NONLINEAR THERMAL INSTABILITY UNDER VARIOUS PHYSICAL CONFIGURATIONS

THESIS

SUBMITTED TO
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PALLE KIRAN

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SUPERVISOR
Prof. B.S. BHADAURIA
Faculty of Sciences
Department of Mathematics
Banaras Hindu University
Varanasi, 221005, India.

CO-SUPERVISOR
Prof. VIPIN SAXENA
Co-ordinator
Department of Applied Mathematics
Babasaheb Bhimrao Ambedkar University
Lucknow, 226025, India.

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Dedicated to My

Beloved Parents

Declaration

I, Palle Kiran, certify that the work embodied in this Ph.D. thesis entitled as "Nonlinear Thermal Instability Under Various Physical Configurations" is my own work carried out by me under the supervisions of Prof. B.S. Bhadauria, Supervisor, Faculty of Sciences, Department of Mathematics, Banaras Hindu University, Varanasi and Prof. Vipin Saxena, Co-Supervisor, Coordinator, Department of Applied Mathematics, Babasaheb Bhimrao Ambedkar University, Lucknow for a period of August 23, 2012 to Dec. 15, 2014 at Babasaheb Bhimrao Ambedkar University (A Central University) Lucknow, India. The information presented in this Ph.D. thesis has not been submitted for any award or any other degree or diploma. I declare that I have faithfully acknowledged, given credit to and referred to the research workers wherever their works have been cited in the text and the body of the thesis. I further certify that I have not willfully lifted up some other's work, para, text, data, results, etc. reported in the journals, books, magazines, reports, dissertations, thesis, etc., or available at web sites and included them in this Ph.D. thesis and cited as my own work.

Date: 15-12-2014

Place: Lucknow

Palle Kiran

Research Scholar

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ESTABLISHED 1996

Babasaheb Bhimrao Ambedkar University
(A Central University)
Department of Applied Mathematics
Vidya Vihar, Rae Bareli Road, Lucknow -226025

CERTIFICATE

This is to certify that the thesis "**Nonlinear Thermal Instability Under Various Physical Configurations**" submitted by Mr. Palle Kiran to fulfill the requirement for the degree of Doctor of Philosophy in Applied Mathematics has been carried out under our supervisions and no part of the thesis has been submitted for any degree or diploma of any other University.

The thesis is forwarded for submission to Babasaheb Bhimrao Ambedkar University.

Prof. B.S. Bhadauria
Research Supervisor
Faculty of Sciences
Department of Mathematics
Banaras Hindu University
Varanasi, 221005, India.

Forwarded for n/s
15/12/14
Prof. Vipin Saxena (Co-Supervisor)
Head of the Department
Department of Applied Mathematics
School for Physical Sciences
Babasaheb Bhimrao Ambedkar University
Lucknow 226025 India.

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Contents

List of Figures

List of Symbols

Latin Symbols

$\vec{g}$	Acceleration due to gravity
$\mathbb{A}$	Amplitude of convection
δ	Amplitude of modulation
Rm	Basic density Rayleigh number, $Rm = \dfrac{[\rho_p\phi_0 + \rho(1-\phi_0)]g_0Kd}{\mu k_T}$
D_B	Brownian diffusion coefficient
Q	Chandrasekhar/Peclet number
Rn	Concentration Rayleigh number, $Rn = \dfrac{(\rho_p - \rho)(\phi_1 - \phi_0)g_0Kd}{\mu k_T}$
R_0	Critical Rayleigh number
Da	Darcy number, $Da = \dfrac{\bar{\mu}K}{\mu d^2}$
$\mathbf{v}_D$	Darcy velocity
d	Depth of the layer
Γ	Diffusivity ratio
d	Dimensional layer depth
$\vec{q}$	Fluid velocilty
(x, y, z)	Horizontal and vertical co-ordinates
R_i	Internal Rayleigh number, $R_i = \frac{Qd^2}{\kappa_T}$
$\vec{H}$	Intensity of the magnetic field
Le	Lewis number
Φ	Magnetic potential
Pm	Magnetic Prandtlnumber, $\frac{\nu_m}{\kappa_T}$

N_A	Modified diffusivity ratio, $N_A = \dfrac{D_T(T_h - T_c)}{D_B T_c(\phi_1 - \phi_0)}$
N_B	Modified particle-density increment, $N_B = \dfrac{\delta(\rho c)_p(\phi_1 - \phi_0)}{(\rho c)_f}$
$\mathbf{v}$	Nanofluid velocity
Nu	Nusselt number
$\mathbf{K}$	Permeability
Pr	Pradtl number, $Pr = \dfrac{\mu}{\rho_f\ \kappa_T}$
Pr_D	Prantdl-Darcy number, $Pr_D = \frac{\delta_1 Pr}{Da}$
p	Reduced pressure
ΔS	Solute difference across the layer
Rs	Solutal Rayleigh number
Ta	Taylor number, $\left(\frac{2\Omega_r d^2}{\nu}\right)^2$
T	Temperature
T_h	Temperature at the lower wall
T_c	Temperature at the upper wall
V_T	Temperature dependant viscosity
ΔT	Temperature difference across the layer
Ra	Thermal Rayleigh number
D_T	Thermophoretic diffusion coefficient
Va	Vadasz number
a	Wavenumber

Greek Symbols

α_T	Coefficient of thermal expansion
ω	Dimensionless oscillatory frequency
Γ	Diffusivity ratio, $\Gamma = \frac{\kappa_S}{\kappa_T}$
μ	Dynamic viscosity of the fluid
$(\rho c)_p$	Effective heat capacity of the nanoparticle material
$(\rho c)_m$	Effective heat capacity of the porous medium
κ_T	Effective thermal diffusivity
$\bar{\mu}$	Effective viscosity of the porous medium

ρ, ρ_f	Fluid density
Ω	Frequency of modulation
$(\rho c)_f$	Heat capacity of the fluid
γ	Heat capacity ratio, $\gamma = \frac{(\rho c)_m}{(\rho c)_f}$
ν	Kinematic viscosity, $\frac{\mu}{\rho_0}$
μ_m	Magnetic permeability
ν_m	Magnetic viscosity
ξ	Mechanical anisotropy parameter
ρ_p	Nanoparticle mass density
ϕ	Nanoparticle volume fraction
χ	Perturbation parameter
θ	Phase angle
δ_1	Porosity
$\vec{\Omega}_r$	Rotation speed vector
s/τ	Slow time
β_S	Solutal expansion coefficient
$\overline{\lambda_2}$	Strain retardation time
ψ	Stream function
$\overline{\lambda_1}$	Stress relaxation time
η	Thermal anisotropy parameter

Other symbols

$\hat{k}$	Vertical unit vector
∇^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
∇_η^2	$\eta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$
∇_ξ^2	$\frac{\partial^2}{\partial x^2} + \frac{1}{\xi} \frac{\partial^2}{\partial z^2}$

Subscripts

b	Basic state
c	Critical

0	Reference value

Superscripts

$*$	Dimensionless quantity
$'$	Perturbed quantity

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Chapter 1

Introduction

Convective heat and mass transfer is one of the major modes of heat and mass transfer in fluids. This phenomenon takes place both by diffusion and by advection, in which matter or heat is transported by the larger scale motion of currents in the fluid. Convection can be qualified in terms of being natural, forced, gravitational, granular, or thermomagnetic. The natural convection has been considered most of our problems since natural convection plays a role in broad range of applications, because in practice, convection occurs for very small temperature gradients. Important in environmental science climate, building ventilation; geophysics e.g. mantle convection; industrial processes e.g. crystal growth. It is known fact that, controlling convection is mainly concerned with space dependent temperature gradients. There are many interesting situations of practical problems in which the temperature gradient is a function of both space and time. This uniform temperature gradient can be determined by solving the energy equation with suitable time dependent thermal boundary conditions and can be used as an effective external mechanism to control the convective flow. However, in practice, the nonuniform temperature gradient finds its origin in transient heating or cooling at the boundaries. Hence the basic temperature profile depends explicitly on position and time, this problem is known thermal modulation problem, involves the solution of the energy equation under suitable time dependent boundary conditions. Predictions exist for a variety of responses to modulation depending on the relative strength and rate of forcing. Among these, there is the upward or downward shift of convective threshold compared to the unmodulated problems. An excellent review related to this problem is given by Davis (1976). The excellent review and analysis of Rayleigh−Bénard convection under various physical configurations given by Chandrasekhar (1961), Drazin and Reid (2004).

Bénard−Rayleigh convection and its porous medium analog, namely, Bénard−Darcy convection. The study of natural convection in a Newtonian fluid saturated porous medium is now well understood and documented phenomenon. A comprehensive account of the problem and their applications are available presently in the excellent books of Ingham and Pop (1998, 2005), Vafai (2000, 2005), Straughan (2004), Nield and Bejan (2013) and Vadasz (2008). However, in most of the above studies the temperature gradient across the porous medium has been considered to be uniform, which is not so in many practical problems. Thus, keeping in mind the industrial applications of the study, it is more appropriate to assume temperature gradient to be a function of both space and time. In the next section the review of literature has been presented according to the types of modulations either by considering Rayleigh−Bénard or Darcy Bénard convection.

1.1 Review of Literature

1.1.1 Temperature modulation

Temperature modulation in fluid layer

Many researchers, under different physical models have investigated thermal instability in a horizontal fluid layer with temperature modulation. Donnelly (1964) investigated the effect of rotation speed modulation on the onset of instability in fluid flow between two concentric cylinders. However, the rotation speed modulation was the originating idea of the temperature, as well as gravity modulation. Venezian (1969) was motivated by the experiment of Donnelly (1964) and performed a linear stability analysis of Rayleigh−Bénard convection for the case of small amplitude temperature modulation.

Using perturbation method and considering free−free surfaces, he calculated the shift in the critical value of the Rayleigh number and found that, the system can be stabilized or destabilized by suitably tuning the frequency of modulation. A similar problem was studied earlier by Gershuni and Zhukhovitskii (1963), for a temperature profile obeying rectangular law. According to them the convective instability of fluid in a gravity field is usually investigated under the assumption that the equilibrium temperature does not depend on time. However, unsteady equilibrium of a fluid is also possible when the equilibrium temperature varies with time according to the law that is determined by unsteady heating conditions. Rosenblat and Herbert (1970), performed a linear stability problem and found an asymptotic solution by considering low frequency modulation and free-free boundaries. Rosenblat and Tanaka (1971), studied the linear stability for a fluid in a classical geometry of Bénard by considering the temperature modulation of rigid−rigid boundaries. Using Galerkin technique and discussed the stability of the system using Floquet theory. Finucane and Kelly (1976), performed a theoretical and experimental investigation of the thermal modulation in a horizontal fluid layer. They found both experimentally and numerically that, at low frequencies the modulation is destabilizing, where as at high frequencies it is stabilizing. Niemela and Donnelly (1987), Schmitt and Lucke (1991), Or and Kelly (1999) and Or (2001) have also investigated the effect of external modulation on the thermal convection in a horizontal fluid layer. The first nonlinear stability problem in a horizontal fluid layer, under temperature modulation of the boundaries was studied by Roppo et al. (1984). Bhadauria and Bhatia (2002), studied the effect of temperature modulation on thermal instability by considering rigid−rigid boundaries and different types of temperature profiles. Bhadauria (2006), studied the effect of temperature modulation under vertical magnetic field by considering rigid boundaries and using Floquet theory. Further, he also found that it is possible to advance or delay the onset of convection by proper tuning of the frequency of modulation of the wall's temperature. Malashetty and Swamy (2008), investigated thermal instability of a heated fluid layer subject to both temperature modulation and rotation effects. It is established that, the instability can be enhanced by the rotation at low frequency symmetric modulation and with moderate to high frequency lower wall temperature modulation, whereas the stability can be enhanced by the rotation in case of asymmetric modulation. They also found that by proper adjusting the system parameter values it is possible to advance or delay the onset of convection. Bhadauria et al. (2009), studied the nonlinear aspects of thermal instability under temperature modulation, considering various temperature profiles. Raju and Bhattacharyya (2010), investigated onset of thermal instability in a horizontal layer of fluid with modulated boundary temperatures by considering rigid boundaries. Bhadauria et al.(2012) studied thermally or gravity modulated nonlinear stability problem in a rotating viscous fluid layer, using Ginzburg−Landau equation for stationary mode of convection. Recently Bhadauria and Kiran (2014d) investigated an oscillatory mode of double diffusive convection under thermal modulation using complex non-autonomous Ginzburg−Landau equation. They have presented a very good results where oscillatory convection advances the convection and enhances the heat transfer than stationary. They also have presented how phase angle and frequency of modulation affects heat and mass transfer in the system.

Temperature modulation in porous medium

Caltaoirone (1976), was the first to study the stability of a horizontal porous layer, the temperature of which at the inner side is a periodical function of time, has been theoretically studied by using the Galerkin technique. He showed a correlation between the Rayleigh number, the reduced frequency of the signal on the wall and its amplitude. Different types of evolution of the instabilities within the porous layer have observed. The stability of a horizontal porous layer bounded by two impermeable planes is investigated by Chhuon and Caltagirone (1979). A time dependent periodic temperature profile is imposed on the lower boundary while the upper plane is kept at constant temperature. Using the linear stability theory, a criterion for the onset of convection is defined as a function of the perturbation wavenumber and of the amplitude and frequency of the temperature oscillation. Experimental work with a setup allowing both the amplitude and the frequency of the thermal signal to vary is done. Rudraiah et al. (1990) have investigated the effect of thermal modulation on the onset of convection in a viscoelastic fluid-saturated porous medium using Oldroyd model. Antohe and Lage (1996), heat and momentum transport is investigated theoretically and numerically considering a rectangular enclosure filled with clear fluid or with fully saturated porous medium, under time-periodic horizontal heating. It is shown that the convection intensity within the enclosure increases linearly with heating amplitude for a wide range of parameters. Moreover, the flow response to pulsating heat is continuously enhanced as the system becomes more permeable. Using Venezian (1969) approach and considering Forchheimer flow model with effective viscosity larger than the fluid viscosity, Malashetty and Wadi (1999), investigated the problem of thermal instability under thermal modulation, and calculated the correction in the critical Rayleigh number as a function of system parameters. It is shown that, the system is most stable when the boundary temperature is modulated out of phase. It is also found that the low frequency thermal modulation can have a significant effect on the stability of the system. Malashetty and Basavaraja (2002, 2003), using Brinkman model with effective viscosity larger than the viscosity of the fluid they have investigated linear theory analysis for anisotropic porous medium. They have found that, it is possible to advance or delay the onset of convection by wall temperature and to advance convection by gravity modulation. They also found, the small anisotropy parameter has a strong influence of the stability of the system. Bhadauria (2007) also studied the convection in a sparsely packed porous medium under temperature modulation. Considering rigid−rigid boundaries and by Galerkin method, he calculated the critical Rayleigh number for the onset of convection. Considering thermal modulation on the boundaries of the porous medium a series of the work has been investigated by Malashetty et al.(2006), Bhadauria (2007), Bhadauria (2008), Suthar and Bhadauria (2009), Bhadauria and Srivastava (2010), Shivakumara et al. (2011), Malashetty and Begum (2011), Siddheshwar et al. (2013), Bhadauria et al. (2013) and Bhadauria and Kiran (2013a). In these papers they have considered the time periodic temperature at the boundaries as proposed by Venezian (1969) and it was discussed onset criteria for linear theory and heat and mass transfer for nonlinear theory for various physical configuration of the problem. Recently Bhadauria and Kiran (2014c,d) investigated an oscillatory mode of convection under thermal modulation, using complex non-autonomous Ginzburg−Landau equation. They have found that, an oscillatory convection advances the convection and enhances the heat transfer than stationary. They also have presented how phase angle and frequency of modulation affects heat transfer in the system.

1.1.2 Gravity modulation

The problem of convection in a fluid layer in the presence of complex body forces has gained considerable attention in recent decades due to its important applications in engineering and technology. The time dependent gravitational field, one of the complex forces, is of interest in space laboratory experiments, in areas of crystal growth and others. It is also of importance in the large scale convection in atmosphere. The random fluctuations of gravity field, both in magnitude and direction, can be seen in space laboratories, significantly influence natural convection. Existence of adverse density variations with in the fluid and a body force are the necessary conditions to initiate natural convection. The idea of using mechanical vibration as a tool to improve the heat transfer rate has received much attention. However, the regulation of convection is important from the applications point of view and thermo−gravitational vibration is known to be an effective means of controlling instabilities. The gravity modulation of the system leads to the variable coefficients in the governing equations of thermal instability and involves the vertical time periodic vibrations of the system and gravity modulation is known as g−jitter in literature.

Gravity modulation in fluid layer

Gershuni and Zhukhovitskii (1963) and Gresho and Sani (1970) were the first to study the effect of gravity modulation in a fluid layer. They studied, the stability of Rayleigh−Bénard convection for the case of a time dependent buoyancy force which is generated by shaking the fluid layer, thus causing a sinusoidal modulation of the gravitational field. A linearized stability analysis is performed to show that gravity modulation can significantly affect the stability limits of the system. Biringen and Peltier (1990), investigated the nonlinear three dimensional Rayleigh−Bénard problem under gravity modulation numerically, and confirmed the result obtained by Gresho and Sani (1970). Wadih and Roux (1988) presented a study on instability of the convection in an infinitely long cylinder with gravity modulation oscillating along the vertical axis. The effect of modulation on the stability limits given by linear theory in the standard steady case is analysed. A method based on Floquet theory is proposed in the case of small values of the modulation amplitude for a fixed value of the frequency of modulation. Saunders et al. (1992) have discussed the effect of gravity modulation on thermosolutal convection in an infinite layer of fluid. Clever et al. (1993) performed a detailed non-linear analysis of Rayleigh−Bénard convection under g−jitter and presented the stability limits to a much wider region of parameter space. Chen and Chen (1999) have studied the effect of gravity modulation on the stability of convection in a vertical slot. They have examined the stability for fluids of different Prandtl numbers. Rogers et al. (2000) have observed superlattice patterns in vertically oscillated Rayleigh−Bénard convection. Rogers et al. (2005), Bhadauria et al. (2005) showed that the gravitational modulation, which can be realized by vertically oscillating a horizontal liquid layer, acts on the entire volume of liquid and may have a stabilizing or destabilizing effect depending on the amplitude and frequency of the forcing. Shu et al. (2005) examined the effects of modulation of gravity and thermal gradients on natural convection in a cavity, numerically as well as experimentally. They found that for low Prandtl number fluids, modulations in gravity and temperature produce the same flow field both in structure and in magnitude. Gravity modulation in a fluid layer has been studied by Bhadauria (2006). Boulal et al. (2007), have given attention on the influence of a quasi periodic gravitational modulation on the convective instability. They predicted that, the threshold of convection corresponds precisely to quasi periodic solutions. Bhadauria et al. (2012), studied thermally or gravity modulated nonlinear stability problem in a rotating viscous fluid layer, using Ginzburg−Landau equation for stationary mode of convection. Siddheshwar et al. (2012a), have investigated thermal/gravitaty modulation on electrically conducting fluid layer as a magneto-convection considering nonlinear analysis of stationary mode. Bhadauria et al. (2013) studied internal heating effects on weak nonlinear Rayleigh−Bénard convection under gravity modulated, using Ginzburg−Landau equation for stationary mode of convection. They found that, the gravity modulation works for both enhancing or diminishing heat transfer in the system. Recently Bhadauria and Kiran (2014a) investigated an oscillatory mode of convection under gravity modulation in a viscoelastic fluid layer using complex non-autonomous Ginzburg−Landau equation. They found that, an oscillatory convection advances the convection and enhances the heat transfer than stationary.

Gravity modulation in porous layer

Malashetty and Padmavathi (1997), studied the effect of small amplitude gravity modulation on the onset of convection in fluid and fluid saturated porous layer. They found that low frequency oscillations have significant effect on the stability of the system. Bardan and Mojtabi (2000), made an analytical and numerical study of convection in a porous cavity in the presence of vertical vibrations. They found that the vibrations stabilize the quiescent state. Rees (2000), considered the boundary layer flow induced by a constant temperature vertical surface embedded in a porous medium is modified by time periodic variations in the gravitational acceleration. Using an amplitude expansion to determine the detailed effect of g−jitter, and the expansion is carried through to fourth order. The numerical and asymptotic solutions show that the g−jitter effect is eventually confined to a thin layer embedded within the main boundary layer, but it becomes weak at increasing distances from the leading edge. Later Rees and Pop (2003), the nonsimilar boundary layer equations are solved using the Keller box method after using a Fourier decomposition in time to reduce the system to parabolic form with only two independent variables. The main effect of such g−jitter is confined mainly to the region near the leading edge and becomes weak at larger distances from the leading edge. Rees and Pop (2001), the effect of periodical gravity modulation on the free convection near the forward stagnation point of a cylindrical surface which is embedded in a porous medium has been investigated while considering nonlinear theory. Govender (2004), made stability analysis to investigate the effect of low amplitude gravity modulation on convection in a porous layer heated from below. It was shown that increasing the frequency of vibration stabilizes the convection. Govender (2005a), using linear theory he demonstrated that increasing the excitation frequency rapidly stabilizes the convection up to the transition point from synchronous to subharmonic convection. Beyond the transition point, the effect of increasing the frequency is to slowly destabilize the convection. Govender (2005b) using weak nonlinear analysis he found that, increasing the vibration frequency causes the convection amplitude to approach zero, i.e., increasing the vibration frequency stabilizes the convection. The effect of vertical vibration on the stability of a dilute suspension of negatively geotactic microorganisms in a fluid layer of finite depth is investigated by Kuznetsov (2005). Kuznetsov (2006) investigated analytically the potential of utilizing the vertical vibration for controlling bioconvection. A shallow horizontal fluid saturated porous layer that contains a suspension of oxytactic bacteria, such as Bacillus subtilis, is considered. He said that, the linear stability analysis indicates that vertical vibration has a stabilizing effect on the suspension. Siddhavaram and Homsy (2006), they studied the effects of gravity modulation on the mixing characteristics of two inter diffusing miscible fluids initially in two vertical regions separated by a thin diffusion layer. Strong (2008), the effect of vertical harmonic vibration on the onset of convection in an infinite horizontal layer of fluid saturating a porous medium is studied. His investigation is carried out that the vertical vibration can significantly affect the stability of the system by increasing or decreasing its susceptibility to convection. The same problem i.e gravity modulation extended for double diffusive convection in porous medium by Strong (2009). Saravanan and Arunkumar (2010), the effect of gravity modulation on the onset of convection in a horizontal fluid saturated porous layer in which the applied temperature gradient is opposite to that of gravity is investigated. The flow through the porous layer is governed by an extended form of Darcys law incorporating Brinkmans boundary layer correction. Their study is focussed on low amplitude gravity modulation and the thresholds are found using Mathieus functions. The emergence of instability via the synchronous and subharmonic modes and the transition between them are discussed as a function of the physical parameters. Malashetty and Swamy (2011), investigated the effect of gravity modulation on the onset of thermal convection in a fluid or porous layer using linear stability. They show that, in general the gravity modulation produces a stabilizing effect in case of viscous fluid layer and both destabilizing and stabilizing effects in case of Brinkman porous medium while it produces a destabilizing effect in case of Darcy porous medium. The low frequency gravity modulation is found to have a significant effect on the stability of the problem. It is also shown that, the onset can be advanced or delayed by proper tuning of various governing parameters. Saravanan and Sivakumara (2010, 2011) studied the effect vibrations on the onset of convection in a horizontal fluid saturated porous layer considering an arbitrary amplitude and frequency. It is demonstrated that vibrations can produce a stabilizing or a destabilizing effect depending on their amplitude and frequency for a porous layer heated from below. The temperature sensitivity of viscous fluid saturated porous medium under vertical vibrations of Rayleigh−Bénard convection is investigated by Siddheshwar et al. (2012b). They found that the variable viscous parameter has a tendency of enhancing heat transfer in the system. Srivastava et al. (2013), have investigated the temperature sensitivity of viscous fluid saturated porous medium under gravity modulation along with internal heating effects. They found that, the variable viscosity and internal Rayleigh number is to enhance the heat transfer in the system. Recently Bhadauria and Kiran (2014b) investigated an oscillatory mode of convection under gravity modulation in a viscoelastic fluid saturated porous medium using complex non-autonomous Ginzburg−Landau equation. They found that, an oscillatory mode of convection enhances the heat transfer more than stationary mode of convection.

1.1.3 Rotation speed modulation

The combined effect of both thermal modulation and rotation on the onset of convection in a rotating fluid layer was made by Rauscher and Kelly (1975). Liu and Ecke (1997), analyzed heat transport of turbulent Rayleigh−Bénard convection under rotational effect. Malashetty and Swamy (2008), investigated thermal instability of a heated rotating fluid layer subjected to temperature modulation. They found that, by proper tuning of modulation frequency, Taylor number and Prandtl number it is possible to advance or delay the onset of convection. Kloosterziel and Carnevale (2003), investigated the effect of rotation on the stability of thermally modulated system. They determined analytically critical points on the marginal stability boundary above which an increase of either viscosity or diffusivity is destabilizing. Finally, they show that, when the fluid has zero viscosity the system is always unstable, in contradiction to Chandrasekhar (1961) conclusion. Malashetty and Swamy (2007), studied the effect of rotation on the stability of thermally modulated system. They found that, the symmetric modulation destabilizes the system at low frequencies while it stabilizes the system at moderate and high frequencies. They also found, an asymmetric modulation is the most stable situations, for all frequencies. Finally the lower wall temperature has stabilizing effect for low and higher values of frequency and destabilizing effect for moderate values of frequency. Bhadauria (2008), investigated rotational influence on Darcy convection and found that both rotation and permeability suppress the onset of thermal instability. Bhadauria et al. (2012) investigated the nonlinear thermal instability in a rotating viscous fluid layer under temperature/gravity Modulation. They found that the effect of rotation is to diminish the heat transfer in the system and modulation is to enhance the heat transfer or diminish the heat transfer depending on the amplitude and frequency.

The rotation speed modulation was the originating idea of the temperature as well as gravity modulation, but not much research work has implemented in this field. The effect of temperature modulation on the Rayleigh−Bénard instability and the effect of modulation of the rotation speed in the Taylor−Couette instability has been investigated in detail both theoretically and experimentally by Ahlers et al. (1985), Niemela and Donnelly (1986), Kumar et al. (1986), Meyer et al. (1988), Walsh and Donnelly (1988). For Rayleigh−Bénard convection the temperature modulation is supposed to stabilize the conduction state. However, since the temperature modulation breaks the reflection symmetry about the midplane and hexagons, rather than cylinders, takes the pattern in which convection occurs immediately above the threshold. The Rayleigh−Bénard problem with rotation, the above problem can be avoided if the rotation speed is modulated in time periodic manner. This leads to a simple problem for the study of the effect of modulation on the threshold. When we study the rotational effect, then one more parameter in the form of rotation speed exists, which can affect the stability of convective flow. From the literature, the study due to Bhattacharjee (1989) is of great importance, in which he studied the effect of rotation speed modulation on Rayleigh−Bénard convection in an ordinary fluid layer. He found that the effect of modulation is stabilizing for most of the configurations. Bhadauria and Suthar (2009) investigated the effect of the rotation speed modulation on the onset of free convection in a rotating porous layer placed farther away from the axis of rotation. Suthar et al. (2011), investigated the time periodic rotational speed on the onset of free convection in a rotating porous layer about z-axis. They conclude that, the effect of modulated rotation speed is found to have a stabilizing effect on the onset of convection for different values of modulation frequency.

1.1.4 Magnetocovection/Magnetic modulation

In general the magnetic fluids are differ from the ordinary fluids by showing magnetic as well as flow properties. The magnetoconvection arises due to the interaction of electrically conducting fluid flow and the applied magnetic field. Convection can also take place in these fluids due to temperature dependence of their magnetization. This property is useful in space research, where the role of gravity can be replaced by a magnetic body force. The magnetic force can be used to create circulation in small passages where natural convection is either absent or ineffective. Generally, the magnetization depends on the magnetic field, temperature and density. Hence, the magnetic force depend on the thermal state of the fluid and may lead to convection.

Thompson (1951) was the first to study magnetoconvection in horizontal fluid layer. Using Galerkin method, Nakagawav (1955, 1957) and Jirlow (1956), investigated that, the vertical magnetic field suppresses the onset of convection. Finlayson (1970) considered a magnetic horizontal fluid layer heated from below in the presence of an uniform vertical magnetic field. He studied the linear stability analysis by considering free−free and rigid−rigid boundaries, and predicted the critical gradient of temperature corresponding to the onset of convection, considering both buoyancy and magnetic forces. Gotoh and Yamada (1982) the linear instability is investigated for a horizontal magnetic fluid layer confined between two ferromagnetic boundaries and heated from below in the presence of a vertical magnetic field. Galerkin method is used for solving the disturbance equations. It is concluded that the magnetization of the boundaries and the nonlinearity of fluid magnetization both reduces critical Rayleigh number, and that the effects of magnetic force and buoyancy compensate each other. Schwab et al. (1983) analyzed the Finlayson's problem experimentally in the presence of strong magnetic field and discussed the onset of convection. Later, Stiles and Kagan (1990) examined the problem reported by Schwab et al. (1983) and generalized the Finlayson's model assuming that under a strong magnetic field, the rotational viscosity augments the shear viscosity. Rudraiah and Sekhar (1991) treated the Finlayson's problem with internal heat source and showed that the variation of temperature, due to heat source, induces a variation in the magnetic field. These variations can be used to control magnetic convection.

Another interesting case consists of applying an external magnetic field of a constant or spatially varying gradient. Aniss et al. (1993) and Souhar et al.(1999) proposed theoretical and experimental investigations of the Rayleigh−Beńard convection in a magnetic fluid layer confined in a horizontal annular Hele-Shaw cell and submitted simultaneously to radial temperature and magnetic field gradients. With their geometrical configuration, they showed the possibility to simulate theoretically and experimentally the Rayleigh−Bénard convection and its control by an external magnetic field gradient in the absence of gravity. Siddheshwar and Pranesh (1999, 2000) examined the effects of time periodic temperature/gravity modulation on the onset of magnetoconvection in electrically conducting fluids with internal angular momentum by making a linear stability analysis. Bhadauria (2006) studied the effect of temperature modulation on magnetoconvection using rigid boundaries and Fluquet theory. Further, Bhadauria (2007, 2008) studied the combined effect of temperature modulation and magnetic field on the onset of convection in an electrically conducting fluid saturated porous medium using rigid−rigid and free−free boundaries. Bhadauria and Sherani (2008, 2010) investigated the onset of Darcy convection in a magnetic fluid saturated porous medium subject to temperature modulation of the boundaries and magnetoconvection. Siddheshwar et al. (2012) investigated heat transport for a stationary magnetoconvection in a Newtonian liquid under temperature or gravity modulation by performing a weak nonlinear stability analysis and using Ginzburg−Landau model.

Aniss et al. (2000) showed that, the vertical oscillations of the cell generate parametric convective instability only for small Prandtl numbers. Consequently, using the similarity between gravitational and magnetic modulations, the parametric convective instability cannot occur in a Hele−Shaw magnetic liquid layer. Most of the above studies are concern with linear stability analysis. It is observed that an imposed vertically time periodic magnetic field is just like gravity modulation. According to Aniss et al. (2001) magnetic modulation is much effective to handle theoretical and experimental investigation rather than the gravity modulation, in which the amplitudes and frequencies are more difficult to control. When we consider the system in the space (where magnetoconvection occurs in stars: near the surface of cool, surrounding nuclear burning shells in the late stages of stellar evolution, in supernova explosions, in accretion disks during the formation of stars and planets and in accretion onto black holes and neutron stars, in the hot plasma in clusters of galaxies etc.) handling temperature modulation is some what difficult.

