INFERENCES ON VARIOUS LIFE TESTING MODELS USED IN RELIABILITY THEORY

THESIS

SUBMITTED TO

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Dedicated to

MY Beloved Parents

DECLARATION

I, **Mayank Vaish**, Enrolment No. 402/11, hereby declare that the work which is being presented in the thesis entitled **"Inferences on Various Life Testing Models Used in Reliability Theory"** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the department of Applied Statistics of the Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow is an authentic record of my own work carried out during a period from August, 2013 to August, 2017 under the supervision of Dr. Surinder Kumar, Associate Professor & Head, Department of Applied Statistics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma of this or any other Institute.

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CERTIFICATE

This is to certify that the thesis titled **"Inferences on Various Life Testing Models Used in Reliability Theory"** submitted by **Mr. Mayank Vaish** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other university.

The thesis submitted to Babasaheb Bhimrao Ambedkar University, Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations -1999 as amended in 2010* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

Date: Supervisor

Head of the Department

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List of Publications

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List of Tables

List of Figures

Contents

Chapter 1

Introduction

1.1 Introduction

In this modern era of science and technology where the complex electrical and mechanical devices are used for medical, military and scientific purposes, there is a need of experts for handling the complicated and multidisciplinary issues of reliability and risk analysis. Now, it is very difficult to take risk in terms of time, cost, human life and national security. These days weapons or any other devices consist of thousand of small components and the failure of any one of them, fails the system completely.

Failure is an unavoidable phenomenon in all technological products and systems. From the scientific and engineering point of view, the investigation of the uncertain and 'obscure' domain of failure entails the exploration of the functional and physical limit of the systems, in an effort to understand how, why and when a device may not function properly. In this respect the required approach is complementary to the traditional engineering view point which focuses on how and when a machine functions in an optimal way.

Whatever particular failure one is considering, proper control and management of it become essential. Areas of application which involve failure-oriented and failure-driven are Reliability, Availability, Maintainability, Safety (RAMS), Quality control, Fault Detection and Identification, Security and others. As such, failure analysis presents a strong connotation of multi-disciplinarily which significantly adds to its inherent difficulty. Hence, these failure oriented disciplines have become more and more important and closely connected so as to require an integrated view. This entails the acquisition of appropriate modelling and analysis tools as complement to the basic and specific engineering knowledge for the technological area of application.

1.1.1 Various Measures of System Performance

1. **Reliability:** The word 'reliability' refers to the ability of a system to perform its stated purpose adequately for a specified period of time under the operational conditions encountered. The problem of increasing reliability of units becomes more important and urgent in connection with complex mechanization and automation of industrial processes in many fields of industry, transport, communications technology etc. The importance of this problem is shown by the fact that insufficient reliability of units engenders great loss in their servicing, partial stoppages of equipment and these may cause accidents with considerable damage to the equipment and even human injuries.

   There is a great variation in the life length of units and it is impossible to predict with complete certain the life length of one particular unit, since X is a random variable. If $F(X)$ is assumed to be the distribution function of the life length X, then the reliability of a new unit corresponding to the mission time of duration 't' is given by

$$\begin{aligned} R(t) &= P(X > t) \\ &= 1 - P(X \leq t) \\ &= 1 - F(t) \end{aligned} \tag{1.1.1}$$

The term Reliability generally focus on Probability, Intended function, Time and Operating conditions. Reliability is always a function of time and it also depends on the environmental conditions which may or may not vary with time. Since, it is a probability so its numerical values lies between 0 and 1.

2. **Availability:** Reliability is closely related to Availability. Pointwise availability is the probability that the system is found to be up at time 't'. It is also known as operational readiness. Mathematically, Pointwise availability is defined as

$$A(t) = P[X(t) = 1] \tag{1.1.2}$$

where, $X(t)$ may take the binary values 1 and 0 respectively for the operation and non-operation of the system.

If in a long run, system operates satisfactorily then its probability is termed as steady state availability and symbolically is defined as

$$A(\infty) = \lim_{t\to\infty} A(t) \tag{1.1.3}$$

3. **Interval Reliability:** It is the probability that, at specified time 't', the system is operating and will continue to operate for an interval of length x. Hence, symbolically interval reliability is defined as

$$R(t, x) = P[X(u) = 1, \text{ for all u}, \ t \leq u \leq t + x]. \tag{1.1.4}$$

The interval reliability $R(t, x)$ is simply called Reliability $R(x)$ when $T = 0$ and further it becomes pointwise availability at time T as $x \to 0$.

4. **Mean time to system failure (MTSF):** The time for which a component is expected to perform its intended task successfully is called its Mean-time-to-system-

failure. It is also called expected life of the system.

$$MTSF = E(T) = \int_0^\infty tf(t)dt \tag{1.1.5}$$

It can be seen that:

$$\begin{aligned} E(T) &= \int_0^\infty R(t)dt \\ &= \int_0^\infty [1 - F(t)]dt \end{aligned}$$

where $f(.)$, $F(.)$ and $R(t)$ are the p.d.f., c.d.f. and reliability function of the random variable T at time t and

$$\lim_{t\to\infty} t\bar{F}(t) = 0 \text{ if } E(T) \text{ exists.}$$

5. **Mean-sojourn-time-in a state:** The expected time taken by the component or system in a particular state before transiting to any other state is known as mean-sojourn-time or mean survival time in that state. If T be the sojourn time in any state then mean-sojourn-time in any state S is given by

$$\mu_i = \int_0^\infty P(T > t)dt \tag{1.1.6}$$

1.2 Stress-Strength Models

The history of the reliability field may be traced back to the early 1930s, when probability principles were applied to electric power generation-related problems in the United States. During World War II, Germany applied the basic reliability concepts to improve reliability of their V1 and V2 rockets. Also during World War II, the United States Department of Defence recognized the need for reliability improvement of its equipment. During the period between 1945-1950, it performed various studies concerning the failure of electronic equipment, equipment maintenance and repair cost, etc. The US Department of Defence established an ad hoc committee on reliability, in 1950.

The term 'stress' has acquired in the second half of the 20^{th} century. The term stress now a days is used in two different meanings: (1) Structural and mechanical stress studied in engineering called the "strength of materials", and more recent concept (2) Psychological stress which is usually defined as any external stimulus such as from threatning words to the sound of gunshot which means anything that is treated as dangerous for the brain. The stress-strength relationship now a days is studied in many branch of science such as psychology, medicine, pedalogy, etc. and the pharmaceutical industry collects billion-dollar profits for assisting us to overcome psychological stresses.

The term 'Stress-Strength' in its simplest term can be described as an assessment of 'reliability' of a 'component' in terms of random variable X representing 'stress' experienced by the component and Y representing the 'strength' of the component available to overcome the stress. If the stress exceeds the strength $(X > Y)$ the component would fail and vice versa. Reliability is then defined as the probability of not failing: $R = P(X < Y)$. Moreover, in a clinical study X is the response of a control group, Y the response of the treatment group and R measures the effectiveness of the treatment. Birnbaum (1956) was perhaps one of the first researchers who dealt with the model $P(X < Y)$ in stress-strength content. Later on developed by Birnbaum and McCarty (1958). His Ideas are in resonance with the observations by Bilikam (1985) which appeared in engineering literature. R. A. Johnson (1988) in his survey paper in *Handbook of Statistics*, Vol. 7, interprets $R = P(X < Y)$ as the probability that a unit in operating environment performs satisfactory when as usual X is the stress placed on the unit, specially X is taken to be the maximum value attained by a 'critical stress'. Church and Harris (1970) used the term stress-strength for the first time.

1.2.1 Mathematical Formulations

The quantity $P(X < Y)$ is one of the simplest expression for the stress-strength reliability. However, this is not the only quantity of interest for variety of practical situations. The construction and operation of complex devices leads to the estimation of system reliability. This happens when the device under consideration is a combination of number of independent components, say k, with strength $Y_1, \ldots, Y_k$ and each component of a device is subject to common shock or stress of a magnitude X. Parallel systems and series systems are the most popular types of models used for such type of systems which are expressed as $P(X < max(Y_1, \ldots, Y_k))$ and $P(X < min(Y_1, \ldots, Y_k))$ respectively. There are more diverse system reliabilities like as, if the system functions properly when at least $s, 1 \leq s \leq k$, components survive the shock, or if it consist of a number of independent subsystems, say m, performing different tasks. Another important quantity of interest is $P(X_1 < X_2 < \ldots < X_k)$ which appears in Isotonic regression problems to obtain the level probabilities. Another interesting particular case is the estimation of $P(X < Y < Z)$ which represents the situation where the strength Y should not only greator than stress X but also be smaller than stress Z. For example, many devices cannot function at high temperatures, neither at very low ones. Similarly, Person's blood pressure should lie within the limits- systolic and diastolic.

In some applications, the data may consist of two independent random vector and one may be intersted in estimating the probability that the linear combination $A'\underline{X} + B'\underline{Y}$ exceeds certain level i.e. $P(A'\underline{X} + B'\underline{Y} + C > 0)$ where A and B are vectors and C is scalar.

1.2.2 Brief History

Wolfe and Hogg (1971) provide a road map to the research by stating that the numerical values of $P(X < Y)$ are of much importance to the practitioners and give better interpretation than the equivalent statement about $\dfrac{\mu_2 - \mu_1}{\sigma}$ (under normal assumptions) and point out that the quantity $P(X < Y)$ can be estimated under many distributional assumptions. This resulted in number of papers starting from Church and Harris (1970) to the beginning of 21^{st} century.

In 20^{th} century, first attempt made to analyse the reliability of a component by applying probabilistic model of failure was initiated in term of inference theory by Mazumdar (1970) in engineering literature. According to this theory, a component fails if the load (stress) exceeds the component strength (resistance). The value of mechanical stress can be calculated at different point of time for given set of initial value. Church and Harris (1970) give the example of a missile flight where initial values of stress corresponds to propulsive force i.e. moved forward force, angle of elevation, atmospheric conditions etc. P.S. Laplace (1812) in his book "Theorie Analytique des Probabilities" states that given the initial condition with some relevant data, one can predict the location of moving particle at given time. Laplace believed that the curve described by a simple molecule of air or any gas can be circulated as the planetary orbits. Laplace claimed that less rigid approach is to use random variables when the initial conditions are random quantities. This amounts stress to be a random variable based on priori consideration and can only be estimated by statistical methods. Bilikam (1985) described in his paper 'Some Stochastic Stress-Strength Processes', the strength is conditioned on stress because realization of strength is found when stress is applied. He also extended his work related to the relationship between the both stress and strength for continuous random processes.

It may be interesting to note that, the stress-strength model initiated not only in parametric but also in non-parametric set up in the path breaking work of Wilcoxon (1945), Mann and Whitney (1947). The main objective of these researchers were to compare two random variables X and Y which describe results of two treatments. Wilcoxon, Mann and Whitney introduced the statistic which is based on ranks of the observations on X and Y in the joint sample. They also state the connection between the hypothesis $F_X = F_Y$ and $P(X < Y) = 1/2$. The initial efforts lead to the series of research papers which studies the point and interval estimation of $P(X < Y)$ in sixties of the last century. We may refer to Birnbaum (1956), Birnbaum and McCarty (1958), Govindarajulu (1967, 1968), Owen et al. (1964), Sen (1960, 1967), Van Dantzig (1951) and Zaremba (1965). Non-parametric methods have no assumptions related to stress and strength variables, so it may be ineffective for practical purpose. The first attempt to study $P(X < Y)$ under certain parametric assumption on X and Y was undertaken by Owen et al. (1964), who constructed the confidence limits for $P(X < Y)$ when X and Y are dependent and independent normally distributed random variables. Other authors that have worked on $P(X < Y)$ for differnt life time distributions are Kelly et al. (1976), Tong (1976), Downton (1973), Woodward and Kelly (1977) and Beg and Singh (1979).

The estimators of $P(X < Y)$ were obtained in majority for common lifetime distributions during 90^{th} century. For brief review, we may refer to Awad and Gharraf (1986), Beg (1979), Constantine et al. (1986), Ismail et al. (1986), Iwase (1987), Reiser and Guttman (1986), Voinov (1984), Ury and Wiggins (1979). At the same time, efforts were made for solving more realistic problems. The introduction of variety of bivariate exponential distributions by Gumbel (1960), Fruend (1961), Marshall and Olkin (1967), Block and Basu (1974) made it possible to study dependent exponential variables with variety of dependence. Raghava Char et al. (1984) studied stress and strength markov model for system reliability. Some other advancements during that period were the extensions of standard stress strength models with categorized data by Brownie (1988), Halperin et al. (1989) and Simonoff et al. (1986).

In this modern era of science and technology, research related to stress-strength models is conducted all over the world and is published in reputed journals like Journal of the American Statistical Association, Canadian Journal of Statistics, Chinese Journal of Statistics, Journal of Korean Statistical Society, Journal of Mathematical Sciences and Journal of Indian Association of Productivity, Quality Control and Reliability, etc. However, most of the results were obtained by American, Russian, Canadian and Indian scientists. Russian school mainly contributed where atmost all foreign publications are available. They worked and developed complete isolation from western and eastern colleagues. Here, useful techniques in estimation theory are developed in late sixties to early eighties. We may refer to Lumelskii (1969), Lumelskii and Sapoznikov (1969), Lumelskii and Pensky (1982). It should be noted that their derivations of the best linear unbiased estimators is based on the general technique.

1.2.3 Applications

The problem of stress-strength models initially originated from a seemingly unrelated problems of classical non-parametric tests of equality of two distribution functions. It then naturally lead to the expression of type $P(X < Y)$ and this resulted in applications in numerous engineering problems under the banner of 'reliability' provided that the random variables under consideration admit appropriate interpretation. The most prominent examples of applications of $P(X < Y)$ relationship in engineering and medicine are presented in Johnson's (1988) survey article. The practical applications of stress-strength problems not only confined to engineering or to military problems but also in medical fields due to the advancement in the medical statistics in the last twenty years. There originate number of medical-oriented problems of which clinical trials are one of the fast growing areas. Next use of stress-strength models are in psychology which required the adjustment of categorical data. It also involve in comparision of two or more random variables representing the state of affairs in two or more situations at different time intervals.

The new area of application is the real-world problems where the model cannot be viewed as consisting involving independent identically distributed random variables and is more appropriately represented by a binary data leading to so called 'ROC approach' with a strong dose of logistic regression. Another challenging recent application is the problem of estimating the strength characteristics from the observable distribution of stress which leads to more interesting probabilistic and statistical theoretical problems. Another possible application is the relation between stress-strength models and the quality control concepts, specifically the so called process capability indices originated in the quality control literature some twenty years ago. It should be noted that as the sources of numerical data are becoming more widely available and statistical calculation becoming more accessible due to rapid advances in computer technology more and more applications are to be expected. The stress-strength model is a powerful tool for comparing and dissecting interrelated situations. The simplicity of the model can be both deceiving and rewarding.

For practical applications of stress-strength models, we may refer to Simon et al. (1986), Oskamp (1962), B.S. Everitt (1977), Morrison (1976), Guttman et al. (1988), Gupta and Gupta (1990), Reiser and Faraggi (1994), Akman et al. (1998), Gupta et al. (1999), Nandi and Aich (1996), weerahandi and Johnson (1992), Halperin et al. (1987, 1989).

1.3 Classical Inferential Procedures Used in Reliability Theory

1.3.1 Maximum Likelihood Estimators

Maximum likelihood estimation (MLE) is one of the most popular and flexible method for the estimation of stress-strength reliability. We can use this method if the joint density of the stress X and strength Y is known with some unknown parameters. Suppose that $(\underline{X}, \underline{Y})$ has a probability density function $f(x, y|\theta)$ with an unknown parameter $\theta \in \Theta$. Let $f(\underline{X}, \underline{Y}; \theta)$ denote the joint pdf of sample observations i.e.

$$f(\underline{X}, \underline{Y}; \theta) = \prod_{i=1}^{n} f(X_i, Y_i|\theta),$$

if X and Y are independent then

$$f(\underline{X}, \underline{Y}; \theta) = \prod_{i=1}^{n_1} f(X_i|\theta) \prod_{j=1}^{n_2} f(Y_j|\theta).$$

Now, in order to derive the MLE of R, we have to follow three steps:

(1) Calculate the stress-strength reliability $R = R(\theta) = P(x < y)$ as a function of θ;

(2) Construct the MLE $\tilde{\theta}$ of parameter θ;

(3) Calculate the MLE $\tilde{R} = R(\tilde{\theta})$ of R.

By definition of stress-strength reliability, $R(\theta)$ can be calculated as

$$R(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y|\theta) I(x < y) dx dy \tag{1.3.1}$$

If X and Y are independent with the pdfs $f_X(x|\theta)$ and $f_Y(y|\theta)$ and the cdfs $F_X(x|\theta)$ and $F_Y(y|\theta)$ respectively, then above equation can be written as

$$R(\theta) = \int_{-\infty}^{\infty} F_X(z|\theta) f_Y(z|\theta) dz = \int_{-\infty}^{\infty} (1 - F_Y(z|\theta)) f_X(z|\theta) dz \tag{1.3.2}$$

After deriving the expression for $R(\theta)$, we now construct the MLE of the unknown parameter θ. Since the logarithmic function is strictly increasing, maximizing $L(\theta|\underline{X}, \underline{Y})$ is equivalent to maximizing the logarithm of the likelihood function $\ln L(\theta|\underline{X}, \underline{Y})$ i.e.

$$\ln L(\tilde{\theta}|\underline{X}, \underline{Y}) = max_{\theta} \ln L(\theta|\underline{X}, \underline{Y}).$$

By invariance property of maximum likelihood estimators, the MLE of R is given by

$$R = R(\tilde{\theta}). \tag{1.3.3}$$

The mean squared error (MSE) of $\tilde{R}$ is given by

$$MSE(\tilde{R}) = E(\tilde{R} - R)^2 = \int \dots \int (\tilde{R} - R(\theta))^2 f(\underline{X}, \underline{Y}|\theta) \prod_{i=1}^{n_1} dX_i \prod_{j=1}^{n_2} dY_j. \tag{1.3.4}$$

1.3.2 Unbiased Estimation

Maximum likelihood estimators are asymptotically unbiased estimators but they may be biased if the sample size is small. In such situation, it is required to obtain an unbiased estimator of $R = P(X < Y)$. For a detailed description of the construction of unbiased estimation and principle of sufficiency, we may refer to Casella and Berger (1990) and Lehmann and Casella (1998). To find the sufficient statistic and UMVUE, we may used the following theorems.

Theorem 1.1: A statistic T is said to be sufficient statistic for $f(x, y|\theta)$ if there exist a function $g(.|\theta)$ and $h(\underline{X}, \underline{Y})$ such that for all the sample points $\underline{X}, \underline{Y}$ and all possible $\theta \in \Theta$ the joint pdf $f(\underline{X}, \underline{Y}|\theta)$ is of the form

$$f(\underline{X}, \underline{Y}|\theta) = g(T(\underline{X}, \underline{Y})|\theta)h(\underline{X}, \underline{Y}). \tag{1.3.5}$$

Since, there are many unbiased estimators of $R = P(X < Y)$ based on sample $(\underline{X}, \underline{Y})$. Our objective is to choose that estimator which has the smallest variance for all values of θ.

Theorem 1.2: If $V(\underline{X}, \underline{Y})$ is any unbiased estimator of $\tau(\theta)$ and T is a sufficient statistic, then $E(V(\underline{X}, \underline{Y})|T)$ is UMVUE of $\tau(\theta)$.

We shall now consider a more general problem of constructing the UMVUE of the joint pdf $f(x_1, \ldots, x_k, y_1, \ldots, yk|\theta)$ of $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ with $k < min(m, n)$. For detailed study, we may refer to Lumelskii and Sapoznikov (1969) or Voinov and Nikulin (1993).

