r+1)} s_r^{-p} \tag{8.3.4}$$

$$\hat{\alpha}_{USB}^{-p} = \frac{\Gamma(r+1-p)}{\Gamma(r+1)} s_r^{p} \tag{8.3.5}$$

Proof: $\hat{\alpha}_{USB}^{p}$ and $\hat{\alpha}_{USB}^{-p}$ are obtained from (8.3.1) and hence, the theorem follows by using (8.2.4) in (8.3.1).

Theorem 8.2: α_{UQB}^{p} and α_{UQB}^{-p} under QLF, are given by:

$$\hat{\alpha}_{UQB}^{p} = \frac{\Gamma(r+1-p)}{\Gamma(r+1-2p)} s_r^{-p} \tag{8.3.6}$$

$$\hat{\alpha}_{UQB}^{-p} = \frac{\Gamma(r+1+p)}{\Gamma(r+1+2p)} s_r^{p} \tag{8.3.7}$$

Proof: $\hat{\alpha}_{UQB}^{p}$ is obtained from (8.3.2) and hence, the theorem follows on using (8.2.4) in (8.3.2).

Theorem 8.3: α_{UGB}^{p} and α_{UGB}^{-p} under GELF are:

$$\hat{\alpha}_{UGB}^{p} = \left\{\frac{\Gamma(r+1-ap)}{\Gamma(r+1)}\right\}^{-1/a} s_r^{-p} \tag{8.3.8}$$

and

$$\hat{\alpha}_{UGB}^{-p} = \left\{\frac{\Gamma(r+1+ap)}{\Gamma(r+1)}\right\}^{-1/a} s_r^{p} \tag{8.3.9}$$

Proof: $\hat{\alpha}_{UGB}^{-p}$ is obtained from (8.3.3) and hence, the theorem follows by utilizing (8.2.4) in (8.3.3).

8.3.2 The Bayes estimators under Natural Conjugate prior (NCP)

The Bayes estimators for different loss functions under natural conjugate prior are obtained in the following theorems.

Theorem 8.4: $\hat{\alpha}_{CSB}^{p}$ and $\hat{\alpha}_{CSB}^{-p}$ under SELF are given by

$$\hat{\alpha}_{CSB}^{p} = \frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}(s_r+\mu)^{-p} \tag{8.3.10}$$

$$\hat{\alpha}_{CSB}^{-p} = \frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}(s_r+\mu)^{p} \tag{8.3.11}$$

Proof: $\hat{\alpha}_{CSB}^{p}$ is obtained from (8.3.1) and hence, the theorem follows on using (8.2.8) in (8.3.1).

Theorem 8.5: α_{QB}^{p} and α_{CQB}^{-p} under QLF are:

$$\hat{\alpha}_{CQB}^{p} = \frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu-2p)}(s_r+\mu)^{-p} \tag{8.3.12}$$

$$\hat{\alpha}_{CQB}^{-p} = \frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu+2p)}(s_r+\mu)^{p} \tag{8.3.13}$$

Proof: $\hat{\alpha}_{CQB}^{p}$ is evaluated from (8.3.2) and hence, the theorem follows on using (8.2.8) in (8.3.2).

Theorem 8.6: α_{CGB}^{p} and α_{CGB}^{-p} under GELF are given by:

$$\hat{\alpha_{CGB}^{p}} = \left\{\frac{\Gamma(r+\nu-ap)}{\Gamma(r+\nu)}\right\}^{-1/a}(s_r+\mu)^{-p} \tag{8.3.14}$$

and

$$\hat{\alpha_{CGB}^{-p}} = \left\{\frac{\Gamma(r+\nu+ap)}{\Gamma(r+\nu)}\right\}^{-1/a}(s_r+\mu)^{p} \tag{8.3.15}$$

Proof: $\alpha^{p\hat{}}_{CGB}$ is evaluated from (8.3.3) and hence, the theorem follows on using (8.2.8) in (8.3.3).

8.3.3 The Bayes estimators under Quasi prior

The Bayes estimators under quasi prior for SELF, QLF and GELF are discussed in the following theorems.

Theorem 8.7: $\hat{\alpha}^{p}_{QSB}$ and $\hat{\alpha}^{-p}_{QSB}$ under SELF are:

$$\hat{\alpha}^{p}_{QSB} = \frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)} s_r^{-p} \tag{8.3.16}$$

$$\hat{\alpha}^{-p}_{QSB} = \frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1)} s_r^{p} \tag{8.3.17}$$

Proof: $\hat{\alpha}^{p}_{QSB}$ is obtained from (8.3.1) and hence, the theorem follows on using (8.2.6) in (8.3.1).

Theorem 8.8: α^{p}_{QQB} and α^{-p}_{QQB} under QLF are given by:

$$\hat{\alpha}^{p}_{QQB} = \frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1-2p)} s_r^{-p} \tag{8.3.18}$$

$$\hat{\alpha}^{-p}_{QQB} = \frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1+2p)} s_r^{p} \tag{8.3.19}$$

Proof: $\hat{\alpha}^{p}_{QQB}$ is obtained from (8.3.2) and hence, the theorem follows on using (8.2.6) in (8.3.2).

Theorem 8.9: α^{p}_{QGB} and α^{-p}_{QGB} under GELF are:

$$\alpha^{p\hat{}}_{QGB} = \left\{\frac{\Gamma(r-d+1-ap)}{\Gamma(r-d+1)}\right\}^{-1/a} s_r^{-p} \tag{8.3.20}$$

and

$$\hat{\alpha_{QGB}^{-p}} = \left\{ \frac{\Gamma(r-d+1+ap)}{\Gamma(r-d+1)} \right\}^{-1/a} s_r^p \tag{8.3.21}$$

Proof: $\hat{\alpha_{QGB}^{p}}$ is evaluated from (8.3.3) and hence, the theorem follows on using (8.2.6) in in (8.3.3).

8.4 Risk for different priors under various loss functions

The risk under SELF, QLF and PLF are

$$R_S(\hat{\alpha}_{SB}^p) = E_{S_r|\alpha}\left[(\alpha^p - \hat{\alpha}_{SB}^p)^2\right] \tag{8.4.1}$$

$$R_Q(\hat{\alpha}_{QB}^p) = E_{S_r|\alpha}\left[\left(1 - \frac{\hat{\alpha}_{QB}^p}{\alpha^p}\right)^2\right] \tag{8.4.2}$$

$$R_G(\hat{\alpha}_{GB}^p) = E_{S_r|\alpha}\left[\left(\frac{\hat{\alpha}_{GB}^p}{\alpha^p}\right)^a - a\left(\frac{\hat{\alpha}_{GB}^p}{\alpha^p}\right) - 1\right] \tag{8.4.3}$$

Theorem 8.10: The Risk $R_S(\hat{\alpha}_{USB}^p)$, $R_Q(\hat{\alpha}_{UQB}^p)$ and $R_G(\hat{\alpha}_{UGB}^p)$ for SELF, QLF and GELF under uniform prior, respectively, are as follows:

$$R_S(\hat{\alpha}_{USB}^p) = \alpha^{2p}\left[\left\{\frac{\Gamma(r+1+p)}{\Gamma(r+1)}\right\}^2 \frac{\Gamma(r-2p)}{\Gamma(r)} - 2\left\{\frac{\Gamma(r+1+p)}{\Gamma(r+1)}\right\}\frac{\Gamma(r-p)}{\Gamma(r)} + 1\right] \tag{8.4.4}$$

$$R_Q(\hat{\alpha}_{UQB}^p) = 1 + \left\{\frac{\Gamma(r+1-p)}{\Gamma(r+1-2p)}\right\}^2 \frac{\Gamma(r-2p)}{\Gamma(r)} - 2\left\{\frac{\Gamma(r+1-p)}{\Gamma(r+1-2p)}\right\}\frac{\Gamma(r-p)}{\Gamma(r)} \tag{8.4.5}$$

$$R_G(\hat{\alpha}_{UGB}^p) = \left\{\frac{r}{(r-ap)}\right\} - log\left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} + \frac{ap}{r}\int_0^\infty z^{r-1}e^{-z} logz dz - 1 \tag{8.4.6}$$