1.2 Equations of hydrodynamics

In fluid mechanics the following are the basic properties that every fluid obeys where the hydrodynamical flow of a viscous fluid of varying density and temperature:

1. ***Equation of continuity***

   A continuity equation is an equation that describes the transport of a conserved quantity. Since mass, momentum, energy, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations. The continuity equation states that, in any steady state process, the rate at which mass enters a system is equal to the rate at which mass leaves the system. The differential form of the continuity equation is

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{q} = 0, \tag{1.2.1}$$

where ρ is fluid density, t is time, $\vec{q}$ is the flow velocity vector field. This equation is one of Euler equations. For incompressible fluid flow, the mass continuity equation simplifies to a volume continuity equation:

$$\nabla \cdot \vec{q} = 0, \tag{1.2.2}$$

which means that the divergence of velocity field is zero everywhere.

2. ***Equation of momentum***

The general form of the equations of fluid motion is:

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q} + F, \tag{1.2.3}$$

where ν is kinetic viscosity, F is the body force term, represents an external forces that act on the fluid; for example: gravity, wind, etc. The time derivative term $\frac{\partial}{\partial t}$ is called the local derivative, which is physically the time rate of change at a fixed point, $(\vec{q} \cdot \nabla)$ is called the convective derivative, which is physically the time rate of change due to the movement of the fluid element from one. This is a statement of the conservation of momentum in a fluid and it is an application of Newton's second law to a continuum.

3. ***Equation of energy***

The energy equation is defined as

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \kappa_T \nabla^2 T. \tag{1.2.4}$$

It is based on the law of conservation of energy, κ_T is the thermal conductivity, which is proportional constant in Fourier's law of heat conduction.

4. ***Boussineq approximation***

The Boussinesq approximation states that, the density differences are sufficiently small to be neglected, except where they appear in terms multiplied by $\vec{g}$, the acceleration due to gravity. The essence of the Boussinesq approximation is that the difference in inertia is negligible but gravity is sufficiently strong to make the specific weight appreciably different between the two fluids. The accounted density according to Boussinesq is given by

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{1.2.5}$$

where α_T denotes the thermal expansion coefficient, and subscript 0 refers to some reference value of the quantity.

1.3 Basics & different models for porous medium

1. ***Porosity***

   Porosity of the porous medium is a basic quantity for characterizing a porous medium. Porosity is defined as the ratio between the volume occupied by the fluid (voids) and the total volume of the material (including voids and solid). It is also defined the ratio of pore volume to its total volume. Let, V_f denotes the volume of the fluid (or voids), and V_m denotes the volume of the material, then the porosity δ_ϕ is given by,

   $$\delta_\phi = \frac{V_f}{V_m}. \tag{1.3.1}$$

   Clearly, the porosity of any porous domain lies in the interval (0,1). The porosities of most commonly occurring porous media are less than 0.6.

2. ***Darcys Law & Permeability***

   Darcy's law describes the flow of a fluid through a porous medium. The law was derived by Henry Darcy (1856) based on the results of experiments on the flow of water through beds of sand. It also forms the scientific basis of fluid permeability used in the earth sciences, particularly in hydrogeology. Henry Darcy found a proportionality between the seepage velocity and the applied pressure gradient, which is given by

   $$\vec{q} = -\frac{K}{\mu}\nabla p + F, \tag{1.3.2}$$

   where K specific permeability or intrinsic permeability of medium, is independent of nature of the fluid but depends on the geometry of the medium. It measures the flow conductance of a porous domain. For an anisotropic porous domain, it is generally a tensor of second order and this permeability K varies, and this variable permeability, enhanced within a region of constant thickness. According to Rees and Pop (2000), near the leading edge the flow enhanced and the rate of heat transfer is more than in non-uniform permeability case. Here μ is the dynamic viscosity of the fluid, p is the pressure and F is the external body force.

3. *Modifications in Darcys Law*

The Darcys law is applicable only when seepage velocity $\vec{q}$ is sufficiently small and is linear or gradually looses it's validity for high velocities, i.e. for high Reynold numbers. Thus, extensions of the Darcys law was modified in the form of Darcy-Forchheimers model and Brinkmans model.

(a) *Brinkman–extended Darcy model*

The Brinkman equation is defined from Darcy equation as

$$\nabla p = -\frac{\mu}{K}\vec{q} + \mu\nabla^2\vec{q} \tag{1.3.3}$$

where K is the permeability and it is the reciprocal of the shear factor. The above equation is the working version of the Brinkman equation where the viscosity associated with the viscous diffusion term is the same as the viscosity of the fluid. The added diffusion term simply to meet the boundary specifications and hence the viscosity was not defined. Brinkmans first version of the flow equation is given by

$$\nabla p = -\frac{\mu}{K}\vec{q} + \overline{\mu}\nabla^2\vec{q}, \tag{1.3.4}$$

where $\overline{\mu}$ is a quantity having the dimension of viscosity and it was named the effective viscosity. One should acknowledge that Eq.(1.3.4) is a general form of volume averaged Stokes equation. In general, the effective viscosity is not expected to be the same as the viscosity of the fluid owing to the effect of tortuosity and the dispersion of viscous diffusion flux.

(b) *Brinkman–Forchheimer–extended Darcy model*

The Darcy−Brinkman−Forchheimer equation is defined as

$$-\nabla p=\frac{\mu}{K}\vec{q}+\frac{c_F\rho|\vec{q}|}{\sqrt{K}}\vec{q}-\mu\nabla^2\vec{q}, \tag{1.3.5}$$

where c_F is the form drag coefficient. Equation (1.3.5) is thought as an extension from the Brinkmans equation by accounting for the inertial effects on the internal shear loss term, however, the dispersion of momentum is not accounted for. Owing to the omission of the momentum dispersion, one should note that Eq.(1.3.5) is useful only for systems where the flow domain is large, that is, when Darcys law is valid at creeping flow. Hence, strict restrictions apply to the use of the Brinkman−Forchheimer equation. When the diffusion term is dropped out, Eq.(1.3.5) becomes the Darcy−Forchheimer equation. That is,

$$-\nabla p=\frac{\mu}{K}\vec{q}+\frac{c_F\rho|\vec{q}|}{\sqrt{K}}\vec{q}. \tag{1.3.6}$$

When an interface is encountered, an additional empirical model on the velocity jump condition needs to be provided in connection with the Darcys law or Darcy−Forchheimer equation to account for the inconsistency of the governing equation and the physical description of the flow.

1.3.1 Fundamental equations for porous medium

1. ***Continuity equation***

$$\nabla\cdot\vec{q}=0. \tag{1.3.7}$$

2. ***Equation of momentum***

$$\frac{\rho}{\delta_1}\frac{\partial\vec{q}}{\partial t}=-\nabla p-\frac{\mu}{K}\vec{q}+\rho_f\vec{g}. \tag{1.3.8}$$

3. ***Equation of energy***

$$(\rho c)_m\frac{\partial T}{\partial t}+(\rho c_p)_f(\vec{q}\cdot\nabla)T=\kappa_m\nabla^2T. \tag{1.3.9}$$

4. ***Oberbeck-Boussinesq approximation***

$$\rho_f = \rho_0[1 - \alpha_T(T - T_0)], \tag{1.3.10}$$

where $\vec{q}$ is seepage velocity, δ_1 is porosity of the porous medium, p is pressure, μ is the dynamic viscosity, K is permeability, c is specific heat, κ_m is overall thermal conductivity, and α_T is the thermal volume expansion coefficient.

1.4 Basics of heat transfer

Heat transfer is nothing but, heat moving from one object to another. Heat generally transfers from a high temperature object to a lower temperature object. Thermal instability happens mainly due to heat transfer. It happens whenever there exists a temperature difference in a medium or between media. Heat transfer can occur in the following three ways.

1. ***Conduction***

   The heat transfer by means of conduction, occurs when the objects are in physically contacts, the heat, in the form of kinetic energy, is transferred at microscopic level, between the molecules by their collisions with each other without any motion of the object as a whole. Thermal conductivity is the property of a material to conduct heat and evaluated primarily in terms of Fourier's Law for heat conduction. The rate of heat transfer via conduction is different for different materials, and is measured by the thermal conductivity of the material.

2. ***Radiation***

   Radiation is heat transfer in the form of emission of electromagnetic waves which carry energy away from the emitting object. This emission may be attributed to changes in the electron configurations of the constituent atoms or molecules. The heat of the Sun is a good example of the radiative heat transfer. The major part of our study is the naturally convective heat transfer and thus we stick to it.

3. *Convection*

In the case of convection, heat transfer occurs via macroscopic motion of the fluid from a hot to a cool region, i.e., when the heated fluid is caused to move away from the source of heat, carrying energy with it. In convection, there are two types of mechanisms. Energy transfer due to random molecular motion (diffusion); in this case energy is transferred by the bulk, or macroscopic, motion of the fluid. Energy is also transformed due to temperature gradient. Boiling of water is a very common example of convective heat transfer. This convective phenomenon divided into; Natural (or free) convection, Forced convection and Mixed convection.

(a) *Natural convection*

If the motion of the fluid is caused by the buoyant force differences, resulting from a temperature or concentration gradient, in a body force field, like a gravitational field, then this process of heat transfer is known as Natural or Free convection. Through the density differences, the buoyancy force come into existence which give rise to the fluid motion.

(b) *Forced convection*

Forced convection is a mechanism, or type of heat transport, in which the fluid motion is generated by any external source (like a pump, fan, suction device, etc.). For example, the use of a fan to provide forced convection air cooling of hot electrical components on a stack of printed circuit boards.

(c) *Mixed convection*

Mixed convection is a combination of both forced and free convection's which is the general case of convection when a flow is determined simultaneously by both an outer forcing system (i.e., outer energy supply to the fluid streamlined body system) and inner volumetric (mass) forces, viz., by the nonuniform density distribution of a fluid medium in a gravity field.

1.4.1 Rayleigh-Bénard convection

Rayleigh−Bénard convection is a type of natural convection (gravity induced free convection), occurring in a plane horizontal fluid layer heated from below, in which the fluid develops a regular pattern of convection cells known as Bénard cells. Buoyancy, and hence gravity, is responsible for the appearance of convection cells. There are three assumptions one can made for the Rayleigh−Bénard convection, viz., (a) the fluid is incompressible, (b) the density of the fluid is the only property that gets affected by the change in the temperature across it (c) an uniform gravitational force it experiences over its entire volume. The second assumptions is further limited to the degree that the density variations are given by the Boussinesq approximation. The Rayleigh−Bénard convection mainly depends on the buoyancy force and the viscous force. The ratio of these two forces is called the Rayleigh number which is defined as

$$Ra = \frac{\alpha_T g \Delta T d^3}{\nu \kappa_m} \tag{1.4.1}$$

where d is the distance between the plates, ΔT is the temperature difference, ν is the kinematic viscosity and κ_m is the thermal diffusivity of the fluid. When the Rayleigh number exceeds a certain value the convection takes place in the system. This value is called critical Rayleigh number.

1.4.2 Hoton−Rogers−Lapwood convection

Rayleigh−Bénard convection for porous media analogue of is known as Horton−Rogers-Lapwood convection it was named by Horton and Rogers (1945) and Lapwood (1948). The assumption other than the Rayleigh−Bénard convection is that the porous layer is assumed to be isotropic and the fluid and solid phases are in local thermal equilibrium. The onset of convection in this case is governed by Rayleigh−Darcy number given by

$$Ra_D = \frac{\alpha_T g \Delta T K d}{\nu \kappa} \tag{1.4.2}$$

In general Ra_D is the product of the Darcy number $Da = \frac{K}{d^2}$ and usual Rayleigh number for clear viscous fluid. Here K is permeability, α is the thermal volume expansion coefficient, κ_T is thermal diffusivity and ν is kinematic viscosity.

1.5 Important definitions

1. ***Double-diffusive convection***

The density stratification in thermal convection is due to the variation of only one component (temperature), then the system is called single diffusive system. If it is due to two components such as temperature and salinity, then it is known as Double diffusive system. These two components will have different diffusivities, and opposite contribution to the density; i.e. one component is to stabilize and other is to destabilize the system. In this case both thermal and concentration (solute) gradients are present, then the Boussinesq equation takes the form

$$\rho = \rho_0[1 - \alpha_T(T - T_0) + \beta_T(S - S_0)], \tag{1.5.1}$$

where α_T and β_T are the thermal and solute expansion coefficients, T and S are the temperature and the solute content, respectively. It is to be noted that when there is direct coupling between the two diffusing components, then it is called cross-diffusion. In this the flux of one component is caused by the another component spatial gradient.

2. ***Magneto-convection***

Thermal convection in the presence of an imposed vertical magnetic field and the fluid is of electrically conducted is known as magneto-convection. The effect of magneticfield on thermal instability is characterized by Chandrasekhar number Q, which is defined as a ratio of Lorentz force exerted by the magnetic field and pressure forces

$$Q = \frac{\mu_m B_0^2 d^2}{\rho_0 \nu \nu_m} \tag{1.5.2}$$

where ν_m is the magnetic viscosity, ν kinematic viscosity and H_0 characteristic magnetic field. If the Lorentz force exerted by the magnetic field is weaker than the force exerted by the moving plasma (turbulent pressure), then the convective motions twist and stretch the magnetic field, which in a turbulent flow increases its strength (dynamo action). If the Lorentz forces are stronger than the turbulent pressure forces, then the magnetic field channels the plasma motions along the field direction and inhibits the convection.

3. ***Rotation***

When we consider the effect of rotation on the thermal instability introduces new elements into the problem as Taylor number Ta. Where Taylor number is defined as

$$Ta = \frac{4\Omega_r^2 d^4}{\nu^2} \tag{1.5.3}$$

where Ω_r is a characteristic angular velocity, d is a characteristic linear dimension perpendicular to the rotation axis, and ν is the kinematic viscosity. And Ta characterizes the importance of centrifugal "forces"or so-called inertial forces due to rotation of a fluid about an axis, relative to viscous forces. Most generally Ta has stabilizing effect for sufficient large values, when it exceeds it's critical value there is an opposite effect and this instability known as K$\ddot{u}$pper$-$Lortz instability.

4. ***Non-Newtonian fluids***

Non-Newtonian fluids are differ from Newtonian fluids most commonly the viscosity of non-Newtonian fluids are dependent of shear rate or shear rate history. The relation between the shear stress and the shear rate is linear, passing through the origin, in the case of a Newtonian fluid. But, in the case of a non-Newtonian fluids the relation is different and can even be time dependent. Viscoelastic fluids are the fluids that behaves as solid and as well as liquid too. They are elastic in nature and they will regain back when applied stress is removed. Some of the examples are as ketchup, custard, toothpaste, starch suspensions, paint, blood, and shampoo. In 19^{th} century, some physicists like Maxwell, Boltzmann, and Kelvin researched and experimented with creep and recovery of glasses, metals, and rubbers. Viscoelasticity was further examined in the late twentieth century when synthetic polymers were engineered and used in a variety of applications. These fluids exhibits the oscillatory nature on thermal instability.

1.6 Boundary conditions

In general for mathematical modeling of any dynamical system, the boundary conditions of the dependent variable is very important. Depending upon the configuration of the problem (flow between two parallel horizontal plates) there are two types of physical boundaries: rigid (impermeable) and free (permeable). We can study different types of boundary conditions such as the upper and lower boundaries are rigid$-$rigid, free$-$free, rigid$-$free and free$-$rigid.

1. Zero normal velocity: $w = 0$, for both rigid and free boundaries.
2. (a) zero tangential velocity (no slip): $\frac{\partial w}{\partial z} = 0$, for rigid boundaries.
   (b) zero tangential stress: $\frac{\partial^2 w}{\partial z^2} = 0$, for free boundaries.

For thermal boundary conditions such as isothermal and adiabatic.

1. Isothermal: For isothermal wall, the temperature disturbances must be zero at the boundary.
2. Adiabatic: For adiabatic wall, the temperature of the wall change, but there should be no through-flow of temperature.

While dealing the convective flow in porous domains, and in the simplest case, we need to have information about the velocity, as well as, temperature at the boundaries. Suppose the boundary of a Darcy-porous domain at $x = 0$ is rigid, then the normal component of the velocity at the boundary must vanish, i.e. $\vec{q}.\hat{e}_x = 0$. Since we consider Darcy flow only one condition applied at a given boundary, the other velocity components may have arbitrary values at the boundary, and thus we have slip at the boundary. In the free boundary case the pressure is constant at the boundary along y, z i.e $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$. Which gives $v = w = 0$ for all y and z, which further implies that $\frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$. And by continuity equation we have $\frac{\partial u}{\partial x} = 0$ at $x = 0$, for the x-component of the fluid velocity.

1.7 Methods

1.7.1 Analytical methods

1. ***Normal mode technique***

   Normal mode analysis is a harmonic analysis. It is one of the major simulation techniques used to probe the large-scale, shape-changing motions in dynamical system. Mainly this method is used to study the oscillations and instability of a dynamical system, under the assumption that all particles move with the same frequency and phase. In our thesis we have used the normal mode expansion to express the perturbations in physical quantities, such that the frequency of perturbation is same in all. The normal mode expansion is expressed in the form $\text{Exp}[i(a_x x + a_y y) + \sigma t]$, where σ is the frequency of perturbation, $a = \sqrt{a_x^2 + a_y^2}$ is the wave number. The frequency of perturbation σ, decides whether the system is stable or unstable. The growth rate σ is in general complex such that $\sigma = \sigma_r + i\sigma_i$. The system with $\sigma_r < 0$ is always stable, while for $\sigma_r > 0$ it will become unstable. When $\sigma = 0$, that is $\sigma_r = 0$ and $\sigma_i = 0$, the system is marginally stable. With $\sigma_r = 0$ and $\sigma_i \neq 0$, the overstable motion may occur.

2. ***Perturbation method***

   In many practical dynamical problems, any mathematical model cannot be solved exactly or, if the exact solution is available, it exhibits such a very complicated dependency on the parameters that it is very hard to use as such. However, to simplify the problem a parameter can be introduced and identified, say χ, such that the solution is available and reasonably simple for $\chi = 0$. This small quantity χ is called perturbation parameter and this method is known as regular perturbation method. There is also a singular perturbation problem, in which a small parameter that cannot be approximated by setting the parameter value to zero. This is in contrast to regular perturbation problems, for which an approximation can be obtained by simply setting the small parameter to zero. More precisely, the solution of the problem cannot be uniformly approximated by an asymptotic expansion. Generally we have for the approximation to the full solution $\mathbb{A}$, a series in the small parameter χ, in the following form

$$\mathbb{A} = \mathbb{A}_0 + \chi\mathbb{A}_1 + \chi^2\mathbb{A}_2 + ... \tag{1.7.1}$$

In the above expression, $\mathbb{A}_0$ would be the known solution to the exactly solvable initial problem and $\mathbb{A}_1, \mathbb{A}_2$ ... represent the higher-order terms which may be found iteratively by some systematic procedure. For small values of χ the higher order terms in the above series become successively smaller. In general an approximate "perturbation solution"is obtained by truncating the series, usually by keeping only the first two terms, the initial solution and the "first-order"perturbation correction.

3. *A truncated representation of Fourier series method*

The linear stability analysis is sufficient only for obtaining the stability condition of the motionless solution and the corresponding eigenfunctions describing qualitatively the convective flow. But, it cannot provide an information about the values of the convective amplitudes, nor even regarding the rate of heat and mass transfer. In order to obtain this additional information about heat and mass transfer in the system, we need to perform the nonlinear analysis, which is useful to understand the physical mechanism with minimum amount of mathematical analysis and is a step forward toward understanding full nonlinear problem.

4. *Nonlinear stability analysis*

The stability of a system is tested by applying infinitesimal disturbances. If the governing equations are approximated just to include the linear terms in the applied disturbances then the theory used to predict the stability of the system is called linear stability analysis. When the non-linear terms are remained in the governing equations, then we call it nonlinear stability analysis. The linear stability theory basically provide us the information that when the flow will be unstable to infinitesimal disturbances. The Nonlinear stability theory provides an information to measure the amount of heat or mass transfer is taking place in the system in terms of amplitudes.

1.7.2 Numerical methods

1. ***Galerkin method***

   Galerkin methods are a class of methods for converting a continuous operator problem (such as a differential equation) to a discrete problem. In principle, it is the equivalent of applying the method of variation of parameters to a function space, by converting the equation to a weak formulation. Galerkin method is basically a weighted residual method used to solve boundary value problems. We apply the following steps:

   (a) Expand the unknown solution in a set of basis functions, with unknown coefficients or parameters; this is called the trial solution.
   (b) Make the trial solution satisfy the boundary conditions (usually) and initial conditions.
   (c) Define the residual.
   (d) Set the weighted residual to zero and solve the equations.
   (e) Examine the error by constructing successive approximations, and show convergence as the number of basis functions increases.

2. ***Runge–Kutta method***

   In numerical analysis, the Runge–Kutta methods are an important family of implicit and explicit iterative methods, which are used in temporal discretization for the approximation of solutions of ordinary differential equations. These techniques were developed by the German mathematicians C. Runge and M. W. Kutta during 19^{th} century. One member of the family of Runge–Kutta methods is so commonly used and often referred as RK4, classical Runge–Kutta method or simply as the Runge–Kutta method. One of our problem in the thesis the Runge–Kutta–Fehlberg method has been used to find accuracy of our results.

1.8 Scope of the thesis

The thesis deals with the thermal instability under different hydrodynamic configurations while considering Rayleigh−Bénard and Darcy−Bénard convection. The problems have been studied analytically/numerically under various physical conditions, for different fluids and boundary conditions. The following assumptions are considered in the present thesis:

1. The systems considered are supposed to have the characteristic length much larger than the mean free path of the fluid molecules, so that the continuum hypothesis can be applied.
2. The density variations are assumed to govern by the Boussinesq approximations for both Newtonian/non-Newtonian fluids.
3. The porous medium considered is assumed to be isotropic unless it is specified.
4. The fluid considered is Newtonian for stationary convection, for oscillatory mode of convection is non-Newtonian fluid.
5. The central part of the thesis is 'thermal instability under nonlinear oscillatory mode of convection', we are the initiators of this model. A lot of scope can be seen in this aspect of the problem.
6. Temperature, Gravity, Rotational speed and Magnetic field modulations have been presented in the current thesis.
7. Nonlinear vertical throughflow effects under oscillatory mode and chaotic convection have been presented.

Since oscillatory mode of convection is strict to additional external constraint to the system (like rotation, magnetic field and non-Newtonian fluids etc..) one can see future scope in this direction (present thesis).

1.9 Preface

The thesis entitled "Nonlinear Thermal Instability Under Various Physical Configurations"comprising of analytical/numerical solutions of some problems related with the topic, is an outcome of the research work carried out by me during the course of investigations under the guidance of Dr. B.S. Bhadauria, Professor, Department of Applied Mathematics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow.

Rayleigh−Bénard convection is a paradigmatic example of convective thermal instability in ordinary fluid layers. The porous media analogue of this problem is known as Horton−Rogers−Lapwood convection, and it is of paramount interest due to its applications in various fields of engineering, thermal sciences and geophysics. Regulating the convective phenomenon in thermal sciences is of considerable importance due to its numerous application in many engineering problems. Keeping in mind the regulations of heat and mass transfer we have presented our results in the following chapters in which some of the work has been published.

The first chapter is of introductory part. The key features of the discipline are stated in this chapter. We describe the governing equation of dynamical systems. This chapter also consists of the basic definitions, relevant to the thesis topic. The literature survey of thermal convection in different hydrodynamic configurations and different kinds of modulations has been explained.

In chapter 2, we have presented thermal instability in anisotropic horizontal porous medium saturated with temperature dependant viscous fluid with time periodic temperature modulation. A weak non−linear stability analysis has been performed for the stationary mode of convection, and heat transport in terms of the Nusselt number is calculated. The effects of thermo rheological parameter, amplitude and frequency of modulation, thermo−mechanical anisotropies and Vadasz number on heat transport have been analyzed and depicted graphically. It is also found that, the heat transport can be controlled effectively by a mechanism that is external to the system.

In chapter 3, using complex non-autonomous Ginzburg−Landau equation, we have investigated nonlinear oscillatory convection in fluid layer (section 3.1) and porous medium (section 3.2) under Gravity modulation, considering viscoelastic fluids in the layer. The influence of (stress) relaxation and (strain) retardation times of viscoelastic fluid on heat transfer has been discussed. The study establishes that the heat transport can be controlled effectively by a mechanism that is external to the system. Modulation has a destabilizing effect at low frequencies and a stabilizing effect at high frequencies, which increases with increasing the amplitude of modulation. We also found that overstability advances the onset of convection, hence increases heat transfer.

In chapter 4, a nonlinear oscillatory convection in viscoelastic fluid saturated porous medium (section 4.1) and double diffusive convection in viscoelastic fluid layer (section 4.2) under temperature modulation has been investigated. The time periodic temperature profile on the boundaries has been considered and its effect on the system has been investigated. The effect of relaxation and retardation times of viscoelastic fluid on heat transfer and mass transfer has been discussed. The average value of Nusselt number is obtained numerically while using the value of Nusselt number and found the good approximation (or combination) of frequency and phase angle where heat and mass transfer is enhances or diminishes.

In chapter 5, the influence of sinusoidally varying magnetic field and rotational speed effects on Rayleigh−Bénard convection is carried out. In section 5.1, we have developed an analytic study of heat transport in an electrically conducting fluid layer under nonuniform time dependent magnetic field. The applied vertical magnetic field consists of two parts; constant part, and a time dependent periodic part, which varies sinusoidally with time. Using weakly nonlinear theory, the Ginzburg−Landau equation is solved through NDSolve Mathematica 8, and the results are verified using Runge−Kutta−Fehlberg method. The Nusselt number is obtained in terms of various system parameters and the effect of each parameter on heat transport is reported in detail. The effect of magnetic Prandtl number Pm, amplitude of modulation δ is to enhance the heat transfer. The Chandrasekhar number Q, modulation frequency Ω is to stabilize the system. Further, it is found that magnetic modulation can effectively be used in either enhancing the heat transfer or diminishing it. In section 5.2, a theoretical investigation has been carried out to study the combined effect of rotation speed modulation and internal heating on thermal instability in a temperature dependent viscous horizontal fluid layer. Using Ginzburg−Landau analysis it is found that, the modulated rotation speed has a stabilizing effect for different values of modulation frequency. Further, internal heating and thermo−rheological parameter is found to destabilize the system.

In chapter 6, in the light of earlier work proposed by Johnathan et al. (2014) motivated us to make a chaotic mode of convection under temperature modulation. The analysis of buoyancy driven convection for moderate Prandtl number in a fluid saturated porous layer heated from below and subject to thermal modulation is presented. It's been investigated a better combination of values of Ω, δ and scaled Rayleigh number R provides a way to chaos. It is also found that temperature modulation of the boundaries is to enhance the behaviour of the chaotic motions.

In chapter 7, thermal instability has been investigated in non-Newtonian fluids. In section 7.1, we study nonlinear convection in a porous medium saturated with nanofluid under gravity modulation, and calculate heat and mass transport across the porous medium. The nonuniform vertical vibrations of the system, which can be realized by oscillating the system vertically, are considered to vary sinusoidally with time. A nonlinear stability analysis has been performed to obtained the Nusselt number, which is found to be the function of thermal Rayleigh number, concentration Rayleigh number, Lewis number, modified diffusivity ratio, amplitude and frequency of modulation. The effects of various physical parameters have been investigated on heat and mass transfer. It is found that gravity modulation can be used effectively to regulate the stability of the system. In section 7.2, the effect of vertical throughflow on oscillatory convection in a viscoelastic fluid saturated porous medium has been investigated. The heat transport is investigated in terms of both the Nusselt and average Nusselt numbers, governed by the non−autonomous complex Ginzburg−Landau equation using weak nonlinear stability analysis. The effect of vertical throughflow is found to stabilize the system irrespective of the direction of throughflow. The time relaxation parameter λ_1 has destabilizing effect, while the time retardation parameter λ_2 has stabilizing effect on the system. Further, it is also found that heat transfer is more in the oscillatory mode of convection rather than stationary.

Chapter 2

Heat transport in an anisotropic porous medium saturated with variable viscosity liquid under temperature modulation

2.1 Introduction

In this chapter we have analyzed an effect of temperature dependant viscosity and anisotropy on the porous medium while considering Darcy flow under temperature modulation. Due to the important applications in engineering and industrial, it is more appropriate to assume temperature gradient to be a function of both space and time. This temperature gradient which can be obtained by solving the energy equation can be used as an effective mechanism to control the convective flow. The modulated temperature field is assumed at the boundaries according to Venezian (1969). Anisotropy of the porous medium is of considerable importance in geological and pedological process, such as sedimentation, compaction, frost action and reorientation of a solid matrix are mainly responsible for creation of anisotropic natural porous media. Anisotropic can also be a characteristic of artificial porous like pelleting used in chemical engineering process and fibre materials used in insulating purposes. Keeping in mind the above, this chapter has been constructed. And it is based on the papers Vanishree et al. (2010) and Srivastava et al.(2013), where they have studied thermal instability in a porous medium with temperature dependant viscosity fluid under gravity modulation. But, what happens in the case of temperature modulation, while considering the relation between temperature and viscosity proposed by Nield (1996). In this aspect the present chapter is to discuss the effect of temperature dependant viscosity on thermal instability.