Theorem 1.3: Let $\theta_0 \in \Theta$ be an arbitrary value of θ. Denote by $g(T|\theta_0)$ and $g_{\theta_0}(T|X_1 = x_1, \ldots, X_k = x_k, Y_1 = y_1, \ldots, Y_k = y_k)$ the pdf of $T(\underline{X}, \underline{Y})$ and the conditional density of T for given $X_j = x_j, Y_j = y_j, j = 1, \ldots, k$ respectively, as $\theta = \theta_0$. Then the UMVUE of $f(x_1, \ldots, x_k, y_1, \ldots, y_k|\theta)$ is of the form

$$\hat{f}(x_1, \ldots, x_k, y_1, \ldots, y_k|\theta) = \prod_{j=1}^{k} f(x_j, y_j|\theta_0) \frac{g_{\theta_0}(T|X_1 = x_1, \ldots, X_k = x_k, Y_1 = y_1, \ldots, Y_k = y_k)}{g(T|\theta_0)} \tag{1.3.6}$$

Theorem 1.4: The UMVUE of R^k is of the form

$$\widehat{R^k} = \int \hat{f}(x_1, \ldots, x_k, y_1, \ldots, y_k) \prod_{j=1}^{k} [I(x_j < y_j)dx_j dy_j]. \tag{1.3.7}$$

In particular, the UMVUEs of R and R^2 are, respectively,

$$\hat{R} = \int I(x < y)\hat{f}(x,y)dxdy, \tag{1.3.8}$$

$$\widehat{R^2} = \int\int I(x_1 < y_1)I(x_2 < y_2)\hat{f}(x_1,x_2,y_1,y_2)dx_1dx_2dy_1dy_2 \tag{1.3.9}$$

and, The UMVUE $\widehat{Var}(\hat{R})$ of $Var(\hat{R})$ is given by

$$\widehat{Var}(\hat{R}) = (\hat{R})^2 - \widehat{R^2} \tag{1.3.10}$$

1.3.3 Interval Estimation

In many cases, point estimation is not sufficient, in such cases we use the methods of interval estimation. For example, in a medical application where X and Y represent responses by an old treatment and a new treatment A and B respectively. The aim is to decide whether one should abandon the old treatment in favour of the new one. If the point estimator $\tilde{R}$ of $R = P(X < Y)$ is obtained and is equal to 0.59, we still cannot confidently recommend the course of action since we don't have information on the variability of $\tilde{R}$. What we really need in such situation is an interval estimation which covers the unknown value of R with high probability of at least $1 - \gamma$ where γ is small.

Let the statistics $L(\underline{X}, \underline{Y})$ and $U(\underline{X}, \underline{Y})$ be such that

$$P(L(\underline{X}, \underline{Y}) < R < U(\underline{X}, \underline{Y})) \geq 1 - \gamma, \quad 0 < \gamma < 1 \tag{1.3.11}$$

The interval $(L(\underline{X}, \underline{Y}), U(\underline{X}, \underline{Y}))$ is called the confidence interval for R with the lower and upper bounds $L(\underline{X}, \underline{Y})$ and $U(\underline{X}, \underline{Y})$, respectively, with the confidence coefficient $1 - \gamma$. There exist at least three approaches of constructing the confidence intervals: exact methods, asymptotic methods and Bayesian methods. Exact confidence intervals have been derived for stress-strength reliability R when vector (X, Y) is normally distributed otherwise it considers non parametric case when distribution of (X, Y) is not properly specified. We may refer to Birnbaum and McCarty (1958), Owen et al. (1964), Enis and Geissar (1971), Ury (1972), Yang and Mo (1985), Reiser and Guttman (1986), Constantine et al. (1986), Teskin and Kostyukova (1991) and Pensky and Takashima (2002) and others.

Asymptotic confidence interval are used when it is not possible to obtain the exact confidence intervals. Confidence intrval for R based on normal approximation have been studied by Church and Harris (1970), Nandy and Aich (1994), Gupta et al. and others. Another asymptotic technique which is less popular in estimating R is based on the well known fact that, for any fixed R, the logarithm of the likelihood $-2\ln L(R|\underline{X}, \underline{Y})$ is distributed asymptotically as χ^2 with 1 degree of freedom. This approach has been used by Madansky (1965) and Easterling (1972). The shortcoming of asymptotic confidence interval is that this technique very often run into serious difficulties and provide crude unreliable results when R is close to zero or one and the sample size are relatively small. Moreover, the recent work by Hallin and Seon (1999) indicates that one ought to take asymptotic results with a grain of salt.

1.4 Transformation Methods

Transformation methods are used to find the point and interval estimation of stress-strength reliability. These methods seem to have been overlooked by the statisticians but are very helpful in finding the stress-strength reliability. Let us consider (X, Y) be a random vector with the probability density function $f(x, y|\theta)$. Suppose that there exist a random variable ε and η and a monotone function $u(.)$ with the inverse $v = u^{-1}$ such that

$$X = u(\varepsilon) \leftrightarrow \varepsilon = v(X), \qquad Y = u(\eta) \leftrightarrow \eta = v(Y). \tag{1.4.1}$$

Also, assume that the function u and v are strictly increasing, so that (ε, η) is the random vector with the pdf

$$g(\varepsilon, \eta|\tau) = f\left(u(\varepsilon), u(\eta)|\nu(\tau)\right) u^{'}(\varepsilon) u^{'}(\eta) \tag{1.4.2}$$

where the scalar or vector valued parameter τ is connected to θ by one to one transformation. Since, the function $u(.)$ is monotonically increasing, $P(\varepsilon < \eta) = P(u(\varepsilon) < u(\eta)) = P(X < Y)$. Hence, the model $R = P(X < Y)$ remains invariant in terms of τ and θ.

Here, we now provide some theorems that serves as a basis for transformation of existing estimators of R into new estimators.

Theorem 1.5: Let $\tilde{\zeta}(\underline{\varepsilon}, \underline{\eta})$ be the MLE of R based on observations $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{n_1})$ and $\underline{\eta} = (\eta_1, \ldots, \eta_{n_2})$, where $n_1 = n_2 = n$ whenever ε and η are dependent. Then the MLE $\tilde{R}$ of R based on $\underline{X}$ and $\underline{Y}$ is given by

$$\tilde{R} = \tilde{\zeta}(v(\underline{X}), v(\underline{Y})), \tag{1.4.3}$$

where, $v(\underline{X}) = (v(X_1), \ldots, v(X_{n_1}))$ and, $v(\underline{Y}) = (v(Y_1), \ldots, v(Y_{n_2}))$.

Theorem 1.6: Let $T_{\varepsilon,\eta} = T_{\varepsilon,\eta}(\underline{\varepsilon}, \underline{\eta})$ be a sufficient statistics for τ based on $(\underline{\varepsilon}, \underline{\eta})$ and let there exist an UMVUE $\hat{\zeta}(T_{\varepsilon,\eta})$ of R based on observations $(\underline{\varepsilon}, \underline{\eta})$. Then, $T_{X,Y} = T_{\varepsilon,\eta}(v(\underline{X}), v(\underline{Y}))$ is a sufficient statistic for θ based on the sample $(\underline{X}, \underline{Y})$ and the UMVUE $\hat{R}$ of R based on $\underline{X}$ and $\underline{Y}$ is given by

$$\hat{R} = \hat{\zeta}(T_{X,Y}). \tag{1.4.4}$$

Moreover, if $\hat{\zeta}_1(T_{\varepsilon,\eta})$ is the UMVUE of the variance of the unbiased estimator $\hat{\zeta}(T_{\varepsilon,\eta})$, then the UMVUE of the variance of $\hat{R}$ is of the form

$$\widehat{Var}(\hat{R}) = \hat{\zeta}_1(T_{X,Y}). \tag{1.4.5}$$

Theorem 1.7: Let $\big(L(\underline{\varepsilon}, \underline{\eta}), U(\underline{\varepsilon}, \underline{\eta})\big)$ be the confidence interval for R with confidence coefficient $(1-\gamma)$. Then, $(L(v(\underline{X}), v(\underline{Y})), U(v(\underline{X}), v(\underline{Y})))$ is the confidence interval for R based on $(\underline{X}, \underline{Y})$ with the same coverage probability.

1.5 Sequential Analysis

The problem of sequential analysis arose in the Statistical Research Group, Columbia University in March 1943. The immediate stimulus was questioned about ordinance testing addresed by Captain G. L. Schuyler of the Bureau of Ordinance, Navy Departement. It was pointed out by Milton Friedman and W. Allen Wallis that the mere notion of sequential analysis could slightly improve the efficiency of some current most powerful test. G. L. Schuyler to W. Allen Wallis, who together with Milton Friedman brought the problem to Wald's attention. It was subsequently published by Wald in (1945, 1947).

Mahalanobis (1940) showed the importance of sampling in steps and developed sampling designs for estimating the acreage of jute crop in the whole state of Bengal. This seminal development in large-scale survey sampling of national importance was regarded by many, including Abraham Wald, as the forerunner of sequential analysis. Wald and his collaborators systematically developed theory and methodology of sequential tests in early 1940 to reduce the number of sampling inspections without compromising the reliability of the terminal decisions. This culminates into a true classic of our time (Wald 1947).

Sequential analysis began its march with deep-rooted applies motivations in response to demands for efficient testing of anti-aircraft gunnery and other weapons during the World War II. Those methodologies were especially crucial at that time. Fewer sampling inspections with accurate outcome were essential to gain an advantage in the front line. These developments were "classified" in the early to mid-1940s. Methodological researchers caught on and began applying sequential analysis to solve a broad range of practical problems from inventory, queuing, reliability, life tests, quality control, designs and optimal stopping rules. The following selection of books and monographs include major account: Armitage (1975), Ghosh and Sen (1991), Jennison and Turnbull (1999), Rosenberger and Lachin (2002), and Whitehead (1997). A number of famous books and monographs already exist. We have mentioned Wald (1947) and Mukhopadhyay et al. (2004) before. Additionally, one will find other volumes including Bechhofer et al. (1968), Chernoff (1972), Chow et al. (1971), Ghosh (1970), Ghosh et al. (1968), Ghosh and Sen (1991), Gibbons et al. (1977), Govindarajulu

(1981), Gupta and Panchapakesan (1979), Gut(1988), Mukhopadhyay and Solanky (1994), Schmitz (1972), Sen (1981, 1985), Shiryaev (1978), Siegmund (1985), Wetherill (1975), and Woodroofe (1982). Some references, for example, Gibbons et al. (1977) and Gupta and Panchapakesan (1979) highlight sequential analysis only partly. Govindarajulu (2004) include codes for some selected computer programmes.

Hager et al. (1971) studied the robustness and testing procedures generalized gamma and Weibull model. Barlow and Proschan (1967) studied the robustness and estimation procedures related to exponential distribution. Dantzing (1940) proved the non existence of the fixed sample size procedure to test student's t hypothesis having a pre-assigned power when the variance is unknown. Stein (1945) developed a two stage procedure. Three stage sequential procedures were developed to construct fixed width confidence interval. Starr (1966a) and Woodroofe (1977) developed a fixed width confidence interval for a normal mean. Basu (1971) and Mukhopadhyay (1974) proposed asymptotically efficient and consistent sequential procedures to construct fixed width confidence interval for the location parameter of two parameter exponential distribution. Swanepoel and Van Wyk (1982) obtained the second order approximation of sequential procedure. Star and Woorfe (1972) worked on one parameter exponential distribution and Mukhopadhyay (1974) also considered sequential minimum risk point estimation procedure for location parameter under a family of loss functions and a cost function of general form. Various multistage procedures for the sequential estimation are available in Ghosh and Mukhopadhyay (1990), Hamdy et al. (1988), Isogai and Uno (1990), Mauromoustakos (1987). Some other estimation problems related to exponential distribution were studied by several authors. We mar refer to Mukhopadhyay (1992), Mukhopadhyay and Hamdy (1984), Mukhopadhyay and Narayan (1981), Singh and Chaturvedi (1991), Chaturvedi et al. (1993, 1996, 1997), Govindarajulu and Sarkar (1991) and others.

Sequential procedures were developed by Abraham Wald (1947) in response to demand for more efficient testing of anti-aircraft gunnery during the second world war. Moreover, the subject grown considerably especially in the areas of sequential estimation and biostatistics, Sequential statistics is concerned with the treatment of data when the number of observations is not fixed in advance i.e. sample size is a random variable. The experimenter has the option of studying at a sequence of observations one or a fixed number at a time and decides whether to stop sampling and take a decision or to continue sampling and take decision sometime later. The order of sequence of observations which the experimenter will take is specified in advance.

1.5.1 Significance of the Study

There are numerous situations in which it is worthwhile to monitor the status of an examination and terminate it early if the conclusion seems manifest. In most of the business, experiment cost money both concerning the actual cost of data collection and regarding the opportunity cost of idling for an investigation, to reach a set number of samples before acting on its outcome which may have been apparent much earlier.

Sequential procedures are of interest because they are economical in the sense that we may reach a decision earlier via a sequential procedure than a fixed sample size problem. Sequential life testing and acceptance sampling procedures have been widely used by government and industries. Sequential analysis is employed in several types of psychometric tests. Some of these are computerised adaptive testing, classroom interaction assesment and intervention, pshycological studies involving longitudinal data, depression diagnosis, and crime suspect identification tests.

1.5.2 Methodology

A. Wald (1947) was first, who developed the sequential analysis to test the simple null hypothesis against the simple alternative hypothesis.

A Procedure for making inference about the distribution of one or more variables in which the size of the sample is a random variable is called a Sequential Procedure.

The principal feature of such a pocedure is sampling scheme which lays down a rule under which has two aspects, (1) A stopping rule or a rule which tells us when to stop sampling and, (2) An action rule which tells us what type of inference to make after sampling has been stopped. Type of sequential procedure that has been studied rather thoroughly concerns hypothesis testing and finding Operating Characteristic (OC) function and Average Sample Number (ASN). Sequential Procedures include the test for Sequential Probability Ratio Test (SPRT), test for the composite hypothesis, sequential estimation.

Sequential analysis is a method of statistical inference whose characteristic feature is that the number of observation required by the procedure is not determined in advance of the experiment. In the theory of testing of hypothesis the number of observation, the size of sample on which the test is based, is treated as a constant for any particular problem. The decision to terminate the experiment depends, at each stage, on the results of the observations previously made. A merit of sequential method, as applied to testing statistical hypothesis is that test procedures can be composed which require, on an average, a substantially smaller number of observations than equally reliable test procedures based on a predetermind number of observations.

The Sequential Procedure of testing a hypothesis H_0 may be represented as follows. A rule is given for making one of the following three decision at any stage of experiment. (a) to accept the hypothesis H_0, (b) to reject the hypothesis H_0, (c) to continue the experiment by making a additional observation. Therefore, such a test procedure is carried out sequentially. On the basis of the first observation, one of the aforementioned three decision is made, else the first and second trial are performed. Over, by the first two observations, one of the three decision is made. If the third decision is made a third trial is performed, and so on. The process is extended until either the first or the second decision is made.

1.5.3 Some Classical Problems and New Challenges

Sequential analysis has been developing steadily but at a somewhat uneven pace during the past six decades. Ghosh (1991) dates the rudiments of sequential analysis to the works of Huyghens, Bernoulli, DeMoivre and Laplace on the gambler's ruin problem. In particular, we reveal how these classical problems are connected to other branches of statistics and probability and applications in other field. That modern sequential testing theory can, in fact, accommodate this adaptive feature in sequential experimentation in addition to providing efficient stopping and terminal decision rules. In fact, the methodology developed and also outwards through active involvement in the biomedical, engineering sciences and socio-economic. Integrating its internal and external growth is a new challenge that will provide it with increasing opportunities and visible role in the changing world of science, technology and socio-economic activities.

1.5.4 Applications of Sequential Testing

Applications of Sequential Analysis Clinical Trials

In a randomized trial with two treatment groups, group sequential testing, may for example be conducted in the following manner:

After n cases in each collection, i.e. a total of $2n$ cases are available, an interim analysis is conducted. That means, a statistical test is performed to compare the two groups, if the null hypothesis is rejected, the trial is terminated. Otherwise, the trial continues. Another n subjects per group are recruited. The statistical test is performed again, now including all $4n$ subjects. If the null is rejected, the trial is terminated. Otherwise, it continues with periodic evaluations until a maximum number of interim analyses have been performed. At this point, the last statistical test is conducted, and the trial is discontinued. Hence, the procedure used here is of the sequential kind.

The sequential analysis also has a connection to the problem of gambler's ruin, Huyghens in 1657.

1.5.5 Two Aspects of Sequential Procedure

Generally sequential procedure has two aspects:

(1) **Stopping rule**, which tells us when to stop sampling.

(2) **Action rule**, which tells us what type of inference to make after sampling has been

stopped or what estimate to adopt for a given parametric function or whether to accept or reject a given hypothesis regarding the parameters.

1.5.6 Wald's Sequential Probability Ratio Test (SPRT)

The **sequential probability ratio test** (SPRT) for testing a simple null hypothesis H_0 against a simple alternative H_1 was first proved to be optimal in a certain sense by Wald and Wolfowitz (1948).

Let X_1, X_2, ..., X_n are identically and independently distributed random variables with the common probability mass function or probability density function f_θ. Let us suppose that there are just two values of θ which are of interest to us, say θ_0 and θ_1. Thus, we have two alternative hypotheses $H_0{:}\theta=\theta_0$ and $H_1{:}\theta=\theta_1$. For any positive integer m, the probability or probability density that observations $x_1, \ldots, x_m$ are obtained is given by when H_0 is true

$$f_{0,m} = \Pi_{\mathrm{i}=1}^{m} f_{\theta_0}(x_i)$$

when H_1 is true

$$f_{1,m} = \Pi_{\mathrm{i}=1}^{m} f_{\theta_1}(x_i)$$

The SPRT for testing H_0 against H_1 is defined in terms of ratio $\frac{f_{1,m}}{f_{0,m}}$ as follows: specify two constant A and B such that $0 \leq B \leq 1 \leq A$.

If $\frac{f_{1,m}}{f_{0,m}} \leq B$, the processes is terminated with acceptance of H_0.

If $\frac{f_{1,m}}{f_{0,m}} \geq A$, the processes is terminated with rejection of H_0.

If $B < \frac{f_{1,m}}{f_{0,m}} < A$, the experiment continued by taking the additional observations.

The constants A and B are to be determined so that the test will have the prescribed strength (α, β). A and B will be given by the relation

$$A \approx \frac{1-\beta}{\alpha} \text{ and } B \approx \frac{\beta}{1-\alpha} \tag{1.5.1}$$

where α and β are the Type I and Type II error.

Operating Characteristic Function (OC)

To judge the relative merits of two or more rival sequential procedures, we make use of two criteria, one being the OC function and other is ASN function. By Operating Characteristic function, the test procedure achieves the objective of making correct decision, we mean the probability of the hypothesis H_0 (Null Hypothesis) being accepted when θ is the true value of the parameter, regarded as a function of θ. It is denoted by $L(\theta)$.The expression for OC function $L(\theta)$ is given by

$$L(\theta) = \frac{A^h - 1}{A^h - B^h} \tag{1.5.2}$$

where A and B are defined in (1.5.1).

After a particular sequential test has been adopted, i.e. a particular choice of the sets has been made, the probability that the process will terminate with the acceptance of the hypothesis H_0 under test depends only on the distribution of the random variable under consideration.

It is assumed that the distribution is known except for the values of a finite number of parameters. Thus the distribution is given by a function, where functional form is known, but the true values of the parameter are unknown. We use the letter θ denote the set of all the parameters. Since the parameters θ determine the distribution, the probability of accepting of H_0 will be a function of θ. This function will be denoted by L(θ) and it will be called OC function. If there is only one unknown parameter θ the function L(θ) can be plotted as a curve, θ being measured along the vertical axis. Since, we shall consider only test for which the probability that the procedure will eventually terminate is equal to 1, the probability of rejecting H_0 is equal to $1 - L(\theta)$. The OC function is very closely related to the notion of the power function. For any value of θ which is not consistent with the hypothesis H_0, the power of the test is defined as the probability of rejecting H_0 when θ is the true value. Hence for any θ not consistent with H_0 the power of the test equal to $1 - L(\theta)$.

Average Sample Number Function (ASN)

In the sequential test the number of observations required by a sequential test is not predetermined, but a random variable, because at any stage of the experiment, the decision to terminate the process depends on the results of the observation made so far. We shall denote by N the number of observations required by the sequential test. Then N is the random variable. For any given test procedure the expected value of N depends only on the distribution of x is determined by parameter point θ, the expected value depends on only θ. For any parameter point θ, We shall denote the expected value of N by $E_\theta(N)$. From the point of view of the ASN, we may then say that the smaller the value of $E(N)$ the better is the sequential procedure. If there is only one unknown parameter θ the function $E_\theta(N)$ can be plotted as a curve, θ being measured along the horizontal axis and $E_\theta(N)$ along the vertical axis. The expression for ASN function is given by

$$E(N|\theta) = \frac{L(\theta)\log B + (1 - L(\theta))\log A}{E(Z)} \qquad \text{Provided } E(Z) \neq 0. \tag{1.5.3}$$

With every sequential test procedure there associates an OC function and ASN function. The former describes how well the procedure achieves its objective of making correct decisions, while the latter represents the price one has to pay to reach a decision, in terms of the number of observations required by the test.