Proof: Under SELF, the risk is obtained by using (8.3.4) in (8.4.1) and on solving, we get

$$R_S(\hat{\alpha}_{USB}^p) = \left\{\frac{\Gamma(r+1+p)}{\Gamma(r+1)}\right\}^2 E_{S_r|\alpha}(s_r^{-2p}) + \alpha^{2p} - 2\alpha^p\left\{\frac{\Gamma(r+1+p)}{\Gamma(r+1)}\right\} E_{S_r|\alpha}(s_r^{-p}) \tag{8.4.7}$$

Now, the value of $E_{S_r|\alpha}(s_r^{-2p})$ and $E_{S_r|\alpha}(s_r^{-p})$ are evaluated as

$$E_{S_r|\alpha}(s_r^{-2p}) = \frac{\alpha^{2p}\Gamma(r-2p)}{\Gamma(r)} \tag{8.4.8}$$

and

$$E_{S_r|\alpha}(s_r^{-p}) = \frac{\alpha^{p}\Gamma(r-p)}{\Gamma(r)} \tag{8.4.9}$$

Substituting (8.4.8) and (8.4.9) in (8.4.7), we get equation (8.4.4).

The risk in case of QLF is obtained by using (8.3.6) in (8.4.2), then we have

$$R_Q(\hat{\alpha}^p_{UQB}) = 1 + \left\{\frac{\Gamma(r+1-p)}{\Gamma(r+1-2p)}\right\}^2 E_{S_r|\alpha}\left\{\alpha s_r\right\}^{-2p} - 2\left\{\frac{\Gamma(r+1-p)}{\Gamma(r+1-2p)}\right\} E_{S_r|\alpha}\left\{\alpha s_r\right\}^{-p}$$

Substituting the values from (8.4.8) and (8.4.9), we obtain the risk for QLF under Uniform prior in (8.4.5).

The risk for GELF is obtained by using (8.3.8) in (8.4.3), we get

$$R_G(\hat{\alpha}^p_{UGB}) = \left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} E_{S_r|\alpha}\left\{\alpha s_r\right\}^{-ap} - log\left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} + apE_{S_r|\alpha}\{log(\alpha s_r)\} - 1 \tag{8.4.10}$$

Now, the value of $E_{S_r|\alpha}\left\{\alpha s_r\right\}^{-ap}$ and $E_{S_r|\alpha}\left\{log(\alpha s_r)\right\}$ are

$$E_{S_r|\alpha}\left\{\alpha s_r\right\}^{-ap} = \frac{\Gamma(r-ap)}{\Gamma(r)} \tag{8.4.11}$$

and

$$E_{S_r|\alpha}\left\{log(\alpha s_r)\right\} = \frac{1}{\Gamma(r)}\int_0^\infty log z z^{r-1}e^{-z}dz \tag{8.4.12}$$

Using (8.4.11) and (8.4.12) in (8.4.10), get the required result given in (8.4.6).

Hence, the theorem follows.

Theorem 8.11: Risk $R_S(\hat{\alpha}^p_{CSB})$, $R_Q(\hat{\alpha}^p_{CQB})$ and $R_G(\hat{\alpha}^p_{CGB})$ for SELF, QLF and GELF, respectively, under natural conjugate prior are as follows:

$$R_S(\hat{\alpha}^p_{CSB}) = \left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}^2 \frac{\alpha^{2p}}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{2p}} + \alpha^{2p} - 2\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}\frac{\alpha^{2p}}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{p}} \tag{8.4.13}$$

$$R_Q(\hat{\alpha}^p_{CQB}) = 1 + \left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu-2p)}\right\}^2 \frac{1}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{2p}} - 2\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu-2p)}\right\}\frac{1}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{p}} \tag{8.4.14}$$

$$\begin{aligned} R_G(\hat{\alpha}^p_{CGB}) = &\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}\frac{1}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{2p}} - log\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\} \\ &+ \frac{ap}{\Gamma(r)}\int_0^\infty log(z+\alpha\mu)z^{r-1}e^{-z}dz - 1 \end{aligned} \tag{8.4.15}$$

Proof: To obtain the risk in case of SELF under NCP, we use the following expression in (8.4.1) and by using (8.3.10), we get

$$R_S(\hat{\alpha}^p_{CSB}) = \left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}^2 E_{S_r|\alpha}(s_r+\mu)^{-2p} + \alpha^{2p} - 2\alpha^p\left\{\frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}\right\}E_{S_r|\alpha}(s_r+\mu)^{-p} \tag{8.4.16}$$

Now, the value of $E_{S_r|\alpha}(s_r+\mu)^{-2p}$ and $E_{S_r|\alpha}(s_r+\mu)^{-p}$ are given as

$$E_{S_r|\alpha}(s_r+\mu)^{-2p} = \frac{\alpha^{2p}}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{2p}} \tag{8.4.17}$$

and

$$E_{S_r|\alpha}(s_r+\mu)^{-p} = \frac{\alpha^{p}}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{p}} \tag{8.4.18}$$

Substituting (8.4.17) and (8.4.18) in (8.4.16), we obtain the risk for SELF.

In case of QLF, the risk under NCP is evaluated by using (8.3.12) in (8.4.2), we get

$$R_Q(\hat{\alpha}^p_{CQB}) = 1 + \left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu-2p)}\right\}^2 E_{S_r|\alpha}\left\{\alpha(s_r+\mu)\right\}^{-2p} - 2\left\{\frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu-2p)}\right\}E_{S_r|\alpha}\left\{\alpha(s_r+\mu)\right\}^{-p} \tag{8.4.19}$$

Now, we know that

$$E_{S_r|\alpha}(s_r+\mu)^{-2p} = \frac{1}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{2p}} \tag{8.4.20}$$

and

$$E_{S_r|\alpha}(s_r+\mu)^{-p} = \frac{1}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(z+\alpha\mu)^{p}} \tag{8.4.21}$$

Further, using (8.4.20) and (8.4.21) in (8.4.19), we get the required result given in (8.4.14).

The risk for GELF under NCP is formulated by using (8.3.14) in (8.4.3), we get

$$\begin{aligned} R_G(\hat{\alpha}^p_{CGB}) = &\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\} E_{\alpha|S_r}\left\{\alpha(s_r+\mu)\right\}^{-ap} - log\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\} \\ &+ apE_{S_r|\alpha}\{log(\alpha(s_r+\mu))\} - 1 \end{aligned} \tag{8.4.22}$$

Now, the values of $E_{S_r|\alpha}\left\{\alpha(s_r+\mu)\right\}^{-ap}$ and $E_{S_r|\alpha}\left[log\{\alpha(s_r+\mu)\}\right]$ are

$$E_{S_r|\alpha}\left\{\alpha(s_r+\mu)\right\}^{-ap} = \frac{\alpha^{-ap}}{\Gamma r}\int_0^\infty \frac{e^{-z}z^{r-1}dz}{(\frac{z}{\alpha}+\mu)^{p}} \tag{8.4.23}$$

and

$$E_{S_r|\alpha}\left[log\{\alpha(s_r+\mu)\}\right] = \frac{1}{\Gamma(r)}\int_0^\infty log(z+\alpha\mu)z^{r-1}e^{-z}dz \tag{8.4.24}$$

On substituting (8.4.23) and (8.4.24) in (8.4.22), we have (8.4.15).

Hence, the theorem follows.