2.2 Mathematical Formulation

We consider an infinitely extended horizontal anisotropic porous medium saturated by Newtonian fluid temperature dependent viscosity, confined between two free$-$free boundaries at $z = 0$ and $z = d$, and heated from below. The temperature of the boundaries vary periodically in a time-dependent manner. The temperature difference across the porous medium is kept at ΔT. We choose Cartesian frame of reference as, origin in the lower boundary and the $z-$axis in vertically upward direction. It is assumed that the mechanical properties and thermal properties in x and y-directions are same. Further Darcy law and the Oberbeck−Boussinesq approximation are taken to be applicable Rajagopal et al.(2010). Under these assumptions, the equations Nield and Bejan (2013) which describe this system are given by:

$$\begin{aligned}
\nabla \cdot \vec{q} &= 0, && (2.2.1)\\
\frac{\rho_0}{\delta_1}\frac{\partial \vec{q}}{\partial t} &= -\nabla p + \rho \vec{g} - \frac{\mu(T)}{\mathbf{K}} \cdot \vec{q}, && (2.2.2)\\
\gamma\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T &= \nabla \cdot (\kappa_T \cdot \nabla T), && (2.2.3)\\
\rho &= \rho_0 \left[1 - \alpha_T (T - T_0)\right], && (2.2.4)\\
\mu(T) &= \frac{\mu_0}{1 + \chi^2 \delta_0 (T - T_0)}. && (2.2.5)
\end{aligned}$$

where $\mu(T)$ is a variable viscosity, $\mathbf{K} = K_x(\hat{i}\hat{i} + \hat{j}\hat{j}) + K_z(\hat{k}\hat{k})$ is permeability tensor, $\kappa_T = \kappa_{Tx}(\hat{i}\hat{i} + \hat{j}\hat{j}) + \kappa_{Tz}(\hat{k}\hat{k})$ is the thermal diffusivity tensor. For simplicity γ is taken to be unity here, $\vec{g} = (0, 0, -g)$ is gravitational acceleration. The externally imposed thermal boundary conditions are taken of the form:

$$\begin{aligned}
T &= T_0 + \Delta T[1 + \chi^2\delta \ \cos(\Omega t)] && \text{at } z = 0\\
&= T_0 + \Delta T \chi^2 \delta \ \cos(\Omega t + \theta) && \text{at } z = d \qquad (2.2.6)
\end{aligned}$$

where the physical variables have their usual meanings given in the list of symbols. The thermo−rheological relationship in Eq.(2.2.5) is guided by Nield (1996). The basic state is assumed to be quiescent and the quantities in this state are given by:

$$\begin{aligned}
q_b = 0, \ \rho = \rho_b(z,t), \ p &= p_b(z,t), \ T = T_b(z,t). && (2.2.7)\\
\frac{\partial p_b}{\partial z} &= -\rho_b g, && (2.2.8)\\
\frac{\partial T_b}{\partial t} &= \kappa_{Tz}\frac{\partial^2 T_b}{\partial z^2}, && (2.2.9)\\
\rho_b &= \rho_0\left[1 - \alpha_T (T_b - T_0)\right]. && (2.2.10)
\end{aligned}$$

The solution of Eq.(2.2.9) subject to the thermal boundary conditions Eq.(2.2.6), is given by:

$$T_b(z,t) = T_s(z) + \chi^2 \delta Re[T_1(z,t)], \qquad (2.2.11)$$

where

$$T_s(z) = T_0 + \Delta T(1 - \frac{z}{d}), \tag{2.2.12}$$

$$T_1(z,t) = [a_1(\zeta)e^{\frac{\zeta z}{d}} + a_1(-\zeta)e^{\frac{-\zeta z}{d}}]e^{-i\Omega t}, \tag{2.2.13}$$

$a_1(\zeta) = \Delta T\frac{(e^{-i\theta}-e^{-\zeta})}{(e^{\zeta}-e^{-\zeta})}$ and $\zeta^2 = \frac{-i\Omega d^2}{\kappa_{Tz}}$. Here $T_s(z)$ steady part and $T_1(z,t)$ oscillatory part of $T_b(z,t)$. The finite amplitude perturbations on the basic state are superposed in the form,

$$\vec{q} = \vec{q_b} + \vec{q}^{'},\ \rho = \rho_b + \rho^{'},\ p = p_b + p^{'},\ T = T_b + T^{'}. \tag{2.2.14}$$

Substituting the Eq.(2.2.14) in Eqs.(2.2.1)-(2.2.5) and using the basic state results, we obtain:

$$\begin{aligned}
\nabla \cdot \vec{q}^{'} &= 0, && (2.2.15)\\
\frac{\rho_0}{\delta_1}\frac{\partial \vec{q}^{'}}{\partial t} &= -\nabla p + \rho^{'}\mathbf{g} - \frac{\mu(T)}{\mathbf{K}} \cdot \vec{q}^{'}, && (2.2.16)\\
\frac{\partial T^{'}}{\partial t} + w^{'}\frac{\partial T_b}{\partial z} + (\vec{q}^{'} \cdot \nabla)T^{'} &= \nabla \cdot \left(\kappa_T \cdot \nabla T^{'}\right), && (2.2.17)\\
\rho^{'} &= -\rho_0 \alpha_T T^{'}, && (2.2.18)\\
\mu(T) &= \frac{\mu_0}{1 + \chi^2 \delta_0 (T_b - T_0)}. && (2.2.19)
\end{aligned}$$

In the case of two-dimensional convection one can introduce, stream functions ψ as $u^{'} = \frac{\partial \psi}{\partial z}$, $w^{'} = -\frac{\partial \psi}{\partial x}$,. Non-dimensionalising the physical variables as $(x, y, z) = d(x^*, y^*, z^*)$, $t = \frac{d^2}{\kappa_{Tz}}t^*$, $\psi = \kappa_{Tz}\psi^*$, $T^{'} = \Delta T\ T^*$, eliminating the pressure term and finally dropping the asterisk, we obtain

$$\begin{aligned}
\frac{1}{Va}\frac{\partial}{\partial t}(\nabla^2\psi) + \overline{\mu}(T)(\nabla_{\xi}^2\psi) &= -\frac{1}{\xi}\frac{\partial \overline{\mu}}{\partial z}\frac{\partial \psi}{\partial z} - Ra\frac{\partial T}{\partial x}, && (2.2.20)\\
-\frac{\partial T_b}{\partial z}\frac{\partial \psi}{\partial x} - {\nabla_{\eta}}^2 T &= -\frac{\partial T}{\partial t} + \frac{\partial(\psi, T)}{\partial(x, z)}, && (2.2.21)
\end{aligned}$$

where $\overline{\mu}(T) = \frac{1}{1+\chi^2 V_T T}$, χ^2 is a small quantity that indicates that the viscosity variation with temperature is weak. The non-dimensionalizing parameters in the above equations are: $Va = \frac{\delta_1 \nu d^2}{K_z \kappa_{Tz}}$ is Vadasz number, $Ra = \frac{\alpha_T g \Delta T d K_z}{\nu \kappa_{Tz}}$ is thermal Rayleigh number and $V_T = \delta_0 \Delta T$ is thermo-rheological parameter or variable viscosity parameter. Eq.(2.2.21) shows that the basic state solution influences the stability problem through the factor $\frac{\partial T_b}{\partial z}$, which is given by:

$$\frac{\partial T_b}{\partial z} = -1 + \chi^2 \delta [f_2(z,t)], \tag{2.2.22}$$

where

$$f_2(z,t) = Re[f(z) e^{(-i\Omega t)}], \tag{2.2.23}$$

$$f(z) = [A(\zeta) e^{\zeta z} + A(-\zeta) e^{-\zeta z}], \quad A(\zeta) = \zeta \frac{(e^{-i\theta} - e^{-\zeta})}{(e^{\zeta} - e^{-\zeta})} \text{ and } \zeta = (1-i)\sqrt{\frac{\Omega}{2}}. \tag{2.2.24}$$

Assuming small variation of time, and re-scaling it as $\tau = \chi^2 t$, we study the stationary convection of the system. We write the non-linear system of equations Eqs.(2.2.20)-(2.2.21) in the form matrix as given below:

$$\begin{bmatrix} \overline{\mu}(T)\nabla_{\xi}^2 & Ra\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla_{\eta}^2 \end{bmatrix} \begin{bmatrix} \psi \\ T \end{bmatrix} = \begin{bmatrix} -\frac{\chi^2}{Va}\frac{\partial}{\partial \tau}(\nabla^2\psi) - \frac{1}{\xi}\frac{\partial \overline{\mu}}{\partial z}\frac{\partial \psi}{\partial z} \\ -\chi^2\frac{\partial T}{\partial \tau} + \frac{\partial(\psi,T)}{\partial(x,z)} + \chi^2\delta f_2(z,\tau)\frac{\partial \psi}{\partial x} \end{bmatrix} \tag{2.2.25}$$

We solve the above Eq.(2.2.25), by using $\overline{\mu} = \overline{\mu}(T_b)$ (Nield, 1996) and considering impermeable boundary conditions as given below:

$$\psi = 0 \;\; \& \;\; T = 0 \quad \text{on} \;\; z = 0 \;\; z = 1. \tag{2.2.26}$$

2.3 Heat transport for stationary instability

We introduce the following asymptotic expansions in Eq.(2.2.25):

$$\begin{aligned} Ra &= R_0 + \chi^2 R_2 + \chi^4 R_4 + ..., \\ \psi &= \chi\psi_1 + \chi^2\psi_2 + \chi^3\psi_3 + ..., \\ T &= \chi T_1 + \chi^2 T_2 + \chi^3 T_3 + ..., \end{aligned} \tag{2.3.1}$$

where R_0 is the critical value of the Rayleigh number at which the onset of convection takes place in the absence of temperature modulation. Now we solve the system for different orders of χ.

At the lowest order, we have

$$\begin{bmatrix} \nabla_{\xi}^2 & R_0\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla_{\eta}^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.3.2}$$

The solution of the lowest order system subject to the boundary conditions Eq.(2.2.26) is:

$$\psi_1 = \mathbb{A}(\tau)\sin(a_c x)\sin(\pi z), \tag{2.3.3}$$

$$T_1 = -\frac{a_c}{{\delta_2}^2}\mathbb{A}(\tau)\cos(a_c x)\sin(\pi z). \tag{2.3.4}$$

where $\delta_1^2 = a_c^2 + \pi^2, \delta_2^2 = a_c^2 + \frac{\pi^2}{\xi}$ and $\delta_3^2 = \eta a_c^2 + \pi^2$.

The critical value of the Rayleigh number and the corresponding wave number for the onset of stationary convection is calculated numerically and the expression for Rayleigh number is given by

$$R_0 = \frac{\delta_2^2\delta_3^2}{a_c^2}, \tag{2.3.5}$$

$$a_c = \frac{\pi}{(\eta\xi)^{\frac{1}{4}}}, \tag{2.3.6}$$

which are the results given by Epherre (1975), Siddheshwar et al. (2012) and Bhadauria et al. (2013). By considering $\xi = 1$, $\eta = 1$, we get the classical results of Lapwood (1948), for the case of isotropic porous media.

At the second order, we have:

$$\begin{bmatrix} \nabla_\xi^2 & R_0\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla_\eta^2 \end{bmatrix}\begin{bmatrix} \psi_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix}, \tag{2.3.7}$$

where

$$R_{21} = 0 \tag{2.3.8}$$

$$R_{22} = \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}. \tag{2.3.9}$$

The second order solutions subjected to the boundary conditions Eq.(2.2.26) is obtained as follows:

$$\psi_2 = 0, \tag{2.3.10}$$

$$T_2 = -\frac{a_c}{8\pi\delta_3^2}\mathbb{A}^2(\tau)\sin(2\pi z). \tag{2.3.11}$$

The horizontally averaged Nusselt number Nu, for the stationary mode of convection is given by

$$\mathrm{Nu}(\tau) = 1 + \frac{\left[\frac{a_c}{2\pi}\int_0^{\frac{2\pi}{a_c}} \left(\frac{\partial T_2}{\partial z}\right)\, dx\right]_{z=0}}{\left[\frac{a_c}{2\pi}\int_0^{\frac{2\pi}{a_c}} \left(\frac{\partial T_b}{\partial z}\right)\, dx\right]_{z=0}} \tag{2.3.12}$$

$$\mathrm{Nu}(\tau) = 1 + \frac{a_c^2}{4\delta_3^2}\mathbb{A}^2(\tau). \tag{2.3.13}$$

At the third order, we have

$$\begin{bmatrix} \nabla_\xi^2 & R_0\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\nabla_\eta^2 \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix}, \tag{2.3.14}$$

where

$$R_{31} = -\frac{1}{Va}\frac{\partial}{\partial \tau}(\nabla^2\psi_1) - V_T T_s(z)\nabla_\xi^2\psi_1 - \frac{V_T}{\xi}\frac{\partial \psi_1}{\partial z} - 2R_0 V_T T_s(z)\frac{\partial T_1}{\partial x} - R_2\frac{\partial T_1}{\partial x}, \tag{2.3.15}$$

$$R_{32} = \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} + \delta f_2(z,\tau)\frac{\partial \psi_1}{\partial x} - \frac{\partial T_1}{\partial \tau}. \tag{2.3.16}$$

Substituting ψ_1, T_1 and T_2 into Eqs.(2.3.15)-(2.3.16) we obtain the expressions for R_{31} and R_{32} easily. Now by applying the solvability condition for the existence of third order solution, we get the Ginzburg−Landau equation for stationary convection with time-periodic coefficients in the form

$$A_1\mathbb{A}'(\tau) - A_2\mathbb{A}(\tau) + A_3\mathbb{A}(\tau)^3 = 0 \tag{2.3.17}$$

where

$A_1 = \frac{1}{\delta_3^2}(1+\frac{\delta_1^2\delta_3^2}{\delta_2^2 Va})$, $A_2 = (\frac{R_2}{R_0}+(\frac{V_T}{2}-2\delta I_1))$, $A_3 = \frac{a_c^2}{8\delta_3^2}$ and $I_1 = \int_0^1 f_2(z,\tau)\sin^2(\pi z)\, dz$. The Ginzburg−Landau equation given by Eq.(2.3.17) is a Bernoulli equation and obtaining its analytical solution is difficult due to its non-autonomous nature. So that it has been solved numerically using the in-built function NDSolve of mathematica subjected to the initial condition $A(0) = b_0$, where b_0 is the chosen initial amplitude of convection. In our calculations we may use $R_2 = R_0$, to keep the parameters to the minimum. For unmodulated case, the analytical solution of the above Eq.(2.3.17) takes the form:

$$\mathbb{A}(\tau) = \frac{1}{\sqrt{\left(\frac{A_3}{A_2} + C_1\mathrm{Exp}\left[-\frac{2A_2}{A_1}\tau\right]\right)}}, \tag{2.3.18}$$

where

$$A_2 = 1 + \frac{V_T}{2}, \tag{2.3.19}$$

and C_1 which appears in Eq.(2.3.18) is an integration constant, can be found by using suitable initial condition.

2.4 Results and discussions

In this chapter, we study the combined effect of temperature modulation and variable viscosity in a fluid saturated closely packed anisotropic porous medium. A weakly non-linear stability analysis has been performed to investigate the effect of temperature modulation on heat transport. The effect of temperature modulation on the Bénard-Darcy system has been assumed to be of order $O(\chi^2)$. This means we consider only small amplitude temperature modulation. Such an assumption will help us in obtaining the amplitude equation of convection in simple and elegant manner and is much easier to obtain than in the case of the Lorenz model. The work of Nield (1996), has been used for the thermo-rheological relationship of temperature dependant viscosity of the fluid. Before writing the discussion of the results, we mention some features of the following aspects of the problem.

1. The need for non-linear stability analysis.
2. The relation of the problem to real application.
3. The selection of all dimensionless parameters utilized in computations.

If one needs to quantify heat transfer, which linear stability analysis is unable to do, this problem needs to perform the non-linear analysis and hence the importance. External regulation of convection is important in the study of convection in porous media. The objective of this article is to consider temperature modulation and variable viscosity for either enhancing or inhabiting convective heat transfer as is required by a real application. The temperature modulation has been considered in the following three cases:

1. In-phase modulation (IPM)($\theta = 0$).

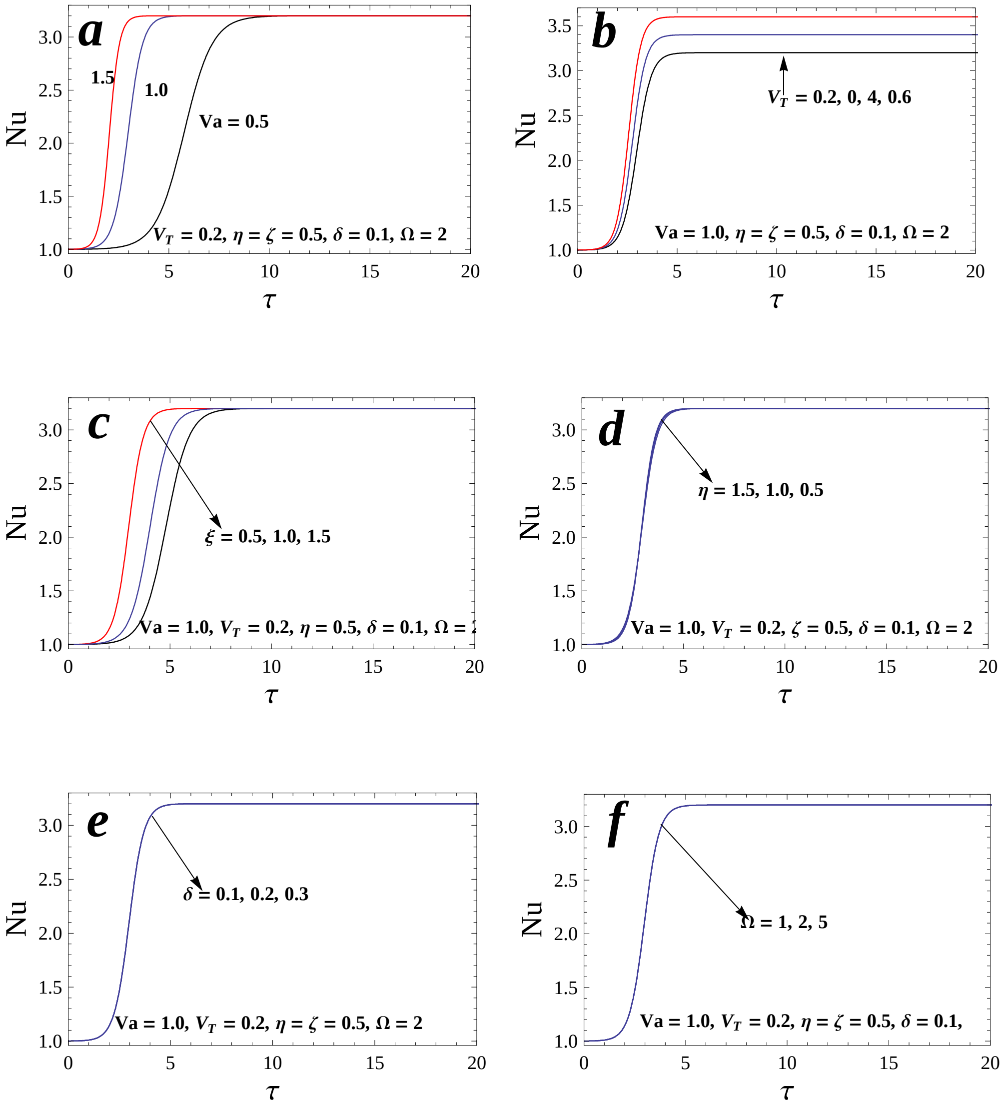

Figure 2.1: Nu versus τ (IPM). $a.Va, b.V, c.\xi, d.\eta, e.\delta, f.\Omega$

2. Out-phase modulation (OPM) ($\theta = \pi$).
3. Only Lower boundary modulated (LBMO) ($\theta = -i\infty$).
   which means that the modulation effect will not be considered in upper boundary but only in lower boundary.

Since the porous medium is assumed to be closely packed, the Darcy-model is considered in governing equation. The parameters that arise in this study of convection and influence the heat transport are $Va, V_T, \xi, \eta, \delta, \Omega$ and θ. The first four are related to the properties of fluid and porous media, and last three are external mechanism for controlling convection. Vadász (1998), pointed that there are some modern porous medium applications, such as mushy layer in solidification of binary alloys and fractured porous medium, where the value of Va may be considered to be unity order, therefore the time-term in the present study has been retained. Further, this is the reason that the values of Va has been kept around one in our calculations. The values of δ is consider very small between 0 and 0.5 since we are studying the effect of small amplitude modulation on the heat transport. Also, since the effect of low frequencies, is maximum, on the onset of convection as well as on the heat transport, therefore the modulation of temperature is assumed to be of low frequency. Further, the value of thermo-rheological parameter, V is also considered to be small.

The numerical results for Nu obtained from the expression in Eq.(2.3.13) by solving the amplitude Eq.(2.3.17) have been presented in the figures 2.1-2.4. It is clear to see the expression in Eq.(2.3.13) in conjunction with Eq.(2.3.17) that Nu(τ) is a function of Vadász number Va, thermo-rheological parameter V_T, thermo-mechanical anisotropy parameters ξ and η, and the amplitude and frequency of modulation, respectively, δ and Ω. The effect of each type of modulation on heat transport is shown in figures 2.1-2.4 wherein the plots of Nusselt number Nu versus τ are presented. It is found from the figures that the value of Nu starts with one and remains constant for quite some time, thus showing the conduction state initially. Then the value of Nu increases with time, thus showing the convection state and finally becomes constant on further increasing τ, thus achieving the steady state.

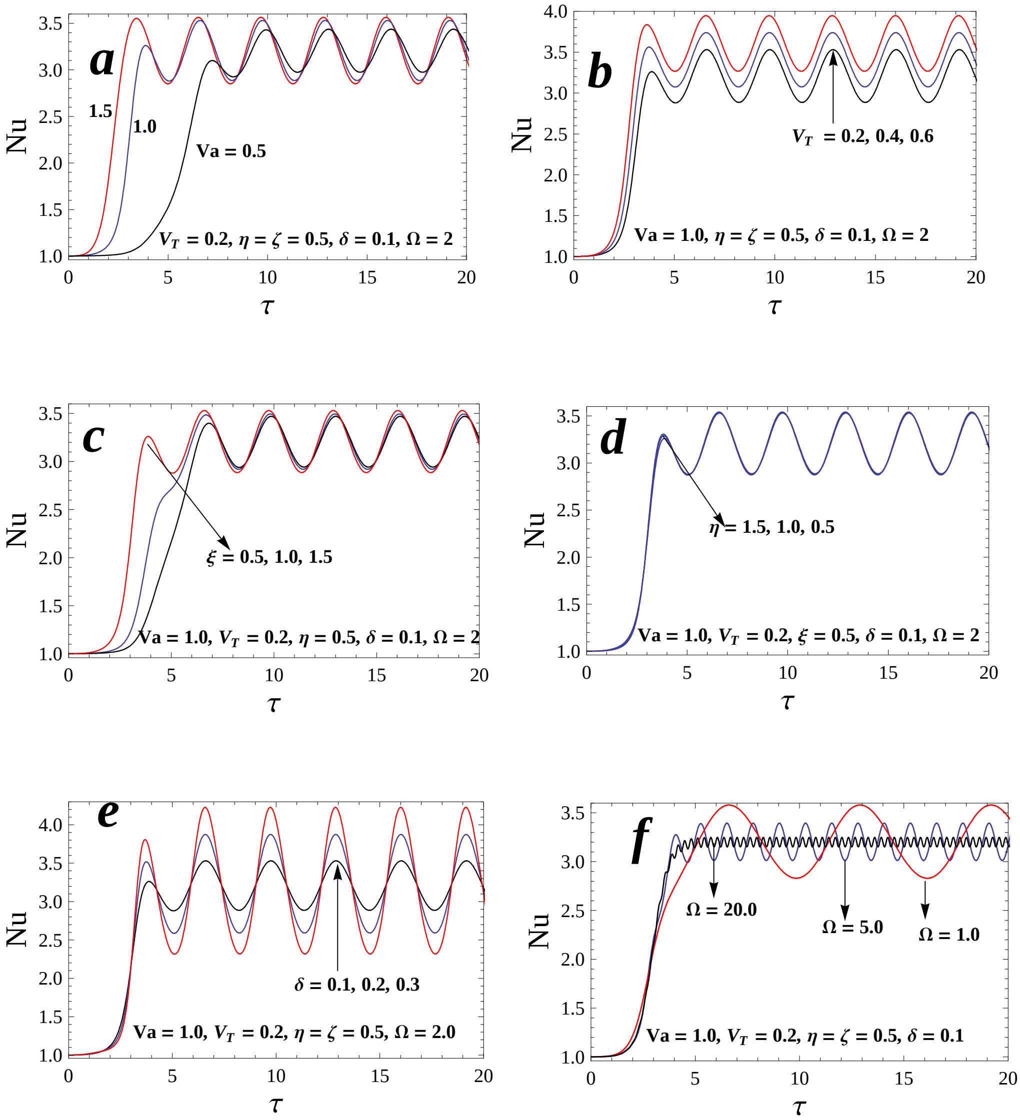

Figure 2.2: Nu versus τ (OPM). $a.Va, b.V_T, c.\xi, d.\eta, e.\delta, f.\Omega$

For IPM, the results are presented in figures 2.1a-f. From the Figures, we observe that Nu increases with individual and collective increases in Vadaśz number and thermo-rheological parameter V_T, but decreases with increase in mechanical anisotropy ξ. Thus, there is appreciable enhancement in heat transport on increasing Va and V_T thereby advancing the onset of convection. However, the heat transport decreases on increasing ξ, thus delaying the convection. The effects of Va and ξ on heat transport diminish at large values of time τ. Further, the amplitude of modulation δ and the frequency modulation Ω both have negligible effects on heat transport in this case. Further, an increment in thermal anisotropic parameter η, decreases Nu initially and then increases with time. Thus the effect of mechanical and thermal anisotropy is found to be opposite at large time, compatible with the results of Epherre (1975), Kuznetsov and Nield (2008) and Bhadauria (2012) obtained for the unmodulated case.

In figures 2.2a-f, we have depicted the variation of Nu with time τ for out of phase modulations. It is found that Nu starts with one, increases with increasing time τ and then becomes oscillatory. However, on further increasing the time, it approaches the steady state. We observe from figures 2.2a-d that the effects of Va, V_T, ξ and η on heat transport are found to be similar to those of IPM. Further, we found in figure 2.2e that the effect of amplitude of modulation is to increase the magnitude of Nu, thus increasing the heat transport and advancing the convection. We note that, the following expression is of the influence of amplitude of modulation on heat transport.

$$\mathrm{Nu}/\delta = 0.1 < \mathrm{Nu}/\delta = 0.2 < \mathrm{Nu}/\delta = 0.3$$

Also, from figure 2.2f, we observe that an increase in the frequency of modulation decreases the magnitude of Nu, and so the effect of frequency of modulation on heat transport diminishes. At high frequency the effect of temperature modulation on thermal instability disappears altogether. This result agrees quite well with the linear theory results Venezian (1969), where the correction in the critical value of Rayleigh number due to temperature modulation becomes almost zero at high frequencies.

In figures 2.3a-f, show the variation of Nu with respect to the time, for lower boundary temperature modulation only. Here we observe that the effect of various parameters and

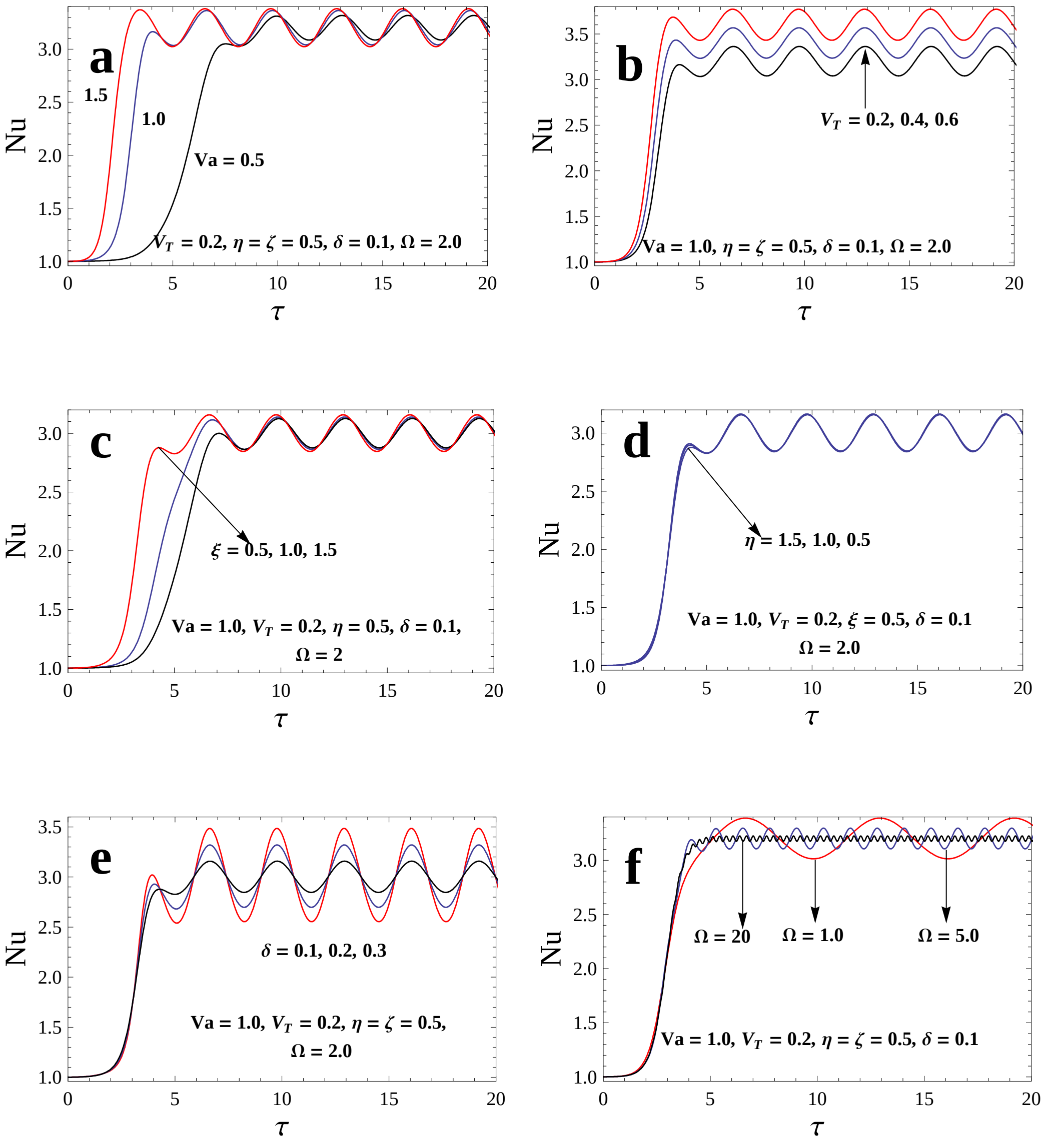

Figure 2.3: Nu versus τ (LBMO). $a.Va, b.V_T, c.\xi, d.\eta, e.\delta, f.\Omega$

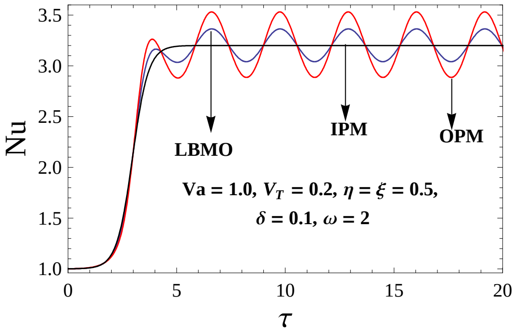

Figure 2.4: Nu versus τ (Comparison).

temperature modulation on the heat transport are qualitatively similar to those obtain in OPM case. However the magnitude of Nu in this case is less than that in the case of OPM. In figure 2.4, a comparison of in phase modulation, out of phase modulation and when only lower boundary temperature, is presented. It is found that the magnitude of Nu for LBMO is greater than that obtained in case of IPM but, less than that of OPM as shown below:

$$\text{Nu/IPM} < \text{Nu/LBMO} < \text{Nu/OPM}$$

On comparing the analytical solution for an amplitude of convection Eq.(2.3.18), results with the present one, we find the similar plots as in the case of IPM shown in figure 2.4. This shows that the results of analytical solution for an amplitude Eq.(2.3.18) and that of IPM are almost same, which confirms that in-phase modulation does not effect heat transport in the system.

2.5 Conclusions

We have analyzed the effect of temperature modulation and temperature dependant viscosity on Bénard−Darcy convection by performing a weakly non-linear stability analysis resulting in the real Ginzburg−Landau amplitude equation. The following conclusions are made by the previous analysis:

1. Effect of IPM is negligible on heat transport in the system.

2. In the case of IPM, the effect of δ and Ω are also found to be negligible on heat transport.
3. Effect of V_T and Va is to enhance the heat transport for all three types of modulations.
4. Effect of mechanical anisotropy ξ is to decrease the heat transport for all three types of modulations.
5. Effect of η on heat transport is negligible for all three types of modulations.
6. In the case of IPM, Nu increase steadily for intermediate value of time τ and ultimately becomes constant when τ is large.
7. In the case of OPM and LBMO, Nu shows an oscillatory nature.
8. The thermo-rheological model of Nield (1996), gives physically acceptable results, namely, the destabilizing effect of variable viscosity on Bénard-Darcy convection and thereby an enhanced heat transport.

The results of this work can be summarized as follows from the figures 2.1-2.4.

1. $[\mathrm{Nu}]_{Va=0.5} < [\mathrm{Nu}]_{Va=1.0} < [\mathrm{Nu}]_{Va=1.5}$.
2. $[\mathrm{Nu}]_{V_T=0.2} < [\mathrm{Nu}]_{V_T=0.4} < [\mathrm{Nu}]_{V_T=0.6}$.
3. $[\mathrm{Nu}]_{\xi=1.5} < [\mathrm{Nu}]_{\xi=1.0} < [\mathrm{Nu}]_{\xi=1.5}$.
4. $[\mathrm{Nu}]_{\eta=0.5} < [\mathrm{Nu}]_{\eta=1.0} < [\mathrm{Nu}]_{\eta=1.5}$.
5. $[\mathrm{Nu}]_{\delta=0.1} < [\mathrm{Nu}]_{\delta=0.2} < [\mathrm{Nu}]_{\delta=0.3}$.
6. $[\mathrm{Nu}]_{\Omega=20.0} < [\mathrm{Nu}]_{\Omega=5.0} < [\mathrm{Nu}]_{\Omega=1.0}$.