1.5.7 Sequential Test in Reliability Analysis

Sequential life testing and acceptance sampling procedures are used by goverments and industries in connection with quality control and procurement activities through the whole word. For testing simple null hypothesis against simple alternative hypothesis. A. Wald (1947) developed the well known sequential probability ratio test (SPRT). Epstein and Sobel (1955) considered sequential life test in exponential case to test the simple hypothesis against a simple alternative, derived formulae for OC and ASN function. Epstein (1960), Woodall and Kurkjian (1962), Aroian (1979) also done the similar type of work. The robustness of the SPRT, when the distribution under consideration has undergone a change, has been studied by various researchers. Harter and Moore (1976) conducted Monto Carlo study to investigate the consequences of using the exponential SPRT procedures when the distribution is a Weibull with shape other than one. By considering the class of distributions having an increasing (decreasing) failure rate Montagne and Singpurwala (1985) generalized the result of Harter and Moore. Similar results for inverted gamma distribution, a family of life time and generalized life distributions have been obtained respectively by Chaturvedi et al. (1998, 2002). Sevial and Dermishan (2008) have developed a group sequential test when the response variable has an Inverse Gaussian distribution with known scale parameter.

In this thesis we work in these directions, we studied the classical inferential procedures of life testing models by using the transformation methods. We also departed from the conventional techniques of obtaining the MLE and UMVUE of Stress Strength reliability and used the estimates of reliability function to derive those results. We also studied the robustness of a life testing model by using the Wald's sequential probability ratio test.

1.6 Thesis Plan

The description of chapters in this Thesis are as follows:

Chapter 1 is Introductory. This Chapter includes some introductory matter on Reliability, Stress-Strength models and Sequential Theory.

In Chapter 2, we have discussed the problem of estimating the $R = P(Y > X)$. We have obtained the point and interval estimation of Stress-Strength Reliability for a class of lifetime distributions. In order to obtain these estimators, the major role is played by the transformation methods.

In Chapter 3, Sequential probability ratio test is developed for the scale parameter of Nakagami Distribution and the robustness of scale parameter is studied when the shape parameter has undergone a change, for testing the hypothesis regarding the parameter of Nakagami Distribution. The expression for the Operating Characteristic (OC) and Average

Sample Number (ASN) functions are derived and the results are presented through Graphs and Tables.

In Chapter 4, the problem of Stress-Strength reliability model is studied for a class of lifetime distributions. Here, deviating from the conventional techniques of obtaining the UMVUE of $R = P(Y > X)$, we have obtained the UMVUE of $R = P(Y > X)$ by using the estimate of the reliability function.

In Chapter 5, the problem of Stress-Strength reliability model is studied for Gompertz distributed stress. We have studied the problem by establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows a Gompertz distribution and Strength follows a Power function distribution. Further, these results are explained with an example and are utilized to get optimum cost of any item when the cost function is linear in terms of parameters.

In Chapter 6, The problem of Stress-Strength reliability model is studied through establishing the relationship among the parameters of the distributions of Stress and Strength of the manufacturing items. It is considered that the Stress follows a Class of lifetime distributions and Strength follows a Power function distribution. Further, these results are used to obtain optimum cost when the cost function is linear in terms of parameters.

In Chapter 7, the problem of Stress-Strength reliability model is studied for exponential distribution under complete sample case, Type I and Type II censoring. Here, we have obtained the MLE and UMVUE of $R = P(Y > X)$. In order to obtain the MLE and UMVUE, first we obtain the estimate of reliability function R(t). Further, the estimates of reliability function R(t) are used to find the MLE and UMVUE for $R = P(Y > X)$.

Chapter 2

On the Estimation of $R = P(Y > X)$ for a Class of Lifetime Distributions by Transformation Method

2.1 Introduction

A lot of work has been done in the literature to deal with various inferential problems related to $R = P(Y > X)$, which represents the reliability of an item of random strength Y subject to a random stress X. The problem of estimating reliability $R = P(Y > X)$ arises in the situation of mechanical reliability of a system with strength Y and stress X and R is known as a system reliability or stress-strength reliability. The system fails if and only if stress exceeds the strength. Many authors have worked on stress-strength reliability for different choices of stress and strength distributions. For a brief review, one may refer to Church and Harris (1970), Enis and Geisser (1971), Downton (1973), Tong (1974, 1975), Kelly, Kelly and Schucany (1976), Sathe and Shah (1981), Chao (1982), Awad and Gharraf (1986), Chaturvedi and Surinder (1999), Chaturvedi and Sharma (2007).

The purpose of present chapter is many-fold. We considered a class of lifetime distributions proposed by Chaturvedi and Rani (1997), which covers many lifetime distributions as specific cases. In Section 2.2, the MLE and UMVUE of 'R' are derived. In Section 2.3, we construct the confidence interval for 'R'. In order to derive the MLE, UMVUE and confidence interval for 'R', the major role is played by the transformation method.

2.2 MLE and UMVUE of $R = P(Y > X)$ for a Class of Lifetime Distributions

Chaturvedi and Rani (1997) has defined the following class of distributions

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}}{\theta^{ab}\Gamma a} exp(-x^c/\theta^b); \quad x,\theta,a,b,c > 0, \tag{2.2.1}$$

where θ is assumed to be unknown and a, b, c are known constants. It is easy to see that $\theta^{\frac{b}{c}}$ is scale parameter and a, b, c are shape parameter. On considering different values for a, b and c, the pdf's of different continuous distributions are obtained as specific cases:

1. For $a = b = c = 1$, (2.2.1) gives the probability density function (pdf) of the one-parameter exponential distribution [see Johnson and Kotz (1970, p.207)].

2. For $b = c = 1$, (2.2.1) becomes the pdf of the gamma distribution [see Johnson and Kotz (1970, p.166)].

3. For $b = c$, (2.2.1) gives the pdf of the generalized gamma distribution [see Johnson and Kotz (1970, p.197)].

4. Taking 'a' as positive integer and $b = c = 1$, (2.2.1) turns out to be the pdf of an Erlang distribution [see Johnson and Kotz (1970, p.166)].

5. For $a = 1$, $b = c$, (2.2.1) represents the pdf of Weibull distribution [see Johnson and Kotz (1970, p.250)].

6. For $a = 1/2$, $b = c = 2$, (2.2.1) is the pdf of halfnormal distribution [see Davis (1952)].

7. For $a = b = 1$, $c = 2$, (2.2.1) turns out to be the pdf of Rayleigh distribution [see Sinha (1986, p.200)].

8. For $a = \alpha/2$, $b = 1$, $c = 2$, (2.2.1) becomes the pdf of the Chi-distribution [see Patel, Kapadia and Owen (1976, p.173)].

9. For $a = 3/2$, $b = 1$, $c = 2$, (2.2.1) gives the pdf of Maxwell's failure distribution [see Tyagi and Bhattacharya (1989a, b)].

Let the random variables X and Y follows the class of lifetime distributions given at (2.2.1).

Theorem 2.1: The MLE of $R = P(Y > X)$ is given by

$$\tilde{R} = \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right), \tag{2.2.2}$$

where, $\bar{\varepsilon} = \dfrac{1}{n_1}\sum\limits_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X(\text{say})$ and, $\bar{\eta} = \dfrac{1}{n_2}\sum\limits_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y(\text{say})$.

Proof: Let us consider the transformation $x^c = \varepsilon$ in (2.2.1), we get

$$f(\varepsilon; a, \lambda) = \frac{\varepsilon^{a-1}}{\lambda^a \Gamma a} exp(-\varepsilon/\lambda); \quad \varepsilon, a, \lambda > 0, \tag{2.2.3}$$

which follows gamma distribution, where $\lambda = \theta^b$.

Now, let us consider ε and η two independent random variables which follows gamma distribution with parameters (a_1, λ_1) and (a_2, λ_2) respectively, where $\varepsilon = x^{c_1}$ and $\eta = y^{c_2}$. Thus, for $R = P(\eta > \varepsilon)$, we have

$$R = P\left(\frac{\eta}{\varepsilon} > 1\right),$$

or,

$$P\left(\frac{\eta/\lambda_2}{\varepsilon/\lambda_1} + 1 > \frac{\lambda_1}{\lambda_2} + 1\right),$$

or,

$$P\left(\frac{\varepsilon/\lambda_1}{\eta/\lambda_2 + \varepsilon/\lambda_1} < \frac{\lambda_2}{\lambda_1 + \lambda_2}\right).$$

Here, random variable $z = \dfrac{\varepsilon/\lambda_1}{\eta/\lambda_2 + \varepsilon/\lambda_1}$, has a beta distribution with pdf

$$f(z, a_1, a_2) = \left[B\left(a_1, a_2\right)\right]^{-1} z^{a_1 - 1}(1 - z)^{a_2 - 1},$$

or,

$$R = I_{\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)}\left(a_1, a_2\right), \tag{2.2.4}$$

On using the relationship between incomplete beta distribution and hypergeometric series, we get

$$R = \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)\right). \tag{2.2.5}$$

On replacing λ_1 and λ_2 by their respective MLE's i.e , $\tilde{\lambda}_1 = \dfrac{\bar{\varepsilon}}{a_1}$ and, $\tilde{\lambda}_2 = \dfrac{\bar{\eta}}{a_2}$ in (2.2.5), we get the MLE of $R = P(\eta > \varepsilon)$, where ε and η are independent gamma distributions. The reliability for the random strength Y and random stress X i.e. $R = P(Y > X)$ is given by (2.2.6) and MLE of R is given by (2.2.7) respectively.

$$R = \left(\frac{\theta_2^{b_2}}{\theta_1^{b_1} + \theta_2^{b_2}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{\theta_2^{b_2}}{\theta_1^{b_1} + \theta_2^{b_2}}\right)\right), \tag{2.2.6}$$

$$\tilde{R} = \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right)\right), \tag{2.2.7}$$

where $\bar{\varepsilon} = \dfrac{1}{n_1}\sum\limits_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X(\text{say})$ and, $\bar{\eta} = \dfrac{1}{n_2}\sum\limits_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y(\text{say})$.

Hence, the theorem follows.

Corollary 2.1

1. For $a = b = c = 1$,

$$\tilde{R} = \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y},$$

where, $\bar{T}_Y = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j$ and, $\bar{T}_X = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i$, which is the MLE of R for exponential distribution.

2. For $b = c = 1$ and ‘a’ as positive integer,

$$\tilde{R} = \left(\frac{a_1\bar{Y}}{a_2\bar{X} + a_1\bar{Y}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \frac{a_1\bar{Y}}{a_2\bar{X} + a_1\bar{Y}}\right),$$

which is the MLE of R for gamma distribution.

3. For $b = c = \alpha$,

$$\tilde{R} = \left(\frac{a_1\bar{T}_Y}{a_2\bar{T}_X + a_1\bar{T}_Y}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \frac{a_1\bar{T}_Y}{a_2\bar{T}_X + a_1\bar{T}_Y}\right),$$

where, $\bar{T}_Y = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\alpha}$ and, $\bar{T}_X = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\alpha}$, which is the MLE of R for generalised gamma distribution.

4. For $a = 1$ and $b = c$,

$$\tilde{R} = \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y},$$

where, $\bar{T}_Y = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c}$ and, $\bar{T}_X = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c}$, which is the MLE of R for Weibul distribution.

5. For $a = 1/2, b = c = 2$,

$$\tilde{R} = \left(\frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right)^{1/2} \frac{1}{\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right),$$

where, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$, which is the MLE of R for Half-Normal distribution.

6. For $a = b = 1$ and $c = 2$,

$$\tilde{R} = \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y},$$

where, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$, which is the MLE of R for Rayleigh distribution.

7. For $a = \alpha/2, b = 1$ and $c = 2$,

$$\tilde{R} = \left(\frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right)^{\alpha/2} \frac{1}{B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)} {}_2F_1\left(\frac{\alpha}{2}, 1 - \frac{\alpha}{2}; \frac{\alpha}{2}; \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right),$$

where, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$, which is the MLE of R for Chi-distribution.

8. For $a = 3/2, b = 1$ and $c = 2$,

$$\tilde{R} = \left(\frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right)^{3/2} \frac{8}{\pi} {}_2F_1\left(\frac{3}{2}, \frac{-1}{2}; \frac{3}{2}; \frac{\bar{T}_Y}{\bar{T}_X + \bar{T}_Y}\right),$$

where, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$ and, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$, which is the MLE of R for Maxwell's failure distribution.

Theorem 2.2: The UMVUE of $R = P(Y > X)$ is given by

$$\hat{R} = \begin{cases} \dfrac{B\left[(n_2-1)a_2+i+1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]}\sum\limits_{i=0}^{\infty}\dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} & \\ \times\sum\limits_{j=0}^{\infty}(-1)^j\binom{(n_1-1)a_1-1}{j}\left(\dfrac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y < T_X \\ \text{where } 0 \le i \le a_1-1 < \infty \text{ and } 0 \le j \le (n_1-1)a_1-1 < \infty & \\ & \\ \dfrac{B\left[(n_1-1)a_1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]}\sum\limits_{i=0}^{\infty}\dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} & \\ \times\sum\limits_{j=0}^{\infty}(-1)^j\binom{(n_2-1)a_2+i}{j}\left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X < T_Y \\ \text{where } 0 \le i \le a_2-1 < \infty \text{ and } 0 \le j \le (n_2-1)a_2+i < \infty & \end{cases} \tag{2.2.8}$$

where, $T_X = \sum\limits_{i=1}^{n_1} X_i^{c_1}$ and, $T_Y = \sum\limits_{j=1}^{n_2} Y_j^{c_2}$.

Proof: Let us consider the transformation $x^c = \varepsilon$ in (2.2.1), we get

$$f(\varepsilon; a, b, c, \theta) = \frac{\varepsilon^{a-1}}{\lambda^a \Gamma a} exp(-\varepsilon/\lambda);$$

which is gamma distribution, where $\lambda = \theta^b$.

Now to obtain $P(\eta > \varepsilon)$, we have to obtain the UMVUE of $f(\varepsilon; a, \lambda)$ i.e. $\hat{f}(\varepsilon; a, \lambda)$ and $f(\eta; a, \lambda)$ i.e. $\hat{f}(\eta; a, \lambda)$ which is given by

$$\hat{f}(\varepsilon; a_1, \lambda) = \frac{1}{B\left(a_1, (n_1-1)a_1\right)}\left\{\frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})_1^a}\right\}\left\{1-\frac{\varepsilon}{n_1\bar{\varepsilon}}\right\}^{(n_1-1)a_1-1}; \quad \text{if } 0 < \varepsilon < n_1\bar{\varepsilon} \tag{2.2.9}$$

On replacing ε by η and n_1 by n_2 in (2.2.9), we get the UMVUE of $f(\eta; a_2, \lambda)$. Now, let us consider ε and η be two independent random variables follows gamma distribution with parameters (a_1, λ_1) and (a_2, λ_2) respectively, where $\varepsilon = x^{c_1}$ and $\eta = y^{c_2}$.

$$\hat{R} = P(\eta > \varepsilon) = \int_0^{\infty}\int_{\varepsilon}^{\infty}\hat{f}(\eta; a_2, \lambda_2)\hat{f}(\varepsilon; a_1, \lambda_1)d\eta d\varepsilon$$

$$= \frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \int_0^{n_1\bar{\varepsilon}} \int_{\varepsilon}^{n_2\bar{\eta}} \frac{\eta^{a_2-1}}{(n_2\bar{\eta})^{a_2}} \left(1-\frac{\eta}{n_2\bar{\eta}}\right)^{(n_2-1)a_2-1} \times \frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}} \left(1-\frac{\varepsilon}{n_2\bar{\varepsilon}}\right)^{(n_1-1)a_1-1} d\eta d\varepsilon$$

let $1-\dfrac{\eta}{n_2\bar{\eta}} = z$, we get

$$= \frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \int_0^{n_1\bar{\varepsilon}} \int_{\varepsilon}^{(1-\frac{\varepsilon}{n_2\bar{\eta}})} z^{(n_2-1)a_2-1}(1-z)^{a_2-1} \times \frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}} \left(1-\frac{\varepsilon}{n_2\bar{\varepsilon}}\right)^{(n_1-1)a_1-1} dz d\varepsilon$$

$$= \frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \int_0^{min(n_1\bar{\varepsilon},n_2\bar{\eta})} \left(1-\frac{\varepsilon}{n_1\bar{\varepsilon}}\right)^{(n_1-1)a_1-1} \frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}} \left(1-\frac{\varepsilon}{n_2\bar{\eta}}\right)^{(n_1-1)a_1-1} dz d\varepsilon$$

Now, consider the case when $n_1\bar{\varepsilon} > n_2\bar{\eta}$, in this situation, let $1-\dfrac{\varepsilon}{n_2\bar{\eta}} = z$

$$= \frac{1}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \int_0^1 z^{(n_2-1)a_2+i} \left(1-\left(\frac{n_2\bar{\eta}}{n_1\bar{\varepsilon}}\right)(1-z)\right)^{(n_1-1)a_1-1} \frac{\left[(n_2\bar{\eta})(1-z)\right]^{(a_1-1)}}{(n_1\bar{\varepsilon})^{a_1}} (n_2\bar{\eta})dz$$

$$= \frac{1}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \int_0^1 \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\frac{n_2\bar{\eta}}{n_1\bar{\varepsilon}}\right)^{a_1+j} (1-z)^{a_1+j-1} z^{(n_2-1)a_2+i} dz$$

$$= \frac{B\left[(n_2-1)a_2+i+1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i}$$
$$\times \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\frac{n_2\bar{\eta}}{n_1\bar{\varepsilon}}\right)^{a_1+j} ;\text{if } n_2\bar{\eta} < n_1\bar{\varepsilon} \quad (2.2.10)$$

Similarly, we can tackle the case when $n_2\bar{\eta} > n_1\bar{\varepsilon}$, we get

$$= \frac{B\left[(n_1-1)a_1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i}$$
$$\times \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\frac{n_1\bar{\varepsilon}}{n_2\bar{\eta}}\right)^{j} ;\text{if } n_2\bar{\eta} > n_1\bar{\varepsilon} \quad (2.2.11)$$

Now, to obtain the UMVUE for the class of distribution, substituting $n_1\bar{\varepsilon} = \sum_{i=1}^{n_1} X_i^{c_1} = T_X$ and $n_2\bar{\eta} = \sum_{j=1}^{n_2} Y_j^{c_2} = T_Y$ in (2.2.10) and (2.2.11), the Theorem follows.

Corollary 2.2

1. For $a = b = c = 1$,

$$\hat{R} = \begin{cases} \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_1-j-1)\Gamma(n_2+j+1)} \sum_{j=0}^{n_1-2} (-1)^j \left(\dfrac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_2-j)\Gamma(n_1+j)} \sum_{j=0}^{n_2-1} (-1)^j \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and, $T_X = \sum_{i=1}^{n_1} X_i$, which is the UMVUE of R for exponential distribution.

2. For $b = c = 1$ and 'a' as positive integer,

$$\hat{R} = \begin{cases} \dfrac{B\left[(n_2-1)a_2+i+1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{a_2-1} \dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times \sum_{j=0}^{(n_1-1)a_1-1} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{B\left[(n_1-1)a_1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{a_2-1} \dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times \sum_{j=0}^{(n_2-1)a_2+i} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X < T_Y \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and, $T_X = \sum_{i=1}^{n_1} X_i$, which is the UMVUE of R for gamma distribution.

3. For $b = c = \alpha$,

$$\hat{R} = \begin{cases} \dfrac{B\left[(n_2-1)a_2+i+1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{a_2-1} \dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times \sum_{j=0}^{(n_1-1)a_1-1} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{B\left[(n_1-1)a_1, a_1+j\right]}{B\left[a_1,(n_1-1)a_1\right]B\left[a_2,(n_2-1)a_2\right]} \sum_{i=0}^{a_2-1} \dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times \sum_{j=0}^{(n_2-1)a_2+i} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X < T_Y \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^{\alpha}$ and, $T_X = \sum_{i=1}^{n_1} X_i^{\alpha}$, which is the UMVUE of R for generalised gamma distribution.

4. For $a = 1$ and $b = c$,

$$\hat{R} = \begin{cases} \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_1-j-1)\Gamma(n_2+j+1)} \sum_{j=0}^{n_1-2} (-1)^j \left(\dfrac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_2-j)\Gamma(n_1+j)} \sum_{j=0}^{n_2-1} (-1)^j \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^c$ and, $T_X = \sum_{i=1}^{n_1} X_i^c$, which is the UMVUE of R for Weibull distribution.

5. For $a = b = 1$ and $c = 2$,

$$\hat{R} = \begin{cases} \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_1 - j - 1)\Gamma(n_2 + j + 1)} \sum_{j=0}^{n_1-2} (-1)^j \left(\dfrac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{\Gamma n_1 \Gamma n_2}{\Gamma(n_2 - j)\Gamma(n_1 + j)} \sum_{j=0}^{n_2-1} (-1)^j \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and, $T_X = \sum_{i=1}^{n_1} X_i^2$, which is the UMVUE of R for Rayleigh distribution.