Theorem 8.12: The Risk $R_S(\hat{\alpha}^p_{QSB})$, $R_Q(\hat{\alpha}^p_{QQB})$ and $R_G(\hat{\alpha}^p_{QGB})$ for SELF, QLF and GELF, respectively, under quasi prior are as follows:

$$R_S(\hat{\alpha}^p_{QSB}) = \alpha^{2p}\left[\left\{\frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)}\right\}^2\frac{\Gamma(r-2p)}{\Gamma(r)} - 2\left\{\frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)}\right\}\frac{\Gamma(r-p)}{\Gamma(r)} + 1\right] \tag{8.4.25}$$

$$R_Q(\hat{\alpha}^p_{QQB}) = 1+\left\{\frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1-2p)}\right\}^2\frac{\Gamma(r-2p)}{\Gamma(r)}-2\left\{\frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1-2p)}\right\}\frac{\Gamma(r-p)}{\Gamma(r)} \tag{8.4.26}$$

$$R_G(\hat{\alpha}^p_{QGB}) = \left\{\frac{r}{(r-ap)}\right\} - log\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} + \frac{ap}{r}\int_0^\infty z^{r-1}e^{-z}logz dz - 1 \tag{8.4.27}$$

Proof: For SELF, the risk under quasi prior is obtained by using (8.3.16) in (8.4.1), we have

$$R_S(\hat{\alpha}^p_{QSB}) = \left\{\frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)}\right\}^2 E_{S_r|\alpha}(s_r^{-2p}) + \alpha^{2p} - 2\alpha^p\left\{\frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)}\right\} E_{S_r|\alpha}(s_r^{-p}) \tag{8.4.28}$$

Substituting (8.4.8) and (8.4.9) in (8.4.28), we get the required result as given in equation (8.4.25).

Risk for QLF under quasi prior is evaluated by using (8.3.18) in (8.4.2), we have

$$R_Q(\hat{\alpha}^p_{QQB}) = 1 + \left\{\frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1-2p)}\right\}^2 E_{S_r|\alpha}\{\alpha s_r\}^{-2p} - 2\left\{\frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1)}\right\} E_{S_r|\alpha}\{\alpha s_r\}^{-p}$$

From (8.4.8) and (8.4.9), the Risk in this case is obtained in (8.4.26).

Risk for GELF is evaluated by using (8.3.18) in (8.4.3) we get

$$\begin{aligned} R_G(\hat{\alpha}^p_{QGB}) = &\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} E_{S_r|\alpha}\{\alpha s_r\}^{-ap} - log\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} \\ &+ apE_{S_r|\alpha}\{log(\alpha s_r)\} - 1 \end{aligned} \tag{8.4.29}$$

Using (8.4.11) and (8.4.12) in (8.5.21), we get the required result given in (8.4.27).

Hence, the theorem follows.

Note: The expressions for $R(\hat{\alpha}^{-p})$ are obtained by replacing p by $-p$ in all the given cases.

8.5 Posterior risk for different loss functions under various priors

The posterior risk $R_{PS}(\hat{\alpha}^p_{SB})$, $R_{PQ}(\hat{\alpha}^p_{QB})$ and $R_{PG}(\hat{\alpha}^p_{GB})$ for SELF, QLF and GELF, respectively, are as follows:

$$R_{PS}(\hat{\alpha}^p_{SB}) = E_{\alpha/|S_r}\left\{\alpha^{2p}\right\} - \left[E_{\alpha/|S_r}\left\{\alpha^p\right\}\right]^2 \tag{8.5.1}$$

$$R_{PQ}(\hat{\alpha}^p_{QB}) = 1 - \frac{\left\{E_{\alpha/|S_r}(\alpha^{-p})\right\}^2}{E_{\alpha/|S_r}(\alpha^{-2p})} \tag{8.5.2}$$

$$R_{PG}(\hat{\alpha}^p_{GB}) = E_{\alpha/S_r}\left[\left(\frac{\hat{\alpha}^p_{GB}}{\alpha^p}\right)^a - a\left(\frac{\hat{\alpha}^p_{GB}}{\alpha^p}\right) - 1\right] \tag{8.5.3}$$

Theorem 8.13: Posterior risk $R_{PS}(\hat{\alpha}^p_{USB})$, $R_{PQ}(\hat{\alpha}^p_{UQB})$ and $R_{PG}(\hat{\alpha}^p_{UGB})$ for SELF, QLF and GELF, respectively, under uniform prior are:

$$R_{PS}(\hat{\alpha}^p_{USB}) = \frac{s_r^{-2p}}{\Gamma(r+1)}\left[\Gamma(r+2p+1) - \frac{\{\Gamma(r+p+1)\}^2}{\Gamma(r+1)}\right] \tag{8.5.4}$$

$$R_{PQ}(\hat{\alpha}^p_{UQB}) = 1 - \frac{\{\Gamma(r+1-p)\}^2}{\Gamma(r+1)\Gamma(r+1-2p)} \tag{8.5.5}$$

$$R_{PG}(\hat{\alpha}^p_{UGB}) = ap\psi(r+1) - log\left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} \tag{8.5.6}$$

Proof: The posterior risk for SELF under uniform prior is obtained from (8.5.1). Now, the values of $E_{\alpha|S_r}(\alpha^p)$ and $E_{\alpha|S_r}(\alpha^{2p})$ are

$$E_{\alpha|S_r}(\alpha^p) = \frac{\Gamma(r+1+2p)}{\Gamma(r+1)}(s_r)^{-p} \tag{8.5.7}$$

and

$$E_{\alpha|S_r}\left(\alpha^{2p}\right) = \frac{\Gamma(r+1+2p)}{\Gamma(r+1)}(s_r)^{-2p} \tag{8.5.8}$$

Since, here we are considering the uniform prior so we substitute the expected values from (8.5.7) and (8.5.8) in (8.5.1) with respect to uniform prior and on solving we get (8.5.4).

For QLF, the posterior risk is obtained by using the values given in (8.5.9) and (8.5.10) in (8.5.2).

$$E_{\alpha|S_r}\left(\alpha^{-p}\right) = \frac{\Gamma(r+1-p)}{\Gamma(r+1)}(s_r)^{p} \tag{8.5.9}$$

and

$$E_{\alpha|S_r}\left(\alpha^{-2p}\right) = \frac{\Gamma(r+1-2p)}{\Gamma(r+1)}(s_r)^{2p} \tag{8.5.10}$$

The required result is obtained in (8.5.5).

For GELF, the posterior risk is evaluated by using (8.3.8) in (8.5.3), we have

$$R_{PG}(\hat{\alpha}^p_{UGB}) = \left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} E_{\alpha|S_r}\left\{\alpha s_r\right\}^{-ap} - log\left\{\frac{\Gamma(r+1)}{\Gamma(r+1-ap)}\right\} + apE_{\alpha|S_r}\{log(\alpha s_r)\} - 1 \quad (8.5.11)$$

Now,

$$E_{\alpha|S_r}\left\{\alpha(s_r)\right\}^{-ap} = \frac{\Gamma(r+1-ap)}{\Gamma(r+1)}(s_r)^{ap} \quad (8.5.12)$$

and

$$E_{\alpha|S_r}\left\{log(\alpha)\right\} = \frac{\Gamma(r+1)}{(s_r)^{-ap}}\left[\psi(r+1) - log(s_r)\right] \quad (8.5.13)$$

Further, substituting (8.5.12) and (8.5.13) in (8.5.11), we get (8.5.6).

Hence, the theorem follows.

Theorem 8.14: Posterior risk for different loss function under natural conjugate prior are as follows:

$$R_{PS}(\hat{\alpha}^p_{CSB}) = \frac{(s_r+\mu)^{-2p}}{\Gamma(r+\nu)}\left[\Gamma(r+\nu+2p) - \frac{\{\Gamma(r+\nu+p)\}^2}{\Gamma(r+\nu)}\right] \quad (8.5.14)$$

$$R_{PQ}(\hat{\alpha}^p_{CQB}) = 1 - \frac{\{\Gamma(r+\nu-p)\}^2}{\Gamma(r+\nu)\Gamma(r+\nu-2p)} \quad (8.5.15)$$

$$R_{PG}(\hat{\alpha}^p_{CGB}) = ap\psi(r+\nu) - log\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\} \quad (8.5.16)$$

Proof: Refering to the expression of posterior risk for SELF from (8.5.1).

$$E_{\alpha|S_r}\left(\alpha^p\right) = \frac{\Gamma(r+\nu+p)}{\Gamma(r+\nu)}(s_r+\mu)^{-p} \quad (8.5.17)$$

and

$$E_{\alpha|S_r}\left(\alpha^{2p}\right) = \frac{\Gamma(r+\nu+2p)}{\Gamma(r+\nu)}(s_r+\mu)^{-2p} \quad (8.5.18)$$

Substituting (8.5.17) and (8.5.18) in (8.5.1), we get (8.5.14).