Chapter 3

Oscillatory convection under gravity modulation

3.1 Weak nonlinear oscillatory convection in a viscoelastic fluid layer under gravity modulation

3.1.1 Introduction

Kim et al.(2003) have performed a weakly nonlinear analysis of Darcy flow for stationary and oscillatory mode of convection. They found that elasticity parameters are destabilizing factor and for a certain parameter range the overstability is a preferred mode. The literature says clearly that numerous data is available for stationary nonlinear convection but lack in oscillatory nonlinear convection. Based on Kim et al.(2003) problem in this section we have performed a weakly nonlinear oscillatory convection in a horizontal fluid layer under gravity modulation using complex non autonomous Ginnzburg−Landau amplitude equation, and in the process quantify the heat transport.

3.1.2 Governing Equations

An infinitely extended horizontal layer of viscoelastic fluid, confined between two stress-free boundaries at $z = 0$ and $z = d$, is considered. The stress-free boundaries are maintained at constant temperature, and the fluid layer is heated from below. The hydrodynamic equations are simplified by assuming Oberbeck−Boussinesq approximation. The constitutive equations for non-Newtonian viscoelastic fluid model with the relaxation time $\overline{\lambda_1}$ and retardation time $\overline{\lambda_2}$ may be represented as (Rajib and Layek 2012):

$$\nabla \cdot \vec{q} = 0, \tag{3.1.1}$$

$$\left(\overline{\lambda_1}\frac{\partial}{\partial t} + 1\right)\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} - \frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\vec{g}\right) - \nu\left(\overline{\lambda_2}\frac{\partial}{\partial t} + 1\right)\nabla^2\vec{q} = 0, \tag{3.1.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \kappa_T\nabla^2 T, \tag{3.1.3}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{3.1.4}$$

where the physical variables have their usual meanings, and are given in Nomenclature. The externally imposed thermal boundary conditions and gravitational fields are given by

$$\begin{aligned} T &= T_0 + \Delta T, && \text{at } z = 0 \\ &= T_0, && \text{at } z = d, && (3.1.5) \\ \vec{g} &= g_0[1 + \chi^2 \delta \cos(\Omega t)]\hat{k}, && && (3.1.6) \end{aligned}$$

where ΔT is the temperature difference across the fluid layer, χ is the smallness of amplitude of modulation, δ, Ω are amplitude and frequency of gravity modulation.

3.1.3 Basic state

The basic state is assumed to be quiescent, and the quantities in this state are given by

$$\vec{q_b} = 0, p = p_b(z,t), \quad T = T_b(z,t), \ \rho = \rho_b(z,t). \tag{3.1.7}$$

Substituting Eq.(3.1.7) in Eqs.(3.1.1)-(3.1.4), we get the following relations, which helps us to define basic state pressure and temperature:

$$\frac{\partial p_b}{\partial z} = -\rho_b g, \tag{3.1.8}$$

$$\kappa_T \frac{d^2 T_b}{dz^2} = 0, \tag{3.1.9}$$

$$\rho_b = \rho_0 \left[1 - \alpha_T \left(T_b - T_0\right)\right]. \tag{3.1.10}$$

The solution of equation (3.1.9), subjected to the boundary conditions (3.1.5), is given by

$$T_b = T_0 + \Delta T \left(1 - \frac{z}{d}\right). \tag{3.1.11}$$

The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q} = \vec{q_b} + q^{'}, \ \rho = \rho_b + \rho^{'}, \ p = p_b + p^{'}, \ T = T_b + T^{'}. \tag{3.1.12}$$

We introduce the Eq.(3.1.12) and the basic state temperature field given by Eq.(3.1.11), and then use the stream function ψ as $u^{'} = \frac{\partial \psi}{\partial z}$, $w^{'} = -\frac{\partial \psi}{\partial x}$, for two dimensional flow. The equations are then non-dimensionalized using the physical variables; $(x, y, z) = d(x^*, y^*, z^*)$, $t = \frac{d^2}{\kappa_T} t^*$, $\psi = \kappa_T \psi^*$, $T^{'} = \Delta T \ T^*$, $\lambda_1 = \frac{\kappa_T \overline{\lambda_1}}{d^2}$, $\lambda_2 = \frac{\kappa_T \overline{\lambda_2}}{d^2}$, and $\Omega = \frac{\kappa_T}{d^2}\Omega^*$.

The resulting non-dimensionalized system of equations can be expressed as (dropping the asterisk)

$$\left(\lambda_1\frac{\partial}{\partial t}+1\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t}\nabla^2\psi-\frac{1}{Pr}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}+g_m\ Ra\frac{\partial T}{\partial x}\right)-\left(\lambda_2\frac{\partial}{\partial t}+1\right)\nabla^4\psi=0, \tag{3.1.13}$$

$$\frac{\partial\psi}{\partial x}+\left(\frac{\partial}{\partial t}-\nabla^2\right)T=\frac{\partial(\psi,T)}{\partial(x,z)}, \tag{3.1.14}$$

where $g_m=(1+\delta\cos(\Omega t))$. The above system will be solved by considering stress free and isothermal boundary conditions as given bellow

$$\psi=\frac{\partial^2\psi}{\partial z^2}=T=0\ \text{ on }\ z=0\ \ z=1. \tag{3.1.15}$$

Introducing a small perturbation parameter χ that show a deviation from the critical state of onset of convection, the variables for a weak nonlinear state may be expanded as power series of χ as in Eq.(2.3.1). Here R_0 is the critical value of the critical Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

3.1.4 Bifurcation of periodic solution

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time τ and the slow time scale s. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}+\chi^2\frac{\partial}{\partial s}$. In the first order problem the nonlinear term in energy equation will vanish therefore, the first order problem reduces to the linear stability problem for overstability.

At the lowest order, we have

$$\begin{bmatrix}\frac{1}{Pr}\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial\tau}\nabla^2-\left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^4 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x}\\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right)\end{bmatrix}\begin{bmatrix}\psi_1\\ T_1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The solution of the lowest order system subject to the boundary conditions Eq.(3.1.15), is assumed to be

$$\psi_1=\left(\mathbb{B}(s)e^{i\omega\tau}+\overline{\mathbb{B}}(s)e^{-i\omega\tau}\right)\sin ax\sin\pi z, \tag{3.1.16}$$

$$T_1=\left(\mathbb{A}(s)e^{i\omega\tau}+\overline{\mathbb{A}}(s)e^{-i\omega\tau}\right)\cos ax\sin\pi z. \tag{3.1.17}$$

The undetermined amplitudes are functions of slow time scale and are related by the following relation:

$$\mathbb{B}(s) = -\frac{c + i\omega}{a}\mathbb{A}(s), \tag{3.1.18}$$

where $c = a^2 + \pi^2$. The values of the critical Rayleigh number and the corresponding wave number for stationary mode of convection are

$$R_0 = \frac{c^3}{a^2}, \tag{3.1.19}$$

$$a = \frac{\pi}{\sqrt{2}}, \tag{3.1.20}$$

which are classical results of Chandrasekhar (1961). We find critical Rayleigh number for oscillatory convection as:

$$R_0 = \frac{c^3}{a^2} - \frac{((\lambda_1 + \lambda_2 Pr)c + 1)\, c\omega^2}{a^2 Pr}, \tag{3.1.21}$$

which is same as obtained by Rajib et al. (2012). Here we calculate the corresponding critical wave number while minimizing critical Rayleigh number with respect to the square of wave number. The critical Rayleigh number and corresponding wave number does not depend on (λ_1, λ_2) in stationary mode but in oscillatory mode. Also we see that the overstability can occur for a particular wave number a only, if the following inequality holds

$$\lambda_1 > \lambda_2 + \frac{1 + Pr}{cPr}. \tag{3.1.22}$$

In the second order, we get the following relations

$$\psi_2 = 0, \tag{3.1.23}$$

$$\left(\frac{\partial}{\partial\tau} - \nabla^2\right) T_2 = \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}. \tag{3.1.24}$$

From the above relation, according to Kim et al. (2003), we can deduce that the velocity and temperature fields have the terms having frequency 2ω and independent of past time scale. Thus, we write the second order temperature term as follows:

$$T_2 = \{T_{20} + T_{22}e^{2i\omega\tau} + \overline{T}_{22}e^{-2i\omega\tau}\}\sin 2\pi z, \tag{3.1.25}$$

where T_{22} and T_{20} are temperature fields having the terms having the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problems are:

$$T_{20} = \frac{a}{8\pi}\{\mathbb{A}(s)\overline{\mathbb{B}}(s) + \overline{\mathbb{A}}(s)\mathbb{B}(s)\}, \tag{3.1.26}$$

and

$$T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega}\mathbb{A}(s)\mathbb{B}(s). \tag{3.1.27}$$

The horizontally averaged Nusselt number, $\mathrm{Nu}(\tau)$, for the oscillatory mode of convection is given by:

$$\mathrm{Nu}(s) = 1 - \chi^2 \left(\frac{\partial T_2}{\partial z}\right)_{z=0} \tag{3.1.28}$$

Using the expression of T_2, given in Eq.(3.1.25), we simplify Eq.(3.1.28) as

$$\mathrm{Nu}(s) = 1 + \left(\frac{c}{2} + 2\pi^2\frac{\sqrt{c^2+\omega^2}}{\sqrt{64\pi^4+16\omega^2}}\right)|\mathbb{A}(s)|^2. \tag{3.1.29}$$

It is clear that the gravity modulation is effective at third order, and affects Nu(s) through $\mathbb{A}(s)$, which is evaluated at third order.

At the third order, we have

$$\begin{bmatrix} \frac{1}{Pr}\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial\tau}\nabla^2 - \left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^4 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right) \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32}, \end{bmatrix}$$

where the expressions for R_{31} and R_{32} are given in the appendix. Now under the stability condition for the existence of third order solution, we obtain the following Landau equation that describes the temporal variation of the amplitude $\mathbb{A}(s)$ of the convection cell

$$\frac{\partial\mathbb{A}(s)}{\partial s} - \gamma_1^{-1}F(s)\mathbb{A}(s) + \gamma_1^{-1}k|\mathbb{A}(s)|^2\mathbb{A}(s) = 0, \tag{3.1.30}$$

where the coefficients $\gamma_1, F(s)$ and k are given in the appendix. Writing $\mathbb{A}(s)$ in the phase-amplitude form, we get

$$\mathbb{A}(s) = |\mathbb{A}(s)|e^{i\phi}. \tag{3.1.31}$$

Now substituting the expression Eq.(3.1.31) in Eq.(3.1.30), we get the following equations for the amplitude $|\mathbb{A}(s)|$:

$$\frac{\partial|\mathbb{A}(s)|^2}{\partial s} - 2p_r|\mathbb{A}(s)|^2 + 2l_r|\mathbb{A}(s)|^4 = 0, \tag{3.1.32}$$

$$\frac{\partial(ph(\mathbb{A}(s)))}{\partial s} = p_i - l_i|\mathbb{A}(s)|^2, \tag{3.1.33}$$

where $\gamma_1^{-1}F(s) = p_r + ip_i$, $\gamma_1^{-1}k = l_r + il_i$ and $ph(.)$ represents the phase shift.

3.1.5 Results and discussion

The bifurcation of a convective layer of a viscoelastic fluid has been analysed by means of weakly nonlinear theory under gravity modulation. The amplitude equations for the bifurcations are also obtained. In order to illustrate the effects of relaxational parameters λ_1, λ_2, the frequency Ω and the amplitude δ of modulation on heat transport, we plot the curves of Nusselt number versus time s. It is observed that the relation Eq.(3.1.22) leads to an interesting result; that for a viscoelastic fluid layer heated underneath; the oscillatory type of instability is possible only when the relaxation parameter λ_1 is greater than the retardation parameter λ_2. Also, it is clear from the relation Eq.(3.1.21) that the oscillatory convection depends on both relaxation and retardation times.

The results corresponding to the gravity modulation has been depicted in figures 3.1-3.4, where we have plotted Nu with respect to the slow time s. It is found that the value of Nu starts with 1, thus showing the conduction state initially that is heat transfer across the fluid layer is taking place through conduction when s is small. The values of Nu increases for intermediate values of s thus showing that convection is in progress and finally when s is very large, the oscillatory state is achieved. The effect of the Prandtl number is important, because many practical available viscoelastic fluids have large Prandtl numbers. It is quite interesting to note that when the Prandtl number is small, the critical value of the Rayleigh number decreases significantly for increasing Prandtl number so that the Prandtl number has a tendency to destabilize the system, compatible with results obtained by Tan et al. (2007), Kim et al. (2003). In figure 3.1a, as Pr increases there is an increment in heat transfer compatible with the results obtained by Bhadauria et al. (2013c) for considering low viscous fluids.

Figure 3.1b, shows the effect of viscoelastic parameter λ_1 on the oscillatory convection. For fixed value of other parameters, the critical Rayleigh number for the onset of oscillatory convection decreases with an increase in the value of λ_1, indicating that the

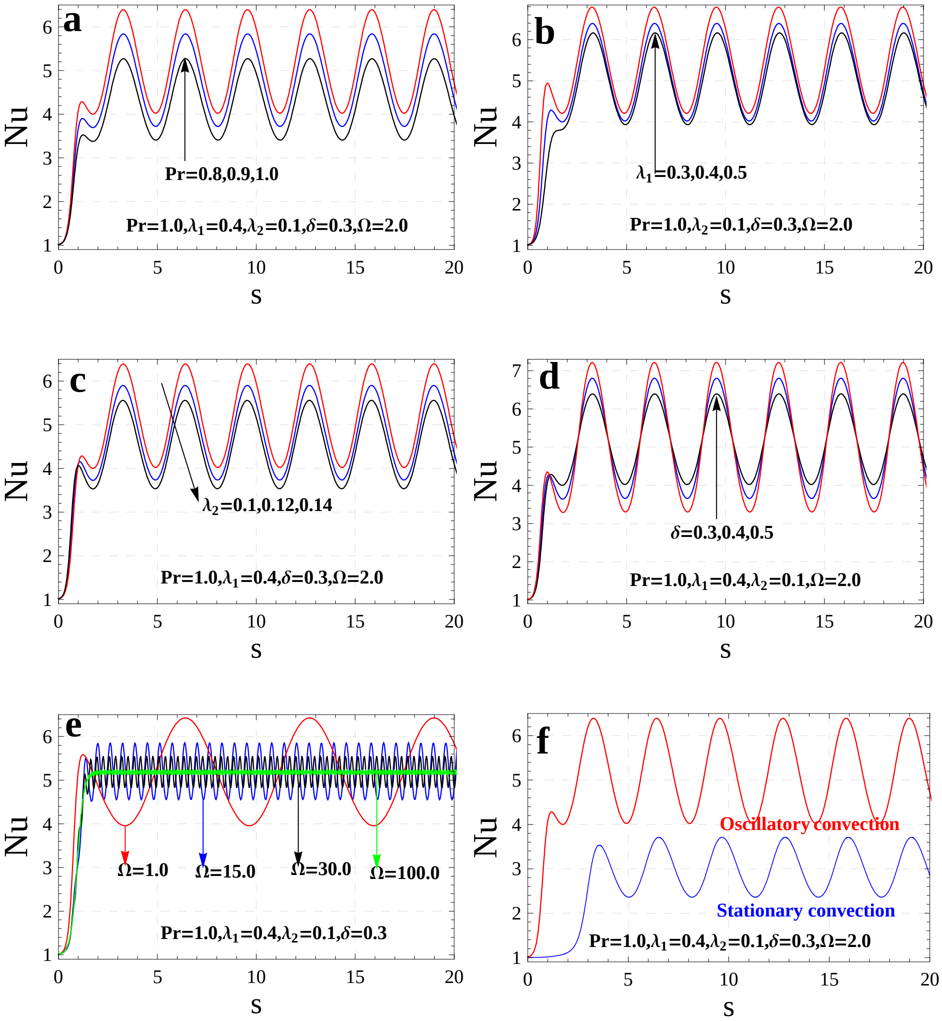

Figure 3.1: Effect of different values of system parameters on Nu

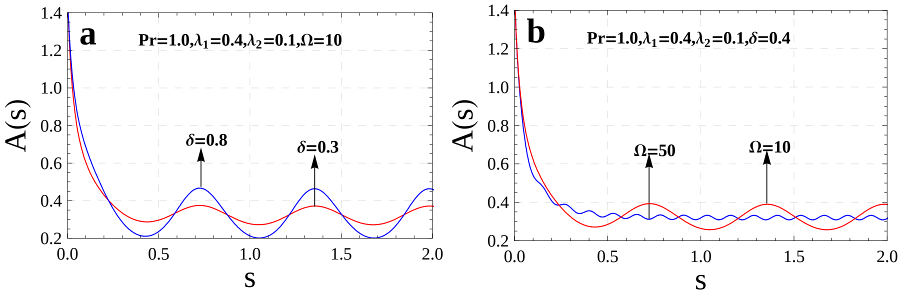

Figure 3.2: Effect of δ and Ω on amplitude of convection

effect of increasing viscoelastic parameter is to advance the onset of oscillatory convection. Thus, it is confirmed that the elastic behavior of the non-Newtonian fluids leads to the oscillatory motions, hence heat transfer increases. Further, the effect of retardation parameter λ_2 is found to stabilize the system as the heat transfer decreases on increasing λ_2, given in figure 3.1c. The effects of frequency Ω and the amplitude of modulation δ on heat transport is given in figures 3.1d-e, respectively. In figure 3.1d, one can see that an increment in amplitude of modulation increases the magnitude of Nu, thus enhances the heat transfer and advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation Ω as given in the figure 3.1e. Hence, we found that the effect of gravity modulation decreases as the frequency of modulation increases, and finally when Ω is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969) and Yang (1997). In figure 3.1f, we compare the results of oscillatory and stationary instabilities. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that ($\mathrm{Nu}^{st} < \mathrm{Nu}^{osc}$) for the same wave number. This implies that oscillatory instability sets in before the stationary instability. Similar results has also been obtained by Rajib and Layek (2012), Kim et al. (2003). In figures 3.2a-b, we plot the amplitude of convection $\mathbb{A}(s)$ versus time s, it is found that amplitude enhances the heat transfer as δ increases but opposite in case of frequency Ω, thus confirming the results obtained by Bhadauria et al. (2012, 2013). and Siddheshwar et al. (2012a,b).

In figures 3.3-3.4, the stream lines and the corresponding isotherms are depicted for gravity modulation, respectively at $s = 0.0, 0.3, 0.6, 0.8, 1.0, 2.0$ for $\lambda_1 = 0.4, \lambda_2 = 0.1, \delta =$

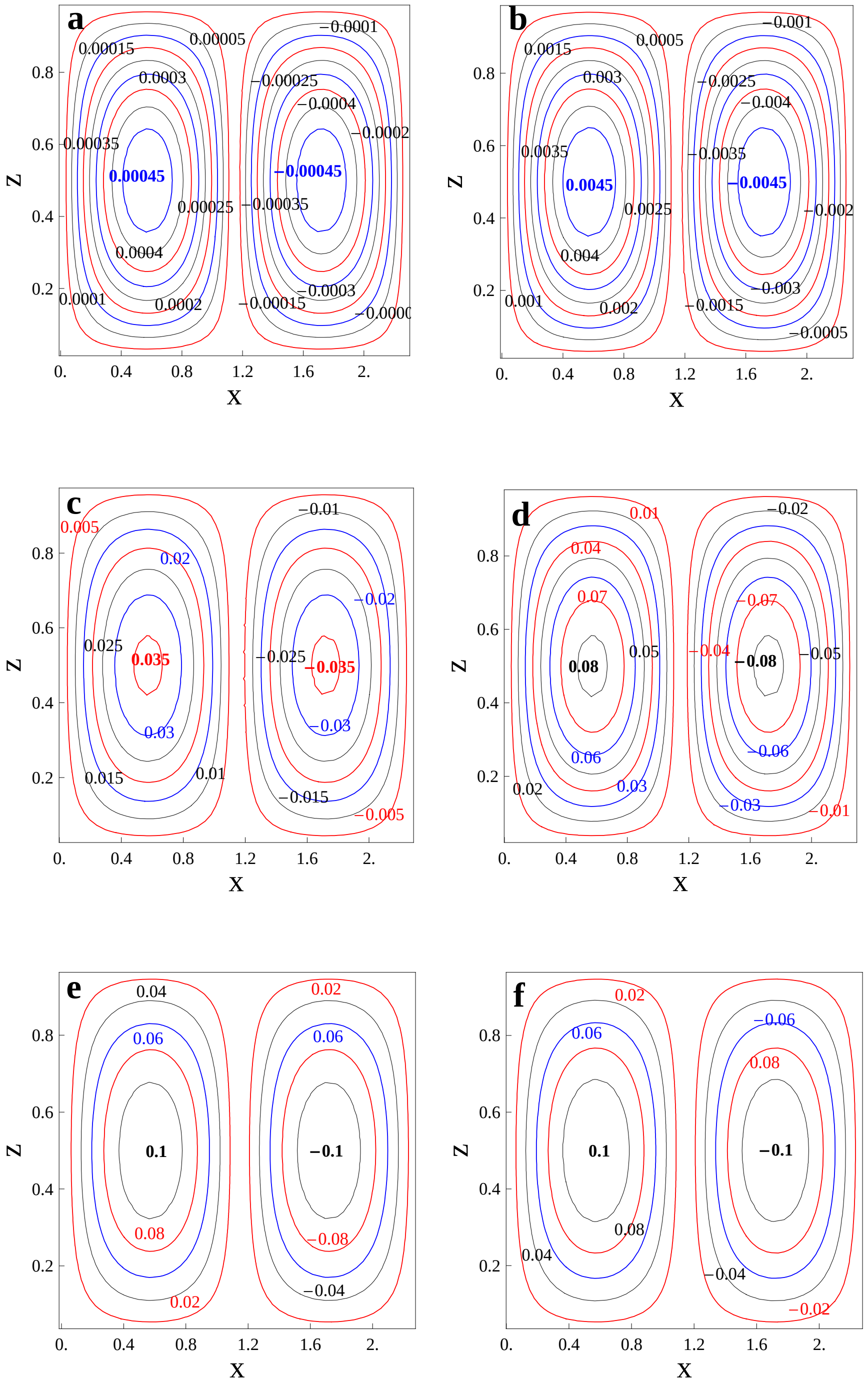

Figure 3.3: Streamlines at (a)s =0(b)s=0.3(c)s=0.6(d)s=0.8(e)s=1(f)s=2

0.1 $\Omega = 2.0$ and $\chi = 0.5$. From the figures, we found that initially when the time is small the magnitude of streamlines is also small given in figures 3.3a-b, and isotherms are straight that is the system is in conduction state, figures 3.4a-b. However, as time increases, the magnitude of streamlines increases and the isotherms loses their evenness, thus showing that the convection is taking place in the system. Convection becomes faster on further increasing the value of time s. However, the system achieves the study state beyond $s = 0.16$ as there is no change in the streamlines and isotherms, figures 3.3-3.4d-f.

3.1.6 Conclusions

We have analyzed the effect of gravity modulation on overstability of Bénard convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg−Landau amplitude equation. The following conclusions are made:

1. Effect of relaxation time λ_1 is to advance the onset of convection and hence enhance the heat transport.

2. Effect of retardation time λ_2 is to delay the onset of convection and hence decrease the heat transport.

3. It is important that for oscillatory convection the relaxation time of fluid must dominant over retardation time.

4. The critical Rayleigh number depends on λ_1, λ_2 for oscillatory mode of convection, but for stationary case it is independent.

5. An increment in the amplitude of modulation δ is to advance the convection and hence heat transfer.

6. The frequency of modulation, Ω is to decrease the heat transfer.

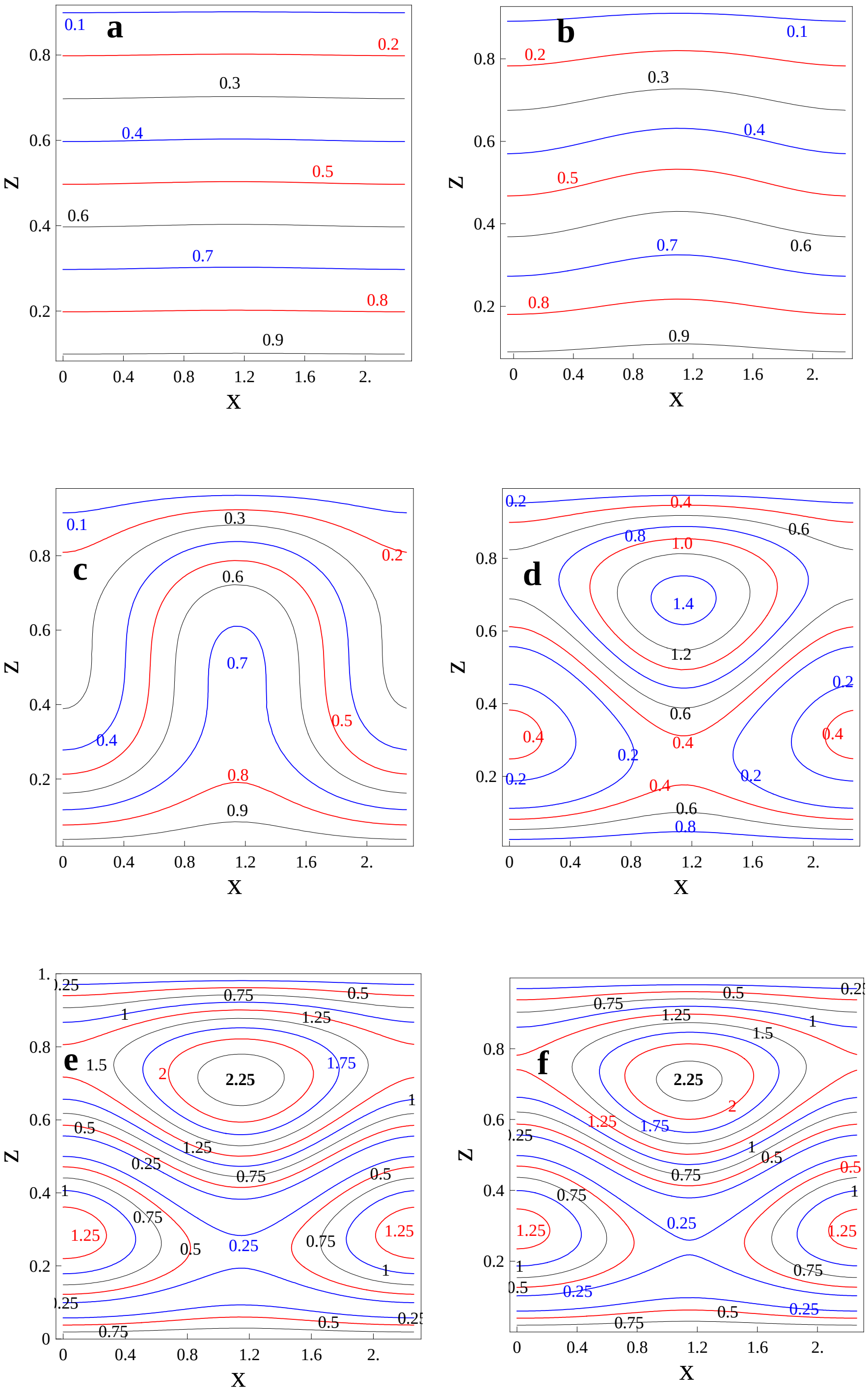

Figure 3.4: Isotherms at (a)s=0(b)s=0.3(c)s=0.6(d)s=0.8(e)s=1(f)s=2

Appendix

The dimensionless frequency of the neutral oscillatory mode is

$$\omega^2 = \frac{cPr(\lambda_1 - \lambda_2) - 1(1 + Pr)}{\lambda_1(\lambda_1 + \lambda_2 Pr)}.$$

The expressions given in Eq.(3.1.30) are

$$\begin{aligned} R_{31} &= \lambda_2 \frac{\partial}{\partial s}(\nabla^4 \psi_1) - R_0 \lambda_1 \frac{\partial}{\partial s}\left(\frac{\partial T_1}{\partial x}\right) - (R_2 + R_0 \delta \cos(\Omega s))\left(\lambda_1 \frac{\partial}{\partial t} + 1\right)\left(\frac{\partial T_1}{\partial x}\right) \\ &- \frac{1}{Pr}\left(\lambda_1 \frac{\partial}{\partial t} + 1\right)\frac{\partial}{\partial s}(\nabla^2 \psi_1) - \frac{1}{Pr}\lambda_1 \frac{\partial}{\partial s}\left(\frac{\partial}{\partial \tau}\nabla^2 \psi_1\right), \\ R_{32} &= \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s}. \end{aligned}$$

The coefficients given in Eq.(3.1.30) are
$\gamma_1 = \left[1 - a\triangle_1 R_0 \lambda_1 + \frac{c^2 \triangle_1 \lambda_2 (c + i\omega)}{a} + \frac{c\triangle_1 (c + i\omega)(1 + 2i\omega\lambda_1)}{aPr}\right]$,
$F(s) = \left[a\triangle_1 R_2 (1 + i\omega\lambda_1)(1 + \delta \cos(\Omega s))\right]$,
$k = \left(\frac{c^2 + ic\omega}{4} + \frac{\pi^2(c^2 + \omega^2)}{(8\pi^2 + 4i\omega)}\right)$ and $\triangle_1 = \frac{aPr}{i\omega cPr(1 + i\omega\lambda_1) + (1 + i\omega\lambda_2)c^2}$.

3.2 Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation

3.2.1 Introduction

Based on Kim et al.(2003) and Bhadauria and Kiran (2014a) in this section we have performed a weakly nonlinear thermal instability in a viscoelastic fluid saturated porous medium under gravity modulation, and quantify the Nusselt number in terms of the amplitude of convection by solving the complex Ginzburg−Landau equation. Finally, till now no experimental work has been found in the literature in support of this viscoelastic model for flow in porous media.

3.2.2 Problem Formulation

An infinitely extended horizontal fluid saturated porous medium of depth 'd' has been considered. The porous medium is homogeneous, isotropic and saturated with viscoelastic fluid. The porous medium is heated slowly from below, the configuration of the problem is given in figure 3.5a. Using modified Darcy's model (Alishaev 1975) and employing the Boussinesq approximation for this system, the governing equations of flow and temperature fields are expressed as:

$$\nabla \cdot \vec{q} = 0, \tag{3.2.1}$$

$$\left(\overline{\lambda_1}\frac{\partial}{\partial t} + 1\right)(-\nabla P + \rho\vec{g}) - \frac{\mu}{K}\left(\overline{\lambda_2}\frac{\partial}{\partial t} + 1\right)\vec{q} = 0, \tag{3.2.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \kappa_T \nabla^2 T, \tag{3.2.3}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{3.2.4}$$

where the physical variables have their usual meanings and are given in Nomenclature. The externally imposed gravitational field and the thermal boundary conditions are given by

$$\begin{aligned} \vec{g} &= g_0[1+\chi^2\delta\cos(\Omega t)]\hat{k}, && && (3.2.5) \\ T &= T_0+\Delta T, && \text{at } z=0 && \\ &= T_0, && \text{at } z=d, && (3.2.6) \end{aligned}$$

where ΔT is the temperature difference across the porous medium, χ is the smallness of amplitude of modulation δ, Ω are amplitude and frequency of gravity modulation.