6. For $a = \alpha/2$, $b = 1$ and $c = 2$,

$$\hat{R} = \begin{cases} \dfrac{B\left[\frac{(n_2-1)\alpha}{2} + i + 1, \frac{\alpha}{2} + j\right]}{B\left[\frac{\alpha}{2}, \frac{(n_1-1)\alpha}{2}\right] B\left[\frac{\alpha}{2}, \frac{(n_2-1)\alpha}{2}\right]} \sum_{i=0}^{\frac{\alpha}{2}-1} \dfrac{(-1)^i}{\frac{(n_2-1)\alpha}{2} + i} \binom{\frac{\alpha}{2}-1}{i} \\ \times \sum_{j=0}^{\frac{(n_1-1)\alpha}{2}-1} (-1)^j \binom{\frac{(n_1-1)\alpha}{2}-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{\frac{\alpha}{2}+j} & ;\text{if } T_Y < T_X \\ \\ \dfrac{B\left[\frac{(n_1-1)\alpha}{2}, \frac{\alpha}{2} + j\right]}{B\left[\frac{\alpha}{2}, \frac{(n_1-1)\alpha}{2}\right] B\left[\frac{\alpha}{2}, \frac{(n_2-1)\alpha}{2}\right]} \sum_{i=0}^{\frac{\alpha}{2}-1} \dfrac{(-1)^i}{\frac{(n_2-1)\alpha}{2} + i} \binom{\frac{\alpha}{2}-1}{i} \\ \times \sum_{j=0}^{\frac{(n_1-1)\alpha}{2}+i} (-1)^j \binom{\frac{(n_2-1)\alpha}{2}+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X < T_Y \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and, $T_X = \sum_{i=1}^{n_1} X_i^2$, which is the UMVUE of R for Chi-distribution.

2.3 Interval Estimation of $R = P(Y > X)$

Theorem 2.3: The Confidence interval for $R = P(Y > X)$ is given by

$$P\left(I_{\left(\frac{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{1-\gamma_2}}{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{1-\gamma_2}+1}\right)}(a_1,a_2) < R < I_{\left(\frac{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{\gamma_1}}{(a_1\bar{T}_Y/a_2\bar{T}_X)F_{\gamma_1}+1}\right)}(a_1,a_2)\right) = 1-\gamma \qquad (2.3.1)$$

where, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c_1}$ and, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c_2}$.

Proof: Let us consider ε and η be two independent random variables follows gamma distribution with parameters (a_1, λ_1) and(a_2, λ_2) respectively, where $\varepsilon = x^{c_1}$ and $\eta = y^{c_2}$. Let us take

$$\lambda = \frac{\lambda_2}{\lambda_1} \text{ and; } \tilde{\lambda} = \frac{a_1\bar{\eta}}{a_2\bar{\varepsilon}} \qquad (2.3.2)$$

Now, it is well known that,

$$\frac{2n_1\bar{\varepsilon}}{\lambda_1} \sim Gamma(n_1a_1, 2) \equiv \chi^2_{2n_1a_1}.$$

Similarly,

$$\frac{2n_2\bar{\eta}}{\lambda_2} \sim Gamma(n_2a_2, 2) \equiv \chi^2_{2n_2a_2},$$

where, χ^2_ν is the p.d.f. of the Chi-Square distribution with ν degree of freedom. Hence,

$$\begin{aligned}\frac{\tilde{\lambda}}{\lambda} &= \frac{2n_2\bar{\eta}/n_2\lambda_2a_2}{2n_1\bar{\varepsilon}/n_1\lambda_1a_1} \\ &= \frac{\chi^2_{2n_2a_2}/2n_2a_2}{\chi^2_{2n_1a_1}/2n_1a_1} \sim F\left(2n_1a_1, 2n_2, a_2\right), \qquad (2.3.3)\end{aligned}$$

where $F(\nu_1, \nu_2)$ denotes the Snedecor's F-distribution with ν_1 and ν_2 degree of freedom. Analogously,

$$\frac{\tilde{\lambda}}{\lambda} \sim F(2n_1a_1, 2n_2a_2)$$

For any δ denote by $F_\delta = F_\delta(2n_1a_1, 2n_2a_2)$, the $1-\delta$ quantile (i.e. the δ cut off points) of $F(2n_1a_1, 2n_2a_2)$ distribution.

Also, the $1-\delta$ quantile of $F(2n_2a_2, 2n_1a_1)$ distribution is related to F_δ as follows.

$$F_\delta(2n_2a_2, 2n_1a_1) = \left[F_{1-\delta}(2n_1a_1, 2n_2a_2)\right]^{-1}$$

Let, γ_1 and γ_2 be non-negative numbers such that $\gamma_1 + \gamma_2 = \gamma$. Then,

$$P\left[\tilde{\lambda}F_{1-\gamma_2} < \lambda < \tilde{\lambda}F_{\gamma_1}\right] = 1-\gamma \tag{2.3.4}$$

Recall now that $R = I_{\frac{\lambda}{\lambda+1}}(a_1, a_2)$. Since, $I_z(a,b)$ is an increasing function of z for any a, b. So, $I_{\frac{\lambda}{\lambda+1}}(a_1, a_2)$ is also a increasing function of λ for any (a_1, a_2). Hence (2.3.4) implies that,

$$P\left(I_{\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1, a_2) < R < I_{\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}}(a_1, a_2)\right) = 1-\gamma \tag{2.3.5}$$

The Confidence interval (2.3.5) has originally derived by Constantine et al. (1986).

Now, in (2.3.5) replacing $\tilde{\lambda} = \dfrac{a_1\bar{\eta}}{a_2\bar{\varepsilon}}$ and $\lambda_1 = \theta_1^{b_1}$, $\lambda_2 = \theta_2^{b_2}$

Thus, $R = I_{\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)}(a_1, a_2) = I_{\left(\frac{\theta_2^{b_2}}{\theta_1^{b_1}+\theta_2^{b_2}}\right)}(a_1, a_2)$ and the confidence interval for R is given as,

$$P\left(I_{\left(\frac{(a_1\bar{\eta}/a_2\bar{\varepsilon})F_{1-\gamma_2}}{(a_1\bar{\eta}/a_2\bar{\varepsilon})F_{1-\gamma_2}+1}\right)}(a_1, a_2) < R < I_{\left(\frac{(a_1\bar{\eta}/a_2\bar{\varepsilon})F_{\gamma_1}}{(a_1\bar{\eta}/a_2\bar{\varepsilon})F_{\gamma_1}+1}\right)}(a_1, a_2)\right) = 1-\gamma \tag{2.3.6}$$

where, $\bar{\varepsilon} = \dfrac{1}{n_1}\sum\limits_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X(\text{say})$ and, $\bar{\eta} = \dfrac{1}{n_2}\sum\limits_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y(\text{say})$,
and, hence the Theorem follows.

Corollary 2.3

1. For $a = b = c = 1$ in (2.3.6), we get confidence interval for one parameter exponential distribution.
$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$
where, $\tilde{\lambda} = \dfrac{\bar{Y}}{\bar{X}}$ and, $R = \dfrac{\theta_2}{\theta_1+\theta_2}$, which coincides with interval derived by Enis and Geiser (1971).

2. For $b = c = 1$ and 'a' as positive integer,
$$P\left(I_{\frac{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}}{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1,a_2) < R < I_{\frac{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}}{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}+1}}(a_1,a_2)\right) = 1-\gamma,$$
where, $\tilde{\lambda} = \dfrac{\bar{Y}}{\bar{X}}$, $\bar{Y} = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j$, $\bar{X} = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i$ and, $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(a_1,a_2)$, which is the Confidence interval of R for gamma distribution.

3. For $b = c = \alpha$,
$$P\left(I_{\frac{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}}{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1,a_2) < R < I_{\frac{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}}{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}+1}}(a_1,a_2)\right) = 1-\gamma,$$
where, $\tilde{\lambda} = \dfrac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\alpha}$, $\bar{T}_X = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\alpha}$ and, $R = I_{\frac{\theta_2^{\alpha}}{\theta_1^{\alpha}+\theta_2^{\alpha}}}(a_1,a_2)$, which is the Confidence interval of R for generalised gamma distribution.

4. For a = 1 and b = c,
$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$
where, $\tilde{\lambda} = \dfrac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \dfrac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c}$, $\bar{T}_X = \dfrac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c}$ and, $R = I_{\frac{\theta_2^{c}}{\theta_1^{c}+\theta_2^{c}}}(a_1,a_2)$, which is the Confidence interval of R for Weibull distribution.

5. For $a = 1/2$, $b = c = 2$,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right)\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$ and, $R = I_{\frac{\theta_2^2}{\theta_1^2+\theta_2^2}}\left(\frac{1}{2},\frac{1}{2}\right)$, which is the Confidence interval of R for Half-Normal distribution.

6. For $a = b = 1$ and $c = 2$,

$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$ and, $R = \frac{\theta_2}{\theta_1+\theta_2}$, which is the Confidence interval of R for Rayleigh distribution.

7. For $a = \alpha/2$, $b = 1$ and $c = 2$,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$ and, $R = I_{\frac{\theta_2}{\theta_2+\theta_1}}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)$, which is the Confidence interval of R for Chi-distribution.

8. For $a = 3/2$, $b = 1$ and $c = 2$,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{3}{2},\frac{3}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{3}{2},\frac{3}{2}\right)\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\bar{T}_Y}{\bar{T}_X}$, $\bar{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$, $\bar{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$ and, $R = I_{\frac{\theta_2}{\theta_2+\theta_1}}\left(\frac{3}{2},\frac{3}{2}\right)$, which is the Confidence interval of R for Maxwell's failure distribution.

Chapter 3

Robustness of SPRT for the Scale Parameter of Nakagami Distribution

3.1 Introduction

The theory of sequential analysis has begun in about 1943 with the work of A. Wald and G.A. Barnard. Sequential analysis has been heavily dominated by the Sequential Probability Ratio Test (SPRT). SPRT for testing the simple null hypothesis against a simple alternative was first time given by Wald (1947). He obtained the expression for the Operating Characteristics and Average Sample Number functions. The robustness of SPRT has been studied by several authors. For a brief review, one may refer to Epstein and Sobel (1955), Johnson(1966), Barlow and Proschan (1967), Phatarfod (1971), Harter and Moore (1976), Montagne and Singpurwalla (1985), Joshi and Shah(1990), Chaturvedi, kumar and Kumar, (1998), Chaturvedi, Kumar and Surinder (2000), Chaturvedi, Tiwari and Tomer (2002), Surinder and Naresh(2009).

Nakagami distribution is a lifetime distribution, given by M. Nakagami (1960) has the probability density function (pdf)

$$f(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma\lambda\beta^{\lambda}}x^{2\lambda-1}e^{-\dfrac{\lambda}{\beta}x^{2}};\quad x,\lambda,\beta>0, \tag{3.1.1}$$

where, λ is a shape parameter and β is scale parameter. Here, the scale parameter β is also known as controlling spread. Nakagami distribution is used for modelling a wider class of fading channel conditions. Simon et al. (1994) have shown that the Nakagami distribution gives the best fit to land-mobile and indoor-mobile multipath propagation as well as scintillating ionospheric radio links. Lakhzouri et al. (2005) has showed that Nakagami distribution gives the best fit for satellite-to-indoor and satellite-to-outdoor radiowaves propagation. Nakagami distribution is related to Rayleigh distribution and one-sided Gaussian distribution when $\lambda = 1$ and $\lambda = \dfrac{1}{2}$ respectively.

In this Chapter, we have developed the SPRT for scale parameter of Nakagami distribution and studied the robustness of the scale parameter when there is change in the shape parameter.

In Section 3.2, we state the problem, and develop SPRT for testing the hypothesis giving expressions for OC and ASN functions in Section 3.3. In Section 3.4, robustness of the SPRT is studied and the results are discussed in Section 3.5. Finally, the conclusions are given in section 3.6.

3.2 Set-up of the Problem

Let the random variable X follows the Nakagami distribution presented by the probability density function (pdf)

$$f(x;\lambda,\beta) = \frac{2\lambda^{\lambda}}{\Gamma\lambda\beta^{\lambda}} x^{2\lambda-1} e^{-\frac{\lambda}{\beta}x^2}; \quad x,\lambda,\beta > 0, \tag{3.2.1}$$

Given a sequence of observations x_1, x_2, x_3,...from (3.2.1), suppose one wish to test the simple null hypothesis $H_0 : \beta = \beta_0$ against the simple alternative $H_1 : \beta = \beta_1 (\beta_1 > \beta_0)$. The expression for OC and ASN function is obtained and their behaviour is studied by plotting graph.

3.3 SPRT for Testing the Hypothesis regarding β

The SPRT for testing the null hypothesis $H_0 : \beta = \beta_0$ against the simple alternative $H_1 : \beta = \beta_1(\beta_1 > \beta_0)$ is defined as

$$\begin{aligned} z_i &= \ln \frac{f(x_i; \lambda, \beta_1)}{f(x_i; \lambda, \beta_0)} && (3.3.1) \\ &= \ln \frac{\frac{2\lambda^\lambda}{\Gamma\lambda\beta_1^\lambda} x_i^{2\lambda-1} e^{-\frac{\lambda}{\beta_1}x_i^2}}{\frac{2\lambda^\lambda}{\Gamma\lambda\beta_0^\lambda} x_i^{2\lambda-1} e^{-\frac{\lambda}{\beta_0}x_i^2}} \\ &= \ln \left(\frac{\beta_0}{\beta_1}\right)^\lambda e^{-\lambda x_i^2\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)}, \end{aligned}$$

or,

$$z_i = \lambda \ln \left(\frac{\beta_0}{\beta_1}\right) - x_i^2\lambda\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right), \tag{3.3.2}$$

or,

$$e^{z_i} = \left(\frac{\beta_0}{\beta_1}\right)^\lambda e^{-\frac{\lambda}{\beta}x_i^2\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)} \tag{3.3.3}$$

Now, we choose two numbers A and B such that $0 < B < 1 < A$. At the n^{th} stage, accept H_0 if $\sum_{i=1}^{n} z_i \leq \log B$, reject H_0 if $\sum_{i=1}^{n} z_i \geq \log A$, otherwise continue sampling by taking the $(n+1)^{th}$ observation.

If $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are TYPE I and TYPE II errors respectively, then according to Wald (1947), A and B are approximately given by

$$A \approx \frac{1-\beta}{\alpha} \text{ and } B \approx \frac{\beta}{1-\alpha} \tag{3.3.4}$$

The Operating Characteristic (OC) function $L(\theta)$ is given by

$$L(\theta) = \frac{A^h - 1}{A^h - B^h}, \tag{3.3.5}$$

where 'h' is the non-zero solution of

$$E[e^{hz_i}] = 1, \tag{3.3.6}$$

or,

$$\int_0^\infty \left[\frac{f(x_i;\lambda,\beta_1)}{f(x_i;\lambda,\beta_0)}\right]^h f(x_i;\lambda,\beta)dx_i \;=\; 1 \tag{3.3.7}$$

$$\Longrightarrow \int_0^\infty \left[\left(\frac{\beta_0}{\beta_1}\right)^\lambda e^{-\lambda\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)}\right]^h \frac{2\lambda^\lambda}{\Gamma\lambda\beta^\lambda} x_i^{2\lambda-1} e^{-\frac{\lambda}{\beta}x_i^2} dx_i \;=\; 1$$

Let, $A = \dfrac{2\lambda^\lambda}{\Gamma\lambda\beta^\lambda}\left(\dfrac{\beta_0}{\beta_1}\right)^{\lambda h}$ and, $B = \lambda\left\{\left(\dfrac{1}{\beta_1}-\dfrac{1}{\beta_0}\right)h + \dfrac{1}{\beta}\right\}$, we get

$$A\int_0^\infty e^{-Bx_i^2} x_i^{2\lambda-1} dx_i = 1$$

On solving this integral, we get

$$\frac{A\Gamma\lambda}{2B^\lambda} = 1$$

Now, substituting the value of A and B, we get

$$E[e_i^z]^h = \frac{\left(\dfrac{\beta_0}{\beta_1}\right)^{\lambda h}}{\left[1+\beta h\left(\dfrac{1}{\beta_1}-\dfrac{1}{\beta_0}\right)\right]^\lambda} = 1, \tag{3.3.8}$$

or,

$$\left[1+\beta h\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)\right]^\lambda = \left(\frac{\beta_0}{\beta_1}\right)^{\lambda h},$$

or,

$$\beta = \frac{1-\left(\dfrac{\beta_0}{\beta_1}\right)^h}{h\left(\dfrac{1}{\beta_1}-\dfrac{1}{\beta_0}\right)} \tag{3.3.9}$$

The ASN function is approximately given by

$$E(N|\theta) = \frac{L(\theta)\log B + (1 - L(\theta))\log A}{E(Z)} \tag{3.3.10}$$

Provided $E(Z) \neq 0$, where

$$\begin{aligned} E(Z) &= E\left[\lambda \ln\left(\frac{\beta_0}{\beta_1}\right) - x^2\lambda\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right)\right] \\ &= \lambda \ln\left(\frac{\beta_0}{\beta_1}\right) - \lambda\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right)E(x^2) \end{aligned}$$

Now,

$$\begin{aligned} E(X^2) &= \frac{2\lambda^\lambda}{\Gamma\lambda\beta^\lambda}\int_0^\infty x^2 x^{2\lambda-1} e^{-\frac{\lambda}{\beta}x^2}dx \\ &= \beta \end{aligned}$$

Therefore,

$$E(Z) = \lambda\left[\ln\left(\frac{\beta_0}{\beta_1}\right) - \beta\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right)\right] \tag{3.3.11}$$

From (3.3.11) ASN function under H_0 and H_1 are given by

$$E_0(N) = \frac{(1-\alpha)\log B + \alpha \log A}{\lambda\left[\ln\left(\frac{\beta_0}{\beta_1}\right) - \beta\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right)\right]} \tag{3.3.12}$$

and

$$E_1(N) = \frac{\beta\log B + (1-\beta)\log A}{\lambda\left[\ln\left(\frac{\beta_0}{\beta_1}\right) - \beta\left(\frac{1}{\beta_1} - \frac{1}{\beta_0}\right)\right]} \tag{3.3.13}$$

3.4 Robustness of SPRT for Parameter of Nakagami Distribution

Let us suppose that the parameter λ has undergone a change then the probability distribution in (3.2.1) becomes $f(x;\lambda^*,\beta)$. To study the robustness of SPRT developed in section 3.3 with respect to OC function, consider 'h' as the solution of the equation

$$E_{\lambda^*}[e^z]^h = 1, \tag{3.4.1}$$

or,

$$\int_0^\infty \left[\frac{f(x_i;\lambda,\beta_1)}{f(x_i;\lambda,\beta_0)}\right]^h f(x_i;\lambda^*,\beta)dx = 1$$

or,

$$\int_0^\infty \left[\left(\frac{\beta_0}{\beta_1}\right)^\lambda e^{-\lambda\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)}\right]^h \frac{2\lambda^{*\lambda^*}}{\Gamma\lambda^*\beta^{\lambda^*}} x_i^{2\lambda^*-1} e^{-\frac{\lambda^*}{\beta}x_i^2} dx_i = 1$$

or,

$$2\left(\frac{\beta_0}{\beta_1}\right)^{\lambda h} \frac{\lambda^{*\lambda^*}}{\Gamma\lambda^*\beta^{\lambda^*}} \int_0^\infty e^{-\left\{\lambda h\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)+\frac{\lambda^*}{\beta}\right\}x_i^2} x_i^{2\lambda^*-1} dx_i = 1$$

Let, $A = 2\left(\frac{\beta_0}{\beta_1}\right)^{\lambda h} \frac{\lambda^{*\lambda^*}}{\Gamma\lambda^*\beta^{\lambda^*}}$, $B = \left\{\lambda h\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)+\frac{\lambda^*}{\beta}\right\}$ and, $C = 2\lambda^*$, we get

$$A\int_0^\infty e^{-Bx_i^2} x_i^{C-1} dx_i = 1$$

or,

$$\frac{A}{2B^{C/2}}\Gamma(C/2) = 1$$

On substituting the values of A, B and C, we get

$$\beta = \frac{1-\left(\frac{\beta_0}{\beta_1}\right)^{\left(\frac{\lambda}{\lambda^*}\right)h}}{h\frac{\lambda}{\lambda^*}\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)} \tag{3.4.2}$$

For different values of β, 'h' is evaluated and the OC function is obtained. The Robustness of SPRT with respect to ASN can be studied by replacing denominator of (3.3.10) by

$$E_{\lambda^*}(z) = \int_0^\infty z f(x;\lambda^*,\beta)dx$$

$$\begin{aligned} E(Z) &= E\left[\lambda \ln\left(\frac{\beta_0}{\beta_1}\right) - x^2\lambda\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)\right] \\ &= \lambda \ln\left(\frac{\beta_0}{\beta_1}\right) - \lambda\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)E(x^2), \end{aligned}$$

where,

$$\begin{aligned} E(X^2) &= \frac{2\lambda^{*\lambda^*}}{\Gamma\lambda^*\beta^{\lambda^*}}\int_0^\infty x^2 x^{2\lambda^*-1}e^{-\frac{\lambda^*}{\beta}x^2}dx \\ &= \beta \end{aligned}$$

Therefore,

$$= \lambda\left[\ln\left(\frac{\beta_0}{\beta_1}\right) - \beta\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)\right] \tag{3.4.3}$$

We consider the cases $\lambda > \lambda^*$ and $\lambda < \lambda^*$ to study the robustness of SPRT.