For QLF, the posterior risk is obtained by using (8.5.19) and (8.5.20) in (8.5.2).

$$E_{\alpha|S_r}\left(\alpha^{-p}\right) = \frac{\Gamma(r+\nu-p)}{\Gamma(r+\nu)}\left(s_r+\mu\right)^p \tag{8.5.19}$$

$$E_{\alpha|S_r}\left(\alpha^{-2p}\right) = \frac{\Gamma(r+\nu-2p)}{\Gamma(r+\nu)}\left(s_r+\mu\right)^{2p} \tag{8.5.20}$$

on subtituting the values and further solving it we get the required result in (8.5.15).

The posterior risk for GELF under NCP is obtained by using (8.3.14) in (8.5.3), we have

$$\begin{aligned} R_{PG}(\hat{\alpha}^p_{CGB}) = &\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}\{\alpha(s_r+\mu)\}^{-ap} - log\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\} \\ &+ apE_{\alpha|S_r}\left\{log(\alpha(s_r+\mu))\right\} - 1 \end{aligned} \tag{8.5.21}$$

We know that,

$$E_{\alpha|S_r}\{\alpha(s_r+\mu)\}^{-ap} = \frac{\Gamma(r+\nu-ap)}{\Gamma(r+\nu)}\left(s_r+\mu\right)^{ap} \tag{8.5.22}$$

and

$$E_{\alpha|S_r}\{log(\alpha)\} = \frac{\Gamma(r+\nu)}{(s_r+\mu)^{-ap}}\left[\psi(r+\nu) - log(s_r+\mu)\right] \tag{8.5.23}$$

Substituting (8.5.22) and (8.5.23) in (8.5.21), we get the required posterior risk of GELF in (8.5.16).

Hence, the theorem follows.

Theorem 8.15: Posterior risk for different loss function under Quasi prior are as follows:

$$R_{PS}(\hat{\alpha}^p_{QSB}) = \frac{s_r^{-2p}}{\Gamma(r-d+1)}\left[\Gamma(r-d+1+2p) - \frac{\{\Gamma(r-d+1+p)\}^2}{\Gamma(r-d+1)}\right] \tag{8.5.24}$$

$$R_{PQ}(\hat{\alpha}^p_{QQB}) = 1 - \frac{\{\Gamma(r-d+1-p)\}^2}{\Gamma(r-d+1)\Gamma(r-d+1-2p)} \tag{8.5.25}$$

$$R_{PG}(\hat{\alpha}^p_{QGB}) = ap\psi(r+1) - log\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} \tag{8.5.26}$$

Proof: We consider the expression of posterior risk for SELF from (8.5.1) in case of quasi prior, we have

$$E_{\alpha|S_r}\left(\alpha^p\right) = \frac{\Gamma(r-d+1+p)}{\Gamma(r-d+1)}\left(s_r\right)^{-p} \tag{8.5.27}$$

and

$$E_{\alpha|S_r}\left(\alpha^{2p}\right) = \frac{\Gamma(r-d+1+2p)}{\Gamma(r-d+1)}\left(s_r\right)^{-2p} \tag{8.5.28}$$

Substituting the values from (8.5.27) and (8.5.28), we get the required result in (8.5.25).

For OLF, the posterior risk is obtained by using (8.5.2). Now, we have

$$E_{\alpha|S_r}\left(\alpha^{-p}\right) = \frac{\Gamma(r-d+1-p)}{\Gamma(r-d+1)}\left(s_r\right)^{p} \tag{8.5.29}$$

and

$$E_{\alpha|S_r}\left(\alpha^{-2p}\right) = \frac{\Gamma(r-d+1-2p)}{\Gamma(r-d+1)}\left(s_r\right)^{2p} \tag{8.5.30}$$

On substituting the values from (8.5.29) and (8.5.30) in (8.5.2), we get the result as given in (8.5.25). The Posterior risk for GELF under quasi prior is obtained by using (8.3.20) in (8.5.3), we get

$$\begin{aligned} R_{PG}(\hat{\alpha}^p_{QGB}) = &\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} E_{\alpha|S_r}\left\{\alpha s_r\right\}^{-ap} - log\left\{\frac{\Gamma(r-d+1)}{\Gamma(r-d+1-ap)}\right\} \\ &+ apE_{\alpha|S_r}\left\{log(\alpha s_r)\right\} - 1 \end{aligned} \tag{8.5.31}$$

Now we have,

$$E_{\alpha|S_r}\{\alpha(s_r)\}^{-ap} = \frac{\Gamma(r-d+1-ap)}{\Gamma(r-d+1)}\left(s_r\right)^{ap} \tag{8.5.32}$$

and

$$E_{\alpha|S_r}\{log(\alpha)\} = \frac{\Gamma(r-d+1)}{\left(s_r\right)^{-ap}}\left[\psi(r-d+1) - log(s_r)\right] \tag{8.5.33}$$

Substituting (8.5.32) and (8.5.33) in (8.5.31), we have the result in (8.5.26).

Hence, the theorem follows.

Note: The expressions for $R_P(\hat{\alpha}^{-p})$ are obtained by replacing p by $-p$ in all the given cases.

8.6 Bayes estimator of F(x) for different loss functions under various priors

Theorem 8.16: Bayes estimator of $F(x)$ for different loss functions under Uniform Prior

$$\hat{F}_{USELF}(x;\alpha,\tau,\mu) = \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r+i} \tag{8.6.1}$$

$$\hat{F}_{UQLF}(x;\alpha,\tau,\mu) = \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r-i} \tag{8.6.2}$$

$$\hat{F}_{UGELF}(x;\alpha,\tau,\mu) = \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r} \tag{8.6.3}$$

Proof: Here we obtain Bayes estimator of (8.1.1) at a specified point x with the help of Bayes estimator of α^p for uniform prior.

$$\hat{F}(x;\alpha,\tau,\mu) = \left\{1 - e^{-\tau x^2}\right\}^{\alpha}$$
$$= exp\left[\alpha log\left\{1 - e^{\tau x^2}\right\}\right]$$

$$\hat{F}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty} \frac{\left[log\left\{1 - e^{\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i \tag{8.6.4}$$

Utilizing Lemma 1, of Chaturvedi and Tomar (2002)[28], we get

$$\hat{F}_{USELF}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty} \frac{\left[log\left\{1 - e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i_{USB}$$

Substituting the value of $\hat{\alpha}^i_{USB}$ in above equation, we get

$$\begin{aligned}\hat{F}_{USELF}(x;\alpha,\tau,\mu) &= \sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\frac{\Gamma(r+1+i)}{\Gamma(r+1)}s_r^{-i}\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(r+1+i\right)}{\Gamma\left(r+1\right)}\right\}\left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^i\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\frac{(r+i)!}{(r)!}\left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^i\end{aligned}$$

Thus, we obtain the result in case of SELF in (8.6.1). Similarly, on solving for QLF and GELF by putting the values of $\hat{\alpha}^i_{UQB}$ and $\hat{\alpha}^i_{UGB}$ in (8.6.4), we get (8.6.2) and (8.6.3), respectively. Hence, the theorem follows.

Theorem 8.17: Bayes estimator of the cdf for different loss functions under NCP Prior

$$\hat{F}_{CSELF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r+i+\nu-1} \tag{8.6.5}$$

$$\hat{F}_{CQLF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r-i+\nu} \tag{8.6.6}$$

$$\hat{F}_{CGELF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r+\nu} \tag{8.6.7}$$

Proof: Here, we obtain Bayes estimator of (8.1.1) at a specified point x with the help of Bayes estimator of α^p for NCP

$$\hat{F}_{CSELF}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i_{CSB}$$

Substituting the value of $\hat{\alpha}^i_{CSB}$ in above equation, we get

$$\hat{F}_{CSELF}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty} \frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!} \frac{\Gamma(r+\nu+i)}{\Gamma(r+\nu)}(s_r+\mu)^{-i}$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left\{\frac{\Gamma(r+\nu+i)}{\Gamma(r+\nu)}\right\}\left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^i$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{(r+\nu+i)!}{(r+\nu)!}\left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^i$$

On solving this expression, we get the Bayes estimator in case of SELF given in (8.6.5). Similarly, solving for QLF and GELF by putting the values of $\hat{\alpha}^i_{CQB}$ and $\hat{\alpha}^i_{CGB}$ in (8.6.4), we get (8.6.6) and (8.6.7), respectively. Hence, the theorem follows.