3.2.3 Basic state

The basic state is assumed to be quiescent, and the quantities in this state are given by

$$\vec{q_b}=0, p=p_b(z,t), \quad T=T_b(z,t), \ \rho=\rho_b(z,t). \tag{3.2.7}$$

Substituting the Eq.(3.2.7) in Eqs.(3.2.1)−(3.2.4), we get the following relations, which helps us to define basic state pressure and temperature:

$$\frac{\partial p_b}{\partial z}=-\rho_b g, \tag{3.2.8}$$

$$\kappa_T\frac{d^2T_b}{dz^2}=0, \tag{3.2.9}$$

$$\rho_b=\rho_0\left[1-\alpha_T\left(T_b-T_0\right)\right]. \tag{3.2.10}$$

The solution of equation (3.2.9), subjected to the boundary conditions (3.2.6), is given by

$$T_b=T_0+\Delta T\left(1-\frac{z}{d}\right). \tag{3.2.11}$$

The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q}=\vec{q_b}+q^{'}, \ \rho=\rho_b+\rho^{'}, \ p=p_b+p^{'}, \ T=T_b+T^{'}. \tag{3.2.12}$$

We introduce the Eq.(3.2.12) and the basic state temperature field given by Eq.(3.2.11) in Eqs.(3.2.1)−(3.2.4), and then use the stream function ψ and non-dimensionalized factors as given chapter 3.1 we obtain the following non-dimensionalized system as

$$\left(\lambda_2\frac{\partial}{\partial t}+1\right)\nabla^2\psi+Ra(1+\chi^2\cos(\Omega t))\left(\lambda_1\frac{\partial}{\partial t}+1\right)\frac{\partial T}{\partial x}=0 \tag{3.2.13}$$

$$\frac{\partial \psi}{\partial x}+\left(\frac{\partial}{\partial t}-\nabla^2\right)T=\frac{\partial(\psi,T)}{\partial(x,z)}, \tag{3.2.14}$$

The above system will be solved by considering stress free and isothermal boundary conditions as given in Eq.(2.2.26). Introduce a small perturbation parameter χ that show deviation from the critical point of onset of convection, then the variables for a weak nonlinear state may be expanded as power series of χ as in Eq.(2.3.1). Here R_0 is the critical value of the Darcy−Rayleigh number at which the onset of convection takes place in the absence of gravity modulation.

3.2.4 Bifurcation of periodic solution

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time τ and the slow time scale of s. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}+\chi^2\frac{\partial}{\partial s}$. In the first order problem the nonlinear term in energy equation will vanish therefore, the first order problem reduces to the linear stability problem for overstability.

At the lowest order, we have

$$\begin{bmatrix} \left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^2 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right) \end{bmatrix}\begin{bmatrix}\psi_1\\ T_1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{3.2.15}$$

The solution of the lowest order system subject to the boundary conditions Eq.(2.2.26), is assumed to be

$$\psi_1 = \left(\mathbb{A}(s)e^{i\omega\tau}+\overline{\mathbb{A}}(s)e^{-i\omega\tau}\right)\sin ax\sin\pi z, \tag{3.2.16}$$

$$T_1 = \left(\mathbb{B}(s)e^{i\omega\tau}+\overline{\mathbb{B}}(s)e^{-i\omega\tau}\right)\cos ax\sin\pi z. \tag{3.2.17}$$

The undetermined amplitudes are functions of slow time scale and are related by the following relation:

$$\mathbb{B}(s)=-\frac{a}{c+i\omega}\mathbb{A}(s) \tag{3.2.18}$$

where $c=a^2+\pi^2$. The values of the Darcy−Rayleigh number and the corresponding wave number for stationary mode of convection

$$R_0=\frac{c^2}{a^2} \tag{3.2.19}$$

$$a_c = \pi, \tag{3.2.20}$$

which are classical results of Horton and Rogers (1945), and Lapwood (1948). We find Darcy−Rayleigh number and corresponding critical wave number for oscillatory convection as given bellow:

$$R_0 = \frac{(\lambda_2\pi^4 + \pi^2 + 2a^2\pi^2\lambda_2 + a^2 + a^4\lambda_2)}{\lambda_1 a^2} \tag{3.2.21}$$

$$a_c^2 = \sqrt{\pi^4 + \frac{\pi^2}{\lambda_2}}, \tag{3.2.22}$$

which are same as obtained by Kim et al. (2003). The critical Darcy Rayleigh number and corresponding wave number does not depend on (λ_1, λ_2) in stationary mode but in oscillatory mode. Also we see that the overstability can occur for a particular wave number only, if the following inequality holds

$$\lambda_1 > \lambda_2 + \frac{1}{c}. \tag{3.2.23}$$

The dimensionless frequency of the neutral oscillatory mode is

$$\omega^2 = \frac{c(\lambda_1 - \lambda_2) - 1}{\lambda_2\lambda_1}. \tag{3.2.24}$$

In the second order, we get

$$\frac{\partial(\psi_1, T_1)}{(x, z)} = \frac{\pi a}{2}\{\mathbb{A}(s)\mathbb{B}(s)e^{2i\omega\tau} + \overline{\mathbb{A}}(s)\overline{\mathbb{B}}(s)e^{-2i\omega\tau} + \mathbb{A}(s)\overline{\mathbb{B}}(s) + \overline{\mathbb{A}}(s)\mathbb{B}(s)\}\sin 2\pi z. \tag{3.2.25}$$

From the above relation, we can deduce that the velocity and temperature fields have the terms having frequency 2ω and independent of past time scale. Thus, we write the second order temperature term as follows:

$$T_2 = \{T_{20} + T_{22}e^{2i\omega\tau} + \overline{T}_{22}e^{-2i\omega\tau}\}\sin 2\pi z \tag{3.2.26}$$

where T_{22} and T_{20} are temperature fields having the terms having the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problems are:

$$T_{20} = \frac{a}{8\pi}\{\mathbb{A}(s)\overline{\mathbb{B}}(s) + \overline{\mathbb{A}}(s)\mathbb{B}(s)\}, \quad \psi_{20} = 0 \tag{3.2.27}$$

and

$$T_{22}=\frac{\pi a}{8\pi^2+4i\omega}\mathbb{A}(s)\mathbb{B}(s). \tag{3.2.28}$$

The horizontally averaged Nusselt number, Nu(s), for the oscillatory mode of convection is given by using the expression of T_2, given in Eq.(3.2.26), one can simplify Eq.(3.1.28) as

$$\mathrm{Nu}(s)=1+\left(\frac{ca_c^2}{2(c^2+\omega^2)}+\frac{2\pi^2a_c^2}{\sqrt{64\pi^4+16\omega^2}\sqrt{c^2+\omega^2}}\right)|\mathbb{A}(s)|^2. \tag{3.2.29}$$

It is clear that the thermal modulation is effective at third order and affects Nu(s) through $\mathbb{A}(s)$ which is evaluated at third order.

At the third order, we have

$$\begin{bmatrix} \left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^2 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right) \end{bmatrix}\begin{bmatrix}\psi_3\\ T_3\end{bmatrix}=\begin{bmatrix}R_{31}\\ R_{32}\end{bmatrix} \tag{3.2.30}$$

where

$$R_{31}=-\lambda_2\frac{\partial}{\partial s}(\nabla^2\psi_1)-R_0\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial T_1}{\partial x}\right)-(R_2+R_0\delta\cos(\Omega s))\left(\lambda_1\frac{\partial}{\partial t}+1\right)\left(\frac{\partial T_1}{\partial x}\right), \tag{3.2.31}$$

$$R_{32}=\frac{\partial\psi_1}{\partial x}\frac{\partial T2}{\partial z}-\frac{\partial T_1}{\partial s}. \tag{3.2.32}$$

Substituting the ψ_1, T_1 and T_2 into Eqs.(3.2.31)-(3.2.32), we obtain the expressions for R_{31} and R_{32} easily. Now under the stability condition for the existence of third order solution, these equations yield the following Landau equation that describes the temporal variation of the amplitude $\mathbb{A}(s)$ of the convection cell

$$\frac{\partial\mathbb{A}(s)}{\partial s}-\gamma_1^{-1}F(s)\mathbb{A}(s)+\gamma_1^{-1}k|\mathbb{A}(s)|^2\mathbb{A}(s)=0 \tag{3.2.33}$$

where $\gamma_1=\left[\lambda_2c-\frac{a^2R_0\lambda_1}{(c+i\omega)}+\frac{a^2R_0(1+i\omega\lambda_1)}{(c+i\omega)^2}\right]$, $F(s)=\left[\frac{a^2R_0(1+i\omega\lambda_1)}{(c+i\omega)}(1+\delta\cos(\Omega s))\right]$, $k=-\left(\frac{R_0a^4c(1+i\omega\lambda_1)}{2(c^2+\omega^2)(c+i\omega)}+\frac{4\pi^2a^4R_0(c^2+i\omega^2\lambda_1)}{(8\pi^2+4i\omega)(c+i\omega)(c^2+\omega^2)}\right)$. Writing $\mathbb{A}(s)$ in the phase-amplitude form, we get

$$\mathbb{A}(s)=|\mathbb{A}(s)|e^{i\phi} \tag{3.2.34}$$

Now substituting the expression Eq.(3.2.34) in Eq.(3.2.33), we get the following equations for the amplitude $|\mathbb{A}(s)|$:

$$\frac{\partial|\mathbb{A}(s)|^2}{\partial s}=2p_r|\mathbb{A}(s)|^2-2l_r|\mathbb{A}(s)|^4 \tag{3.2.35}$$

$$\frac{\partial(ph(\mathbb{A}(s)))}{\partial s} = p_i - l_i|\mathbb{A}(s)|^2 \tag{3.2.36}$$

where $\gamma_1^{-1}F(s) = p_r + ip_i$, $\gamma_1^{-1}k = l_r + il_i$ and $ph(.)$ represents the phase shift. One can observe from the Eq.(3.2.33) for the case $l_r > 0$ and $Ra > Ra_c$ i.e. $p_r > 0$, the solution gives as $\mathbb{A} \sim \mathbb{A}_0 e^{p_r s}$ as $s \longrightarrow -\infty$, and $\mathbb{A} \longrightarrow 0$ is unstable solution, and a new stable solution develops, $\mathbb{A} = \sqrt{\frac{p_r}{l_r}}$ as $s \longrightarrow \infty$, whatever be the value of $\mathbb{A}_0$. This is called supercritical pitch fork bifurcation, the base system being linearly unstable for $Ra > Ra_c$ but settling down as a new laminar flow. The steady state amplitude exists when Ra_c takes positive values. Supercritical pitch fork bifurcation diagram has been shown in the figure 3.5b.

3.2.5 Results and discussion

In this work, we carried out a study of heat transport for oscillatory convection in an horizontal porous medium saturated with viscoelastic fluid under gravity modulation. In order to illustrate the effects of relaxational parameters λ_1, λ_2, the frequency Ω and the amplitude δ of modulation on heat transport, we plot the curves of Nusselt number versus time s. It is observed that the relation Eq.(3.2.24) leads to an interesting result; that for a horizontal porous layer heated underneath; the oscillatory type of instability is possible only when the relaxation parameter λ_1 is greater than the retardation parameter λ_2. Also, it is clear from the relation Eq.(3.2.21) that the oscillatory convection depends on both relaxation and retardation times. The marginal stability curves for the stationary and oscillatory modes are plotted in the figure 3.6. For comparison, the curves representing exchange of stabilities and overstability at the marginal state are drawn. The solid curve represents the Rayleigh number for oscillatory convection as a function of wave number while the broken curves represent the same for stationary convection. To illustrate the effects of relaxational parameter and the retardation parameter on the onset of convection, we plot the curves of the Rayleigh number versus the wave number. One can see in figures 3.6a-b that the marginal overstability curve deviates from the stationary Newtonian curve by showing a bifurcation point on the Newtonian curve and we observe that in this case the onset of convection is characterized by stationary convection. To study the effect

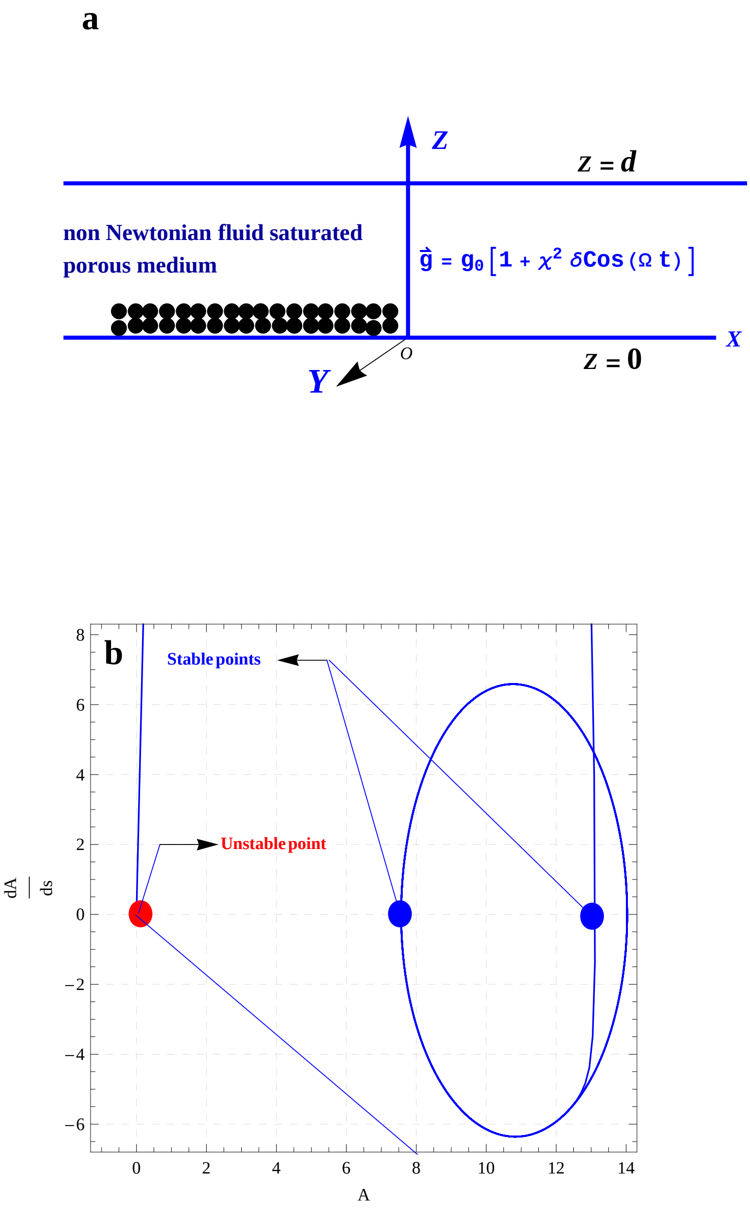

Figure 3.5: a.Physical configuration of the problem b.Supercritical pitch fork bifurcation diagram.

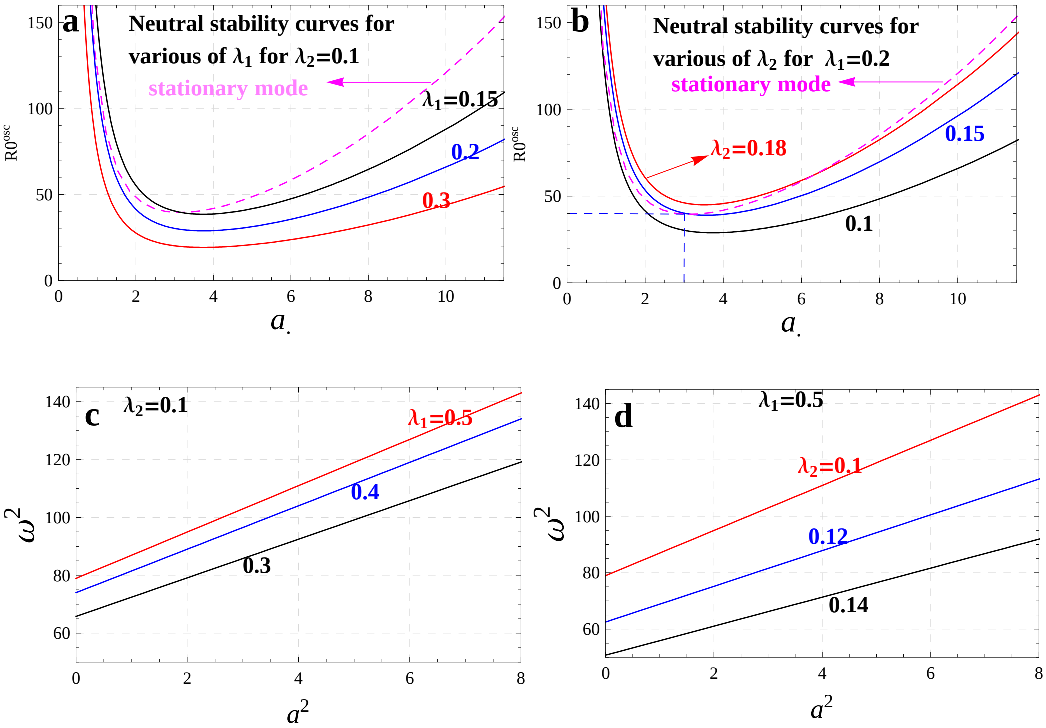

Figure 3.6: Effect of λ_1 and λ_2 on R_0^{osc} and ω^2

of relaxation time of the fluid on the onset of overstability, from the figure 3.6a, it can be seen that the critical Rayleigh number decreases with the increasing value of the relaxation time λ_1 for fixed values of λ_2, indicating that the effect of increasing relaxation time is to destabilize the system. The effect of retardation time λ_2 on the onset of overstability is shown in the figure 3.6b, where we observe that viscoelastic fluids with higher value of retardation time exhibits overstability at higher Rayleigh number and the critical Rayleigh number increases with increasing retardation time. Thus the effect

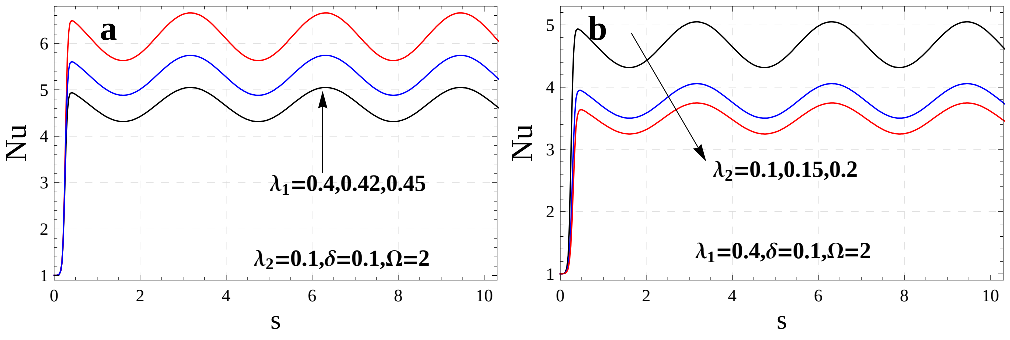

Figure 3.7: Effect of λ_1 and λ_2 on Nu for fixed values of other parameters

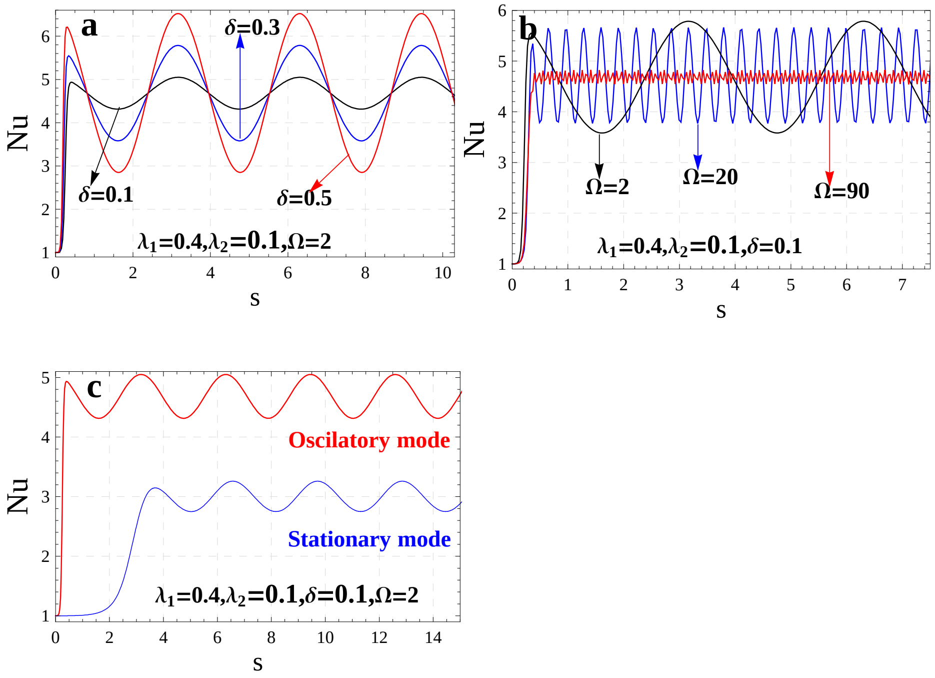

Figure 3.8: Effect of δ and Ω on Nu: c.Comparison

of increasing retardation time has a stabilizing effect on the system. The effect of time relaxation and time retardation parameter on the critical value of dimensionless frequency for marginally oscillatory modes is obtained from the relation Eq.(3.2.24). The results are given by plotting square of frequency against square of wave number in figure 3.6c and figure 3.6d, the critical value of the frequency increases with increasing relaxation time figure 3.6c but with decreasing retardation time figure 3.6d.

The results corresponding to the gravity modulation has been depicted in figures 3.7−3.9, where we have plotted Nu with respect to the slow time s. It is found that the value of Nu starts with 1 thus showing the conduction state initially that is heat transfer across the porous medium is taking place through conduction when s is small. The values of Nu increases for intermediate values of s thus showing that convection is in progress and finally when s is very large, the oscillatory state is achieved. As in figure 3.7a, the effect of an increment in the value of relaxation parameter λ_1 is destabilizing as the value of Nu increases on increasing λ_1. Further, the effect of retardation parameter λ_2 is found to stabilize the system as the heat transfer decreases on increasing λ_2, given in figure 3.7b. The effects of frequency Ω and the amplitude of modulation δ on heat transport is given in figures 3.8a-b. In figure 3.8a, one can see that, an increment in amplitude of modulation increases the magnitude of Nu, thus enhances the heat transfer and advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation as Ω increases given in figure 3.8b. Hence we found that the effect of gravity modulation decreases as the frequency of modulation increases, and finally when Ω is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969). In figure 3.8c, we compare the results of oscillatory and stationary instabilities. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that ($\mathrm{Nu}_{st} < \mathrm{Nu}_{osc}$). This implies that oscillatory instability sets in before the stationary instability. Similar results have also been obtained by Rajib and Layek (2012), Kim et al. (2003). In figures 3.9-3.10, the stream lines and the corresponding isotherms are depicted for gravity modulation, respectively at $s = 0.0, 0.12, 0.14, 0.15, 0.16, 0.17$ for $\lambda_1 = 0.4, \lambda_2 = 0.1, \delta = 0.1$ $\Omega = 2.0$ and $\chi = 0.5$. From the figures we found that initially when the time is small the magnitude of streamlines is also small given in figures 3.9a-b, and isotherms are straight that is the system is in conduction state figures 3.10a-b. However, as time increases, the magnitude of streamlines increases and the isotherms loses their evenness. This shows that the convection is taking place in the system. Convection becomes faster on further increasing the value of time s. However, the system achieves the study state beyond $s = 0.16$ as there is no change in the streamlines and isotherms figures 3.9-3.10d-f.

3.2.6 Conclusions

We have analyzed the effect of gravity modulation on overstability of Bénard−Darcy convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg−Landau amplitude equation. The following conclusions are made:

1. Effect of relaxation time λ_1 is to advance the onset of convection and hence enhance the heat transport.
2. Effect of retardation time λ_2 is to delay the onset of convection and hence decrease the heat transport.

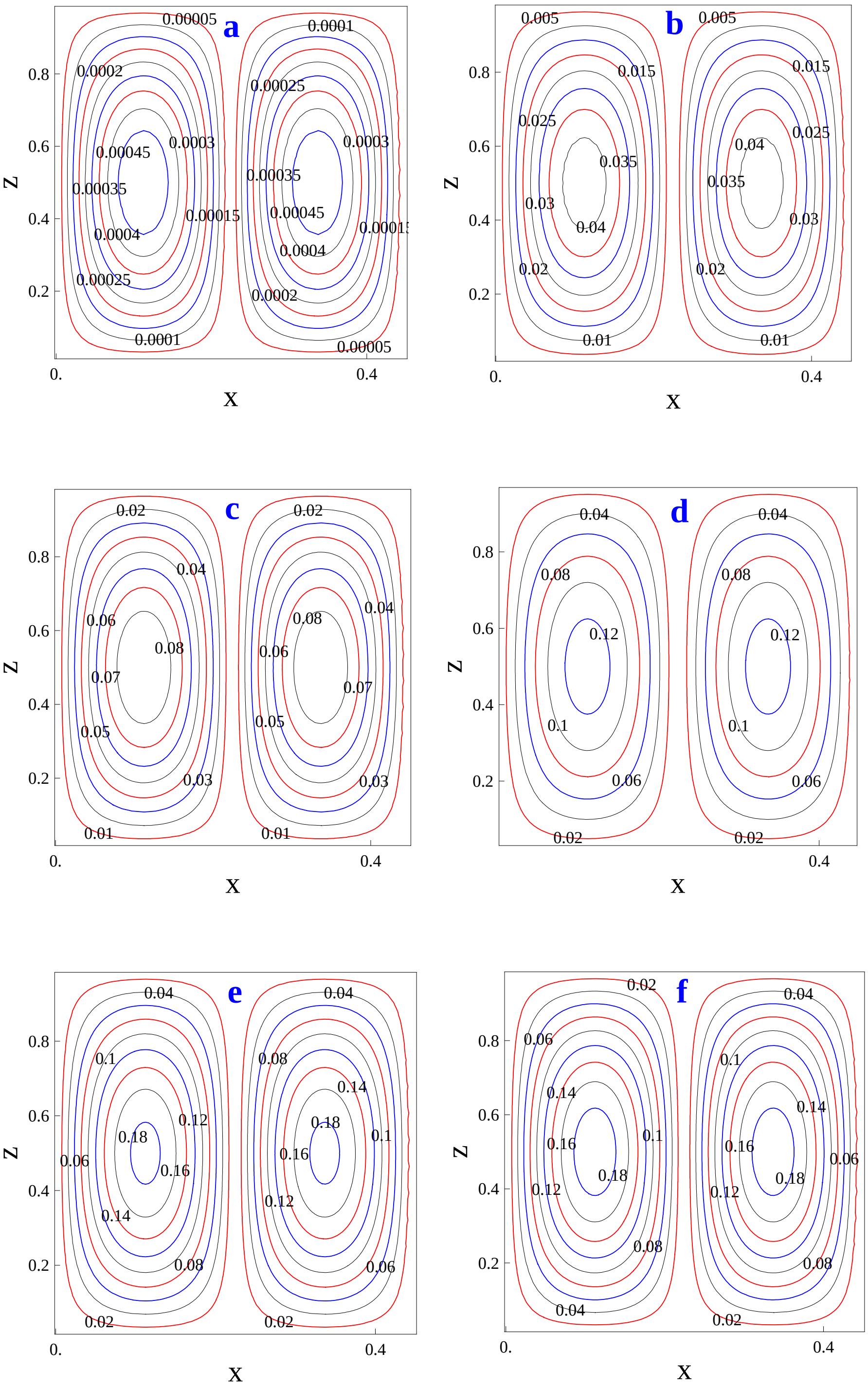

Figure 3.9: Streamlines at (a)s =0.0(b)s=0.12(c)s=0.14(d)s=0.15(e)s=0.16(f)s=0.17

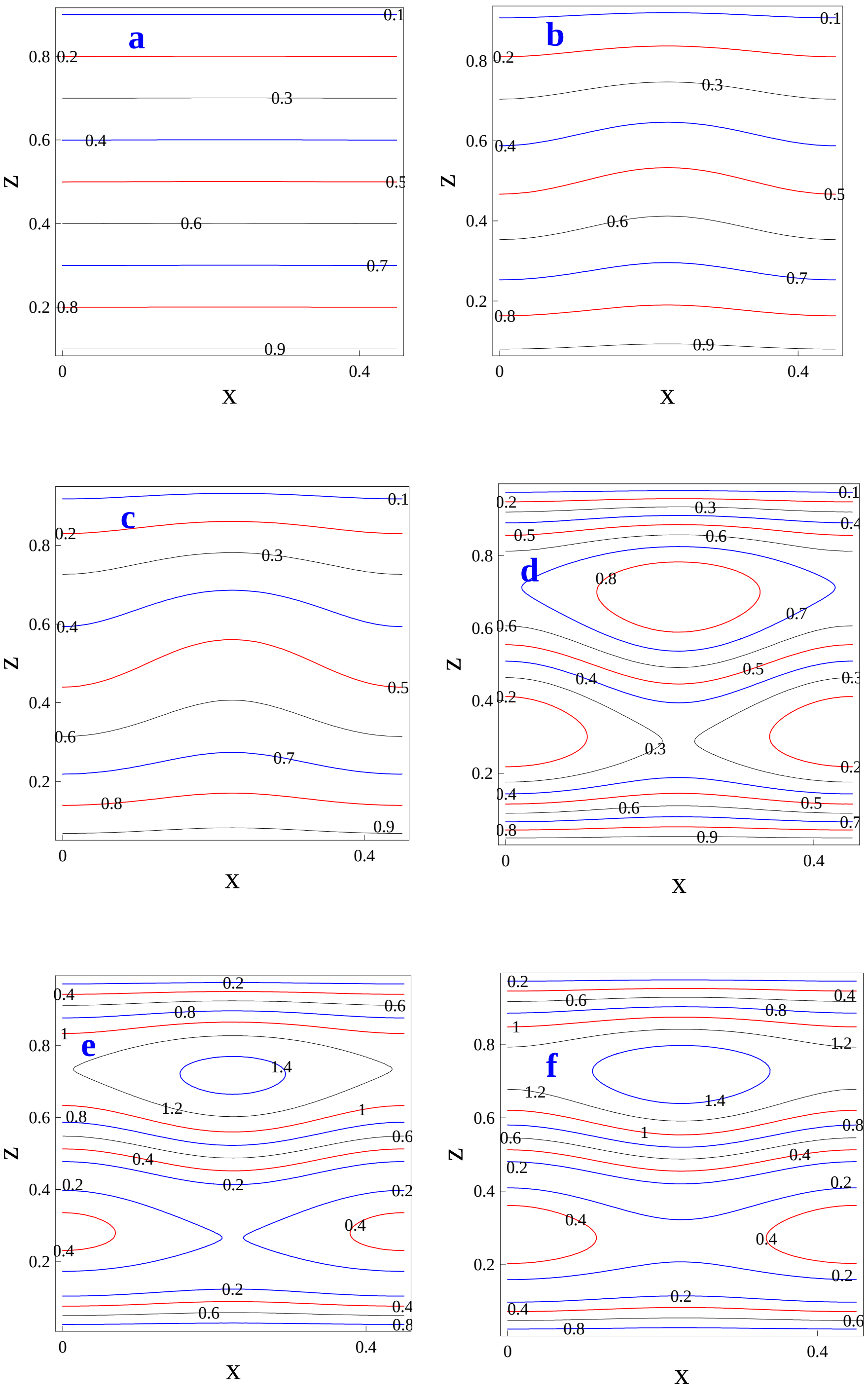

Figure 3.10: Isotherms at (a)s =0.0(b)s=0.12(c)s=0.14(d)s=0.15(e)s=0.16(f)s=0.17

3. The oscillatory critical Rayleigh−Darcy number depends on λ_1, λ_2, but in stationary case it is independent.
4. An increment in the amplitude δ of modulation is to advance the convection and hence heat transfer.
5. The frequency Ω of modulation decreases the heat transfer as its value increases.
6. Supercritical pitch fork bifurcation exits for Eq.(3.2.33).

Chapter 4

Oscillatory convection under thermal modulation

4.1 Weakly nonlinear oscillatory convection in a viscoelastic fluid saturating porous medium under temperature modulation

4.1.1 Introduction

Based on the following papers (where Bhadauria and Kiran (2013a) studied temperature modulation (Venezian 1969) in an anisotropic porous medium while performing a weakly nonlinear study for stationary mode of convection, Bhadauria and Kiran (2014a,b) studied gravity modulation in fluid and porous medium considering viscoelastic fluid, while performing a weakly nonlinear study for oscillatory mode of convection) in this section we study a weakly nonlinear thermal instability in a viscoelastic fluid saturated porous medium under temperature modulation, and quantify the heat transfer in terms of the amplitude of convection which is evaluated by complex Ginzburg−Landau equation.