3.5 Results and Discussions

Consider the equation (3.4.2) and taking the logarithms of both sides, we get

$$\left(\frac{\lambda}{\lambda_*}\right) h \log\left(\frac{\beta_0}{\beta_1}\right) = \log\left[1 + \beta h \frac{\lambda}{\lambda_*}\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)\right]$$

Expanding and retaining the terms up to third degree, we get

$$\left\{\beta^3 P^3\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)^3\right\}\frac{h^2}{3} - \left\{\beta^2 P^2\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right)^2\right\}\frac{h}{2} + \left\{\beta\left(\frac{1}{\beta_1}-\frac{1}{\beta_0}\right) - ln\left(\frac{\beta_0}{\beta_1}\right)\right\}P = 0 \tag{3.5.1}$$

where $P = \dfrac{\lambda}{\lambda^*}$,

For testing $H_0 : \beta = 13$ verses $H_1 : \beta = 15$ and taking $\alpha = \beta = 0.05$, the real roots of 'h' are obtained by using (3.5.1) for the different values of β. The OC and ASN functions are evaluated by using the equations (3.3.5) and (3.3.10) by considering the cases $\lambda > \lambda^*$, $\lambda < \lambda^*$ and $\lambda = \lambda^*$ respectively and are tabulated in **Table 3.1** and **Table 3.2**. The graph for OC and ASN functions are plotted in **Fig. 3.1** and **Fig. 3.2** respectively.

Table 3.1: Values of OC Function for Scale Parameter of Nakagami Distribution

$(H_0 : \beta = 13, H_1 : \beta = 15, \alpha = \beta = 0.05)$

L(β)							
β	**P=0.5**	**P=1**	**P=1.5**	**β**	**P=0.5**	**P=1**	**P=1.5**
12.0	0.999998	0.998612	0.987699	14.0	0.430317	0.464988	0.476637
12.1	0.999996	0.997951	0.984101	14.1	0.296256	0.393507	0.428397
12.2	0.999991	0.996997	0.979569	14.2	0.190791	0.326856	0.381868
12.3	0.999981	0.995632	0.973898	14.3	0.117107	0.266968	0.337742
12.4	0.999960	0.993692	0.966851	14.4	0.069694	0.214889	0.296546
12.5	0.999917	0.990956	0.958155	14.5	0.040728	0.170848	0.258633
12.6	0.999830	0.987128	0.947507	14.6	0.023563	0.134457	0.224191
12.7	0.999657	0.981816	0.934578	14.7	0.013561	0.104946	0.193261
12.8	0.999316	0.974510	0.919019	14.8	0.007784	0.081368	0.165766
12.9	0.998652	0.964562	0.900481	14.9	0.004462	0.062748	0.141541
13.0	**0.997371**	**0.951168**	**0.878632**	**15.0**	**0.002555**	**0.048174**	**0.120358**
13.1	0.994931	0.933375	0.853186	15.1	0.001461	0.036846	0.101959
13.2	0.990337	0.910101	0.823936	15.2	0.000834	0.028087	0.086067
13.3	0.98182	0.880222	0.790789	15.3	0.000475	0.021343	0.072407
13.4	0.966332	0.842704	0.753801	15.4	0.00027	0.016167	0.060713
13.5	0.938946	0.796814	0.713205	15.5	0.000153	0.012206	0.050738
13.6	0.892517	0.742377	0.669422	15.6	0.000086	0.009182	0.042256
13.7	0.818716	0.680014	0.623060	15.7	0.000048	0.006879	0.035063
13.8	0.71206	0.611282	0.574883	15.8	0.000027	0.005129	0.028978
13.9	0.57676	0.538609	0.525768	15.9	0.000015	0.003803	0.023843

Figure 3.1: Graph of OC Function for Scale Parameter of Nakagami Distribution

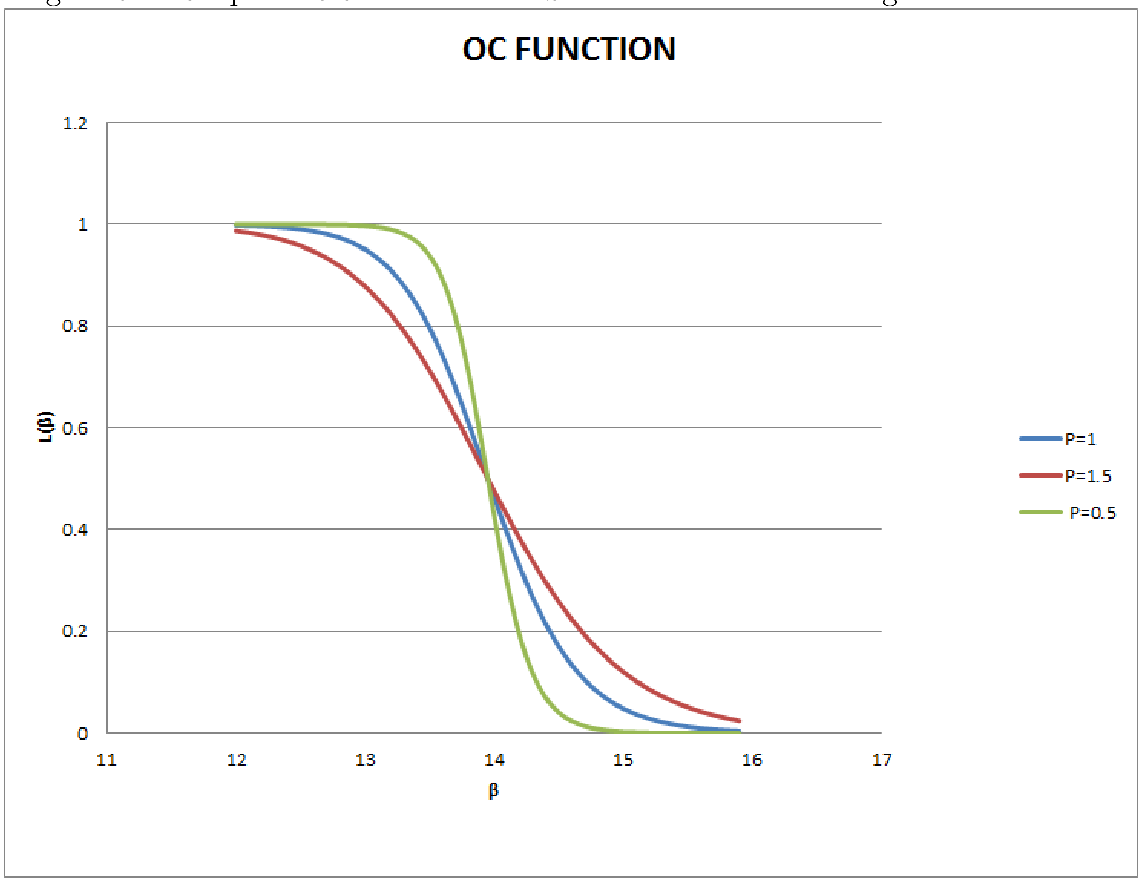
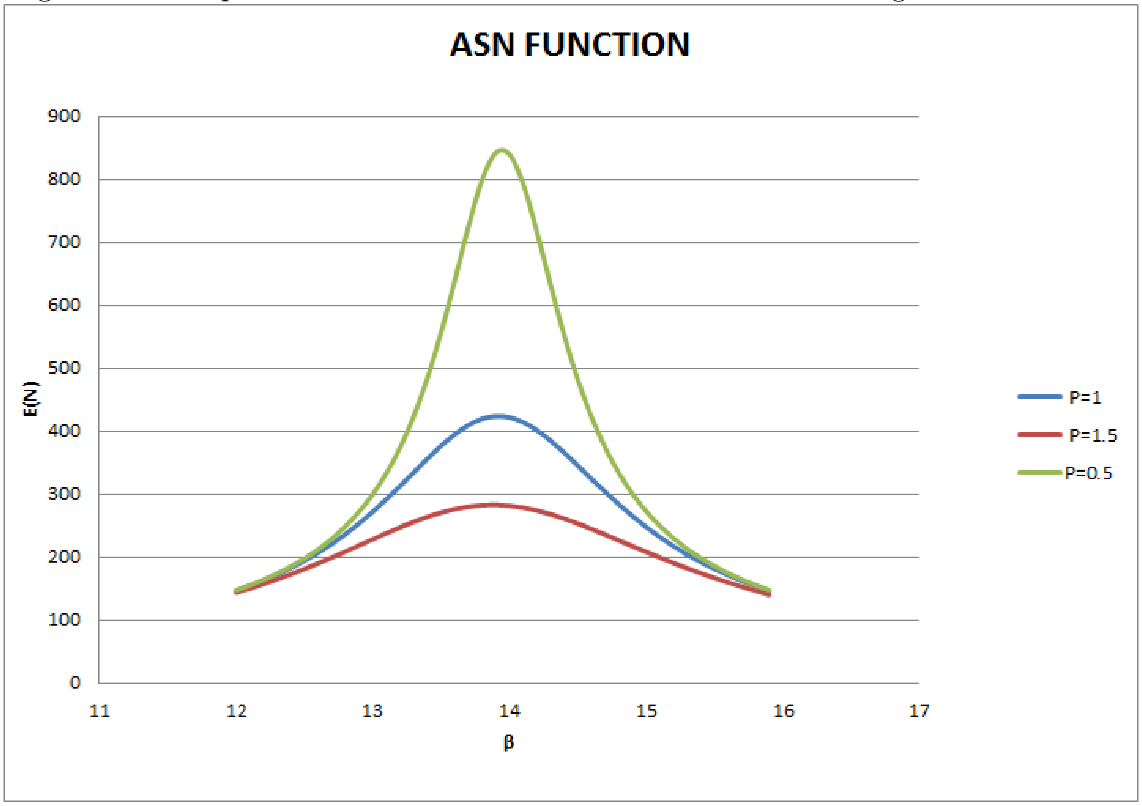


Table 3.2: Values of ASN Function for scale parameter of Nakagami Distribution

$(H_0 : \beta = 13, H_1 : \beta = 15, \alpha = \beta = 0.05)$

E(N)							
$\boldsymbol{\beta}$	**P=0.5**	**P=1**	**P=1.5**	$\boldsymbol{\beta}$	**P=0.5**	**P=1**	**P=1.5**
12.0	147.0455	146.6378	143.4283	14.0	839.3402	421.7279	281.4078
12.1	154.9832	154.3493	150.0561	14.1	792.2010	414.0683	278.4097
12.2	163.8260	162.8449	157.1345	14.2	716.8359	401.3978	273.8638
12.3	173.7373	172.2259	164.6739	14.3	632.3396	384.8466	267.9650
12.4	184.9215	182.6031	172.6755	14.4	551.8982	365.6746	260.9443
12.5	197.6373	194.0947	181.1271	14.5	481.4929	345.0775	253.0451
12.6	212.2150	206.8220	190.0001	14.6	422.3672	324.0585	244.5083
12.7	229.0813	220.9015	199.2439	14.7	373.5567	303.3786	235.5578
12.8	248.7913	236.4311	208.7821	14.8	333.4009	283.5593	226.3924
12.9	272.0708	253.4705	218.5076	14.9	300.2326	264.9189	217.1806
13.0	**299.86749**	**272.0115**	**228.2793**	**15.0**	**272.6206**	**247.6194**	**208.0594**
13.1	333.4053	291.9388	237.9207	15.1	249.4134	231.7108	199.1354
13.2	374.2158	312.9812	247.2219	15.2	229.7115	217.1699	190.4882
13.3	424.0849	334.6612	255.9449	15.3	212.8194	203.9286	182.1736
13.4	484.7660	356.2504	263.8338	15.4	198.2000	191.8949	174.2276
13.5	557.1738	376.7594	270.6304	15.5	185.4371	180.9654	166.6707
13.6	639.6517	394.9803	276.0926	15.6	174.2059	171.0361	159.5109
13.7	725.2166	409.6095	280.0148	15.7	164.2505	162.0063	152.7470
13.8	799.2898	419.4413	282.2477	15.8	155.3675	153.7820	146.3708
13.9	842.1760	423.5988	282.7132	15.9	147.3939	146.2772	140.3695

Figure 3.2: Graph of ASN Function for Scale Parameter of Nakagami Distribution



3.6 Conclusion

The values of OC and ASN functions for the cases $\lambda > \lambda^*$, $\lambda < \lambda^*$ and $\lambda = \lambda^*$ are plotted in **Fig. 3.1** and **Fig. 3.2** respectively. From the **Fig. 3.1** we observe that for $\lambda < \lambda^*\,(\lambda > \lambda^*)$, the OC curve shifts to the right side (left side) of the curve when $\lambda^* = \lambda \pm 0.5$. From the figure it is clear that SPRT is non-robust for $\lambda = \lambda^*$ as the deviation in OC function is significant. Again, from **Fig. 3.2** we observe that for $\lambda < \lambda^*\,(\lambda > \lambda^*)$, the ASN curve shifts above (below) of the curve when $\lambda = \lambda^*$. Both the curves are highly sensitive for the changes in λ. Thus we conclude that for the present model, the SPRT for testing the hypothesis regarding β, is highly non-robust for changes in λ.

Chapter 4

UMVUE of the Stress-Strength Reliability for a Class of Distributions by using the Estimates of Reliability

4.1 Introduction

The problem of increasing reliability of any system is now a well-recognized and rapidly developing branch of engineering. Reliability study is considered essential for proper utilization and maintenance of engineering systems and equipments. Thus, it has become more significant in many fields of industry, transport, communications technology, etc., with the complex mechanization and automation of industrial processes. Under estimation and over estimation of factors associated with reliability may engender great losses. In the reliability engineering the practice of Stress-Strength testing is an important and interesting topic of connotation for the researchers. One of the statistical models of the Stress-Strength testing is the probability $P = Pr(Y > X)$, which represents the performance of an item of Strength Y subject to a Stress X, where X and Y are taken to be non-negative independent continuous random variables. The term Stress-Strength was first introduced by Church and Harries (1970). Since then a lot of work has been done in this direction by various authors.

For a brief review, one may refer to Downton (1973), Tong (1974), Kelly (1976), Sathe and Vande (1981), Chao (1982), Awad (1986), Chaturvedi and Surinder (1999).

In the present Chapter, we consider the following class of distributions proposed by Chaturvedi and Rani (1997), which covers many life time distributions as specific cases.

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}}{\theta^{ab}\Gamma a} exp(-x^c/\theta^b); \quad x,\theta,a,b,c > 0, \tag{4.1.1}$$

where θ is assumed to be unknown and a, b, c are known constants. It is easy to see that $\theta^{\frac{b}{c}}$ is scale parameter and a, b, c are shape parameters.

Note that (4.1.1) represents a Class of life time distributions as it covers the following distributions as the specific cases:

1. For $a = b = c = 1$, (4.1.1) gives the probability density function (pdf) of the one-parameter exponential distribution [see Johnson and Kotz (1970, p.207)].

2. For $b = c = 1$, (4.1.1) becomes the pdf of the gamma distribution [see Johnson and Kotz (1970, p.166)].

3. For $b = c$, (4.1.1) gives the pdf of the generalized gamma distribution [see Johnson and Kotz (1970, p.197)].

4. Taking 'a' as positive integer and $b = c = 1$, (4.1.1) turns out to be the pdf of an Erlang distribution [see Johnson and Kotz (1970, p.166)].

5. For $a = 1$, $b = c$, (4.1.1) represents the pdf of Weibull distribution [see Johnson and Kotz (1970, p.250)].

6. For $a = 1/2$, $b = c = 2$, (4.1.1) is the pdf of halfnormal distribution [see Davis (1952)].

7. For $a = b = 1$, $c = 2$, (4.1.1) turns out to be the pdf of Rayleigh distribution [see Sinha (1986, p.200)].

8. For $a = \alpha/2$, $b = 1$, $c = 2$, (4.1.1) becomes the pdf of the Chi-distribution [see Patel, Kapadia and Owen (1976, p.173)].

9. For $a = 3/2$, $b = 1$, $c = 2$, (4.1.1) gives the pdf of Maxwells failure distribution [see Tyagi and Bhattacharya (1989a, b)].

We conclude from (1) to (9) that the different combinations of a, b and c provides different distributions useful in reliability theory.

In Section 4.2, we have obtained the UMVUE of $P = Pr(Y > X)$ for a Class of distributions. In order to derive the UMVUE of the sampled pdf $\hat{f}(x; a, b, c, \theta)$, we have used the results of the Theorem 4.1 and Theorem 4.2 as given by Chaturvedi and Rani (1997).

Theorem 4.1: The Reliability function $R(t)$ for a specified mission time $(t > 0)$ is

$$R(t) = P(X > t) = \frac{\gamma(a, t^c/\theta^b)}{\Gamma a} \tag{4.1.2}$$

where Γa and $\gamma(a, y)$ are respectively the gamma and incomplete gamma functions given by

$$\Gamma a = \int_0^\infty x^{a-1} exp(-x) dx,$$

and,

$$\gamma(a, y) = \int_y^\infty x^{a-1} exp(-x) dx; \qquad a, y > 0$$

Theorem 4.2: The UMVUE of Reliability function $R(t)$ is given by

$$\hat{R}(t) = \begin{cases} 1 - I_{t^c/T}\left(a, (n-1)a\right) & ;\text{if } T > t^c \\ 0 & ;\text{if } T \leq t^c \end{cases} \tag{4.1.3}$$

Where $I_z(p, q)$ is a incomplete beta function ratio given by

$$I_z(p, q) = \frac{1}{B(p, q)} \int_0^z w^{p-1}(1-w)^{q-1} dw; \qquad 0 < z < 1; p, q > 0$$

4.2 UMVUE of $P = P(Y > X)$ for a Class of Distributions

In order to derive the UMVUE of $P = P(Y > X)$, first we have obtained the UMVUE of sampled pdf which is given by the following Lemma.