Theorem 8.18: Bayes estimator of the cdf for different loss functions under quasi prior

$$\hat{F}_{QSELF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r+i-d} \quad (8.6.8)$$

$$\hat{F}_{QQLF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r-i-d} \quad (8.6.9)$$

$$\hat{F}_{QGELF}(x;\alpha,\tau,\mu) = \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r-d} \quad (8.6.10)$$

Proof: Here, we obtain Bayes estimator of (8.1.1) at a specified point x with the help of Bayes estimator of α^p for quasi prior

$$\hat{F}_{QSELF}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty} \frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i_{QSB}$$

Substituting the value of $\hat{\alpha}^i_{CSB}$ in above equation, we get

$$\begin{aligned}
\hat{F}_{QSELF}(x;\alpha,\tau,\mu) &= \sum_{i=0}^{\infty} \frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!} \frac{\Gamma(r+1-d+i)}{\Gamma(r+1-d)} s_r^{-i} \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left\{\frac{\Gamma(r-d+1+i)}{\Gamma(r-d+1)}\right\} \left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(r-d+i)!}{(r-d)!} \left[\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^i
\end{aligned}$$

On solving this expression, we get the Bayes estimator in case of SELF given in (8.6.8). Similarly, solving for QLF and GELF by putting the values of $\hat{\alpha}^i_{CQB}$ and $\hat{\alpha}^i_{CGB}$ in (8.6.4) we get (8.6.9) and (8.6.10), respectively.

Hence, the theorem follows.

8.7 Bayes estimator of f(x) for different loss functions under various priors

Theorem 8.19: Bayes estimator of the pdf for different loss functions under uniform prior

$$\hat{f}_{USELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+i+1)}{s_r\left\{1-e^{-\tau x^2}\right\}} \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r+i} \tag{8.7.1}$$

$$\hat{f}_{UQLF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r-i+1)}{s_r\left\{1-e^{-\tau x^2}\right\}} \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r-i} \tag{8.7.2}$$

$$\hat{f}_{UGELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+1)}{s_r\left\{1-e^{-\tau x^2}\right\}} \left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r} \tag{8.7.3}$$

Proof: Here, we obtain Bayes estimator of the pdf (8.1.2) at a specified point x with the help of Bayes estimator of α^p and using the Bayes estimate of $F(x;\alpha,\tau,\mu)$ defined in (8.1.1) is

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\hat{F}\left(x;\alpha,\tau,\mu\right)$$

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\left[\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i\right]$$

$$= \sum_{i=0}^{\infty}\frac{i\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{(i-1)!}\frac{2\tau x e^{-\tau x^2}}{\{1-e^{-\tau x^2}\}}\hat{\alpha}^i$$

$$\hat{f}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{i!}\frac{2\tau x e^{-\tau x^2}}{\{1-e^{-\tau x^2}\}}\hat{\alpha}^{i+1} \tag{8.7.4}$$

where,

$$\hat{\alpha}^{i+1} = \frac{\Gamma(r+1+i+1)}{\Gamma(r+1)}s_r^{-(i+1)}$$

which is obtained by putting $p = i+1$ in (8.3.4), (8.3.6) and (8.3.8) for different loss functions in Section 8.3 and on solving (8.7.4) in all three cases, we get Bayes estimator for the pdf under SELF, QLF and GELF, respectively. Hence, the theorem follows.

Theorem 8.20: Bayes estimator of the pdf for different loss functions under NCP

$$\hat{f}_{CSELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+i+\nu)}{(s_r+\mu)\{1-e^{-\tau x^2}\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r+i+\mu-1} \tag{8.7.5}$$

$$\hat{f}_{CQLF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+\nu-i)}{(s_r+\mu)\{1-e^{-\tau x^2}\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r-i+\mu-1} \tag{8.7.6}$$

$$\hat{f}_{CGELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+\nu)}{(s_r+\mu)\{1-e^{-\tau x^2}\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{(s_r+\mu)}\right]^{r+\mu-1} \tag{8.7.7}$$

Proof: Here, we obtain Bayes estimator of (8.1.2) at a specified point x with the help of Bayes estimator of α^p and using the Bayes estimate of $F(x;\alpha,\tau,\mu)$ defined in (8.1.1) is

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\hat{F}\left(x;\alpha,\tau,\mu\right)$$

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\left[\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i\right]$$

$$= \sum_{i=0}^{\infty}\frac{i\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{(i-1)!}\frac{2\tau x e^{-\tau x^2}}{\left\{1-e^{-\tau x^2}\right\}}\hat{\alpha}^i$$

$$\hat{f}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{i!}\frac{2\tau x e^{-\tau x^2}}{\left\{1-e^{-\tau x^2}\right\}}\hat{\alpha}^{i+1} \tag{8.7.8}$$

where,

$$\hat{\alpha}^{i+1} = \frac{\Gamma(r+1+i+1+\nu)}{\Gamma(r+\nu)}(s_r+\mu)^{-(i+1)}$$

which is obtained by substituting $p = i+1$ in (8.3.10), (8.3.12) and (8.3.14) for different loss functions in Section 8.3 and on solving (8.7.4) in all three cases, we get Bayes estimator for the pdf under SELF, QLF and GELF, respectively.

Hence, the theorem follows.

Theorem 8.21: Bayes estimator of the pdf for different loss functions under Quasi Prior

$$\hat{f}_{QSELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+i+1-d)}{s_r\left\{1-e^{-\tau x^2}\right\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r+i-d} \tag{8.7.9}$$

$$\hat{f}_{QQLF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r-i+1-d)}{s_r\left\{1-e^{-\tau x^2}\right\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r-i-d} \tag{8.7.10}$$

$$\hat{f}_{QGELF}(x;\alpha,\tau,\mu) = \frac{2\tau x e^{-\tau x^2}(r+1-d)}{s_r\left\{1-e^{-\tau x^2}\right\}}\left[1+\frac{log\left\{1-e^{-\tau x^2}\right\}}{s_r}\right]^{r-d} \tag{8.7.11}$$

Proof: Here, we obtain Bayes estimator of (8.1.2) at a specified point x with the help of Bayes estimator of α^p and using the Bayes estimate of $F(x;\alpha,\tau,\mu)$ defined in (8.1.1) is

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\hat{F}\left(x;\alpha,\tau,\mu\right)$$

$$\hat{f}(x;\alpha,\tau,\mu) = \frac{d}{dx}\left[\sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^i}{i!}\hat{\alpha}^i\right]$$

$$= \sum_{i=0}^{\infty}\frac{i\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{(i-1)!}\frac{2\tau x e^{-\tau x^2}}{\left\{1-e^{-\tau x^2}\right\}}\hat{\alpha}^i$$

$$\hat{f}(x;\alpha,\tau,\mu) = \sum_{i=0}^{\infty}\frac{\left[log\left\{1-e^{-\tau x^2}\right\}\right]^{i-1}}{i!}\frac{2\tau x e^{-\tau x^2}}{\left\{1-e^{-\tau x^2}\right\}}\hat{\alpha}^{i+1} \qquad (8.7.12)$$

where,

$$\hat{\alpha}^{i+1} = \frac{\Gamma(r+1+i+1-d)}{\Gamma(r+1-d)}s_r^{-(i+1)}$$

which is evaluated by putting $p = i+1$ in (8.3.16), (8.3.18) and (8.3.20) for different loss functions in Section 8.3 and on solving (8.7.4) in all three cases, we get Bayes estimator for the pdf under SELF, QLF and GELF, respectively.

Hence, the theorem follows.