4.1.2 Governing Equations

We considered an infinitely extended horizontal fluid saturated porous layer of depth 'd' as given in figure 4.1a. The porous layer is homogeneous and isotropic, and saturated with viscoelastic fluid. The porous medium is heated slowly from below. Using modified Darcy's model (Alishaev 1975) and employing the Boussinesq approximation for this system, the governing equations of flow and temperature fields are expressed as

$$\nabla \cdot \vec{q} = 0, \tag{4.1.1}$$

$$\left(\overline{\lambda_1}\frac{\partial}{\partial t}+1\right)(-\nabla P+\rho\vec{g})-\frac{\mu}{K}\left(\overline{\lambda_2}\frac{\partial}{\partial t}+1\right)\vec{q}=0, \tag{4.1.2}$$

$$\frac{\partial T}{\partial t}+(\vec{q}\cdot\nabla)T=\kappa_T\nabla^2 T, \tag{4.1.3}$$

$$\rho=\rho_0[1-\alpha_T(T-T_0)], \tag{4.1.4}$$

where the physical variables have their usual meanings as given in Nomenclature. The externally imposed thermal boundary conditions are considered as (Venezian 1969)

$$\begin{aligned} T &= T_0 + \frac{\Delta T}{2}[1 + \chi^2\delta\cos(\Omega t)], && \text{at } z = 0 \\ &= T_0 - \frac{\Delta T}{2}[1 - \chi^2\delta\cos(\Omega t + \theta)], && \text{at } z = d \end{aligned} \tag{4.1.5}$$

where ΔT is the temperature difference across the porous medium, δ, Ω are amplitude and frequency of temperature modulation, and θ is the phase angle.

4.1.3 Basic state

The basic state is assumed to be quiescent and the quantities in this state are given

$$\vec{q_b} = 0, p = p_b(z,t), \;\; T = T_b(z,t), \;\; \rho = \rho_b(z,t). \tag{4.1.6}$$

Substituting the Eq.(4.1.6) in Eqs.(4.1.1)-(4.1.4), we get the following relations which helps us to define basic state pressure and temperature:

$$\frac{\partial p_b}{\partial z} = -\rho_b\vec{g}, \tag{4.1.7}$$

$$\frac{\partial T_b}{\partial t} = \kappa_T\frac{\partial^2 T_b}{\partial z^2}, \tag{4.1.8}$$

$$\rho_b = \rho_0\left[1 - \alpha_T\left(T_b - T_0\right)\right]. \tag{4.1.9}$$

The solution of equation (4.1.8), subjected to the boundary conditions (4.1.5), is given by

$$T_b(z,t) = T_s(z) + \chi^2\delta\mathrm{Re}[T_1(z,t)], \tag{4.1.10}$$

where

$$T_s(z) = T_0 + \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right), \tag{4.1.11}$$

$$T_1(z,t) = \left(\{a_1(\zeta)e^{\frac{\zeta z}{d}} + a_1(-\zeta)e^{\frac{-\zeta z}{d}}\}e^{-i\Omega t}\right), \tag{4.1.12}$$

and $a_1(\zeta) = \frac{\Delta T}{2}\frac{(e^{-i\theta} - e^{-\zeta})}{(e^{\zeta} - e^{-\zeta})}$ and $\zeta^2 = \frac{-i\Omega d^2}{\kappa_T}$. Here $T_s(z)$ is the steady part, while $T_1(z,t)$ is the oscillatory part of the basic state temperature field $T_b(z,t)$. The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q} = \vec{q_b} + q', \;\; \rho = \rho_b + \rho', \;\; p = p_b + p', \;\; T = T_b + T'. \tag{4.1.13}$$

We introduce the Eq.(4.1.13) and the basic state temperature field in Eqs.(4.1.1)−(4.1.4), and then using stream function and non-dimensionalized quantities as in chapter 3, the resulting non-dimensionalized system of equations are

$$\left(\lambda_2\frac{\partial}{\partial t}+1\right)\nabla^2\psi+Ra_D\left(\lambda_1\frac{\partial}{\partial t}+1\right)\frac{\partial T}{\partial x}=0 \tag{4.1.14}$$

$$-\frac{\partial T_b}{\partial z}\frac{\partial\psi}{\partial x}+\left(\frac{\partial}{\partial t}-\nabla^2\right)T=\frac{\partial(\psi,T)}{\partial(x,z)}. \tag{4.1.15}$$

The basic state solution which appears in Eq.(4.1.15), influences the stability problem through the factor $\frac{\partial T_b}{\partial z}$, which is given by

$$\frac{\partial T_b}{\partial z}=-1+\chi^2\delta\left(f_2(z,t)\right), \tag{4.1.16}$$

where

$$f_2(z,t)=\mathrm{Re}\left(f(z)e^{(-i\Omega t)}\right), \tag{4.1.17}$$

$$f(z)=\left(A(\zeta)e^{\zeta z}+A(-\zeta)e^{-\zeta z}\right),\ A(\zeta)=\frac{\zeta}{2}\frac{\left(e^{-i\theta}-e^{-\zeta}\right)}{\left(e^{\zeta}-e^{-\zeta}\right)}\ \&\ \zeta=(1-i)\sqrt{\frac{\Omega}{2}}. \tag{4.1.18}$$

We write the nonlinear system of Eqs.(4.1.14)-(4.1.15), in the matrix form as given bellow

$$\begin{bmatrix}\left(\lambda_2\frac{\partial}{\partial t}+1\right)\nabla^2 & Ra_D\left(\lambda_1\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial t}-\nabla^2\right)\end{bmatrix}\begin{bmatrix}\psi\\ T\end{bmatrix}=\begin{bmatrix}-Ra_D\left(\lambda_1\frac{\partial}{\partial t}+1\right)\frac{\partial T}{\partial x}\\ \frac{\partial(\psi,T)}{\partial(x,z)}+\chi^2\delta f_2(z,t)\frac{\partial\psi}{\partial x}\end{bmatrix} \tag{4.1.19}$$

The above system will be solved by considering stress-free and isothermal boundary conditions as given in Eq.(2.2.26). In order to seek the solution of the above system Eq.(4.1.19) we introduce an asymptotic series or Poincaré expansion given in Eq.(2.3.1) for Ra, ψ and T in terms of a small perturbation parameter χ that show a deviation from the critical state of onset of convection.

4.1.4 Bifurcation of periodic solution

In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time τ and the slow time scale of s. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}+\chi^2\frac{\partial}{\partial s}$. In the first order problem the nonlinear term in energy equation will be vanished therefore, the first order problem reduces to the linear stability problem for supercritical flow.

At the lowest order, we have

$$\begin{bmatrix} \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right) \nabla^2 & R_{0c} \left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial \tau} - \nabla^2\right) \end{bmatrix} \begin{bmatrix} \psi_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.1.20}$$

The solution of the lowest order system subject to the boundary conditions Eq.(2.2.26), is assumed to be

$$\psi_1 = \left(\mathbb{A}_1(s)e^{i\omega\tau} + \overline{\mathbb{A}_1}(s)e^{-i\omega\tau}\right) \cos ax \sin \pi z, \tag{4.1.21}$$

$$T_1 = \left(\mathbb{B}_1(s)e^{i\omega\tau} + \overline{\mathbb{B}_1}(s)e^{-i\omega\tau}\right) \sin ax \sin \pi z. \tag{4.1.22}$$

The undetermined amplitudes are functions of slow time scale and are related by the following relation:

$$\mathbb{B}_1(s) = -\frac{c + i\omega}{a} \mathbb{A}_1(s) \tag{4.1.23}$$

where $c = a^2 + \pi^2$. The values of the Darcy-Rayleigh number and the corresponding wave number for stationary mode of convection

$$Ra_D^{st} = \frac{(\pi^2 + a^2)^2}{a^2} \tag{4.1.24}$$

$$a_c = \pi, \tag{4.1.25}$$

which are classical results of Horton and Rogers(1945) and Lapwood (1948). We find Darcy-Rayleigh number and corresponding critical wave number for oscillatory convection as given bellow:

$$Ra_D^{osc} = \frac{(\lambda_2 \pi^4 + \pi^2 + 2a^2\pi^2\lambda_2 + a^2 + a^4\lambda_2)}{\lambda_1 a^2} \tag{4.1.26}$$

$$a_c^2 = \sqrt{\pi^4 + \frac{\pi^2}{\lambda_2}}, \tag{4.1.27}$$

which are same as obtained by Kim et al.(2003). The critical Darcy−Rayleigh number and corresponding wave number does not depend on (λ_1, λ_2) in stationary mode but in oscillatory mode. Also we see that the supercritical flow can occur for a particular wave number only if the following inequality holds

$$\lambda_1 > \lambda_2 + \frac{1}{c}. \tag{4.1.28}$$

The dimensionless frequency of the neutral oscillatory mode is

$$\omega^2 = \frac{c(\lambda_1 - \lambda_2) - 1}{\lambda_2 \lambda_1}. \tag{4.1.29}$$

In the second order, we get

$$\frac{\partial(\psi_1, T_1)}{(x, z)} = \frac{\pi a}{2}\{\mathbb{A}_1(s)\mathbb{B}_1(s)e^{2i\omega\tau} + \overline{\mathbb{A}_1}(s)\overline{\mathbb{B}_1}(s)e^{-2i\omega\tau} + \mathbb{A}_1(s)\overline{\mathbb{B}_1}(s) + \overline{\mathbb{A}_1}(s)\mathbb{B}_1(s)\}\sin 2\pi z. \tag{4.1.30}$$

From the above relation, we can deduce that the velocity and temperature fields have the terms having frequency 2ω and independent of past time scale. Thus, we write the second order temperature term as follows:

$$T_2 = \{T_{20} + T_{22}e^{2i\omega\tau} + \overline{T}_{22}e^{-2i\omega\tau}\}\sin 2\pi z \tag{4.1.31}$$

where T_{22} and T_{20} are temperature fields having the terms having the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problems are:

$$T_{20} = \frac{a}{8\pi}\{\mathbb{A}_1(s)\overline{\mathbb{B}_1}(s) + \overline{\mathbb{A}_1}(s)\mathbb{B}_1(s)\}, \quad \psi_{20} = 0 \tag{4.1.32}$$

and

$$T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega}\mathbb{A}_1(s)\mathbb{B}_1(s). \tag{4.1.33}$$

The horizontally averaged Nusselt number, Nu(s), for the oscillatory mode of convection is given by using the expression of T_2, given in Eq.(4.1.31), one can simplify Eq.(3.1.28) and obtain Nu as

$$\mathrm{Nu}(s) = 1 + \left(\frac{c}{2} + \frac{2\pi^2\sqrt{c^2 + \omega^2}}{\sqrt{64\pi^4 + 16\omega^2}}\right)|\mathbb{A}_1(s)|^2. \tag{4.1.34}$$

It is clear that the thermal modulation is effective at third order and affects Nu(s) through $\mathbb{A}_1(s)$ which is evaluated at third order.

At the third order, we have

$$\begin{bmatrix} \left(\lambda_2\frac{\partial}{\partial\tau} + 1\right)\nabla^2 & R_{0c}\left(\lambda_1\frac{\partial}{\partial\tau} + 1\right)\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau} - \nabla^2\right) \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix} \tag{4.1.35}$$

where

$$R_{31} = -\lambda_2\frac{\partial}{\partial s}(\nabla^2\psi_1) - R_{0c}\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial T_1}{\partial x}\right) - R_2\left(\lambda_1\frac{\partial}{\partial t} + 1\right)\left(\frac{\partial T_1}{\partial x}\right), \tag{4.1.36}$$

$$R_{32} = \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} + \delta f_2(z,s)\frac{\partial \psi_1}{\partial x} - \frac{\partial T_1}{\partial s}. \tag{4.1.37}$$

Substituting ψ_1, T_1 and T_2 into Eqs.(4.1.36-4.1.37), we obtain the expressions for R_{31} and R_{32} easily. Now under the stability condition for the existence of third order solution, these equations yield the following Landau equation that describes the temporal variation of the amplitude $\mathbb{A}_1(s)$ of the convection cell

$$\frac{\partial \mathbb{A}_1(s)}{\partial s} - \gamma_1^{-1}F(s)\mathbb{A}_1(s) + \gamma_1^{-1}k|\mathbb{A}_1(s)|^2\mathbb{A}_1(s) = 0, \tag{4.1.38}$$

where $\gamma_1 = \left[\frac{a^2R_{0c}\lambda_1}{c(1+i\omega\lambda_2)} - \frac{\lambda_2(c+i\omega)}{(1+i\omega\lambda_2)} - 1\right]$, $F(s) = \left[\frac{a^2R_{0c}(1+i\omega\lambda_1)}{c(1+i\omega\lambda_2)} - 2\delta I_1(c+i\omega)\right]$, $k = -\pi^2\left(\frac{2c^2+2ci\omega}{8\pi^2} + \frac{c^2+\omega^2}{8\pi^2+4i\omega}\right)$ and $I_1 = \int_0^1 f_2(z,s)\sin^2(\pi z)\,dz$. Writing $\mathbb{A}_1(s)$ in the phase amplitude form, we get

$$\mathbb{A}_1(s) = |\mathbb{A}_1(s)|e^{i\phi} \tag{4.1.39}$$

Now substituting the expression Eq.(4.1.39) in Eq.(4.1.38), we get the following equation for the amplitude $|\mathbb{A}_1(s)|$:

$$\frac{\partial |\mathbb{A}_1(s)|^2}{\partial s} = 2p_r|\mathbb{A}_1(s)|^2 - 2l_r|\mathbb{A}_1(s)|^4 \tag{4.1.40}$$

$$\frac{\partial (ph(\mathbb{A}_1(s)))}{\partial s} = p_i - l_i|\mathbb{A}_1(s)|^2 \tag{4.1.41}$$

where $\gamma_1^{-1}F(s) = p_r + ip_i$, $\gamma_1^{-1}k = l_r + il_i$ and $ph(.)$ represents the phase shift. One can observe here from Eq.(4.1.38) for the case $l_r > 0$ and $Ra_D^{osc} > Ra_c$ i.e. $p_r > 0$, the solution gives as $\mathbb{A} \sim \mathbb{A}_0 e^{p_r s}$ as $s \longrightarrow -\infty$, and $\mathbb{A} \longrightarrow 0$ is unstable solution, and a new stable solution develops, $\mathbb{A} = \sqrt{\frac{p_r}{l_r}}$ as $s \longrightarrow \infty$, whatever be the value of $\mathbb{A}_0$. This is called supercritical pitch fork bifurcation, the base system being linearly unstable for $Ra_D^{osc} > Ra_c$ but settling down as a new laminar flow. The steady state amplitude exists when Ra_c takes positive values. Supercritical pitch fork bifurcation diagram has been explained in the figure 4.1b. We have calculated the mean value of Nusselt number (MNu) for better understanding the effect of temperature modulation on heat transport, a representative time interval that allows a clear comprehension of the modulation effect needs to be chosen. The interval $(0, 2\pi)$ seemed an appropriate interval to calculate MNu.

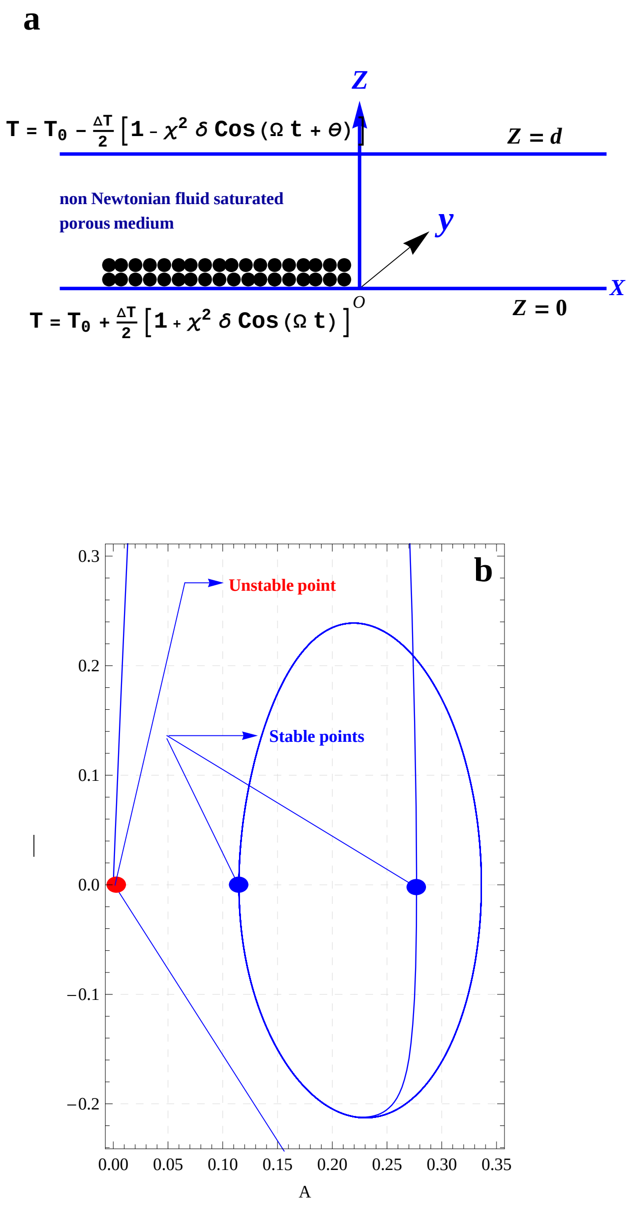

Figure 4.1: a.Physical configuration of the problem: b.Supercritical pitch fork bifurcation diagram

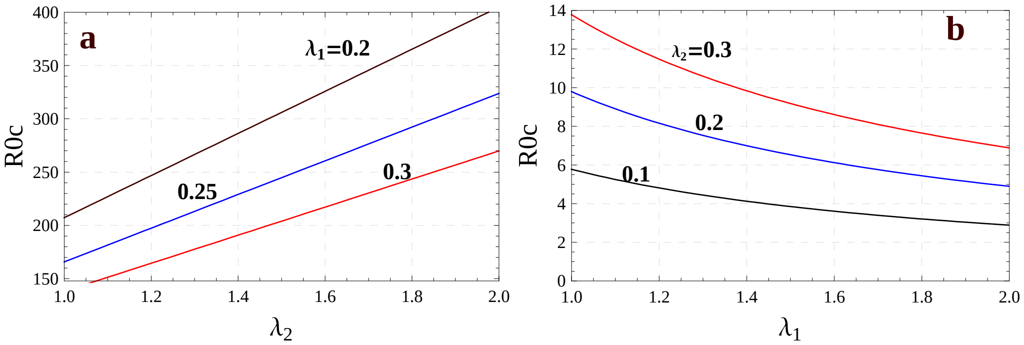

Figure 4.2: Variation of $R0c$ with respect to λ_2 and λ_1

The time-averaged Nusselt number MNu is defined as

$$\text{MNu} = \frac{1}{2\pi}\int_0^{2\pi} \text{Nu}\ ds. \tag{4.1.42}$$

The amplitude $\mathbb{A}_1(s)$ is obtained numerically and hence MNu is also to be numerically evaluated. An interesting observation that can be observed in I_1, which determines whether the modulation amplifies or diminishes the amplitude of convection. A discussion of the results now follows culminating in a listing of conclusions.

4.1.5 Results and Discussion

In this section we made an attempt to investigate oscillatory mode of convection in a viscoelastic fluid saturated porous medium under temperature modulation. We derive the non autonomous complex Ginzburg−Landau equation to evaluate an amplitude of convection under solvability condition. We consider a weakly nonlinear theory to study heat transport in the porous medium. It quite interest that the oscillatory mode of convection is possible only when the values of λ_1, λ_2 are consider as given in the Eq.(4.1.28). To exhibit supercritical flow in the system it is important to take λ_1 more than λ_2. For small amplitude and lower values of frequency of modulation the maximum the heat transfer hence we consider δ around 0.1 and Ω at 2.0. In order to find out the effect of temperature modulation on the system we consider the following three types of temperature profiles at the boundaries of the problem:

1. In-phase modulation (IPM)($\theta = 0$).

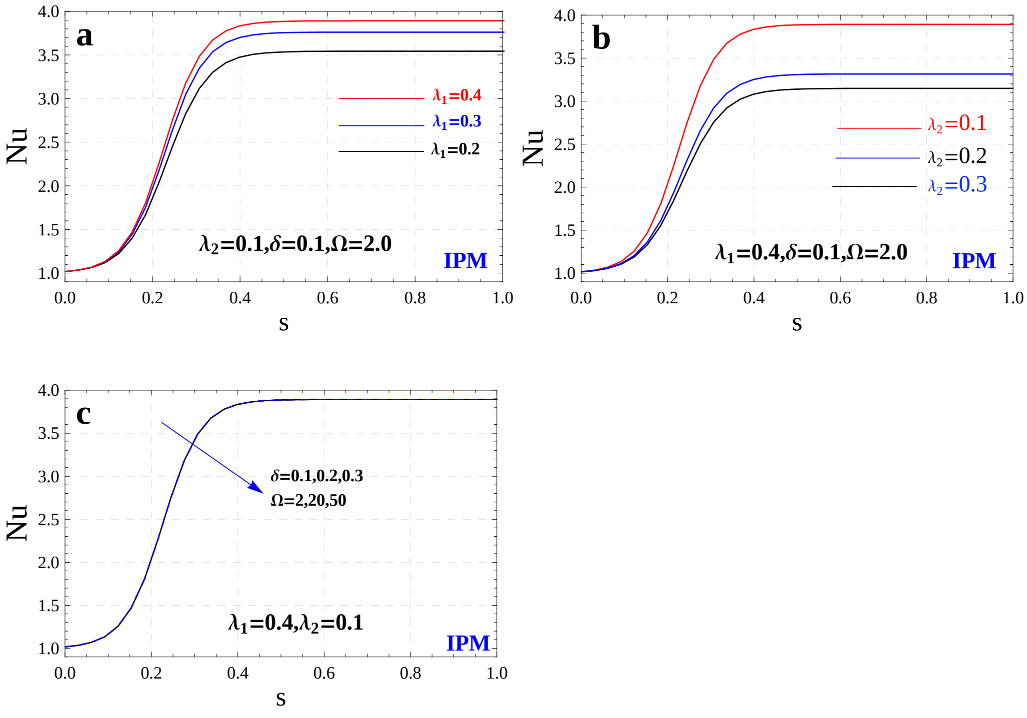

Figure 4.3: Effect of $(\lambda_1, \lambda_2, \delta, \Omega)$ on Nu IPM case

2. Out-phase modulation (OPM) $(\theta = \pi)$.

3. Only Lower boundary modulated (LBMO) $(\theta = -i\infty)$,
   which means that the modulation effect will not be considered in upper boundary but only in lower boundary.

In order to illustrate the effects of relaxation parameters λ_1, λ_2, the frequency Ω and the amplitude δ of modulation on heat transport, we obtain Nu numerically as a function of slow time scale s and depicted the curves of the Nusselt number versus time s. It is clear from the relation Eq.(4.1.26) that the oscillatory convection depends on both stress relaxation and strain retardation times. The stability curves corresponding to the oscillatory critical Rayleigh number R_{0c} has been presented in figure 4.2. In figure 4.2a it is found that the effect of increasing the value of stress relaxation parameter λ_1 is to reduce R_{0c} hence it has destabilizing effect on the system. However, opposite effect is found for strain retardation parameter λ_2 as it is increases R_{0c} increase hence it delays the onset of convection and enhance the stability of the system shown in figure 4.2b.

The results corresponding to the temperature modulation on the system shown in figures 4.3-4.5, where we have plotted Nu versus s. From these figures it is observed that Nu starts with 1 thus showing the conduction state initially, which means heat transfer across the porous medium is taking place through conduction when s is small. The value of Nu starts increase for intermediate values of s thus showing the convection is in progress and finally when s is very large, the steady state is achieved. The results has been presented in figure 4.3 for in phase modulation case. The results of this case is just as unmodulated system. Where the modulation effect will not be taking place. The effect of an increment in the value of stress relaxation parameter λ_1 given in figure 4.3a is to enhance the value of Nu on increasing λ_1. For lower values of slow time s there is an enhancement in Nu, further values of time s, Nu achieve steady state. The opposite effect is achieved for strain retardation parameter λ_2, i.e an increment in λ_2 is to suppress the heat transfer in the system given in figure 4.3b. There is no effect is obtained for an amplitude and frequency of modulation given in figure 4.3c. Hence in phase modulation is of negligible effect on heat transfer.

For out of phase modulation (OPM) same results obtained for the case of λ_1, λ_2 given in figures 4.4a-b. Here one can notice that effect of modulation reflect on Nu, initially Nu start with 1 at conduction state and increases further in time shows sinusoidal behaviour. The effects of frequency Ω and the amplitude δ of modulation on heat transport is clearly visible in the case OPM given in figures 4.4c-d. Figure 4.4c show an effect of amplitude of modulation on heat transport as δ increases heat transfer enhances in the system. For frequency of modulation Ω reduces heat transfer as it is increases given in figure 4.4d. Which means the effect of temperature modulation decreases as the frequency of modulation increases, and finally when Ω is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969), Bhadauria et al.(2010,2013c,d) and Siddheshwar et al.(2012a,b). Here we are not presenting results for LBMO case as they are similar to OPM case. On comparing the results in figure 4.5a, it is found that $\mathrm{Nu}_{\mathrm{LBMO}}$ is lower than $\mathrm{Nu}_{\mathrm{OPM}}$ but higher than $\mathrm{Nu}_{\mathrm{IPM}}$ as given below:

$$\mathrm{Nu}_{\mathrm{IPM}} < \mathrm{Nu}_{\mathrm{LBMO}} < \mathrm{Nu}_{\mathrm{OPM}}$$

This is also clear due to the fact that in phase modulation of the boundary temperature

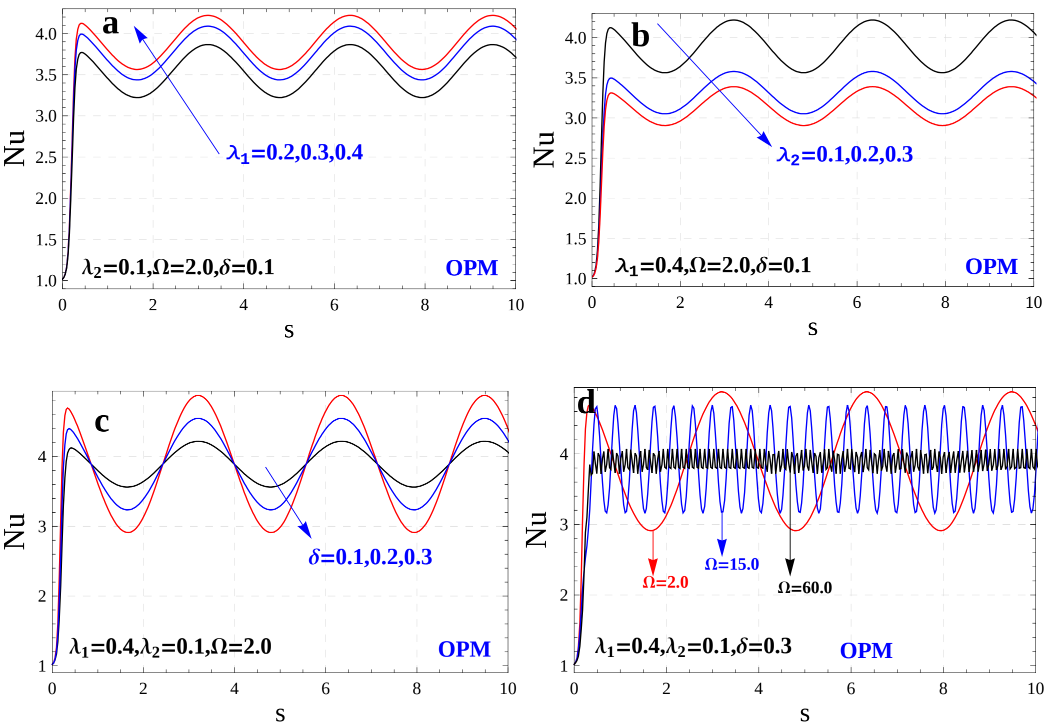

Figure 4.4: Variation of Nu with respect to s OPM case

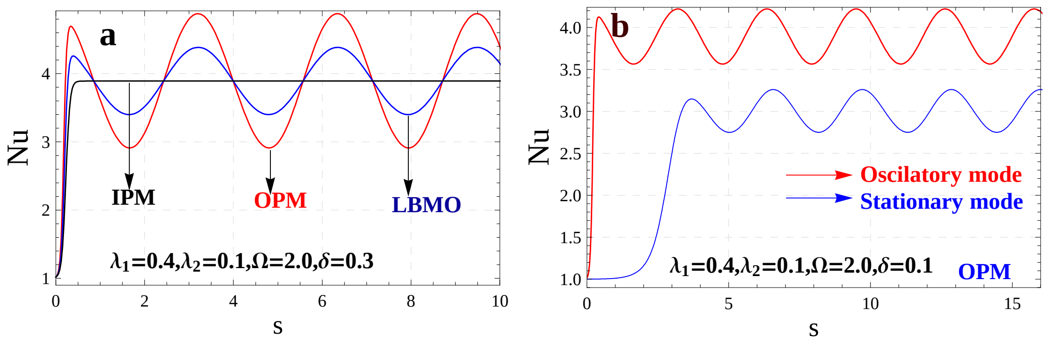

Figure 4.5: Comparison

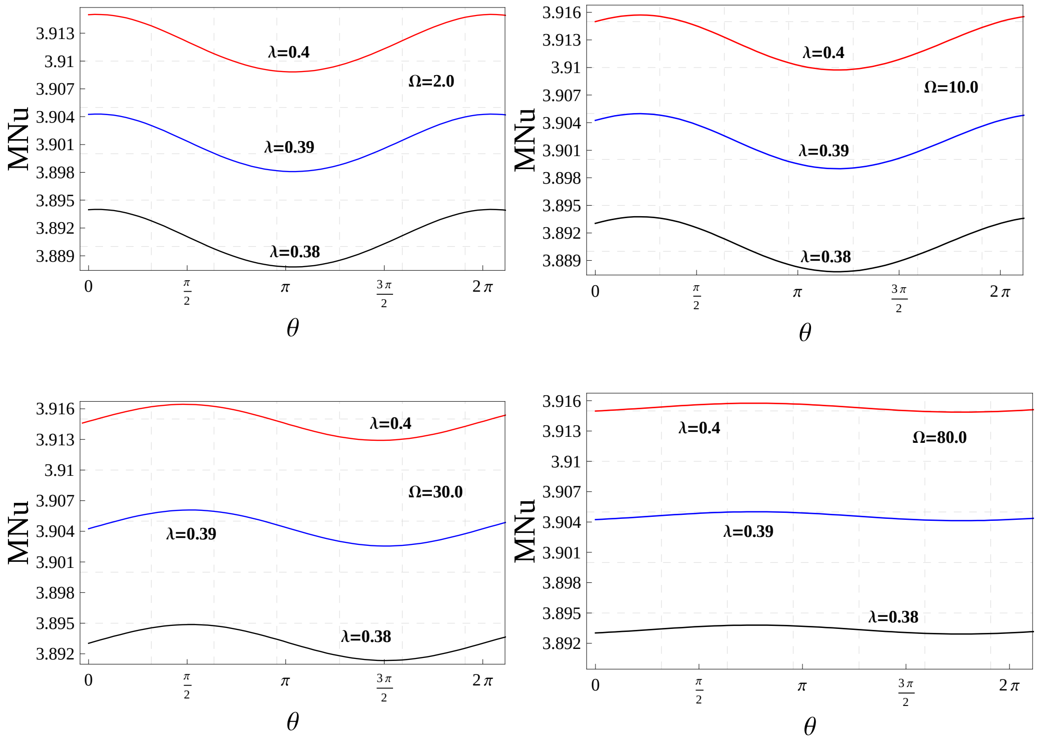

Figure 4.6: Effect of θ on MNu for various values of Ω and λ_1

does not substantially modify the temperature gradient across the porous medium, therefore not much effect on heat transfer. However, in cases of OPM and LBMO, the effect of temperature modulation on heat transfer is quite visible [figures 4.4a-d, figure 4.5a], and is oscillatory in nature. Here we can find certain frequencies where the value of Nu is high thus destabilizing the system and low thus stabilizing the system. In figure 4.5b we compare the results of oscillatory and stationary mode of convection for OPM case. It is found that heat transfer is more in oscillatory mode of convection than in stationary mode. It can be observed that ($\mathrm{Nu}^{st} < \mathrm{Nu}^{osc}$) for the same wave number. This implies that oscillatory instabilities can set in before stationary mode.

