ON SOME ASPECTS OF ORDERED RANDOM VARIABLES

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ABSTRACT

Order statistics:

If we draw a random sample $X_1, X_2, ..., X_n$ of size n from a continuous population with density function *(pdf)* $f(x)$ and distribution function *(cdf)* $F(x)$ and arrange sample observations in an ascending order of their magnitude i.e. $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ such that

$X_{1:n} = \min(X_1, X_2, ..., X_n)$, is the smallest observation

and

$X_{n:n} = \max(X_1, X_2, ..., X_n)$, is the largest observation

Then the r^{th} observation $X_{r:n}$ $(1 \leq r \leq n)$ is called the r^{th} order statistics.

Order statistics is a specialized branch of statistics which deals with the theory and practice of ordered random variables. Though, Wilks (1942) was the first to introduce the term "Order Statistics" but this section of statistics is considerably much older as it was used by the astronomers, who were interested in estimation of location beyond sample mean. Also, it was Laplace (1818) who obtained the first derivation of distribution of r^{th} order statistics. He also derived a condition for the parent density, where he proved median to be more efficient (asymptotically) than mean. The up growing theories and techniques involving order statistics and their applications have been recognized, deserving its study separately.

Many authors have made remarkable contribution in the area of order statistics. Wilks (1948) compiled the procedures and theories propounded by him

under the title of "Order Statistics". Sarhan and Greenberg (1962) presented a volume containing their contribution towards the development of theory of order statistics along with many tables of great statistical importance. Galambos (1978) presented his work as an asymptotic theory of extreme order statistics. Apart from that, articles by many authors on various models of ordered random variables and their applications have appeared in last few decades. Castillo (1988) made an excellent application of extreme value theory in the problems related to engineering. Recurrence relations between moments of order statistics and between product moment of order statistics is given by Khan *et al.* (1983a) and Khan *et al.* (1983b) respectively. On the basis of complete and censored samples, various estimation procedures are well explained by Balakrishnan and Cohen (1991). Balakrishnan and Sandhu (1996) used general progressive type II censored samples to BLUE and MLE for exponential distribution. Khan and Athar (2000) obtained ratio and inverse moments of order statistics from doubly truncated Weibull distribution. Bounds and approximation for moments of order statistics, linear functions of order statistics, moving order statistics, characterizations, concomitant of order statistics, etc. are the topics which are very well compiled as a complete text by David and Nagaraja (2003). Sengupta *et al.* (2009) used time censored sample data to calculate the mean life and reliability for exponential life distribution. Singh *et al.* (2011) used stress-strength reliability model based on order statistics to obtain an inferential analysis with the real data. Khan and Khan (2012) worked on ratio and inverse moment of generalized order statistics from Burr distribution.

Order statistics also finds application in measurement of extremity of floods, droughts, ocean waves and ocean tides; in measurement of longevity of devices and components. It is also used for signal and image processing and radar target detection.

Concomitants of order statistics:

Let us consider a continuous bivariate population $\left(X, Y\right)$ with joint distribution function $F\left(X, Y\right)$ and joint density as $f\left(x, y\right)$. Let $\left(X_1, Y_1\right), ..., \left(X_n, Y_n\right)$ be an *iid* sample of size n from this population. If we arrange variable X in ascending order of their magnitude as $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ then the variable Y corresponding to r^{th} order statistics of variable X i.e. $X_{r:n}$ is known as concomitant of r^{th} order statistics. It is denoted as $Y_{[r:n]}$.

The evidences of the evolution of the theory of concomitants of order statistics are found about four decades ago, when David (1973) paved the way for the authors to make a noteworthy contribution towards the development of the theory. A year later, Bhattacharya (1974) independently defined the theory of concomitants of order statistics in the name of Induced Order Statistics, but the name concomitants of order statistics remained more common in general practice. In the series of development, Yang (1977) explained the general distribution theory of the concomitants of order statistics.