Lemma 4.1: The UMVUE of the sampled pdf $f(x; a, b, c, \theta)$ at a specified point x is

$$\hat{f}(x; a, b, c, \theta) = \begin{cases} \dfrac{1}{B(a, (n-1)a)} \dfrac{cx^{ac-1}}{T^a} \left(1 - \dfrac{x^c}{T}\right)^{(n-1)a-1} & ; \text{if } x^c < T \\ 0 & ; \text{otherwise} \end{cases} \tag{4.2.1}$$

Proof: We note that the expectation of $\int_t^\infty \hat{f}(x; a, b, c, \theta)dx$, with respect to T is $R(t)$. Hence,

$$\hat{R}(t) = \int_t^\infty \hat{f}(x; a, b, c, \theta)dx$$

or,

$$\begin{aligned} -\frac{d}{dt}\hat{R}(t) &= -\frac{d}{dt}\left[1 - \int_0^{t^c/T} \frac{1}{B(a, (n-1)a)} {v_1}^{a-1}(1 - v_1)^{(n-1)a-1} dv1\right] \\ &= \frac{1}{B(a, (n-1)a)} \frac{ct^{c-1}}{T^a} \left(1 - \frac{t^c}{T}\right)^{(n-1)a-1} ; \text{if } t^c < T. \end{aligned}$$

Theorem 4.3: Let X and Y be two independent random variables following a class of life-time distributions $f_1(x; a_1, b_1, c_1, \theta_1)$ and $f_2(y; a_2, b_2, c_2, \theta_2)$ respectively, then the UMVUE of $R = P(Y > X)$ is given by

$$\hat{R} = \begin{cases} \dfrac{B\left[\dfrac{c_2}{c_1}j + a_1, (n_1-1)a_1\right]}{B\left[a_1, (n_1-1)a_1\right] B\left[(n_2-1)a_2, a_2\right]} \sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} & \\ \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\dfrac{{T_X}^{c_1/c_2}}{T_Y}\right)^j & ; \text{if } T_X < T_Y \\ & \\ \dfrac{B\left[\dfrac{c_1}{c_2}(j + a_1), (n_2-1)a_2+i+1\right]}{B\left[a_1, (n_1-1)a_1\right] B\left[(n_2-1)a_2, a_2\right]} \sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} & \\ \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \frac{c_1}{c_2} \left(\dfrac{{T_Y}^{c_1/c_2}}{T_X}\right)^{a_1+j} & ; \text{if } T_Y < T_X \end{cases} \tag{4.2.2}$$

Proof: We know that,

$$\begin{aligned} \hat{R} &= P(Y > X) = \int_0^\infty \int_{y=x}^\infty f_2(y; a_2, b_2, c_2, \theta_2) f_1(x; a_1, b_1, c_1, \theta_1) dy dx \\ &= \int_{x=0}^\infty \hat{R}_2(x; a_2, b_2, c_2, \theta_2) \left\{ -\frac{d}{dx} \hat{R}_1(x; a_1, b_1, c_1, \theta_1) \right\} dx \end{aligned}$$

$$= \frac{1}{B[a_1, (n_1-1)a_1]} \int_{x=0}^{min\left(T_X^{1/c_1}, T_Y^{1/c_2}\right)} \left\{1 - I_{x^{c_2}/T_Y}(a_2, (n_2-1)a_2)\right\} \times \frac{c_1 x^{a_1 c_1 - 1}}{{T_X}^{a_1}} \left(1 - \frac{x^{c_1}}{T_X}\right)^{(n_1-1)a_1-1} dx$$

Since $I_X(a, b) = 1 - I_{1-X}(b, a)$, therefore,

$$= \frac{1}{B[a_1, (n_1-1)a_1]} \int_{x=0}^{min\left(T_X^{1/c_1}, T_Y^{1/c_2}\right)} \left\{I_{1-x^{c_2}/T_Y}((n_2-1)a_2, a_2)\right\} \times \frac{c_1 x^{a_1 c_1 - 1\, a_1}}{T_X^{a_1}} \left(1 - \frac{x^{c_1}}{T_X}\right)^{(n_1-1)a_1-1} dx$$

$$= \frac{1}{B[(n_2-1)a_2, a_2]B[a_1,(n_1-1)a_1]} \int_{x=0}^{min\left(T_X^{1/c_1}, T_Y^{1/c_2}\right)} \frac{c_1 x^{a_1c_1-1}}{T_X^{a_1}} \left(1-\frac{x^{c_1}}{T_X}\right)^{(n_1-1)a_1-1} \times \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \left(1-\frac{x^{c_2}}{T_Y}\right)^{(n_2-1)a_2+i} dx \quad (4.2.3)$$

If $T_X < T_Y$ then,

$$= \frac{1}{B[(n_2-1)a_2, a_2]B[a_1,(n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \times \int_{x=0}^{T_X^{1/c_1}} \left(\frac{x^{c_2}}{T_Y}\right)^j \frac{c_1 x^{a_1c_1-1}}{T_X^{a_1}} \left(1-\frac{x^{c_1}}{T_X}\right)^{(n_1-1)a_1-1} dx$$

Let $\dfrac{x^{c_1}}{T_X} = z$ then,

$$= \frac{1}{B[(n_2-1)a_2, a_2]B[a_1,(n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\frac{T_X^{c_2/c_1}}{T_Y}\right)^j \int_0^1 Z^{\frac{c_2}{c_1}j+a_1-1} (1-z)^{(n_1-1)a_1-1} dz$$

$$= \frac{B\left[\frac{c_2}{c_1}j + a_1, (n_1-1)a_1\right]}{B[(n_2-1)a_2, a_2]B[a_1,(n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\frac{T_X^{c_2/c_1}}{T_Y}\right)^j \quad (4.2.4)$$

If $T_Y < T_X$ then,

$$= \frac{1}{B[(n_2-1)a_2, a_2]B[a_1,(n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \times \int_{x=0}^{T_Y^{1/c_2}} \left(\frac{x^{c_1}}{T_X}\right)^j \frac{c_1 x^{a_1c_1-1}}{T_X^{a_1}} \left(1-\frac{x^{c_2}}{T_Y}\right)^{(n_2-1)a_2+i} dx$$

Let $\dfrac{x^{c_2}}{T_Y} = z$ then,

$$= \frac{1}{B[(n_2-1)a_2, a_2]B[a_1, (n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i}$$
$$\times \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \frac{c_1}{c_2} \left(\frac{T_Y^{c_1/c_2}}{T_X}\right)^{j+a_1} \int_0^1 Z^{\frac{c_1}{c_2}(j+a_1)-1}(1-z)^{(n_2-1)a_2+i} dz$$

$$= \frac{B\left[\frac{c_1}{c_2}(j+a_1), (n_2-1)a_2+i+1\right]}{B[(n_2-1)a_2, a_2]B[a_1, (n_1-1)a_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i}$$
$$\times \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \frac{c_1}{c_2} \left(\frac{T_Y^{c_1/c_2}}{T_X}\right)^{j+a_1} \tag{4.2.5}$$

and, hence the Theorem follows.

Chapter 5

A Study of Strength-Reliability for Gompertz Distributed Stress

5.1 Introduction

Gompertz distribution plays a very important role in modelling survival times, human mortality and actuarial tables. In reliability and survival studies, many equipment life are characterized by a increasing hazard rate and having Gompertz distribution with the probability density function

$$f(x;c,\beta)=\beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\};\quad x,c,\beta>0 \tag{5.1.1}$$

This distribution has been studied by various authors. We may refer to Wu et al. (2003, 2004, 2006), Jaheen (2003), Al-Khedhair and El-Gohary (2008) and Saracoglu et al. (2009).

The problem of increasing reliability of any system is now a well-recognized and rapidly developing branch of engineering. Reliability study is considered essential for proper utilization and maintenance of engineering systems and equipments. Thus, it has become more significant in many fields of industry, transport, communications technology, etc., with the complex mechanization and automation of industrial processes. Under estimation and over estimation of factors associated with reliability may engender great losses. In the reliability engineering the practice of Stress-Strength testing is an important and interesting topic of connotation for the researchers. One of the statistical models of the Stress-Strength testing is the probability $P = Pr(Y > X)$, which represents the performance of an item of Strength Y subject to a Stress X, where X and Y are taken to be non-negative independent continuous random variables. The term Stress-Strength was first introduced by Church and Harries (1970). Since, then a lot of work has been done in this direction by various authors. For a brief review, one may refer to Downton (1973), Tong (1974), Kelly (1976), Sathe and Vande (1981), Chao (1982), Awad (1986), Chaturvedi and Surinder (1999).

In the present Chapter, we consider Gompertz distribution given in (5.1.1). It is assumed that the random variable X represents the stress that an item faces, follows the distribution having the pdf given at (5.1.1) and Strength Y follows Power function distribution with pdf given by

$$g(y) = \frac{\mu}{\theta}\left(\frac{y}{\theta}\right)^{\mu-1}; \quad 0 < y < \theta, \mu > 0 \tag{5.1.2}$$

where θ and μ are scale and shape parameters respectively.

In Section 5.2, we obtain the strength reliability for finite strength. In Section 5.3, Stress-Strength reliability is obtained when strength follows Power function distribution and stress faces Gompertz distribution. Section 5.4 is a discussion followed by an illustrative example in Section 5.5.

5.2 Strength Reliability for Finite Strength

A finite time distribution should be capable of describing the random variations in failure time of equipment. The designed lifetime (Strength) of equipment should only be limited to a finite range. This is because the Strength of manufactured product is always a function of the combination of a set of sub-components and the sub-components are not likely to have an infinite lifetime. Since, the maximum possible value of Strength distribution is θ. Y cannot exceed X if X exceeds θ. The total unreliability of the items is therefore, obtained by $P(X > \theta)$. Alam and Roohi (2003) have termed it as probability of "disaster".

Theorem 5.1: If the random variable X and Y follows the Gompertz distribution given at (5.1.1) and Power function distribution given at (5.1.2) respectively, then $\alpha = P(X > \theta)$ is given by

$$\alpha \equiv P(X > \theta) = \exp\left\{-\frac{\beta}{c}\left(e^{m} - 1\right)\right\} \tag{5.2.1}$$

where $m = c\theta$.

Proof: We know that

$$\alpha \equiv P(X > \theta) = \int_{\theta}^{\infty} \beta e^{cx} \exp\left\{-\frac{\beta}{c}\left(e^{m} - 1\right)\right\} dx \tag{5.2.2}$$

On substituting $t = \frac{\beta}{c}\left(e^{cx} - 1\right)$, we get

$$\alpha = \int_{\frac{\beta}{c}\left(e^{c\theta}-1\right)}^{\infty} e^{-t} dt$$

$$\alpha = \exp\left\{-\frac{\beta}{c}\left(e^{m} - 1\right)\right\}, \text{ where } m = c\theta, \tag{5.2.3}$$

and, hence the Theorem follows.

5.2.1 Numerical Values for the Probability of Disaster and 'm' for Different Values of 'c' and 'β'

Here, the numerical values of Probability of Disaster $\alpha = P(X > \theta)$ for different combinations of m, c and β are computed from equation (5.2.1). It can be interpreted from Table 5.1 that Probability of Disaster decreases with the increase in m.

Alternatively, we may also obtain in table 5.2, the numerical values of m for fixed β, c and at different tolerance levels α from equation (5.2.1). Further, these values are used to obtain the optimum cost for manufacturing of items at desired tolerance level.

Table 5.1: Numerical values for Probability of disaster $\alpha = P(X > \theta)$ for different values of m, c and β

m	c=0.05, $\beta = 0.05$	c=0.03, $\beta = 0.05$	c=0.05, $\beta = 0.03$
0.5	0.5227	0.3392	0.6776
1.0	0.1794	0.0571	0.3567
1.5	0.0308	0.003	0.1238
2.0	0.0017	0.00002	0.0216
2.5	0.00001	0	0.0012
3.0	0	0	0.00001

Table 5.2: Values of m at different levels of α

c=0.05, $\beta = 0.03$						
α	0.05	0.02	0.01	0.001	0.0001	0.00001
$m = c\theta$	1.7906	2.0176	2.1605	2.5268	2.7943	3.0051

Remarks:

1. Table 5.1 depicts the probability of disaster α, for Gompertz distribution. It is interested to note that the probability of disaster decreases with the increase in 'm'.

2. Table 5.2 shows the values of 'm' for different values of α for fixed 'β' and 'c'. It is obvious that the value of 'm' increases as α decrease i.e. the ultimate strength capacity must increase if we wish to have a small tolerance-level.

5.3 Stress and Strength Reliability

For the Stress-Strength model the probability $P = Pr(Y > X)$, when the random variable X and Y follows the pdfs (5.1.1) and (5.1.2), respectively is given by the following theorem.

Theorem 5.2: $P = Pr(Y > X)$ is given by

$$P = 1 - \exp\left\{-\frac{\beta}{c}\left(e^{m}-1\right)\right\} - \frac{1}{m^{\mu}}\int_0^{\frac{\beta}{c}(e^{m}-1)}\left[\log\left(\frac{c}{\beta}t+1\right)\right]^{\mu} e^{-t}dt, \tag{5.3.1}$$

where $m = c\theta$.

Proof:

$$P(Y > X) = \int_0^{\theta}\int_x^{\theta} f(x;\beta,c)g(y;\mu,\theta)dydx \tag{5.3.2}$$

Substituting $y = vx$ in (5.3.2), we get

$$= \int_0^{\theta}\int_1^{\frac{\theta}{x}} xf(x;\beta,c)g(vx;\mu,\theta)dvdx$$

or,

$$= \int_0^{\theta}\int_1^{\frac{\theta}{x}} x\beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\}\left(\frac{\mu}{\theta}\right)\left(\frac{vx}{\theta}\right)dvdx$$

$$= \frac{\mu}{\theta^{\mu}}\int_0^{\frac{m}{c}}\int_1^{\frac{m}{cx}} x^{\mu}\beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\}v^{\mu-1}dvdx$$

$$= \frac{1}{\theta^{\mu}}\int_0^{\frac{m}{c}} x^{\mu}\beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\}\left[\left(\frac{m}{cx}\right)^{\mu}-1\right]dx$$

$$= \int_0^{\frac{m}{c}} \beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\}dx - \frac{1}{\theta^{\mu}}\int_0^{\frac{m}{c}} x^{\mu}\beta e^{cx}\exp\left\{-\frac{\beta}{c}(e^{cx}-1)\right\}dx \tag{5.3.3}$$

Now, substituting $\frac{\beta}{c}(e^{cx}-1) = t$ in (3.3), we get

$$= \int_0^{\frac{\beta}{c}(e^{m}-1)} e^{-t}dt - \frac{1}{m^{\mu}}\int_0^{\frac{\beta}{c}(e^{m}-1)}\left[\log\left(\frac{c}{\beta}t+1\right)\right]^{\mu} e^{-t}dt$$

or,

$$P(Y > X) = 1 - \exp\left\{-\frac{\beta}{c}\left(e^{m}-1\right)\right\} - \frac{1}{m^{\mu}}\int_0^{\frac{\beta}{c}(e^{m}-1)}\left[\log\left(\frac{c}{\beta}t+1\right)\right]^{\mu} e^{-t}dt \tag{5.3.4}$$

The expression obtained in equation (5.3.4) is complicated since, the integral cannot be solved easily. However, the numericle values can be obtained by using Mathematica Software.

Table 5.3: Strength-reliability of an item for c=0.05, $\beta = 0.03$ and varying values of m and μ

$\downarrow m \overrightarrow{\mu}$	2	4	6	8
0.5	0.21211	0.25582	0.27473	0.28528
1.0	0.43138	0.51835	0.55518	0.57538
1.5	0.62398	0.73855	0.78373	0.80726
2.0	0.76210	0.87920	0.91922	0.93785
2.5	0.84503	0.94648	0.97425	0.98490
3.0	0.89229	0.97406	0.99122	0.99633
3.5	0.92087	0.98599	0.99652	0.99893
4.0	0.93928	0.99095	0.99631	0.99560
4.5	0.95213	0.99487	0.99922	0.99985
5.0	0.96122	0.99663	0.99959	0.99993
5.5	0.96795	0.99770	0.99976	0.99997
6.0	0.97307	0.99837	0.99986	0.99998

5.4 Discussion

While manufacturing an item, if the strength of an item follows Power function distribution, it is likely that the possible values of θ may have an upper limit say θ_0. For example, the capacity of accelerating an engine must be subject to maximum possible speed. For a fixed tolerance level α, suppose θ_α is the desired value of θ. In case $\theta_\alpha < \theta_0$, we may obtain the required value of μ say μ_α, by using Table 5.3, so that the item is manufactured with the strength distribution having parameters $(\mu_\alpha, \theta_\alpha)$ and consequently the desired strength reliability is achieved. However, if $\theta_\alpha > \theta_0$, we will have to either adjust α or look for an alternate item.

5.5 An Illustrative Example

Without loss of generality, let us suppose that the maximum possible value of m is 5. For $\alpha \leq 0.01$ we must have $m \geq 2$. Since m cannot exceed 5 we have the option of fixing the item in such a way that $2 \leq m \leq 5$ i.e. $40 \leq \theta \leq 100$ and corresponding value of μ leads to a maximum of $P(Y > X)$. The cost factor of adjusting the parameters may be taken into consideration here as the cost of varying θ and μ may be different. Theoretically the costs here may be increasing or decreasing function of θ and μ depending upon the nature of the parameters. Usually C(Y) is an increasing function of Y if Y is the mean strength. In our case $E(Y) = \dfrac{\mu\theta}{\mu + 1}$ implies that the mean strength increases by increasing either of the two parameters. Hence, we may assume the two costs to be an increasing function of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem may be formulated as follows:

Let C_1 be the cost of adjusting one unit of θ and C_2 be the cost of adjusting one unit of μ.

Minimize $C = C_1\theta + C_2\mu$ subject to $40 \leq \theta \leq 100$ and $P(Y > X) \geq 0.99$

The problem may be solved analytically as follows:

Using **Table 5.3** for $m =$ 2, 2.5, 3, 3.5, 4, 4.5, 5 i.e. $\theta =$ 40, 50, 60, 70, 80, 90, 100 and find those values of μ for which $P(Y > X) \geq 0.99$. Evaluate the cost function for each pair of (θ, μ): Clearly, the minimum of the cost lies at $60C_1 + 6C_2$ depending upon the numerical

Table 5.4: Table for obtaining the Optimum cost of manufacturing item

θ	μ	$C_1\theta + C_2\mu$	θ	μ	$C_1\theta + C_2\mu$
60	6	$60C_1 + 6C_2$	90	4	$90C_1 + 4C_2$
60	8	$60C_1 + 8C_2$	90	6	$90C_1 + 6C_2$
70	6	$70C_1 + 6C_2$	90	8	$90C_1 + 8C_2$
70	8	$70C_1 + 8C_2$	100	4	$100C_1 + 4C_2$
80	4	$80C_1 + 4C_2$	100	6	$100C_1 + 6C_2$
80	6	$80C_1 + 6C_2$	100	8	$100C_1 + 8C_2$
80	8	$80C_1 + 8C_2$			

values of C_1 and C_2.

Chapter 6

Study of Stress-Strength Reliability Relationship among the Parameters of a Class of Distributions

6.1 Introduction

The problem of increasing reliability of any system is now a well recognized and rapidly developing branch of engineering. Reliability study is considered essential for proper utilization and maintenance of engineering systems and equipments. Thus, it has become more significant in many fields of industry, transport, communications technology, etc., with the complex mechanization and automation of industrial processes. Under estimation and over estimation of factors associated with reliability may engender great losses. In the reliability engineering the practice of Stress-Strength testing is an important and interesting topic of connotation for the researchers. One of the statistical models of the Stress-Strength testing is the probability $P = Pr(Y > X)$, which represents the performance of an item of Strength Y subject to a Stress X, where X and Y are taken to be non-negative independent continuous random variables. The term Stress-Strength was first introduced by Church and Harries (1970). Since then a lot of work has been done in this direction by various authors.

For a brief review, one may refer to Downton (1973), Tong (1974), Kelly (1976), Sathe and Vande (1981), Chao (1982), Awad(1986), Chaturvedi and Surinder (1999).

In the present Chapter, we consider the following class of distributions proposed by Chaturvedi and Rani(1997), which covers many life time distributions as specific cases.

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}}{\theta^{ab}\Gamma a}\exp\left(-x^c/\theta^b\right); \quad x,\theta,a,b,c>0, \tag{6.1.1}$$

where θ is assumed to be unknown and a, b, c are known constants. It is easy to see that $\theta^{\frac{b}{c}}$ is scale parameter and a, b, c are shape parameters.

Note that (6.1.1) represents a Class of life time distributions as it covers the following distributions as the specific cases:

1. For $a = b = c = 1$, (6.1.1) gives the probability density function (pdf) of the one-parameter exponential distribution [see Johnson and Kotz (1970, p.207)].

2. For $b = c = 1$, (6.1.1) becomes the pdf of the gamma distribution [see Johnson and Kotz (1970, p.166)].

3. For $b = c$, (6.1.1) gives the pdf of the generalized gamma distribution [see Johnson and Kotz (1970, p.197)].

4. Taking 'a' as positive integer and $b = c = 1$, (6.1.1) turns out to be the pdf of an Erlang distribution [see Johnson and Kotz (1970, p.166)].

5. For $a = 1$, $b = c$, (6.1.1) represents the pdf of Weibull distribution [see Johnson and Kotz (1970, p.250)].

6. For $a = 1/2$, $b = c = 2$, (6.1.1) is the pdf of halfnormal distribution [see Davis (1952)].

7. For $a = b = 1$, $c = 2$, (6.1.1) turns out to be the pdf of Rayleigh distribution [see Sinha (1986, p.200)].

8. For $a = \alpha/2$, $b = 1$, $c = 2$, (6.1.1) becomes the pdf of the Chi-distribution [see Patel, Kapadia and Owen (1976, p.173)].

9. For $a = 3/2$, $b = 1$, $c = 2$, (6.1.1) gives the pdf of Maxwells failure distribution [see Tyagi and Bhattacharya (1989a, b)].

We conclude from (1) to (9) that the different combinations of a, b and c provides different distributions useful in reliability theory.

It is assumed that the random variable X represents the strength that an item faces, follows the distribution having the pdf given at (6.1.1) and strength Y follows Power function distribution with the pdf given by

$$g(y) = \left(\frac{\mu}{\sigma}\right)\left(\frac{y}{\sigma}\right)^{\mu-1}; \qquad 0 < y < \sigma, \mu > 0 \tag{6.1.2}$$

where, σ and μ are the scale and shape parameters respectively.

6.2 Strength Reliability for Finite Strength

A finite time distribution should be capable of describing the random variations in failure time of equipment. The designed lifetime (Strength) of equipment should only be limited to a finite range. This is because the Strength of manufactured product is always a function of the combination of a set of subcomponents and the subcomponents are not likely to have an infinite lifetime. Since, the maximum possible value of Strength distribution is σ. Y cannot exceed X if X exceeds σ. The total unreliability of the items is therefore, obtained by $P(X > \sigma)$. Alam and Roohi (2003) have termed it as probability of "disaster".

Theorem 6.1: If the random variables X and Y follow a class of distributions given at (6.1.1) and Power function distribution given at (6.1.2) respectively, then $P(X > \sigma)$ is given by

$$\alpha \equiv P(X > \sigma) = 1 - \frac{\gamma(a, m)}{\Gamma a}, \tag{6.2.1}$$

where, $\gamma(a, m)$ is an incomplete gamma function given by $\gamma(a, m) = \int_0^m e^{-t} t^{m-1} dt$ and, $m = \dfrac{\sigma^c}{\theta^b}$.