A better way of presenting our results according to Siddheshwar et al. (2013) the effect of modulation on mean Nusselt number depends on both the phase difference θ and frequency Ω of modulation than only on the choice of the small amplitude modulation. We present the effect of Ω in the figure 4.6 and the effect of θ in figure 4.7. From the figures 4.6-4.7 it is evident that for a given frequency of modulation there is a range of θ in which MNu increases with increasing θ and another range in which MNu decreases.

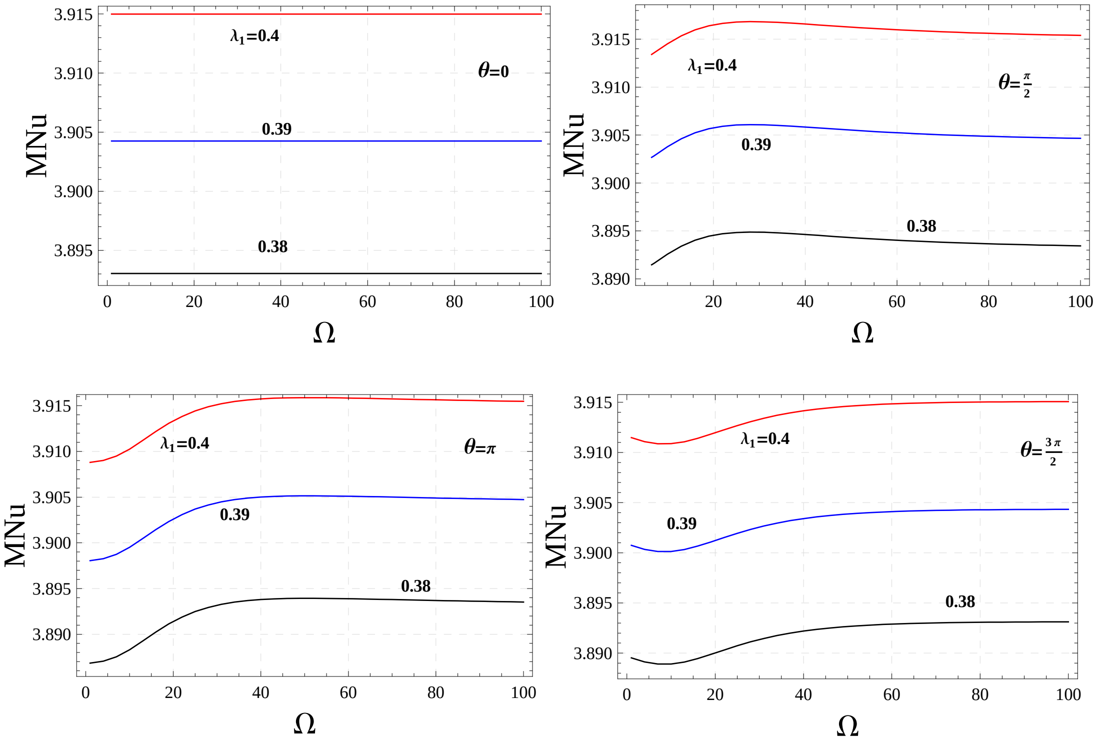

Figure 4.7: Effect of Ω on MNu for various values of θ and λ_1

Thus, one can conclude that, the combination of choices of Ω and θ can be made depending on the demands on heat transport in an application situation. Heat transfer can be regulated (enhanced or reduced) with the external mechanism of temperature modulation effectively. Our results are compatible with results of Malashetty et al.(2002). We also can observe our results in figures 4.6-4.7 are the results which are similar to Siddheshwar et al. (2013) for the Newtonian fluid case. It is clear that for temperature modulation the boundary temperatures should not be in in-phase modulation (synchronized), where the effect of modulation is negligible on heat transport.

4.1.6 Conclusions

We have analyzed the effect of temperature modulation on supercritical flow of Bénard−Darcy convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg−Landau amplitude equation. The following conclusions are made by the previous analysis

1. Effect of IPM is negligible on heat transport in the system.
2. In the case of IPM, the effect of δ and Ω are also found to be negligible on heat transport.
3. Effect of λ_1 is to enhance the heat transport for all three types of modulations.
4. Effect of λ_2 is to decrease the heat transport for all three types of modulations.
5. In the case of IPM, Nu increase steadily for intermediate value of time s and ultimately becomes constant when s is large.
6. In the case of OPM and LBMO, Nu shows an oscillatory nature.
7. The results upon MNu follows as in the case of Nu.

4.2 Heat and mass transfer for oscillatory convection in a binary viscoelastic fluid layer subjected to temperature modulation at the boundaries

4.2.1 Introduction

Thermohaline convection is an important fluid dynamics phenomenon that involves motions driven by two different density gradients diffusing at different rates. In two component fluid convection, the buoyancy force is affected not only by the difference of temperatures, but also by the difference of concentration of the fluid. The best example of double diffusive convection can be seen in oceanography, under ground water and lakes, atmospheric pollution, modeling of solar ponds, electrochemistry, chemical processes, laboratory experiments, magma chambers and sparks (Huppert and Sparks 1984), Fernando and Brandt (1994) formation of microstructure during the cooling of molten metals, migration of moisture through air contained in fibrous insulations, fluid flows around shrouded heat dissipation fins, grain storage system, the dispersion of contaminants through water saturated soil, solidification of binary mixtures, crystal P growth, and the underground disposal of nuclear wastes. In the case of non-Newtonian fluids; in particular viscoelastic fluids are important with a lot of industrial applications. The convection in viscoelastic fluids are important in chemical processing industries. The understanding of convective motion and its behaviour is important for controlling many industrial processes e.g. geothermal reservoirs, enhanced oil recovery, filtration, polymer filament package and composite impregnations. Malashetty and Swamy (2010) investigated linear and weak nonlinear thermal instability of double diffusive convection in a viscoelastic fluid. The onset criterion for both stationary and oscillatory convection is derived while using finite amplitude analysis analytically. The truncated representation of Fourier series method is used to find the heat and mass transfers for weak nonlinear theory. Using Runge−Kutta method they solved finite amplitude equations and quantified heat and mass transfer in terms of the Nusselt and Sherwood number. In this section the double diffusive oscillatory thermal convection in a viscoelastic fluid layer has been investigated and found much more effective results than in the case of stationary convection.

4.2.2 Mathematical Formulation

We consider an infinitely extended horizontal layer of incompressible binary viscoelastic fluid mixture, confined between two free free boundaries at $z = 0$ and $z = d$, and subjected to a time periodic temperature at the boundaries. A cartesian frame of reference is chosen with the origin at the lower boundary and the z axis vertically upward. The fluid layer is heated from below. The hydrodynamic equations are simplified by assuming Oberbeck−Boussinesq approximation. The disturbances are varied in the vertical direction. The viscoelastic fluid of the Oldroyd type is used to model the momentum equation. The constitutive equations are given bellow

$$\nabla \cdot \vec{q} = 0, \tag{4.2.1}$$

$$\left(\overline{\lambda_1}\frac{\partial}{\partial t} + 1\right)\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} + \frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\vec{g}\right) - \nu\left(\overline{\lambda_2}\frac{\partial}{\partial t} + 1\right)\nabla^2 \vec{q} = 0, \tag{4.2.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \kappa_T \nabla^2 T, \tag{4.2.3}$$

$$\frac{\partial S}{\partial t} + (\vec{q} \cdot \nabla)S = \kappa_S \nabla^2 S, \tag{4.2.4}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0) + \beta_S(S - S_0)], \tag{4.2.5}$$

where the physical variables have their usual meanings and are given in Nomenclature. The externally imposed, thermal and solutal boundary conditions are given by

$$\begin{aligned} T &= T_0 + \frac{\Delta T}{2}[1 + \chi^2 \delta \ \cos(\Omega t)] && \text{at } z = 0 \\ &= T_0 - \frac{\Delta T}{2}[1 - \chi^2 \delta \cos(\Omega t + \theta)] && \text{at } z = d \end{aligned} \tag{4.2.6}$$

$$\begin{aligned} S &= S_0 + \Delta S && \text{at } z = 0 \\ &= S_0 && \text{at } z = d, \end{aligned} \tag{4.2.7}$$

where ΔT, ΔS are the temperature, concentration difference across the fluid layer, χ is the smallness of amplitude of modulation, δ, Ω are amplitude and frequency of thermal modulation. The basic state is assumed to be quiescent, and the quantities in this state are given by

$$\vec{q_b} = 0, p = p_b(z,t), \quad T = T_b(z,t), \ \rho = \rho_b(z,t) \ S = S_b(z). \tag{4.2.8}$$

Substituting the Eq.(4.2.8) in Eqs.(4.2.1)-(4.2.5), we get the following relations, which help us in defining the basic state pressure, temperature and solute:

$$\frac{\partial p_b}{\partial z} = -\rho_b g, \tag{4.2.9}$$

$$\frac{\partial T_b}{\partial t} = \kappa_T \frac{\partial^2 T_b}{\partial z^2}, \tag{4.2.10}$$

$$\kappa_S \frac{d^2 S_b}{dz^2} = 0, \tag{4.2.11}$$

$$\rho_b = \rho_0 \left[1 - \alpha_T \left(T_b - T_0\right) + \beta_S \left(S_b - S_0\right)\right]. \tag{4.2.12}$$

The solution of Eqs.(4.2.10)-(4.2.11), subjected to the boundary conditions given in Eqs.(4.2.6)-(4.2.7), are given by

$$T_b(z,t) = T_s(z) + \chi^2 \delta \mathrm{Re}[T_1(z,t)], \tag{4.2.13}$$

$$S_b = S_0 + \Delta S \left(1 - \frac{z}{d}\right). \tag{4.2.14}$$

The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q} = \vec{q_b} + q^{'}, \ \rho = \rho_b + \rho^{'}, \ p = p_b + p^{'}, \ T = T_b + T^{'}. \tag{4.2.15}$$

We introduce the perturbed quantities Eq.(4.2.15) in Eqs.(4.2.1)−(4.2.5) and then non-dimensionalizing, one can obtained the following equations:

$$\left(\lambda_1 \frac{\partial}{\partial t} + 1\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t}\nabla^2\psi + \ Ra\frac{\partial T}{\partial x} - \ Rs\frac{\partial S}{\partial x} - \frac{1}{Pr}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,z)}\right) = \left(\lambda_2 \frac{\partial}{\partial t} + 1\right)\nabla^4\psi, \tag{4.2.16}$$

$$-\frac{\partial T_b}{\partial z}\frac{\partial \psi}{\partial x} + \left(\frac{\partial}{\partial t} - \nabla^2\right)T = \frac{\partial(\psi, T)}{\partial(x,z)}, \tag{4.2.17}$$

$$\frac{\partial \psi}{\partial x} + \left(\frac{\partial}{\partial t} - \Gamma\nabla^2\right)S = \frac{\partial(\psi, S)}{\partial(x,z)}, \tag{4.2.18}$$

where $Ra = \frac{\alpha_T g \Delta T d^3}{\nu \kappa_T}$ is thermal Rayleigh number, $Rs = \frac{\beta_S g \Delta S d^3}{\nu \kappa_S}$ is the solutal Rayleigh number. The basic state solution which appears in Eq.(4.2.17), influences the stability problem through the factor $\frac{\partial T_b}{\partial z}$, which is given by

$$\frac{\partial T_b}{\partial z} = -1 + \chi^2 \delta \left(f_2(z,t) \right), \tag{4.2.19}$$

where

$$f_2(z,t) = \text{Re}\left(f(z) e^{(-i\Omega t)} \right), \tag{4.2.20}$$

$$f(z) = \left(A(\zeta) e^{\zeta z} + A(-\zeta) e^{-\zeta z} \right),\ A(\zeta) = \frac{\zeta}{2} \frac{\left(e^{-i\theta} - e^{-\zeta} \right)}{\left(e^{\zeta} - e^{-\zeta} \right)}\ \&\ \zeta = (1-i)\sqrt{\frac{\Omega}{2}}. \tag{4.2.21}$$

The above system Eqs.(4.2.16)-(4.2.18) will be solved by considering stress free, isothermal and isosolutal boundaries. Hence the boundary conditions for perturbed variables are given by

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = S = T = 0 \ \text{ on } \ z = 0 \ \ z = 1. \tag{4.2.22}$$

4.2.3 Finite amplitude equation

Using an asymptotic expansion given in Eq.(2.3.1) for physical quantities (Ra, ψ, T, S), the above system Eqs.(4.2.16-4.2.18) is solved for every order of χ. In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time τ and the slow time scale of s. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \chi^2 \frac{\partial}{\partial s}$. In the first order problem, the nonlinear term in energy equation will vanish, therefore, the first order problem reduces to the linear stability problem for over-stability.

At the lowest order, we have

$$\begin{bmatrix} \frac{1}{Pr}\left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial}{\partial \tau} \nabla^2 - \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right) \nabla^4 & R_0 \left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial}{\partial x} & -Rs \left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial}{\partial x} \\ -\frac{\partial T_b}{\partial z} \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial \tau} - \nabla^2\right) & 0 \\ \frac{\partial}{\partial x} & 0 & \left(\frac{\partial}{\partial \tau} - \Gamma \nabla^2\right) \end{bmatrix}$$

$$\begin{bmatrix} \psi_1 \\ T_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.2.23}$$

The solution of the lowest order system subject to the boundary conditions Eq.(4.2.22), is assumed to be

$$\psi_1 = \left(\mathbb{A}(s)e^{i\omega\tau} + \overline{\mathbb{A}}(s)e^{-i\omega\tau}\right)\sin ax \sin \pi z, \tag{4.2.24}$$

$$T_1 = \left(\mathbb{B}(s)e^{i\omega\tau} + \overline{\mathbb{B}}(s)e^{-i\omega\tau}\right)\cos ax \sin \pi z, \tag{4.2.25}$$

$$S_1 = \left(\mathbb{C}(s)e^{i\omega\tau} + \overline{\mathbb{C}}(s)e^{-i\omega\tau}\right)\cos ax \sin \pi z. \tag{4.2.26}$$

The undetermined amplitudes are functions of slow time scale, and are related by the following relation:

$$\mathbb{B}(s) = -\frac{a}{c + i\omega}\mathbb{A}(s), \tag{4.2.27}$$

$$\mathbb{C}(s) = -\frac{a}{\Gamma c + i\omega}\mathbb{A}(s), \tag{4.2.28}$$

where $c = a^2 + \pi^2$ is total wavenumber. The critical Rayleigh number for oscillatory case is given by

$$R_0 = \frac{c}{a^2}\left[\frac{c^2(1+\omega^2\lambda_1\lambda_2)}{1+\lambda_1^2\omega^2} - \omega^2\left(\frac{1}{Pr} + \frac{(\lambda_2-\lambda_1)c}{1+\lambda_1^2\omega^2}\right)\right] + \frac{Rs(\Gamma c^2+\omega^2)}{\Gamma^2c^2+\omega^2}. \tag{4.2.29}$$

Here we calculate the corresponding critical wave number by minimizing critical Rayleigh number with respect to the wave number. The growth rate ω^2 can be obtained from:

$$a_1\omega^4 + a_2\omega^2 + a_3 = 0. \tag{4.2.30}$$

Observing closely on a_1, a_2, a_3 (are given in appendix) it reveals that the necessary condition for the occurrence of the oscillatory convection is that the following inequalities hold:

$$\lambda_1 > \lambda_2 \,\&\, \Gamma < 1. \tag{4.2.31}$$

Also, the value of critical Rayleigh number and the corresponding wavenumber for stationary mode of convection is given by

$$R_0 = \frac{c^3}{a^2} + \frac{Rs}{\Gamma}, \tag{4.2.32}$$

$$a = \frac{\pi}{\sqrt{2}}. \tag{4.2.33}$$

The critical Rayleigh number and corresponding wave number does not depend on (λ_1, λ_2) in stationary mode but in oscillatory mode.

In the second order, we get the following relations

$$\begin{aligned} \psi_2 &= 0, & (4.2.34)\\ \left(\frac{\partial}{\partial\tau} - \nabla^2\right) T_2 &= \frac{\partial(\psi_1, T_1)}{\partial(x,z)}, & (4.2.35)\\ \left(\frac{\partial}{\partial\tau} - \Gamma\nabla^2\right) S_2 &= \frac{\partial(\psi_1, S_1)}{\partial(x,z)}. & (4.2.36) \end{aligned}$$

From the above relation, according to Kim et al. (2003), we can deduce that the velocity, temperature and solutal fields have the terms having frequency 2ω and independent of fast time scale. Thus, we write the second order temperature, solutal terms as follows:

$$T_2 = \{T_{20} + T_{22}e^{2i\omega\tau} + \overline{T}_{22}e^{-2i\omega\tau}\}\sin 2\pi z, \tag{4.2.37}$$

$$S_2 = \{S_{20} + S_{22}e^{2i\omega\tau} + \overline{S}_{22}e^{-2i\omega\tau}\}\sin 2\pi z, \tag{4.2.38}$$

where (T_{20}, T_{22}) and (S_{20}, S_{22}) are temperature and solutal fields having the terms having the frequency 2ω and independent of fast time scale, respectively. The second order solutions can be defined using T_2, S_2 in Eqs.(4.2.35)-(4.2.36). The horizontally averaged Nusselt, Sherwood numbers, for the oscillatory mode of convection is given by:

$$\begin{aligned} \text{Nu}(s) &= 1 - \chi^2\left(\frac{\partial T_2}{\partial z}\right)_{z=0}, & (4.2.39)\\ \text{Sh}(s) &= 1 - \chi^2\left(\frac{\partial S_2}{\partial z}\right)_{z=0}. & (4.2.40) \end{aligned}$$

At the third order, we have

$$\begin{bmatrix} \frac{1}{Pr}\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial\tau}\nabla^2 - \left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^4 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x} & -Rs\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x}\\ -\frac{\partial T_b}{\partial z}\frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right) & 0\\ \frac{\partial}{\partial x} & 0 & \left(\frac{\partial}{\partial\tau}-\Gamma\nabla^2\right) \end{bmatrix}$$

$$\begin{bmatrix} \psi_3\\ T_3\\ S_3 \end{bmatrix} = \begin{bmatrix} R_{31}\\ R_{32}\\ R_{33} \end{bmatrix} \tag{4.2.41}$$

where the expressions for R_{31}, R_{32} and R_{33} are given in the appendix. Now under the stability condition for the existence of third order solution, we obtain the following Landau equation that describes the temporal variation of the amplitude $\mathbb{A}(s)$ of the convection cell

$$\frac{\partial \mathbb{A}(s)}{\partial s} - \gamma_1^{-1} F(s)\mathbb{A}(s) + \gamma_1^{-1} k |\mathbb{A}(s)|^2 \mathbb{A}(s) = 0. \tag{4.2.42}$$

where the coefficients $\gamma_1, F(s)$ and k are given in the appendix. Writing $\mathbb{A}(s)$ in the phase-amplitude form, we get

$$\mathbb{A}(s) = |\mathbb{A}(s)| e^{i\theta}. \tag{4.2.43}$$

Now substituting the expression Eq.(4.2.43) in Eq.(4.2.42), we get the following equations for the amplitude $|\mathbb{A}(s)|$:

$$\frac{\partial |\mathbb{A}(s)|^2}{\partial s} - 2p_r |\mathbb{A}(s)|^2 + 2l_r |\mathbb{A}(s)|^4 = 0, \tag{4.2.44}$$

$$\frac{\partial (ph(\mathbb{A}(s)))}{\partial s} = p_i - l_i |\mathbb{A}(s)|^2, \tag{4.2.45}$$

where $\gamma_1^{-1} F(s) = p_r + ip_i$, $\gamma_1^{-1} k = l_r + il_i$ and $ph(.)$ represents the phase shift. We have calculated the mean value of Nusselt (MNu), Sherwood (MSh) numbers for better understanding the effect of thermal modulation on heat and mass transports, a representative time interval that allows a clear comprehension of the modulation effect needs to be chosen. The interval $(0, 2\pi)$ seemed an appropriate interval to calculate MNu and MSh. The time averaged Nusselt and Sherwood numbers are defined as

$$\text{MNu} = \frac{1}{2\pi} \int_0^{2\pi} \text{Nu} ds, \tag{4.2.46}$$

$$\text{MSh} = \frac{1}{2\pi} \int_0^{2\pi} \text{Sh} ds. \tag{4.2.47}$$

An interesting observation that can be seen in I_1, which determines whether the modulation amplifies or diminishes the amplitude of convection. A discussion of the results now follows culminating in a listing of conclusions.

4.3 Results and discussions

Effect of temperature modulation on double diffusive thermal convection in a viscoelastic fluid layer has been analysed, for overstable mode, by means of a weakly nonlinear theory. The amplitude equations for the bifurcations states are also obtained. In order to illustrate the effects of various parameters of the system on heat and mass transports, we plot the curves of Nusselt, Sherwood numbers versus slow time s. It is observed that the relation given in Eq.(4.2.31) leads to an interesting result; that for a viscoelastic fluid layer heated and salted underneath; the oscillatory type of instability is possible only when the relaxation parameter λ_1 is greater than the retardation parameter λ_2, and the diffusivity ratio less than 1. Also, it is clear from the relation Eq.(4.2.29) that the oscillatory convection depends on both relaxation and retardation times. It can be notified that the modulation effect enters into the system at the third order $O(\chi^2)$. Without loss of generality $R_2 = R_0$ is assumed in the calculations and this is done to keep the parameters to a minimum. The results that we obtain and present here are only for particular set of parameter values. We observe here that the stationary critical Rayleigh number Eq.(4.2.30) does not depend on λ_1, λ_2, therefore for stationary case, the study reduces to double diffusive convection in Newtonian fluid. In order to analyze the effect of modulated temperature field, we discuss the following three cases:

1. In-phase modulation (IPM)($\theta = 0$).
2. Out-phase modulation (OPM) ($\theta = \pi$).
3. Only Lower boundary modulated (LBMO) ($\theta = -i\infty$),
   which means only lower boundary temperature will be modulated and the upper boundary will be at fixed constant temperature.

The Nusselt and Sherwood numbers have been obtained as functions of the system parameters and an amplitude of convection. The parameters of the system are $Rs, Pr, \lambda_1, \lambda_2, \Gamma, \delta$ and Ω. To see the effect of each parameter on the system, we fix all other parameters and study individual parameter. The results corresponding to the

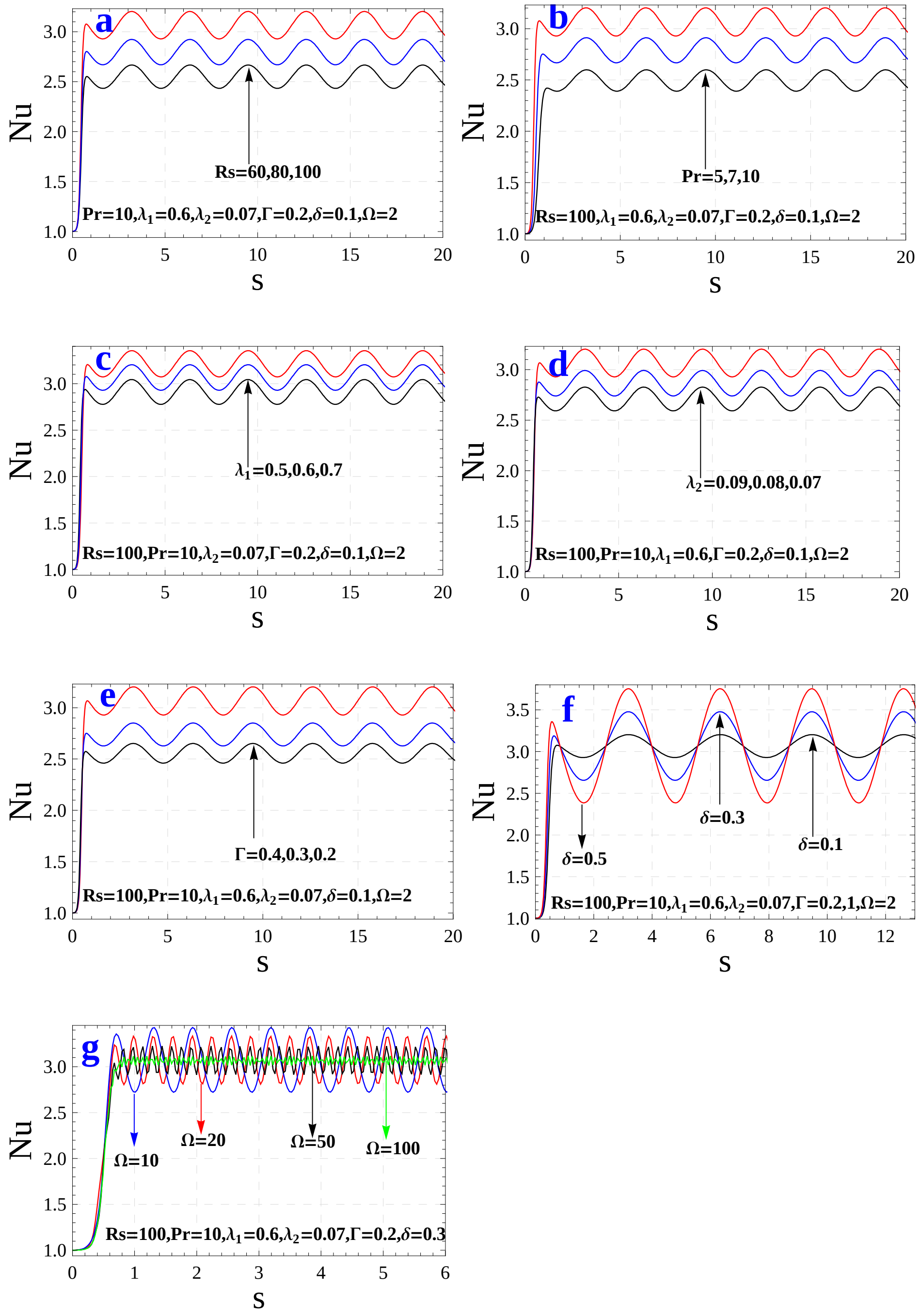

Figure 4.8: Effect of various parameters on heat transport OPM case

thermal modulation have been depicted in figures 4.8-4.11. In figures 4.8-4.9 we have plotted Nu, Sh with respect to the slow time s and discussed heat and mass transports. It is found that the value of Nu starts with 1, thus showing the conduction state initially that is heat transfer across the fluid layer is taking place through conduction when s is small. The values of Nu, Sh increases for intermediate values of s thus showing that convection is in progress and finally when s is very large, the oscillatory state is achieved. The effect of the Prandtl number is important, because many practical available viscoelastic fluids have large Prandtl numbers. It is quite interesting to note that when the Prandtl number increases, the critical value of the Rayleigh number decreases significantly so that the Prandtl number has a tendency to destabilize the system, compatible with results obtained by Kim et al.(2003), Tan et al.(2007) and Bhadauria et al.(2012, 2014)

‘ One can observe from Eq.(4.2.42) for the case $l_r > 0$ and $Ra > Ra_c$ i.e. $p_r > 0$, the solution gives as $\mathbb{A} \sim \mathbb{A}_0 e^{p_r s}$ as $s \longrightarrow -\infty$, and $\mathbb{A} \longrightarrow 0$ is unstable solution, and a new stable solution develops, $\mathbb{A} = \sqrt{\frac{p_r}{l_r}}$ as $s \longrightarrow \infty$, whatever be the value of $\mathbb{A}_0$. This is called supercritical pitch fork bifurcation, the base system being linearly unstable for $Ra > Ra_c$ but settling down as a new laminar flow. The steady state amplitude exists when Ra_c takes positive values. Supercritical pitch fork bifurcation diagram has been presented in the figure 4.1a for the case of OPM. Similar results are also obtained for IPM and LBMO but not included here. thus conforms the results of Bhadauria and Kiran (2014c).

First we discuss our results for out of phase modulation (OPM): The effect of solutal Rayleigh number Rs in figures 4.8a and 4.9a is to increase Nu and Sh so heat and mass transfer, hence it has a destabilizing effect on the system. Though the presence of a stabilizing gradient of solute will prevent the onset of convection, the strong finite amplitude motions, which exist for large Rayleigh numbers, tend to mix the solute and redistribute it so that the interior layers of the fluid are more neutrally stratified. As a consequence, the inhibiting effect of the solutal gradient is greatly reduced, and hence fluid will convect more and more heat and mass when Rs is increased. The effect of Prandtl number on heat and mass transfer is shown in figures 4.8b and 4.9b and found that Pr has a destabilizing effect and advance the convection, hence increases heat and mass transfer. In

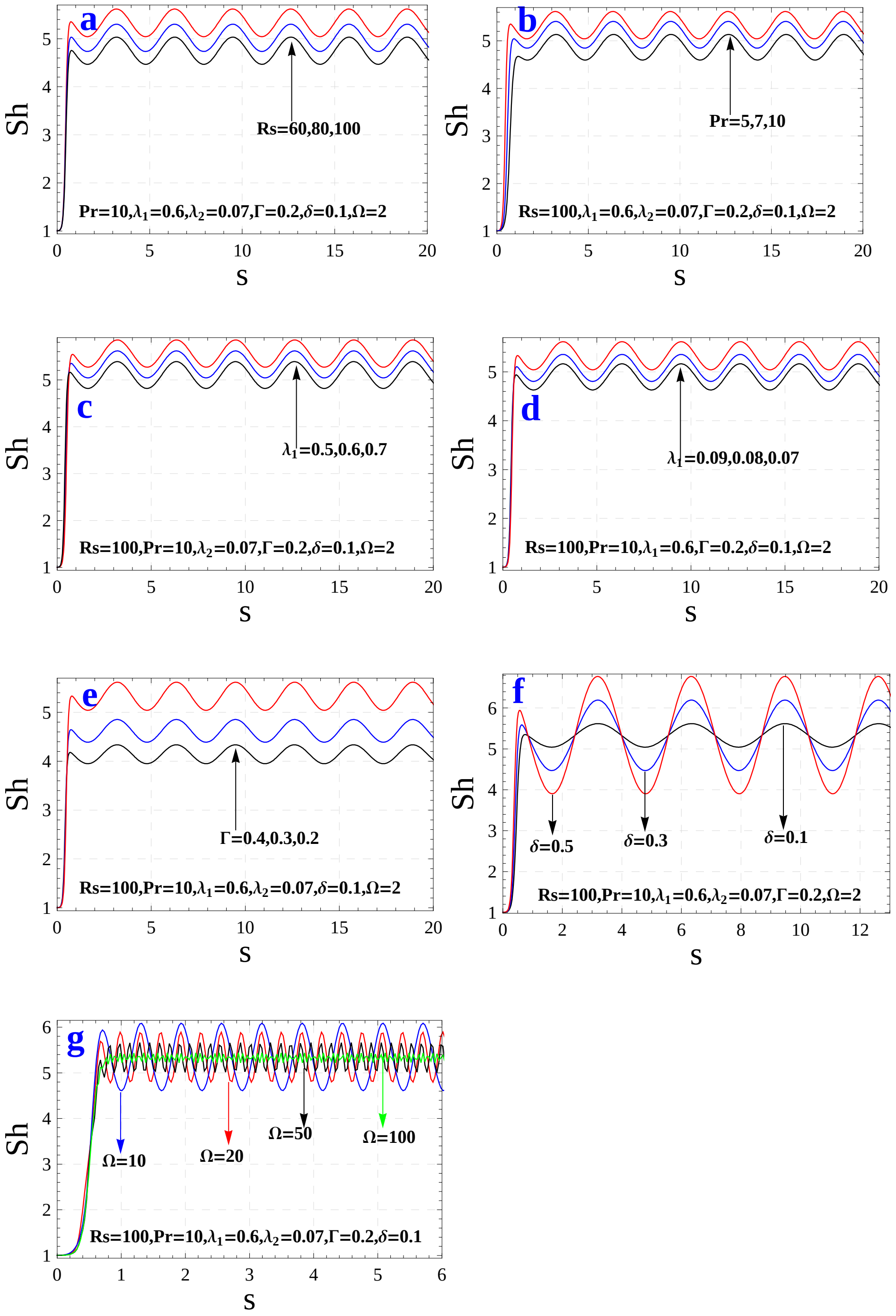

Figure 4.9: Effect of various parameters on mass transport OPM case

figures 4.8c and 4.9c, we depict the effect of viscoelastic parameter λ_1 on the oscillatory convection. For fixed values of other parameters, the critical Rayleigh number for the onset of oscillatory convection decreases with an increase in the value of λ_1, indicating that the effect of increasing viscoelastic parameter is to advance the onset of oscillatory convection. Thus, it confirms that the elastic behavior of the non-Newtonian fluids leads to the oscillatory motions, hence heat and mass transfer increases. Further, the effect of retardation parameter λ_2 given in figures 4.8d and 4.9d, is found to stabilize the system as the heat and mass transfer decreases on increasing λ_2. The effect of diffusivity ratio Γ is delay the onset of convection and hence heat and mass transfer given in figures 4.8e and 4.9e. The effects of frequency Ω and the amplitude of modulation, δ on heat and mass transport are given in the figures 4.8f and 4.9f and figures 4.8g and 4.9g, respectively. It is found that an increment in amplitude of modulation increases the magnitude of Nu, Sh, thus enhances heat and mass transfer, and thus advancing the onset of convection. An opposite effect is obtained in the case of frequency of modulation as Ω increases. Hence, we found that the effect of thermal modulation decreases as the frequency of modulation increases, and finally when Ω is very large, the effect of modulation disappears altogether, thus confirming the results of Venezian (1969), Bhadauria and Kiran (2013a, 2014c, g).