8.8 Bayes estimator of R(t) for different loss functions under various priors

8.8.1 Bayes estimator of R(t) for different loss functions under uniform prior

Let $\hat{R}(t)$ and $\hat{F}(t)$ be the Bayes estimator of $R(t)$ and $F(t)$, respectively. The relation between both is given as

$$\hat{R}(t) = 1 - \hat{F}(t) \qquad (8.8.1)$$

In order to obtain $\hat{R}(t)$ for SELF, QLF and GELF, we substitute the value of $\hat{F}(t)$ for the three loss functions from (8.6.1), (8.6.2) and (8.6.3), respectively.

$$\hat{R}_{USELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r+i} \tag{8.8.2}$$

$$\hat{R}_{UQLF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r-i} \tag{8.8.3}$$

$$\hat{R}_{UGELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r} \tag{8.8.4}$$

8.8.2 Bayes estimator of R(t) for different loss functions under NCP

From (8.8.1), we obtain $\hat{R}(t)$ for SELF, QLF and GELF, we substitute the value of $\hat{F}(t)$ for the three loss functions from (8.6.5), (8.6.6) and (8.6.7), respectively.

$$\hat{R}_{SELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{(s_r + \mu)}\right]^{r+i+\nu-1} \tag{8.8.5}$$

$$\hat{R}_{QLF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{(s_r + \mu)}\right]^{r-i+\nu} \tag{8.8.6}$$

$$\hat{R}_{GELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{(s_r + \mu)}\right]^{r+\nu} \tag{8.8.7}$$

8.8.3 Bayes estimator of R(t) for different loss functions under quasi prior

From (8.8.1), we obtain $\hat{R}(t)$ for SELF, QLF and GELF, we substitute the value of $\hat{F}(t)$ for the three loss functions from (8.6.8), (8.6.9) and (8.6.10), respectively.

$$\hat{R}_{SELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r+i-d} \tag{8.8.8}$$

$$\hat{R}_{QLF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r-i-d} \tag{8.8.9}$$

$$\hat{R}_{GELF}(t) = 1 - \left[1 + \frac{log\left\{1 - e^{-\tau x^2}\right\}}{s_r}\right]^{r-d} \tag{8.8.10}$$

8.9 Bayes estimator of $R = P(Y > X)$ for different loss functions under various prior

8.9.1 Bayes estimator of R for different loss functions under uniform prior

The expression for the Bayes estimator of $R = P(Y > X)$ is as follows:

$$\hat{R} = \int_{y=0}^{\infty}\int_{x=0}^{\infty} \hat{f}(x;\alpha_1,\tau_1,\mu_1)\hat{f}(y;\alpha_2,\tau_2,\mu_2)dy$$

$$\hat{R} = \int_{y=0}^{\infty} \hat{R}(y;\alpha_1,\tau_1,\mu_1)\hat{f}(y;\alpha_2,\tau_2,\mu_2)dy \tag{8.9.1}$$

In case of uniform prior, by using the Bayes estimator of $R(t)$ and pdf for SELF, QLF and GELF and on evaluating, we get the results for $\hat{R}_{USELF}$, $\hat{R}_{UQLF}$ and $\hat{R}_{UGELF}$, respectively in the following expressions

$$\hat{R}_{USELF} = 1 - \int_{y=0}^{\infty} \frac{2(s+2)\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r+1} \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s+1} dy \tag{8.9.2}$$

$$\hat{R}_{UQLF} = 1 - \int_{y=0}^{\infty} \frac{2s\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r-1} \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s-1} dy \tag{8.9.3}$$

$$\hat{R}_{UGELF} = 1 - \int_{y=0}^{\infty} \frac{2(s+1)\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r} \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s} dy \tag{8.9.4}$$

8.9.2 Bayes estimator of $R = P(Y > X)$ for different loss functions under NCP

Using (8.9.1), in case of NCP prior, by using the Bayes estimator of $R(t)$ and pdf for SELF, QLF and GELF and further evaluating we get, the results for $\hat{R}_{CSELF}$, $\hat{R}_{CQLF}$ and $\hat{R}_{CGELF}$, respectively in the following expression

$$\begin{aligned}\hat{R}_{CSELF} = 1 - \int_{y=0}^{\infty} & \frac{2(s+1+\nu_2)\tau_2 y e^{-\tau_2 y^2}}{(t_s+\mu_2)\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{(s_r+\mu_1)}\right]^{r+\nu_1} \\ & \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{(t_s+\mu_2)}\right]^{s+\nu_2} dy\end{aligned} \tag{8.9.5}$$

$$\begin{aligned}\hat{R}_{CQLF} = 1 - \int_{y=0}^{\infty} & \frac{2(s+\nu_2-1)\tau_2 y e^{-\tau_2 y^2}}{(t_s+\mu_2)\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{(s_r+\mu_1)}\right]^{r+\nu_1-1} \\ & \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{(t_s+\mu_2)}\right]^{s-d_2-2} dy\end{aligned} \tag{8.9.6}$$

$$\hat{R}_{CGELF} = 1 - \int_{y=0}^{\infty} \frac{2(s+\nu_2)\tau_2 y e^{-\tau_2 y^2}}{(t_s+\mu_2)\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{(s_r+\mu_1)}\right]^{r+\nu_1} \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{(t_s+\mu_2)}\right]^{s+\nu_2-1} dy \tag{8.9.7}$$

8.9.3 Bayes estimator of the $R = P(Y > X)$ for different loss functions under quasi prior

Using (8.9.1), in case of quasi prior, by using the Bayes estimator of $R(t)$ and pdf for SELF, QLF and GELF and further evaluating we get the results for $\hat{R}_{QSELF}$, $\hat{R}_{QQLF}$ and $\hat{R}_{QGELF}$, respectively in the following expression

$$\hat{R}_{QSELF} = 1 - \int_{y=0}^{\infty} \frac{2(s+2-d_2)\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r+1-d_2} \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s+1-d_1} dy \tag{8.9.8}$$

$$\hat{R}_{QQLF} = 1 - \int_{y=0}^{\infty} \frac{2(s-d_2-1)\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r-1-d_1} \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s-1-d_2} dy \tag{8.9.9}$$

$$\hat{R}_{QGELF} = 1 - \int_{y=0}^{\infty} \frac{2(s-d_2)\tau_2 y e^{-\tau_2 y^2}}{t_s\left\{1-e^{-\tau_2 y^2}\right\}} \left[1 + \frac{log\left\{1-e^{-\tau_1 y^2}\right\}}{s_r}\right]^{r-d_1} \times \left[1 + \frac{log\left\{1-e^{-\tau_2 y^2}\right\}}{t_s}\right]^{s-d_2-1} dy \tag{8.9.10}$$

8.10 Simulation Study

In this section, we are interested in comparing the performance of the estimators with the help of the simulation process. The steps used in the simulation process are as follows:

STEP I: For $n = 10$, α=5 and τ=0.5, we generate random numbers presented in Data Set I and get s_r= 3.517841. Using, $\nu = 0.5$ as the value of the posterior parameter. The prior and posterior densities under NCP are plotted in the Figure 8.1.

Data Set I				
2.097107	1.706324	3.112203	2.434314	2.500450
2.237584	1.751606	2.767507	2.429740	1.952407

STEP II: Now, we generate another set of random numbers i.e. Data Set II with sample size n=50, using these values the Bayes estimates are obtained for $R(t)$ in the Table 8.2.

Data Set II				
2.891029	1.978671	2.485286	1.960415	1.568828
1.864591	2.683489	1.708386	1.428110	2.799518
1.201171	2.241747	1.567015	1.775008	2.005853
1.690921	2.393680	1.710244	2.016199	2.428479
2.229687	1.325276	1.986825	3.672582	2.286996
2.982632	2.030895	1.896112	1.744301	1.520201
2.239931	2.611822	1.853334	2.367566	1.723117
2.573123	1.417590	1.211948	1.917024	1.848686
2.104132	3.346246	2.427658	1.772620	1.992676
1.229574	2.960595	2.421354	1.838500	1.269435

STEP III: For the different values of r i.e. r=5, 10, 20, 35, 50, we obtain the values of $\hat{f}(x)$ and the value of $f(x)$ is obtained for n=50. The graphs presenting $\hat{f}(x)$ are displayed in the following Figure 8.2. Each row displays the pdf for different loss functions in case of uniform, NCP and quasi, respectively.