Proof: Here,

$$\alpha \equiv P(X > \sigma) = \int_{\sigma}^{\infty} \frac{cx^{ac-1}}{\theta^{ab}\Gamma a} \exp\left(-x^c/\theta^b\right) dx$$

On substituting $\dfrac{x^c}{\theta^b} = t$, we get

$$\alpha = \int_{\sigma^c/\theta^b}^{\infty} \exp\left(-t\right) t^{a-1} dt$$

or,

$$\alpha = 1 - \frac{\gamma(a, m)}{\Gamma a},$$

where $m = \dfrac{\sigma^c}{\theta^b}$, and, hence the Theorem follows.

Remarks:

1. Table 6.1 depicts the probability of disaster α, for various life time distributions. It is interested to note that the probability of disaster decreases with the increase in 'm' and probability of disaster increases with the increase in 'a'.

2. Table 6.2 shows the values of 'm' for the case $a = b = c = 1$. It is obvious that the value of m increases as α decrease i.e. the ultimate strength capacity σ must increase if we wish to have a small tolerance-level.

Table 6.1: Numerical values for Probability of disaster $\alpha = P(X > \sigma)$ for different values of m, c and β

Exponential distribution		weibull distribution	Rayleigh distribution	Half Normal distribution	Maxwell failure distribution	Chi-distribution ($a = \alpha/2$), Gamma and Generalised gamma distribution ($a > 0$)				
	a=1	a=1	a=1	a=0.5	a=1.5	a=1	a=2	a=3	a=4	a=5
0.5	0.6065	0.6065	0.6065	0.3173	0.8013	0.6065	0.9098	0.9856	0.9982	0.9998
1.0	0.3679	0.3679	0.3679	0.1573	0.5724	0.3679	0.7358	0.9197	0.9810	0.9963
1.5	0.2231	0.2231	0.2231	0.0833	0.3916	0.2231	0.5578	0.8088	0.9344	0.9814
2.0	0.1353	0.1353	0.1353	0.0455	0.2615	0.1353	0.4060	0.6767	0.8571	0.9473
3.0	0.0498	0.0498	0.0498	0.0143	0.1116	0.0498	0.1991	0.4232	0.6472	0.8153
4.0	0.0183	0.0183	0.0183	0.0047	0.046	0.0183	0.0916	0.2381	0.4335	0.6288
6.0	0.0025	0.0025	0.0025	0.000532	0.0074	0.0025	0.0174	0.0620	0.1512	0.2851
8.0	0.000335	0.000335	0.000335	0.0000633	0.0011	0.0003	0.0030	0.0138	0.0424	0.0996
10.0	0.0000454	0.0000454	0.0000454	0.0000077	0.000170	0.0000	0.0005	0.0028	0.0103	0.0293

Table 6.2: Values of m for selected tolerance level α.

$\theta = 1$ and $a = b = c = 1$							
α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
$m = \sigma$	2.3026	2.9957	3.9120	4.6052	6.90775	9.2103	11.5129

6.3 Stress and Strength Reliabilty

For the Stress-Strength model the probability $P = Pr(Y > X)$, when the random variable X and Y follows the pdf's (6.1.1) and (6.1.2), respectively is given by the following theorem.

Theorem 6.2: $P = Pr(Y > X)$ is given by

$$\hat{P} = \frac{\gamma(a,m)}{\Gamma a} - \frac{\gamma\left(a + \frac{\mu}{c}, m\right)}{m^{\mu/c}\Gamma a} \tag{6.3.1}$$

where, $\gamma(a,m)$ and $\gamma\left(a + \frac{\mu}{c}, m\right)$ are incomplete gamma functions.

Proof: $P(Y > X)$ is given by

$$P(Y > X) = \int_0^{\sigma}\int_0^{\sigma} f(x)g(y)dydx \tag{6.3.2}$$

Substituting $y = vx$ in (6.3.2), we get

$$\begin{aligned} P(Y > X) &= \int_0^{\sigma}\int_0^{\sigma/x} xf(x)g(vx)dvdx \\ &= \int_0^{\left(m\theta^b\right)^{1/c}}\int_1^{\left(\frac{m\theta^b}{x}\right)^{1/c}} x\frac{cx^{ac-1}}{\theta^{ab}\Gamma a}\exp\left(-x^c/\theta^b\right)\frac{\mu}{\sigma}\left(\frac{vx}{\sigma}\right)^{\mu-1} dvdx \\ &= \frac{1}{\left(m\theta^b\right)^{\mu/c}}\int_0^{\left(m\theta^b\right)^{1/c}} \frac{cx^{ac+\mu-1}}{\theta^{ab}\Gamma a}\exp\left(-x^c/\theta^b\right)\left[\left(\frac{\left(m\theta^b\right)^{1/c}}{x}\right)^{\mu} - 1\right] dx \end{aligned}$$

$$= \frac{c}{\theta^{ab}\Gamma a}\int_0^{\left(m\theta^b\right)^{1/c}} x^{ac-1}\exp\left(-x^c/\theta^b\right)dx - \frac{c}{\left(m\theta^b\right)^{\mu/c}\theta^{ab}\Gamma a}\int_0^{\left(m\theta^b\right)^{1/c}} x^{ac+\mu-1}\exp\left(-x^c/\theta^b\right)dx$$

On substituting $\dfrac{x^c}{\theta^b} = t$ and solving, we get

$$= \frac{1}{\Gamma a}\int_0^m \exp(-t)t^{a-1}dt - \frac{1}{m^{\mu/c}\Gamma a}\int_0^m \exp(-t)t^{a+\mu-1}dt$$

$$= \frac{\gamma(a,m)}{\Gamma a} - \frac{\gamma\left(a+\frac{\mu}{c},m\right)}{m^{\mu/c}\Gamma a}.$$

and, hence the theorem follows.

Table 6.3: Strength-reliability of an item for selected values of m and μ

	Exponential distribution ($a=1$)				Half normal distribution ($a=1/2$)				Maxwell Failure distribution ($a=3/2$)			
m	$\mu=2$	$\mu=4$	$\mu=6$	$\mu=8$	$\mu=2$	$\mu=4$	$\mu=6$	$\mu=8$	$\mu=2$	$\mu=4$	$\mu=6$	$\mu=8$
1.0	0.5518	0.6285	0.632	0.6321	0.4151	0.6918	0.8025	0.8342	0.2767	0.3874	0.4191	0.4261
1.5	0.6919	0.7732	0.7768	0.7769	0.5112	0.7834	0.8827	0.9097	0.4084	0.5573	0.5978	0.6065
2.0	0.7838	0.8614	0.8646	0.8647	0.5852	0.8419	0.927	0.949	0.5132	0.6835	0.7275	0.7367
2.5	0.8449	0.9151	0.9179	0.9179	0.6434	0.8812	0.9529	0.9704	0.5946	0.7738	0.8176	0.8264
3.0	0.8861	0.9479	0.9502	0.9502	0.6896	0.9086	0.9686	0.9825	0.6571	0.8373	0.8788	0.8868
3.5	0.9144	0.968	0.9698	0.9698	0.7267	0.9282	0.9785	0.9894	0.7054	0.8815	0.9196	0.9268
4.0	0.9341	0.9802	0.9817	0.9817	0.7568	0.9426	0.9849	0.9935	0.743	0.9123	0.9467	0.9529
4.5	0.9481	0.9877	0.9889	0.9889	0.7816	0.9533	0.9891	0.9959	0.7727	0.9338	0.9645	0.9698
5.0	0.9582	0.9924	0.9932	0.9933	0.8021	0.9614	0.9919	0.9974	0.7965	0.949	0.9762	0.9807
5.5	0.9658	0.9952	0.9959	0.9959	0.8194	0.9677	0.9939	0.9983	0.8158	0.9598	0.9839	0.9877
6.0	0.9715	0.997	0.9975	0.9975	0.834	0.9727	0.9953	0.9989	0.8317	0.9676	0.989	0.9921

6.4 Discussion

While manufacturing an item, if the strength of an item follows Power function distribution, it is likely that the possible values of σ may have an upper limit say σ_0. For example, the capacity of accelerating an engine must be subject to maximum possible speed. For a fixed tolerance level α, suppose σ_α is the desired value of σ. In case $\sigma_\alpha < \sigma_0$, we may obtain the required value of μ say μ_α, by using Table 6.3, so that the item is manufactured with the strength distribution having parameters $(\mu_\alpha, \sigma_\alpha)$ and consequently the desired strength reliability is achieved. However, if $\sigma_\alpha > \sigma_0$, we will have to either adjust α or look for an alternate item.

6.5 An Illustrative Example

Without loss of generality, let us assume that the mean stress $\theta = 1$ and for the case $a = b = c = 1$ so that $m = \sigma$. Let us suppose that the maximum possible value of σ is 6. For $\alpha \leq 0.01$ we must have $m = \sigma \geq 5$. Since σ cannot exceed 6 we have the option of fixing the item in such a way that $5 \leq \sigma \leq 6$ and corresponding value of μ leads to a maximum of $P(Y > X)$. The cost factor of adjusting the parameters may be taken into consideration here as the cost of varying σ and μ may be different. Theoretically the costs here may be increasing or decreasing function of σ and μ depending upon the nature of the parameters. Usually $C(X)$ is an increasing function of X if X is the mean strength. In our case $E(Y) = \mu\sigma/(\mu + 1)$ implies that the mean strength increases by increasing either of the two parameters. Hence we may assume the two costs to be an increasing function of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem may be formulated as follows:

Let C_1 be the cost of adjusting one unit of 'μ' and C_2 be the cost of adjusting one unit of 'σ'.

Minimize $C = C_1\mu + C_2\sigma$ Subject to $5 \leq \sigma \leq 6$ and $P(Y > X) \geq 0.99$.

The problem may be solved analytically as follows:

Look into Table (6.3) for $\sigma = 5, 5.5$ and 6 and find those values of 'μ' for which $P(Y > X) \geq 0.99$. Evaluate the cost function for each pair of (σ, μ):

Table 6.4: Table for obtaining the Optimum cost of manufacturing item

σ	μ	$C_1\mu + C_2\sigma$	σ	μ	$C_1\mu + C_2\sigma$
60	6	$60C_1 + 6C_2$	90	4	$90C_1 + 4C_2$
60	8	$60C_1 + 8C_2$	90	6	$90C_1 + 6C_2$
70	6	$70C_1 + 6C_2$	90	8	$90C_1 + 8C_2$
70	8	$70C_1 + 8C_2$	100	4	$100C_1 + 4C_2$
80	4	$80C_1 + 4C_2$	100	6	$100C_1 + 6C_2$
80	6	$80C_1 + 6C_2$	100	8	$100C_1 + 8C_2$
80	8	$80C_1 + 8C_2$			

Chapter 7

Study of $P(Y > X)$ for Exponential Distribution under Type I and Type II Censoring by using the Estimates of Reliability

7.1 Introduction

In life testing and reliability experiments there are several situations where the complete information on failure items is not possible to obtain. Such incomplete data are called censored data. Type I and Type II schemes are the most commonly used censoring schemes. In Type I censoring scheme an experiment is stopped at a pre determined time T and in Type II Censoring scheme an experiment is stopped at a pre determined r^{th} failure (r is prefixed). The problem of estimating the reliability function $R(t) = P(X > t)$ and the reliability under stress-strength set-up i.e. $R = P(Y > X)$ is considered by various authors. For a brief review, one may refer to Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Chaturvedi and Surinder (1999), Sathe and Shah (1981), Constantine, Karson and Tse (1986).

In the present Chapter, we consider the problem of point estimation of the stress-strength reliability for exponential distribution under complete sample, Type I and Type II censoring. In order to obtain the MLE and UMVUE of $R = P(Y > X)$, we first obtain the estimates of $R(t)$. Further, the estimates of $R(t)$ are used to find the MLE and UMVUE for $R = P(Y > X)$.

In Section 7.2, we obtain the MLE and UMVUE of $R = P(Y > X)$ under complete sample. In Section 7.3 and 7.4, the MLE and UMVUE of R are obtained under Type II and Type I censoring schemes, respectively.

7.2 Point Estimation of $R = P(Y > X)$ under Complete Sample

Let the random variable X follow the exponential distribution given by

$$f(x;\lambda) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right); \qquad x;\lambda > 0 \tag{7.2.1}$$

Theorem 7.1: The UMVUE of $R(t)$ is given by

$$\hat{R}(t) = \begin{cases} \left(1 - \dfrac{t}{n\bar{x}}\right)^{n-1} & ; \quad t \le n\bar{x} \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{7.2.2}$$

Proof: Let $X_1, X_2, ..., X_n$ be a random failure time following the exponential distribution with scale parameter λ. We know that $\bar{x}$ is sufficient for λ and also $\bar{x}$ belongs to exponential family, it is also complete. Let us define a random variable

$$T = \begin{cases} 1; & x_1 \ge t \\ 0; & x_1 < t \end{cases} \tag{7.2.3}$$

It is easy to check that T is unbiased for $R(t)$. Now applying Rao-Blackwellization, it follows from (7.2.3) that

$$\begin{aligned}\hat{R}(t) &= E(T|\bar{x}) \\ &= \int_t^\infty f(x_1, \lambda|\bar{x})dx_1 \\ f(x_1, \lambda|\bar{x}) &= \frac{f(x_1, \bar{x})}{f(\bar{x})}\end{aligned}$$

Now, the joint pdf of $f(x_1, \bar{x})$ is given by

$$f(x_1, \bar{x}) = \frac{n(n-1)^{n-2}}{\lambda^n \Gamma(n-1)} \left(\frac{n\bar{x} - x_1}{n-1}\right)^{n-2} \exp(-n\bar{x}/\lambda); 0 < x < n\bar{x}$$

and,

$$f(\bar{x}) = \frac{n^n}{\Gamma(n)\lambda^n} \bar{x}^{n-1} \exp(-n\bar{x}/\lambda)$$

Thus,

$$f(x_1, \lambda|\bar{x}) = \frac{n-1}{n\bar{x}} \left(1 - \frac{x_1}{n\bar{x}}\right)^{n-2}$$

Therefore, the UMVUE of $R(t)$ is given by

$$\begin{aligned}\hat{R}(t) &= \int_t^{n\bar{x}} \frac{n-1}{n\bar{x}} \left(1 - \frac{x_1}{n\bar{x}}\right)^{n-2} dx_1 \\ &= \left(1 - \frac{t}{n\bar{x}}\right)^{n-1}\end{aligned}$$

Lemma 7.2(a): The UMVUE of the sampled pdf $f(x, \lambda)$ at a specified point x is

$$\hat{f}(x, \lambda) = \begin{cases} \frac{n-1}{n\bar{x}} \left(1 - \frac{x}{n\bar{x}}\right)^{n-2}; & x \leq n\bar{x} \\ 0 \quad ; & \text{otherwise} \end{cases} \tag{7.2.4}$$

Proof: We note that the expectation of $\int_t^\infty f(x,\lambda)dx$ with respect to $\bar{x}$ is Rt.

Hence,

$$\hat{R}(t) = \int_t^\infty f(x,\lambda)dx$$

Or,

$$-\frac{d\hat{R}(t)}{dt} = \hat{f}(t,\lambda)$$

The result now follows from theorem 7.1.

Theorem 7.2: The UMVUE of $R = P(Y > X)$ is given by

$$\hat{R} = \begin{cases} \sum_{i=0}^{n_2-1} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2-i)\Gamma(n_1+i)} \left(\frac{n_1\bar{x}}{n_2\bar{y}}\right)^i ; & \text{if } n_1\bar{x} < n_2\bar{y} \\ \sum_{i=0}^{n_1-2} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-i+1)\Gamma(n_2+i+1)} \left(\frac{n_2\bar{y}}{n_1\bar{x}}\right)^{i+1} ; & \text{if } n_2\bar{y} < n_1\bar{x} \end{cases} \tag{7.2.5}$$

Proof: It follows from lemma 7.2(a) that the UMVUEs of $f_1(x;\lambda_1)$ and $f_2(y;\lambda_2)$ at specified point 'x' and 'y' are respectively,

$$\hat{f}_1(x,\lambda) = \begin{cases} \frac{n_1-1}{n_1\bar{x}}\left(1-\frac{x}{n_1\bar{x}}\right)^{n_1-2} ; & x \le n_1\bar{x} \\ 0 ; & \text{otherwise} \end{cases}$$

and,

$$\hat{f}_2(y,\lambda_2) = \begin{cases} \frac{n_2-1}{n_2\bar{y}}\left(1-\frac{y}{n_2\bar{y}}\right)^{n_2-2} ; & y \le n_2\bar{y} \\ 0 ; & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} \hat{R} &= \int_0^\infty \int_x^\infty \hat{f}_2(y;\lambda_2)\hat{f}_1(x;\lambda_1)dydx \\ &= \int_0^\infty \hat{R}(x;\lambda_2)\left\{-\frac{d}{dx}\hat{R}(x,\lambda_1)\right\}dx \\ &= \frac{n_1-1}{n_1\bar{x}} \int_0^{min(n_1\bar{x},n_2\bar{y})} \left(1-\frac{x}{n_2\bar{y}}\right)^{n_2-1}\left(1-\frac{x}{n_1\bar{x}}\right)^{n_1-2} dx, \end{aligned}$$

if $n_1\bar{x} < n_2\bar{y}$ then,

$$= \frac{n_1 - 1}{n_1\bar{x}} \int_0^{n_1\bar{x}} \left(1 - \frac{x}{n_2\bar{y}}\right)^{n_2-1} \left(1 - \frac{x}{n_1\bar{x}}\right)^{n_1-2} dx$$

$$= \sum_{i=0}^{n_2-1} (-1)^i \left(\frac{n_1 - 1}{n_1\bar{x}}\right) \binom{n_2 - 1}{i} \int_0^{n_1\bar{x}} \left(\frac{x}{n_2\bar{y}}\right)^i \left(1 - \frac{x}{n_1\bar{x}}\right)^{n_1-2} dx$$

Let, $1 - \dfrac{x}{n_1\bar{x}} = z$

$$= \sum_{i=0}^{n_2-1} (-1)^i \left(\frac{n_1 - 1}{n_1\bar{x}}\right) \binom{n_2 - 1}{i} \left(\frac{n_1\bar{x}}{n_2\bar{y}}\right)^i \int_0^1 z^{n_1-2}(1-z)^{i+1-1} dz$$

$$= \sum_{i=0}^{n_2-1} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_2 - i)\Gamma(n_1 + i)} \left(\frac{n_1\bar{x}}{n_2\bar{y}}\right)^i; \qquad \text{if } n_1\bar{x} < n_2\bar{y}$$

Now, if $n_2\bar{y} < n_1\bar{x}$ then,

$$\hat{R} = \frac{n_1 - 1}{n_1\bar{x}} \int_0^{n_2\bar{y}} \left(1 - \frac{x}{n_2\bar{y}}\right)^{n_2-1} \left(1 - \frac{x}{n_1\bar{x}}\right)^{n_1-2} dx$$

Proceeding in similar way as above, we get

$$= \sum_{i=0}^{n_1-2} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1 - i + 1)\Gamma(n_2 + i + 1)} \left(\frac{n_2\bar{y}}{n_1\bar{x}}\right)^{i+1}; \qquad \text{if } n_2\bar{y} < n_1\bar{x}$$

Theorem 7.3: The MLE of $R(t)$ is given by

$$\tilde{R}(t) = \exp(-t/\bar{x}) \tag{7.2.6}$$

Lemma 7.2(b): The MLE of $f(x;\lambda)$ at a specified point x is

$$\tilde{f}(x;\lambda) = \frac{1}{\bar{x}} \exp(-x/\bar{x}) \tag{7.2.7}$$

Proof: The proof follows from the fact that

$$-\frac{d}{dt}\tilde{R}(t) = \tilde{f}(t;\lambda).$$

Theorem 7.4: The MLE of 'R' is given by

$$\tilde{R} = \frac{\bar{Y}}{\bar{X}+\bar{Y}}. \tag{7.2.8}$$

Proof: We have

$$\begin{aligned}
\tilde{R} &= \int_0^\infty \int_x^\infty \tilde{f}_2(y;\lambda_2)\tilde{f}_1(x;\lambda_1)dydx \\
&= \int_0^\infty \tilde{R}(x;\lambda_2)\left\{-\frac{d}{dx}\tilde{R}(x;\lambda_1)\right\}dx \\
&= \int_0^\infty \exp(-x/\bar{y})\frac{1}{\bar{x}}\exp(-x/\bar{x})dx \\
&= \frac{\bar{Y}}{\bar{x}+\bar{Y}}.
\end{aligned}$$

7.3 Point Estimators under Type II Censoring

Let n items are to be tested and we terminate the experiment when a pre-assigned number of items say $(r < n)$ have failed. Let $0 \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(r)}$ be the lifetimes of the first r ordered observations. The failed items are not replaced.