Balasubramanian and Beg (1996) and (1997), obtained expressions of concomitant of order statistics in bivariate exponential distribution of Marshall and Olkin and concomitant of order statistics in Morgenstern type bivariate exponential distribution. Concomitants of order statistics in Gumble's bivariate exponential distribution are discussed by Balasubramanian and Beg (1998). In this sequence of study, Begum and Khan (2000) and Begum (2003) obtained the expressions of concomitant of order statistics from Marshall and Olkin's bivariate Weibull distribution and from bivariate Pareto II distribution. Aleem (2006) obtained the expression for concomitant of order statistics from bivariate inverse Rayleigh distribution. In a similar manner, Shahbaz and Ahmad (2009) and Shahbaz *et al.* (2009) worked on concomitant of order statistics for bivariate Pseudo-Weibull distribution and bivariate Pseudo exponential distribution. Das *et al.* (2012) made a comparative study on concomitant of order statistics and record statistics for weighted inverse Gaussian distribution. Balakrishnan and Stepanov (2015) gave the limit results for concomitants of order statistics.

Concomitants of order statistics finds successful application in several real life situations where individuals or items are selected or chosen not only on the basis of their primary characteristic denoted by random variable X but another characteristic, denoted by random variable Y, associated with X is also taken into consideration.

Ranked-set sampling introduced by Mac Intyre (1952) is another important application of concomitants of order statistics. This sampling scheme is very effective and widely applicable because of its observational economy consideration. Chen *et al.* (2004) made a detailed study of application of ranked set sampling for a population involving joint distribution of an auxiliary variable X and a variable of primary interest Y.

Articles on the application of concomitants of order statistics in estimation procedures and in the problems of testing of hypothesis are presented by Chacko (2007), Chacko and Thomas (2004, 2006, 2008) and Chacko and Thomas (2011). Their application in characterizing family of bivariate distributions are well explained by Veena (2008) and Thomas and Veena (2011). For a recent comprehensive review of these applications, references can be made from Sections 9.8 and 11.7 of David and Nagaraja (2003).

The thesis entitled **On Some Aspects of Ordered Random Variables** embodies a unified approach to the study of single and joint density function *(pdf)* of the concomitants of order statistics, their single moments and moment generating function from some bivariate continuous distribution.

The work contained in this thesis is spread over in five chapters and a comprehensive list of references relevant to the study has been given at the end.

Chapter 1 is expository in nature in which we intend to define and explain various concepts and results concerning order statistics and concomitants of order statistics, which are helpful in understanding and appreciating the ideas presented in subsequent chapters.

In Chapter 2, the distribution of concomitant of order statistics has been obtained from bivariate log-exponentiated Kumarswamy distribution, proposed by Elsherpieny *et al*. (2014). The distribution of the r^{th} order statistics and its concomitant arising from bivariate log-exponentiated Kumarswamy distribution has been deduced. The properties of concomitant arising from the corresponding order statistics are used to derive the results. The moment generating function (*mgf*) of concomitant of order statistics is also derived. Further, we obtained the expression for joint distribution of concomitants of two non-adjacent order statistics for bivariate log-exponentiated Kumarswamy distribution.

In Chapter 3, the distribution of concomitant of order statistics and joint distribution of two concomitant of order statistics has been obtained for four parameter weighted Marshall-Olkin bivariate exponential distribution, proposed by Jamalizadeh and Kundu (2013). Further, explicit expression for single moment of concomitant of order statistics and explicit expression for moment generating function of concomitant of order statistics, for the above said distribution has been obtained.

In Chapter 4, the expression for *pdf* of r^{th} order statistics and *pdf* of concomitant arising from r^{th} order statistics has been derived for four parameter new bivariate Gompertz distribution, proposed by Elsherpieny *et al*. (2013). Further, exact expression for moment generating function (*mgf*) of concomitant of r^{th} order statistics is derived and mean of concomitant arising from r^{th} order statistics is computed using the *mgf* of concomitant of r^{th} order statistics. Also, the exact expression for joint density of concomitant of two non- adjacent order statistics is obtained.

In Chapter 5, expression for the *pdf* of r^{th} order statistics and *pdf* of concomitant arising from r^{th} order statistics has been obtained for a new bivariate generalized Gompertz distribution, proposed by Elsherpieny *et al*. (2013). Assuming parameters to be natural numbers, the expression for moment generating function *(mgf)* of concomitant arising from r^{th} order statistics is obtained using which mean of concomitant arising from r^{th} order statistics is computed, as a particular case for the above said distribution. Further, expression for the joint density of concomitant of two order statistics at various mutually exclusive conditions are obtained as a part of complete joint density of concomitant of two order statistics for the given distribution.

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