STEP IV: For the different values of r i.e. r=5, 10, 20, 35, 50, we obtain the values of $\hat{F}(x)$ and the value of $F(x)$ at n=50. The graphs of $\hat{F}(x)$ are also displayed in the following Figure 8.3, respectively. Each row displays the cdf for different loss functions in case of uniform, NCP and quasi, respectively.

STEP V: For R_{SELF}, R_{QLF} and R_{GELF}, we generate two samples of sizes r=30 and s=35 given on Data Set III and IV, respectively. In case of uniform prior $\mu_1 = 5$, $\mu_2 = 4$, $\tau_1 = \tau_2 = 1$; for NCP $\mu_1 = 5$, $\mu_2 = 4$, $\tau_1 = \tau_2 = 1$ $\nu_1 = 5$ and $\nu_2 = 3$; in case of quasi prior $\mu_1 = 5$, $\mu_2 = 4$, $\tau_1 = \tau_2 = 1$, $d_1 = 1$ and $d_2 = 2$, respectively. The values of $s_r = 16.88855$ and $t_s = 18.96123$ which are obtained from (8.2.2) are used in all the above cases for getting the value of $\hat{R}$.

Data Set III				
2.625276	1.950587	2.467171	2.455341	1.252427
1.822391	1.102986	2.382754	1.607448	1.754747
1.343295	1.439131	1.488832	2.220681	1.828188
1.373219	2.237508	1.750748	1.509423	2.170858
1.776714	3.082229	2.771269	1.569442	1.666161
2.748354	2.430942	2.088908	2.481159	1.550883

Data Set IV				
2.178046	2.671675	1.857314	2.570315	1.971740
2.048440	1.562137	0.657608	1.848366	2.104324
1.991163	2.948588	2.073798	1.021837	1.358501
1.953237	2.104454	1.731697	1.761853	1.863773
2.189392	1.698630	3.121350	2.199901	2.298727
1.868388	1.265720	2.290040	1.442819	2.174263
2.194839	2.146829	2.069397	3.436780	2.212066

8.11 Conclusion

1. The complete sample case can be easily done by substituting $r = n$ and $s = m$.

2. From Table 8.1, we can conclude that among NCP gives the minimum values of the posterior risk and risk. Among the loss functions, GELF provides the appropriate values.

3. Figure 8.1, shows that the posterior distribution has its significant peak and covers more than 50 percent of the prior distribution.

4. Table 8.2, concludes that as the value of t increases, the values of $R(t)$ and $\hat{R}(t)$ i.e $\hat{R}_{SELF}(t)$, $\hat{R}_{QLF}(t)$ and $\hat{R}_{GELF}(t)$ decreases. The Bayes estimator of $\hat{R}_{QLF}(t)$ are less

than the corresponding estimates of SELF and GELF ∀ values of t. $R(t)$ has maximum values in all cases.

5. Figure 8.2, shows that the values $f(x;\alpha,\tau,\mu)$ and $\hat{f}(x;\alpha,\tau,\mu)$ for different loss functions under different priors. The figures indicate that in each of the cases as the value of r increases, the curve $\hat{f}(x;\alpha,\tau,\mu)$ shifts towards $f(x;\alpha,\tau,\mu)$ and in some cases even overlaps it.

6. Similarly, Figure 8.3, shows that the values $F(x;\alpha,\tau,\mu)$ and $\hat{F}(x;\alpha,\tau,\mu)$ for different loss functions under differenr priors. The figures indicate that in each of the cases as the value of r increases, the curve $\hat{F}(x;\alpha,\tau,\mu)$ shifts towards $F(x;\alpha,\tau,\mu)$ and in some cases even overlaps it.

7. Table 8.3 indicates the values of P for different cases and we observe a slight change in the simulated values.

8.12 Tables and Graphs

Table 8.1:Bayes estimator, Posterior risk and Risk for G2PRD

Prior	Loss function	Bayes estimate		Posterior risk		Risk	
		Positive	Negative	Positive	Negative	Positive	Negative
Uniform	SELF	3.12692	0.35178	0.88887	0.01375	0.23611	0.10000
	QLF	2.55838	0.29315	0.10000	0.08333	0.12500	0.09722
	GELF	2.84265	0.31980	0.04985	0.04545	0.06027	0.05523
NCP	SELF	1.87841	0.56785	0.22053	1.67112	0.11493	0.04889
	QLF	1.64361	0.50105	0.06667	0.05882	0.05952	0.06574
	GELF	1.76101	0.53236	0.03328	0.03125	0.22048	0.33588
Quasi	SELF	2.78581	0.39975	0.79191	0.02048	0.15611	0.14772
	QLF	2.21729	0.32573	0.11363	0.09259	0.11167	0.09122
	GELF	2.50153	0.35896	0.05661	0.05102	0.05477	0.05103

Table 8.2:Values of R(t) for different loss functions				
Uniform Prior				
t	R(t)	$\hat{R}_{SELF}(t)$	$\hat{R}_{QLF}(t)$	$\hat{R}_{GELF}(t)$
0.5	0.9597214	0.9179981	0.8707856	0.8970639
0.7	0.8987078	0.8214576	0.7557761	0.7911835
0.9	0.8078184	0.7027665	0.6294062	0.6681071
1.1	0.6941722	0.5758821	0.5043009	0.5414851
0.3	0.5691581	0.4526476	0.3892622	0.4218229
0.5	0.4450026	0.3415881	0.2896081	0.3160917
0.7	0.3318769	0.2476315	0.2076852	0.2279166
1.9	0.2362704	0.1725051	0.2076853	0.1581374
2.1	0.1607302	0.1154908	0.2076854	0.1055675
2.3	0.1045944	0.0743111	0.2076855	0.0677901
2.5	0.0651761	0.0459522	0.2076856	0.0418635
NCP				
t	R(t)	$\hat{R}_{SELF}(t)$	$\hat{R}_{QLF}(t)$	$\hat{R}_{GELF}(t)$
0.5	0.9597214	0.9695867	0.9626496	0.9695867
0.7	0.8987078	0.9105883	0.8969431	0.9105883
0.9	0.8078184	0.8180323	0.7988482	0.8180323
1.1	0.6941722	0.7007376	0.6787282	0.7007376
1.3	0.5691581	0.5719647	0.5500572	0.5719647
1.5	0.4450026	0.4450511	0.4254908	0.4450511
1.7	0.3318769	0.3304385	0.3144516	0.3304385
1.9	0.2362704	0.2343671	0.2222448	0.2343671
2.1	0.1607302	0.1589692	0.1503604	0.1589692
2.3	0.1045944	0.1032261	0.0974601	0.1032261
2.5	0.0651761	0.0642271	0.0605658	0.0642271
Quasi Prior				
t	R(t)	$\hat{R}_{SELF}(t)$	$\hat{R}_{QLF}(t)$	$\hat{R}_{GELF}(t)$
0.5	0.9597214	0.7444119	0.5972578	0.6791634
0.7	0.8987078	0.6092861	0.4655514	0.5430355
0.9	0.8078184	0.4840605	0.3567213	0.4238986
1.1	0.6941722	0.3736572	0.2679489	0.3228627
1.3	0.5691581	0.2801591	0.1967988	0.2396204
1.5	0.4450026	0.2038431	0.1409882	0.1730125
1.7	0.3318769	0.1437541	0.0982925	0.1213173
1.9	0.2362704	0.0981283	0.0665383	0.0824692
2.1	0.1607302	0.0647482	0.0436452	0.0542555
2.3	0.1045944	0.0412437	0.0276883	0.0344898
2.5	0.0651761	0.0253326	0.0169605	0.0211555

Table 8.3: Values R for different loss functions			
Prior	Uniform	NCP	Quasi
P_{SELF}	0.4851455	0.4850759	0.4960493
P_{QLF}	0.4823958	0.4909935	0.4908575
P_{GELF}	0.4838125	0.4915684	0.4971129