Theorem 7.5: The UMVUE of $R(t)$ under Type II Censoring is given by

$$\hat{R}_{II}(t) = \begin{cases} \left(1-\frac{t}{rS_r}\right)^{r-1} & ; \quad 0 < t < rS_r \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{7.3.1}$$

where, $S_r = \frac{1}{r}\left\{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}\right\}$.

Proof: From (7.2.1) the joint pdf of $0 \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(r)}$ is given by

$$f(x_{(1)}, x_{(2)}, ..., x_{(r)}; \lambda) = \frac{n!}{(n-r)!}\frac{1}{\lambda^r}\exp\left\{-\frac{1}{\lambda}\left(\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}\right)\right\}.$$

taking log and partially differentiating with respect to λ and equating to zero, we get the MLE of λ given by

$$\tilde{\lambda_{II}} = \frac{1}{r}\left\{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}\right\} = S_r(say).$$

We know that S_r is sufficient for λ and also S_r belongs to exponential family, it is also complete. Let us define a random variabe

$$T = \begin{cases} 1; & x_1 \geq t \\ 0; & x_1 < t \end{cases} \tag{7.3.2}$$

It is easy to check that T is unbiased for $R(t)$. Now applying Rao-Blackwellization, it follows from (7.3.2) that

$$\begin{aligned} \hat{R}_{II}(t) &= E(T|S_r) \\ &= \int_t^{\infty} f(x_1, \lambda|S_r)dx_1 \\ f(x_1, \lambda|S_r) &= \frac{f(x_1, S_r)}{f(S_r)} \end{aligned}$$

Now, the joint pdf of $f(x_1, S_r)$ is given by

$$f(x_1, S_r) = \frac{r(r-1)^{r-2}}{\lambda^r\Gamma(r-1)}\left(\frac{rS_r - x_1}{r-1}\right)^{r-2}\exp(-rS_r/\lambda); 0 < x < rS_r$$

and,

$$f(S_r) = \frac{r^r}{\Gamma(r)\lambda^r} {S_r}^{r-1} \exp(-rS_r/\lambda)$$

Thus,

$$f(x_1, \lambda | S_r) = \frac{r-1}{rS_r}\left(1 - \frac{x_1}{rS_r}\right)^{r-2}$$

Therefore, the UMVUE of $R_{II}(t)$ is given by

$$\begin{aligned} \hat{R}_{II}(t) &= \int_t^{rS_r} \frac{r-1}{rS_r}\left(1 - \frac{x_1}{rS_r}\right)^{r-2} dx_1 \\ &= \left(1 - \frac{t}{rS_r}\right)^{r-1} \end{aligned}$$

Lemma 7.3(a): The UMVUE of the sampled pdf $f(x, \lambda)$ at a specified point x is

$$\hat{f}_{II}(x, \lambda) = \begin{cases} \dfrac{r-1}{rS_r}\left(1 - \dfrac{x}{rS_r}\right)^{r-2} & ; \quad x \leq rS_r \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{7.3.3}$$

Proof: We note that the expectation of $\int_t^{\infty} f(x, \lambda)dx$ with respect to S_r is $R_{II}t$.

Hence,

$$\hat{R}_{II}(t) = \int_t^{\infty} f(x, \lambda)dx$$

or,

$$-\frac{d\hat{R}_{II}(t)}{dt} = \hat{f}(t, \lambda)$$

The result now follows from theorem 7.5.

Theorem 7.6: The UMVUE of $R = P(Y > X)$ is given by

$$\hat{R}_{II} = \begin{cases} \displaystyle\sum_{i=0}^{r_2-1} (-1)^i \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_2 - i)\Gamma(r_1 + i)} \left(\frac{r_1 S_{r_1}}{r_2 T_{r_2}}\right)^i ; & \text{if } r_1 S_{r_1} < r_2 T_{r_2} \\ \displaystyle\sum_{i=0}^{r_1-2} (-1)^i \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1 - i + 1)\Gamma(r_2 + i + 1)} \left(\frac{r_2 T_{r_2}}{r_1 S_{r_1}}\right)^{i+1} ; & \text{if } r_2 T_{r_2} < r_1 S_{r_1} \end{cases} \tag{7.3.4}$$

where, $S_{r_1} = \dfrac{1}{r_1}\left\{\sum_{i=1}^{r_1} x_{(i)} + (n_1 - r_1)x_{(r_1)}\right\}$ and, $T_{r_2} = \dfrac{1}{r_2}\left\{\sum_{i=1}^{r_2} y_{(i)} + (n_2 - r_2)y_{(r_2)}\right\}$.

Proof: It follows from lemma 7.3(a) that the UMVUEs of $f_1(x;\lambda_1)$ and $f_2(y;\lambda_2)$ at specified point 'x' and 'y' are respectively,

$$\hat{f}_{1II}(x,\lambda) = \begin{cases} \dfrac{r_1-1}{r_1 S_{r_1}}\left(1 - \dfrac{x}{r_1 S_{r_1}}\right)^{r_1-2} & ; \quad x \le r_1 S_{r_1} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and,

$$\hat{f}_{2II}(y,\lambda_2) = \begin{cases} \dfrac{r_2-1}{r_2 T_{r_2}}\left(1 - \dfrac{y}{r_2 T_{r_2}}\right)^{r_2-2} & ; \quad y \le r_2 T_{r_2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} \hat{R}_{II} &= \int_0^\infty \int_x^\infty \hat{f}_{2II}(y;\lambda_2)\hat{f}_{1II}(x;\lambda_1)dydx \\ &= \int_0^\infty \hat{R}_{II}(x;\lambda_2)\left\{-\frac{d}{dx}\hat{R}_{II}(x,\lambda_1)\right\}dx \\ &= \frac{r_1-1}{r_1 S_{r_1}}\int_0^{min(r_1 S_{r_1}, r_2 r_2 T_{r_2})}\left(1 - \frac{x}{r_2 T_{r_2}}\right)^{r_2-1}\left(1 - \frac{x}{r_1 S_{r_1}}\right)^{r_1-2} dx, \end{aligned}$$

if $r_1 S_{r_1} < r_2 T_{r_2}$ then,

$$\begin{aligned} &= \frac{r_1-1}{r_1 S_{r_1}}\int_0^{r_1 S_{r_1}}\left(1 - \frac{x}{r_2 T_{r_2}}\right)^{r_2-1}\left(1 - \frac{x}{r_1 S_{r_1}}\right)^{r_1-2} dx \\ &= \sum_{i=0}^{r_2-1}(-1)^i\left(\frac{r_1-1}{r_1 S_{r_1}}\right)\binom{r_2-1}{i}\int_0^{r_1 S_{r_1}}\left(\frac{x}{r_2 T_{R_2}}\right)^i\left(1 - \frac{x}{r_1 S_{r_1}}\right)^{r_1-2} dx \end{aligned}$$

Let, $1 - \dfrac{x}{r_1 S_{r_1}} = z$

$$= \sum_{i=0}^{r_2-1} (-1)^i \left(\frac{r_1-1}{r_1 S_{r_1}}\right) \binom{r_2-1}{i} \left(\frac{r_1 S_{r_1}}{r_2 T_{r_2}}\right)^i \int_0^1 z^{r_1-2}(1-z)^{i+1-1} dz$$

$$= \sum_{i=0}^{r_2-1} (-1)^i \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_2-i)\Gamma(r_1+i)} \left(\frac{r_1 S_{r_1}}{r_2 T_{r_2}}\right)^i; \qquad \text{if } r_1 S_{r_1} < r_2 T_{r_2}$$

Now, if $r_2 T_{r_2} < r_1 S_{r_1}$ then,

$$\hat{R} = \frac{r_1-1}{r_1 S_{r_1}} \int_0^{r_2 T_{r_2}} \left(1 - \frac{x}{r_2 T_{r_2}}\right)^{r_2-1} \left(1 - \frac{x}{r_1 S_{r_1}}\right)^{r_1-2} dx$$

Proceeding in similar way as above, we get

$$= \sum_{i=0}^{r_1-2} (-1)^i \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1-i+1)\Gamma(r_2+i+1)} \left(\frac{r_2 T_{r_2}}{r_1 S_{r_1}}\right)^{i+1}; \qquad \text{if } r_2 T_{r_2} < r_1 S_{r_1}.$$

Lemma 7.3(b): The MLE of $f(x;\lambda)$ at a specified point x is

$$\tilde{f}_{II}(x;\lambda) = \frac{1}{S_r} \exp(-x/S_r) \tag{7.3.5}$$

Proof: The proof follows from the fact that

$$-\frac{d}{dt}\tilde{R}_{II}(t) = \tilde{f}_{II}(t;\lambda).$$

Theorem 7.7: The MLE of 'R' is given by

$$\tilde{R}_{II} = \frac{T_{r_1}}{S_{r_1} + T_{r_2}}. \tag{7.3.6}$$

where, $S_{r_1} = \dfrac{1}{r_1}\left\{\sum_{i=1}^{r_1} x_{(i)} + (n_1 - r_1)x_{(r_1)}\right\}$ and, $T_{r_2} = \dfrac{1}{r_2}\left\{\sum_{i=1}^{r_2} y_{(i)} + (n_2 - r_2)y_{(r_2)}\right\}$.

Proof: We have

$$\begin{aligned}
\tilde{R}_{II} &= \int_0^\infty \int_x^\infty \tilde{f}_{2II}(y;\lambda_2)\tilde{f}_{1II}(x;\lambda_1)dydx \\
&= \int_0^\infty \tilde{R}_{II}(x;\lambda_2)\left\{-\frac{d}{dx}\tilde{R}_{II}(x;\lambda_1)\right\}dx \\
&= \int_0^\infty \exp(-x/T_{r_2})\frac{1}{S_{r_1}}\exp(-x/S_{r_1})dx \\
&= \frac{T_{r_2}}{S_{r_1}+T_{r_2}}.
\end{aligned}$$

7.4 Point Estimators under Type I Censoring

Here, the number of items failed before time t_0 is a random variable which we denote by M. Let $P(t_0)$ be the probability of failure before time t_0.

Let $X_{(1)}, X_{(2)}, ..., X_{(m)}$ be the ordered sample. We will put n samples on test upto 't_0', m will failed out of n. Therefore,

$$P(M=m) = \binom{n}{m} p^m q^{n-m}; m \leq n, m = 0, 1, 2, ..., n$$

where, $p = p(X|x < t_0) = 1 - exp(-t_0/\lambda)$ and, $q = 1 - p$.

Suppose that the items are not replaced. The data consist of the lifetimes as $X_{(1)} < X_{(2)} < ... < X_{(m)}$ of m items failed before t_0 and (n-m) items survived beyond t_0.

$$\begin{aligned}
h(x) &= p(X|x < t_0) \\
&= \frac{p(x, x < t_0)}{p(x < t_0)} \\
&= \frac{\frac{1}{\lambda}\exp(-x/\lambda)}{1-\exp(-x/\lambda)}
\end{aligned}$$

The joint distribution of $X_{(1)}, X_{(2)}, ..., X_{(m)}$ is

$$g(x_{(1)}, ..., x_{(m)}) = \frac{m!}{\lambda^m} \frac{\exp\left(-\sum_{i=1}^{m} x_{(i)}/\lambda\right)}{(1-\exp(-t_0/\lambda))^m}$$

Therefore, the likelihood of $X_{(1)}, X_{(2)}, ..., X_{(m)}$ and m is

$$\begin{aligned} L(x_{(1)}, X_{(2)}, ..., X_{(m)}, m|\lambda) &= g(x_{(1)}, X_{(2)}, ..., X_{(m)})P(M=m) \\ &= \frac{m!}{\lambda^m} \frac{\exp\left(-\sum_{i=1}^{m} x_{(i)}/\lambda\right)}{(1-\exp(-t_0/\lambda))^m} \binom{n}{m} (1-\exp(-t_0/\lambda))^m \left(\exp(-t_0/\lambda)\right) \\ &= \frac{n!}{(n-m)!\lambda^m} \left(\exp(-t_0/\lambda)\right)^{(n-m)} \exp\left(-\sum_{i=1}^{m} x_{(i)}/\lambda\right) \end{aligned}$$

Taking log on both sides and partially differentiating with respect to λ and equating to zero, we get the MLE of λ as

$$\tilde{\lambda} = \frac{1}{m}\left[\sum_{i=1}^{m} x_{(i)} + (n-m)t_0\right]$$

Case 1: If number of failures observed during one test is zero i.e. if $m=0$ then

$$\tilde{\lambda} = \infty$$

Therefore, the likelihood function for $m=0$ is

$$L = \exp(-nt_0/\lambda)$$

Therefore, $P[m=0] = \exp(-nt_0/\lambda)$ would be quite small for large and t_0/λ is not too small.

In such a case, we follow the recommendation of Bartholomew (1957) that

$$\tilde{\lambda} = \begin{cases} \dfrac{1}{m}\left[\sum\limits_{i=1}^{m} x_{(i)} + (n-m)t_0\right]; & m > 0 \\ nt_0 \qquad ; & m = 0 \end{cases} \tag{7.4.1}$$

Theorem 7.8: The MLE of $R(t)$ in Type I Censoring is

$$\tilde{R}(t) = \exp(-t/\tilde{\lambda}) \tag{7.4.2}$$

Where, $\tilde{\lambda} = \begin{cases} \dfrac{1}{m}\left[\sum\limits_{i=1}^{m} x_{(i)} + (n-m)t_0\right]; & m > 0 \\ nt_0 \qquad ; & m = 0 \end{cases}$.

Lemma 7.4(a): The MLE of sampled pdf $f(x, \lambda)$ at a specified point x under Type I Censoring is

$$\tilde{f}_I(x; \lambda) = \frac{1}{\tilde{\lambda}} \exp(-x/\tilde{\lambda}) \tag{7.4.3}$$

Where, $\tilde{\lambda} = \begin{cases} \dfrac{1}{m}\left[\sum\limits_{i=1}^{m} x_{(i)} + (n-m)t_0\right]; & m > 0 \\ nt_0 \qquad ; & m = 0 \end{cases}$.

Proof: The Proof follows from the fact that

$$-\frac{d}{dt}\tilde{R}_I(t) = \tilde{f}_I(t; \lambda).$$

Theorem 7.9: The MLE of ‘R’ is given by

$$\tilde{R}_I = \frac{\tilde{\lambda}_2}{\tilde{\lambda}_1 + \tilde{\lambda}_2} \tag{7.4.4}$$

Where,

$$\tilde{\lambda}_1 = \begin{cases} \dfrac{1}{m_1}\left[\sum_{i=1}^{m_1} x_{(i)} + (n_1 - m_1)t_0\right]; & m_1 > 0 \\ n_1 t_0 & ; \quad m_1 = 0 \end{cases}$$

and,

$$\tilde{\lambda}_2 = \begin{cases} \dfrac{1}{m_2}\left[\sum_{i=1}^{m_2} y_{(i)} + (n_2 - m_2)t_0\right]; & m_2 > 0 \\ n_2 t_0 & ; \quad m_2 = 0 \end{cases}$$

Proof: We have,

$$\begin{aligned} \tilde{R}_I &= \int_0^\infty \int_0^\infty \tilde{f}_{2I}(y;\lambda_2)\tilde{f}_{1I}(x;\lambda_1)dydx, \\ &= \int_0^\infty \tilde{R}_I(x;\lambda_2)\left\{-\frac{d}{dx}\tilde{R}_I(x;\lambda_1)\right\}dx \\ &= \int_0^\infty \exp(-x/\tilde{\lambda}_2)\frac{1}{\tilde{\lambda}_1}\exp(-x/\tilde{\lambda}_1)dx \\ &= \frac{\tilde{\lambda}_2}{\tilde{\lambda}_1 + \tilde{\lambda}_2}. \end{aligned}$$

Theorem 7.10: The UMVUE of $R(t)$ under Type I Censoring is given by

$$\hat{R}_I(t) = \begin{cases} \left(1 - \dfrac{t}{nt_0}\right)^m & ; \quad 0 < t < nt_0 \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{7.4.5}$$

Lemma 7.4(b): The UMVUE of sampled pdf $f(x;\lambda)$ at a specified point x under Type I Censoring is given by

$$\hat{f}_I(x;\lambda) = \begin{cases} \dfrac{m}{nt_0}\left(1 - \dfrac{t}{nt_0}\right)^{m-1} & ; \quad x \le nt_0 \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{7.4.6}$$

Proof: We note that the expectation of $\int_0^\infty dx$ with respect to $\tilde{\lambda}$ is $R_I(t)$.

Hence, $\hat{R}_I(t) = \int_t^\infty f(x;\lambda)dx$

Or,

$$-\frac{d}{dt}\hat{R}_I(t) = \hat{f}(t;\lambda).$$

The result now follows from Theorem 7.10.

Theorem 7.11: The UMVUE of $R = P(Y > X)$ is given by

$$\hat{R}_I = \begin{cases} \sum_{i=0}^{m_2}(-1)^i \dfrac{\Gamma(m_1+1)\Gamma(m_2+1)}{\Gamma m_2 - i + 1\Gamma(m_1+i+1)}\left(\dfrac{n_1t_0}{n_2t_0}\right)^i; & \text{if } n_1t_0 < n_2t_0 \\ \\ \sum_{i=0}^{m_1}(-1)^i \dfrac{\Gamma(m_1+1)\Gamma(m_2+1)}{\Gamma m_1 - i\Gamma(m_2+i+2)}\left(\dfrac{n_2t_0}{n_1t_0}\right)^{(i+1)}; & \text{if } n_2t_0 < n_1t_0 \end{cases} \tag{7.4.7}$$

Proof: It follows from lemma 7.4(a) that the UMVUEs of $f_1(x;\lambda_1)$ and $f_2(y;\lambda_2)$ at specified point 'x' and 'y' are respectively,

$$\hat{f}_{1I}(x,\lambda_1) = \begin{cases} \dfrac{m_1}{n_1t_0}\left(1-\dfrac{t}{n_1t_0}\right)^{m_1-1}; & x \leq n_1t_0 \\ 0 ; & \text{otherwise} \end{cases}$$

and,

$$\hat{f}_{2I}(y,\lambda_2) = \begin{cases} \dfrac{m_2}{n_2t_0}\left(1-\dfrac{t}{n_2t_0}\right)^{m_2-1}; & x \leq n_2t_0 \\ 0 ; & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} \hat{R}_I &= \int_0^\infty \int_x^\infty \hat{f}_{2I}(y;\lambda_2)\hat{f}_{1I}(x;\lambda_1)dydx \\ &= \int_0^\infty \hat{R}_I(x;\lambda_2)\left\{-\frac{d}{dx}\hat{R}_I(x,\lambda_1)\right\}dx \\ &= \frac{m_1}{n_1t_0}\int_0^{min(n_1t_0,n_2t_0)}\left(1-\frac{x}{n_2t_0}\right)^{m_2}\left(1-\frac{x}{n_1t_0}\right)^{m_1-2}dx, \end{aligned}$$

if $n_1t_0 < n_2t_0$ then,

$$= \frac{m_1}{n_1t_0}\int_0^{n_1t_0}\left(1-\frac{x}{n_2t_0}\right)^{m_2}\left(1-\frac{x}{n_1t_0}\right)^{m_1-2}dx$$

$$= \sum_{i=0}^{m_2}(-1)^i\left(\frac{m_1}{n_1t_0}\right)\binom{m_2}{i}\int_0^{n_1t_0}\left(\frac{x}{n_2t_0}\right)^i\left(1-\frac{x}{n_1t_0}\right)^{m_1-1}dx$$

Let, $1-\dfrac{x}{n_1t_0}=z$

$$= \sum_{i=0}^{m_2}(-1)^i m_1\binom{m_2}{i}\left(\frac{n_1t_0}{n_2t_0}\right)^i\int_0^1 z^{m_1-1}(1-z)^{i+1-1}dz$$

$$= \sum_{i=0}^{m_2}(-1)^i\frac{\Gamma(m_1+1)\Gamma(m_2+1)}{\Gamma(m_2-i+1)\Gamma(m_1+i+1)}\left(\frac{n_1t_0}{n_2t_0}\right)^i; \qquad \text{if } n_1t_0 < n_2t_0$$

Now, if $n_2t_0 < n_1t_0$ then,

$$\hat{R} = \frac{m_1}{n_1t_0}\int_0^{n_2t_0}\left(1-\frac{x}{n_2t_0}\right)^{m_2}\left(1-\frac{x}{n_1t_0}\right)^{m_1-2}dx$$

Proceeding in similar way as above, we get

$$= \sum_{i=0}^{m_1-1}(-1)^i\frac{\Gamma(m_1+1)\Gamma(m_2+1)}{\Gamma(m_1-i)\Gamma(m_2+i+2)}\left(\frac{n_2t_0}{n_1t_0}\right)^{i+1}; \qquad \text{if } n_2t_0 < n_1t_0.$$

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