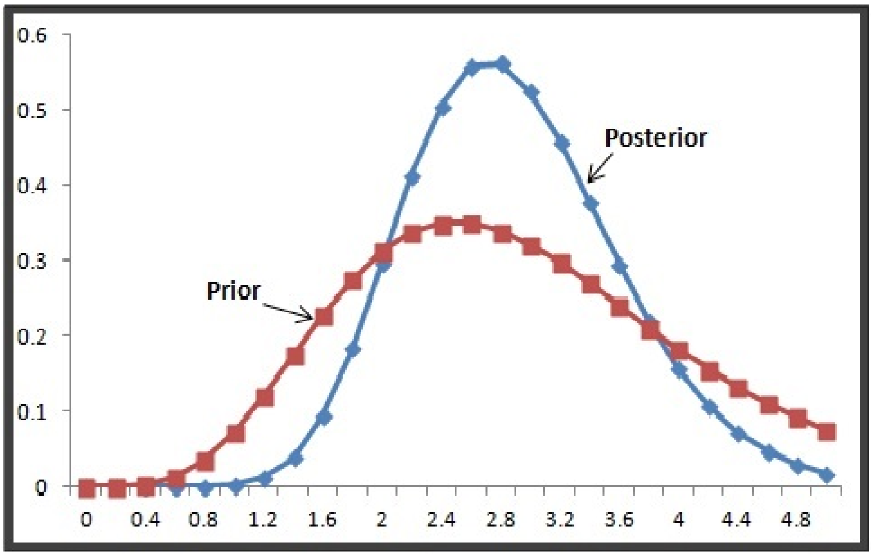


Figure 8.1 Plot of Prior and Posterior densities

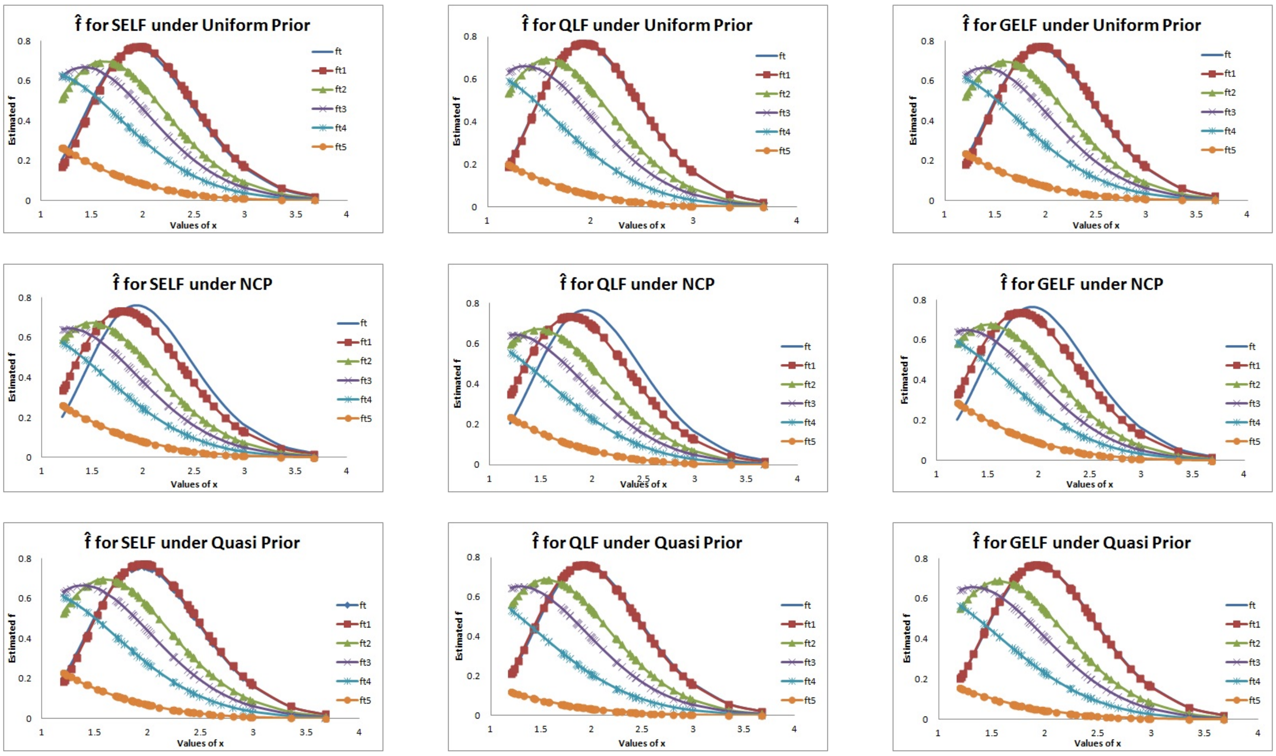


Figure 8.2 Plots showcasing $f(x)$ and $\hat{f}(x)$ for different loss functions under various prior.

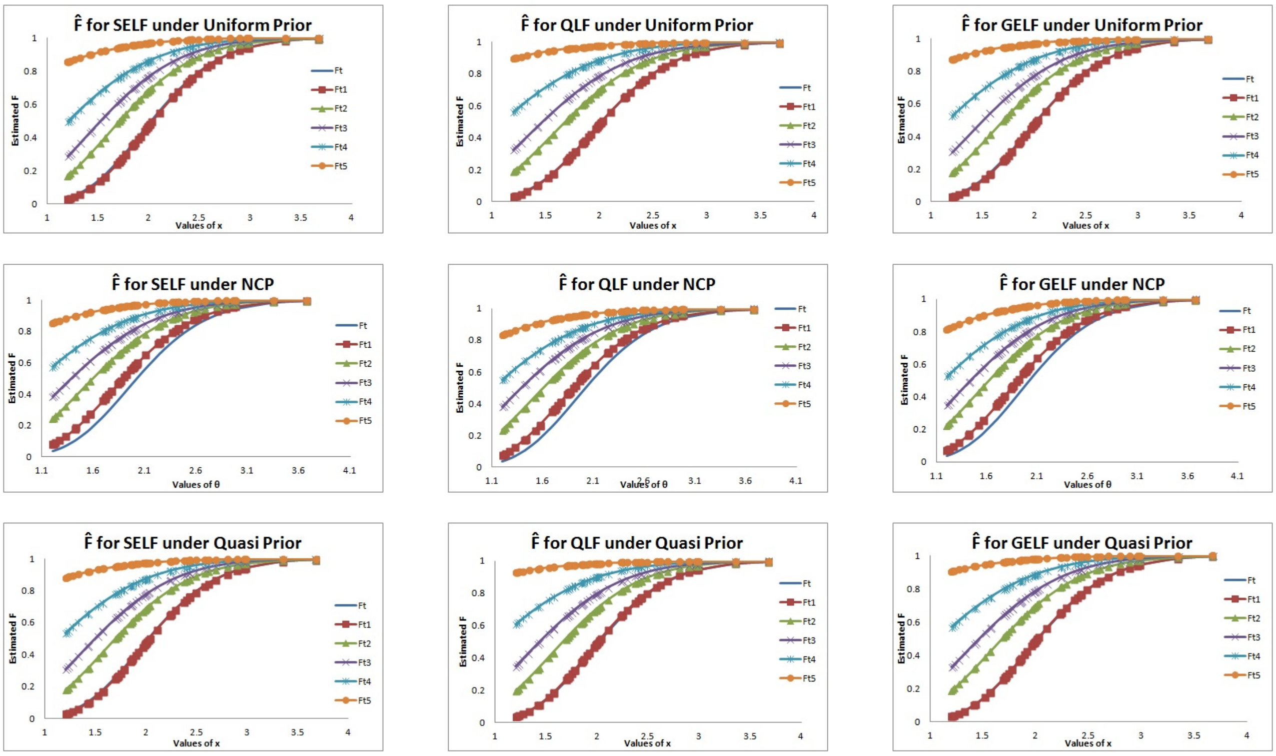


Figure 8.3 Plots showcasing $F(x)$ and $\hat{F}(x)$ for different loss functions under various prior.

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Annexure-I

URKUND

Urkund Analysis Result

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