In the case of IPM, the parameters' effects are same as in OPM, however, in−phase modulation (IPM) of the boundary temperature does not substantially modify the temperature gradient across the layer, therefore not much effect of modulation on heat transfer. Thus, the results in the case of IPM figure 4.10 are same as that are in unmodulated case, compatible with the results of Malashetty and Swamy (2010), Kumar and Bhadauria (2011) and Rajib and Layak (2012) for unmodulated case. The effect of individual parameter is same as in the case of OPM. The similar results can be obtained for mass transfer. Here we do not present the results corresponding to LBMO, as they are similar to the results obtained in case of OPM. However, on comparing the results of all three cases it is found that $\mathrm{Nu/Sh}^{LBMO}$ is lower than $\mathrm{Nu/Sh}^{OPM}$ but higher than $\mathrm{Nu/Sh}^{IPM}$ as given below:

$$\mathrm{Nu/Sh}^{IPM} < \mathrm{Nu/Sh}^{LBMO} < \mathrm{Nu/Sh}^{OPM}$$

Finally in figures 4.11a-h we have presented our results in terms of averaged Nusselt

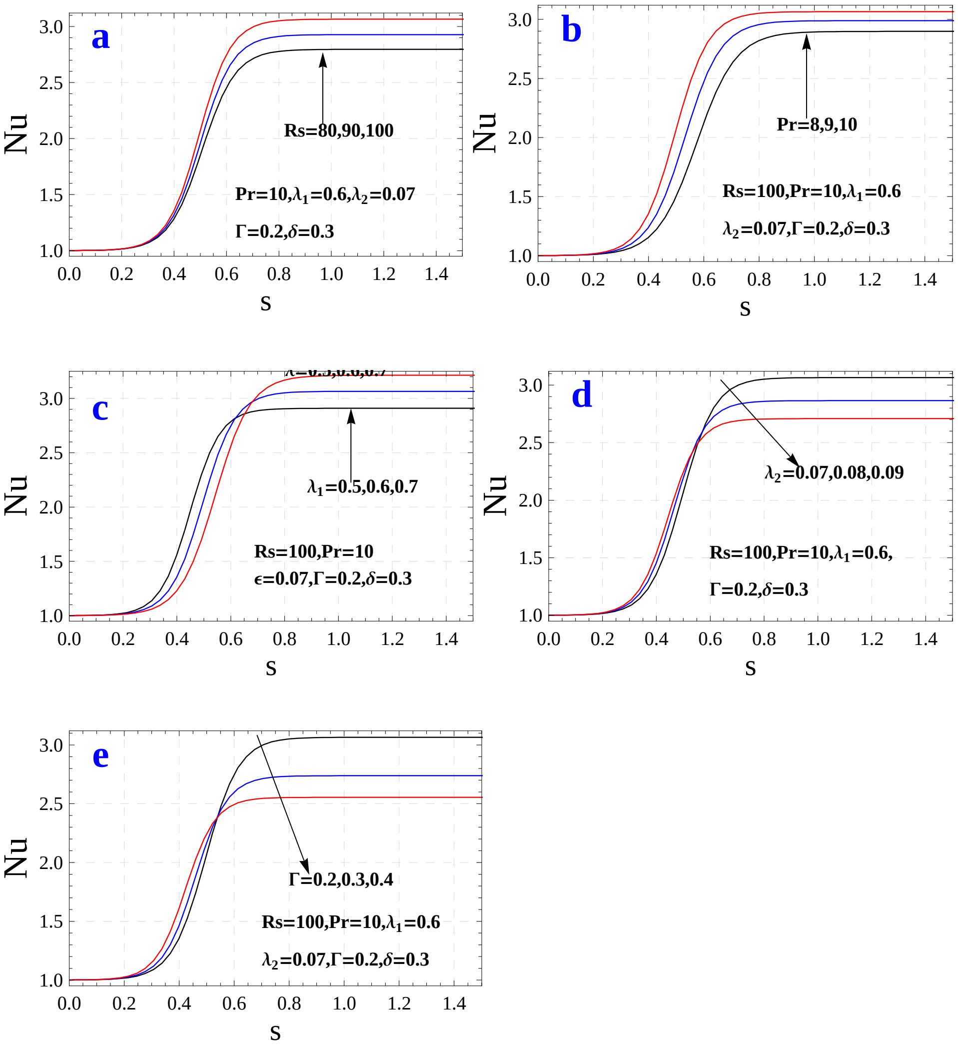

Figure 4.10: Effect of various parameters on heat transport IPM case

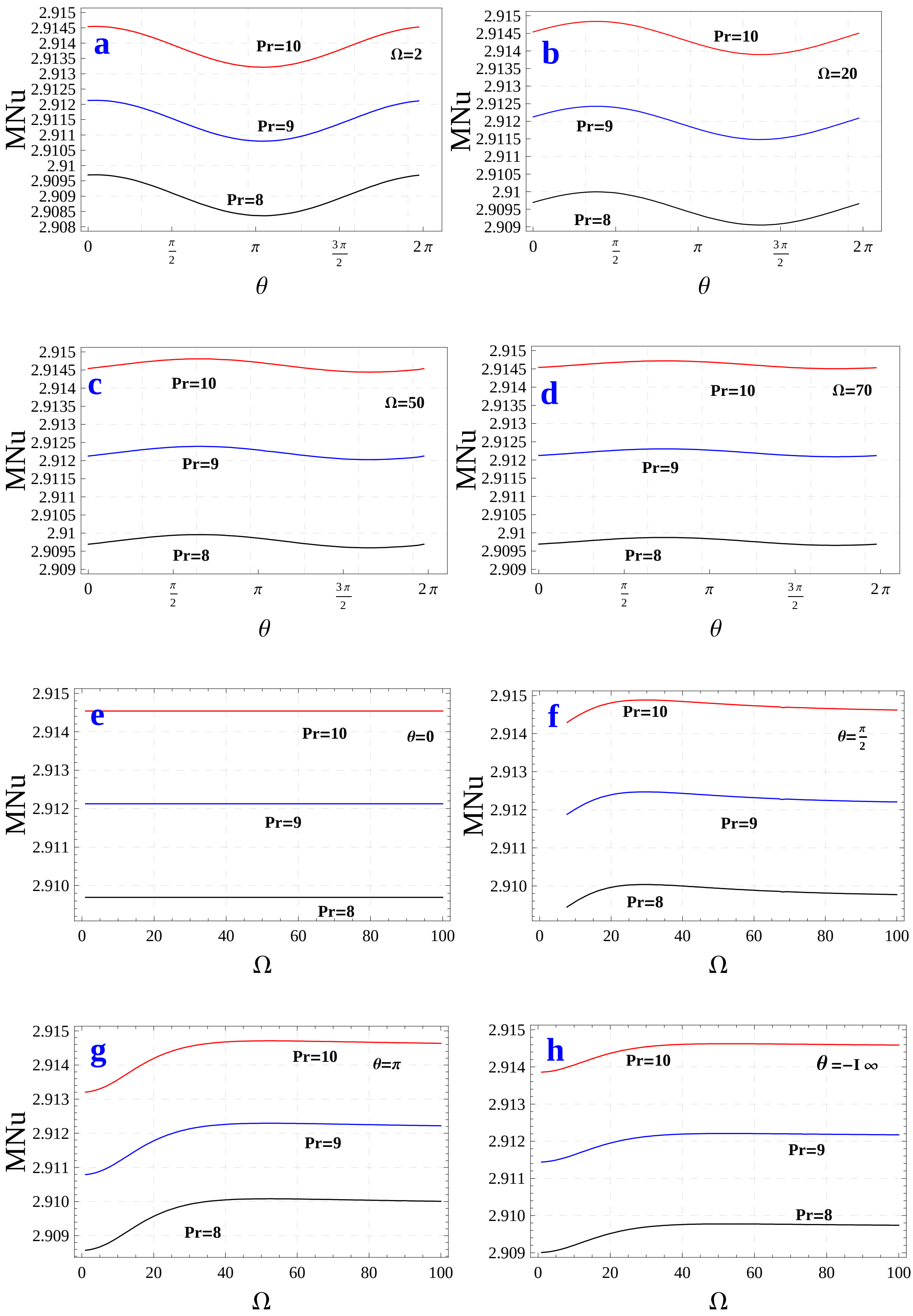

Figure 4.11: Effect of θ on MNu for different values of Ω and Pr (a-d): Effect of Ω on MNu for different values of θ and Pr (e-h).

number to see the effect of temperature modulation. In figures 4.11a-d we observe the effect of phase angle θ and in figures 4.11e-h the effect of frequency of modulation. Instead of choosing the frequency of small amplitude modulation, it is to be noted that there is good combination of selecting the range of θ, Ω in which the rate of heat transfer regulated effectively. From the figures it is evident that for a given frequency of modulation there is a range of θ in which (MNu) increases with increasing θ and another range in which (MNu) decreases. Thus, one can conclude that, the combination of choices of Ω and θ can be made depending on the applications on heat and mass transports. Heat and Mass transfer can be regulated (enhanced or reduced) with the external mechanism of temperature modulation. One can notice here that the effect of modulation is negligible when both the boundary temperatures are synchronized, hence for temperature modulation the boundary temperatures should not be synchronized. Only asynchronized temperature boundaries are effective for temperature modulation for either enhancing or diminishing heat and mass transfer. Our results are compatible with results which are similar to Siddheshwar et al.(2013) and Bhadauria and Kiran (2014c). The similar results can be obtained for the case of averaged Sherwood number MSh.

4.4 Conclusions

We have analyzed the effect of thermal modulation on overstability of viscoelastic double diffusive convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg−Landau amplitude equation. The following conclusions are made by the previous analysis

1. Effect of IPM is negligible on heat and mass transport in the system.

2. In the case of IPM, the effect of δ and Ω are also found to be negligible on heat and mass transport.

3. Effect of Rs, Pr, λ_1 is to enhance the heat and mass transport for all three types of modulations.

4. Effect of λ_2, Γ is to decrease the heat and mass transport for all three types of modulations.
5. In the case of IPM, Nu increase steadily for intermediate value of time s and ultimately becomes constant when s is large.
6. In the case of OPM and LBMO, Nu shows an oscillatory nature at intermediate and large values of time s.
7. Supercritical pitch fork bifurcation exits for Eq.(4.2.42).
8. The effects of θ, Ω observed in terms of MNu, similar results obtained for MSh.

Our results can be summarised as (in the case of OPM, LBMO)

1. $[\mathrm{Nu/Sh}]_{Rs=60} < [\mathrm{Nu/Sh}]_{Rs=80} < [\mathrm{Nu/Sh}]_{Rs=100}$
2. $[\mathrm{Nu/Sh}]_{Pr=5} < [\mathrm{Nu/Sh}]_{Pr=7} < [\mathrm{Nu/Sh}]_{Pr=10}$
3. $[\mathrm{Nu/Sh}]_{\lambda_1=0.5} < [\mathrm{Nu/Sh}]_{\lambda_1=0.6} < [\mathrm{Nu/Sh}]_{\lambda_1=0.7}$
4. $[\mathrm{Nu/Sh}]_{\lambda_2=0.09} < [\mathrm{Nu/Sh}]_{\lambda_2=0.08} < [\mathrm{Nu/Sh}]_{\lambda_2=0.07}$
5. $[\mathrm{Nu/Sh}]_{\Gamma=0.4} < [\mathrm{Nu/Sh}]_{\Gamma=0.3} < [\mathrm{Nu/Sh}]_{\Gamma=0.2}$
6. $[Nu/Sh]_{\delta=0.1} < [\mathrm{Nu/Sh}]_{\delta=0.3} < [\mathrm{Nu/Sh}]_{\delta=0.5}$
7. $[Nu/Sh]_{\Omega=100} < [\mathrm{Nu/Sh}]_{\Omega=50} < [\mathrm{Nu/Sh}]_{\Omega=20} < [\mathrm{Nu/Sh}]_{\Omega=10}$

Appendix

The coefficients in Eq.(4.2.30) are defined as:

$$\begin{aligned}
a_1 &= (\lambda_1 + Pr\lambda_2)\lambda_1 c,\\
a_2 &= c\left[Pr(1+\lambda_1\lambda_2\Gamma^2c^2) + (1+\lambda_1^2\Gamma^2c^2) + Pr(\lambda_2-\lambda_1)c\right] - a^2Pr(1-\Gamma)\lambda_1^2Rs,\\
a_3 &= c^3\Gamma^2(1+Pr) + Pr\Gamma^2c^4(\lambda_2-\lambda_1) - a^2Pr(1-\Gamma)Rs.
\end{aligned}$$

The expressions given in Eqs.(4.2.39)-(4.2.40), one can simplify as

$$\begin{aligned}
\mathrm{Nu}(s) &= 1 + a^2\left(\frac{c}{2(c^2+\omega^2)} + \frac{\pi^2}{\sqrt{16\pi^4 + 4\omega^2(\sqrt{c^2+\omega^2})}}\right)|\mathbb{A}(s)|^2,\\
\mathrm{Sh}(s) &= 1 + a^2\left(\frac{c}{2(\Gamma^2c^2+\omega^2)} + \frac{\pi^2}{\sqrt{16\Gamma^2\pi^4 + 4\omega^2(\sqrt{\Gamma^2c^2+\omega^2})}}\right)|\mathbb{A}(s)|^2.
\end{aligned}$$

The expressions given in Eq.(4.2.41) are

$$\begin{aligned}
R_{31} &= \lambda_2\frac{\partial}{\partial s}(\nabla^4\psi_1) - R_0\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial T_1}{\partial x}\right) - R_2\left(\lambda_1\frac{\partial}{\partial t}+1\right)\left(\frac{\partial T_1}{\partial x}\right) + Rs\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial S_1}{\partial x}\right)\\
&- \frac{1}{Pr}\left(\lambda_1\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial s}(\nabla^2\psi_1) - \frac{1}{Pr}\lambda_1\frac{\partial}{\partial s}\left(\frac{\partial}{\partial \tau}\nabla^2\psi_1\right),\\
R_{32} &= \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s} + \delta f_2\frac{\partial \psi}{\partial x},\\
R_{33} &= \frac{\partial \psi_1}{\partial x}\frac{\partial S_2}{\partial z} - \frac{\partial S_1}{\partial s}.
\end{aligned}$$

The coefficients given in Eq.(4.2.42) are

$$\gamma_1 = \left[\lambda_2c^2 + \frac{c(1+2i\omega\lambda_1)}{Pr} + a^2\lambda_1\left(\frac{Rs}{c\Gamma+i\omega} - \frac{R_0}{c+i\omega}\right) + \frac{R_0a^2(1+i\omega\lambda_1)}{(c+i\omega)^2} - \frac{Rsa^2(1+i\omega\lambda)}{(\Gamma c+i\omega)^2}\right],$$

$$F(s) = \left[\frac{R_2a^2(1+i\omega\lambda_1)}{(c+i\omega)} - \frac{2R_0a^2(1+i\omega\lambda_1)}{(c+i\omega)}\delta I_1\right], \text{ where } I_1 = \textstyle\int_0^1 f_2\sin^2(\pi z)dz,$$

$$k = \frac{a^4cR_0(1+i\omega\lambda_1)}{4(c^2+\omega^2)(c+i\omega)} + \frac{a^4\pi^2R_0(1+i\omega\lambda_1)}{(8\pi^2+4i\omega)(c+i\omega)^2} - \frac{a^4cRs(1+i\omega\lambda_1)}{4(\Gamma^2c^2+\omega^2)(\Gamma c+i\omega)} - \frac{a^4\pi^2Rs(1+i\omega\lambda_1)}{(8\pi^2\Gamma+4i\omega)(\Gamma c+i\omega)^2}.$$

Chapter 5

Weak nonlinear thermal instability under magnetic field and rotational speed modulation

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5.1 Weak nonlinear analysis of magneto−convection under magnetic field modulation

5.1.1 Introduction

Aniss et al. (2001) studied a magnetic field modulation in an electrically conducting fluid layer with linear stability analysis and show that magnetic field modulation is much effective to handle theoretical and experimental investigation rather than the gravity modulation (Gresho and Sani 1970). In which the amplitudes and frequencies of modulations are more difficult to control. Keeping in mind the linear theory analysis could give only onset of convection but, fails at heat transfer in the system. Due to this reason we have investigated a weakly nonlinear analysis in an electrically conducting fluid layer under magnetic field modulation where we have found external regulations to regulate heat transfer in the system. Using the non-autonomous Ginzburg−Landau equation, we obtain an amplitude equation for convection as a function of system parameters and quantify heat transport in terms of the Nusselt number.

5.1.2 Mathematical Formulation

We consider an electrically conducting liquid of depth d, confined between two infinite, parallel, horizontal planes at $z = 0$ and $z = d$. Cartesian co-ordinates have been taken with the origin at the bottom of the liquid layer, and the z-axis vertically upwards. The surfaces are maintained at a constant temperature gradient $\frac{\Delta T}{d}$. Under the Boussinesq approximation, the dimensional governing equations for the study of magnetoconvection in an electrically conducting liquid are (Siddheshwar et al. 2012)

$$\nabla \cdot \vec{q} = 0, \tag{5.1.1}$$

$$\nabla \cdot \vec{H} = 0, \tag{5.1.2}$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = \frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\vec{g} - \frac{\mu}{\rho_0}\nabla^2 \vec{q} + \frac{\mu_m}{\rho_0}\vec{H} \cdot \nabla \vec{H}, \tag{5.1.3}$$

$$\gamma\frac{\partial T}{\partial t}+(\vec{q}\cdot\nabla)T=\kappa_T\nabla^2T, \tag{5.1.4}$$

$$\frac{\partial\vec{H}}{\partial t}+(\vec{q}\cdot\nabla)\vec{H}-(\vec{H}\cdot\nabla)\vec{q}=\nu_m\nabla^2\vec{H}, \tag{5.1.5}$$

$$\rho=\rho_0\left[1-\alpha_T\left(T-T_0\right)\right]. \tag{5.1.6}$$

where the physical variables have their usual meanings given in list of symbols. The externally imposed thermal boundary conditions are given by:

$$\left.\begin{array}{lll} T & =T_0+\Delta T & \text{at } z=0 \\ & =T_0 & \text{at } z=1 \end{array}\right\} \tag{5.1.7}$$

Vertically imposed sinusoidally varying time dependent magnetic field is given by (Aniss et al. 2001)

$$H_0\left[1+\delta\chi^2\cos\left(\Omega t\right)\right] \tag{5.1.8}$$

where δ is the small amplitude of magnetic modulation, Ω is modulation frequency and χ indicates the smallness of magnetic modulation. The basic state is assumed to be quiescent and the quantities in this state are given by

$$\vec{q_b}=0,\ \rho=\rho_b(z,t),\ p=p_b(z,t),\ T=T_b(z,t),\vec{H}=H_0, \tag{5.1.9}$$

$$\frac{\partial p_b}{\partial z}=-\rho_b g, \tag{5.1.10}$$

$$T_b=T_0+\Delta T\left(1-\frac{z}{d}\right) \tag{5.1.11}$$

$$\rho_b=\rho_0\left[1-\alpha_T\left(T_b-T_0\right)\right] \tag{5.1.12}$$

Now, we impose finite perturbations to the basic state given in Eq.(5.1.9) as:

$$\vec{q}=q_b+\vec{q}^{'},\ \rho=\rho_b+\rho^{'},\ p=p_b+p^{'},\ T=T_b+T^{'},\vec{H}=H_0+\vec{H}^{'}, \tag{5.1.13}$$

where primes denote the quantities at the perturbations. Substituting the Eq.(5.1.13) in Eqs.(5.1.1)-(5.1.6) and using the basic state results, consider only two dimensional disturbances and hence the stream function ψ and magnetic potential Φ are introduced as $(\vec{u}^{'},\vec{w}^{'})=\left(\frac{\partial\psi}{\partial z},-\frac{\partial\psi}{\partial x}\right)$ and $(\vec{H}_x^{'},\vec{H}_z^{'})=\left(\frac{\partial\Phi}{\partial z},-\frac{\partial\Phi}{\partial x}\right)$. Further, eliminating density and pressure terms from Eqs.(5.1.1)-(5.1.6) then we obtain the following non-dimensionalized governing equations are:

$$-\nabla^4\psi + Ra\frac{\partial T}{\partial x} - QPm\ \mathrm{g}_m\frac{\partial \nabla^2\Phi}{\partial z} = -\frac{1}{Pr}\frac{\partial \nabla^2\psi}{\partial t} + \frac{1}{Pr}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} - QPm\frac{\partial(\Phi,\nabla^2\Phi)}{\partial(x,z)}, \tag{5.1.14}$$

$$-\frac{\partial\psi}{\partial x}\frac{\partial T_b}{\partial z} - (\nabla^2)T = -\frac{\partial T}{\partial t} + \frac{\partial(\psi,T)}{\partial(x,z)}, \tag{5.1.15}$$

$$-\mathrm{g}_m\frac{\partial\psi}{\partial z} - Pm\nabla^2\Phi = -\frac{\partial\Phi}{\partial t} + \frac{\partial(\psi,\Phi)}{\partial(x,z)}, \tag{5.1.16}$$

where $\mathrm{g}_m = [1 + \delta\chi^2\cos(\Omega t)]$ and $Q = \frac{\mu_m H_0^2 d^2}{\rho_0 \nu \nu_m}$ is the Chandrasekhar number. Since, we assume small variations of time, therefore re-scaling it as $\tau = \chi^2 t$. Now, to study the stationary mode of convection of the system, we write the nonlinear Eqs.(5.1.14)-(5.1.16) in the matrix form as given bellow

$$\begin{bmatrix} -\nabla^4 & Ra\frac{\partial}{\partial x} & -QPm\ \mathbf{g}_m\frac{\partial\nabla^2}{\partial z} \\ -\frac{\partial}{\partial x} & -\nabla^2 & 0 \\ -\mathbf{g}_m\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 \end{bmatrix} \begin{bmatrix} \psi \\ T \\ \Phi \end{bmatrix} = \begin{bmatrix} \frac{1}{Pr}\left(\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} - \frac{\partial\nabla^2\psi}{\partial t}\right) - QPm\frac{\partial(\Phi,\nabla^2\Phi)}{\partial(x,z)} \\ -\frac{\partial T}{\partial t} + \frac{\partial(\psi,T)}{\partial(x,z)} \\ -\frac{\partial\Phi}{\partial t} + \frac{\partial(\psi,\Phi)}{\partial(x,z)} \end{bmatrix} \tag{5.1.17}$$

The considered stress free and isothermal boundary conditions to solve the Eq.(5.1.17) are

$$\psi = D^2\psi = 0 \text{ and } \Phi = D\Phi = 0 \text{ at } z = 0,\ z = 1, \tag{5.1.18}$$

where $D = \frac{\partial}{\partial z}$.

5.1.3 Finite amplitude equation and heat transport

Using an asymptotic expansion given in Eq.(2.3.1) for physical quantities (Ra, ψ, T, Φ) the above system Eq.(5.1.17) is solved for every order of χ.

At the lowest order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}(\nabla^2) \\ -\frac{\partial}{\partial x} & -\nabla^2 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ T_1 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.1.19}$$

The solutions of the lowest order system subject to the boundary conditions Eq.(5.1.18) is

$$\begin{aligned} \psi_1 &= \mathbb{A}(\tau)\sin(a_c x)\sin(\pi z), && (5.1.20)\\ T_1 &= -\frac{a_c}{\delta^2}\mathbb{A}(\tau)\cos(a_c x)\sin(\pi z), && (5.1.21)\\ \Phi_1 &= \frac{\pi}{Pm\delta^2}\mathbb{A}(\tau)\sin(a_c x)\cos(\pi z), && (5.1.22) \end{aligned}$$

where $\delta^2 = a_c^2 + \pi^2$. The critical value of the Rayleigh number for the onset of magneto-convection in the absence of temperature modulation is:

$$R_0 = \frac{\delta^2(\delta^4 + Q\pi^2)}{a_c^2}.$$

If $Q = 0$, we obtained the classical results of Rayleigh−Bénard convection obtained by Chandrasekhar (1961).

At the second order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}(\nabla^2) \\ -\frac{\partial}{\partial x} & -\nabla^2 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_2 \\ T_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \\ R_{23} \end{bmatrix} \qquad (5.1.23)$$

$$\begin{aligned} R_{21} &= 0, && (5.1.24)\\ R_{22} &= \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}, && (5.1.25)\\ R_{22} &= \frac{\partial\psi_1}{\partial x}\frac{\partial \Phi_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial \Phi_1}{\partial x}. && (5.1.26) \end{aligned}$$

The second order solutions subjected to the boundary conditions Eq.(5.1.18) is obtained as follows:

$$\begin{aligned} \psi_2 &= 0 && (5.1.27)\\ T_2 &= -\frac{a_c^2}{8\pi\delta^2}\mathbb{A}^2(\tau)\sin(2\pi z), && (5.1.28)\\ \Phi_2 &= -\frac{\pi^2}{8a_c Pm^2\delta^2}\mathbb{A}^2(\tau)\sin(2a_c x). && (5.1.29) \end{aligned}$$

The horizontally averaged Nusselt number, $\mathrm{Nu}(\tau)$, for the stationary mode of convection is given by using Eq.(2.3.12) as

$$\mathrm{Nu}(\tau) = 1 + \frac{a_c^2}{4\delta^2}\mathbb{A}^2(\tau). \qquad (5.1.30)$$

Here, we notice that $\delta\cos(\Omega\tau)$ is effective at second order and affects the above Nusselt number through factor $\mathbb{A}(\tau)$ as shown later.

At the third order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -QPm\frac{\partial}{\partial z}(\nabla^2) \\ -\frac{\partial}{\partial x} & -\nabla^2 & 0 \\ -\frac{\partial}{\partial z} & 0 & -Pm\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \\ R_{33} \end{bmatrix} \tag{5.1.31}$$

where

$$R_{31} = \frac{-1}{Pr}\frac{\partial\nabla^2\psi_1}{\partial\tau} + QPm\delta\cos(\Omega\tau)\frac{\partial\nabla^2\Phi_1}{\partial z} - R_2\frac{\partial T_1}{\partial x} - QPm\left(\frac{\partial\Phi_1}{\partial z}\frac{\partial\nabla^2\Phi_2}{\partial x} - \frac{\partial\Phi_2}{\partial x}\frac{\partial\nabla^2\Phi_1}{\partial z}\right), \tag{5.1.32}$$

$$R_{32} = -\frac{\partial T_1}{\partial\tau} + \frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}, \tag{5.1.33}$$

$$R_{33} = -\frac{\partial\Phi_1}{\partial\tau} - \frac{\partial\psi_1}{\partial z}\frac{\partial\Phi_2}{\partial x} + \delta\cos(\Omega\tau)\frac{\partial\psi_1}{\partial z}. \tag{5.1.34}$$

Substituting ψ_1, T_1 and T_2 into Eqs.(5.1.32)-(5.1.34), we can obtain expressions for R_{31}, R_{32} and R_{33} easily. Now by applying the solvability condition for the existence of third order solution, we get the Ginzburg−Landau equation for stationary convection with time-periodic coefficients in the form:

$$A_1\mathbb{A}'(\tau) = A_2\mathbb{A}(\tau) - A_3\mathbb{A}(\tau)^3 \tag{5.1.35}$$

where

$A_1 = \frac{\delta^2}{Pr} + \frac{R_0 a_c^2}{\delta^4} - \frac{Q\pi^2}{Pm\delta^2}$, $A_2 = \left[\frac{R_2 a_c^2}{\delta^2} - Q\pi^2\delta\cos(\Omega t)\right]$ $A_3 = \frac{Q\pi^4 a_c^2}{2Pm^2\delta^2} + \frac{R_0 a_c^4}{8\delta^4} - \frac{Q\pi^4}{4Pm^2\delta^4}$.

The Ginzburg-Landau equations given in Eq.(5.1.35) is Bernoulli equation and obtaining its analytical solution is not an easy task, due to its non-autonomous nature. So it has been solved numerically using the in-built function NDSolve of Mathematica, subjected to the initial condition $A(0) = b_0$, where b_0 is the chosen initial amplitude of convection. In our calculations we may use $R_2 = R_0$, to keep the parameters to the minimum.

5.1.4 Analytical solution for Unmodulated case

In the case of unmodulated fluid layer, the above Ginzburg−Landau equation can be written as

$$A_1\mathbb{A}_u'(\tau) = A_2\mathbb{A}_u(\tau) - A_3\mathbb{A}_u(\tau)^3, \tag{5.1.36}$$

where $\mathbb{A}_u(\tau)$ is an amplitude of convection for unmodulated case and A_1, A_3 have the same expression as given in the Eq.(5.1.35) and $A_2 = \frac{R_2 a_c^2}{\delta^2}$. The solution of Eq.(5.1.36) is given by

$$\mathbb{A}_u(\tau) \;=\; \frac{1}{\sqrt{\left(\frac{A_3}{2A_2} + C_1 \text{Exp}\left[\frac{-2A_2}{A_1}\right]\right)}}, \tag{5.1.37}$$

where C_1 is a parameter, it can be calculated for given suitable initial condition. The horizontal averaged Nusselt number in this case is obtained from Eq.(5.1.30) by using the value of $\mathbb{A}_u(\tau)$ in the place of $\mathbb{A}(\tau)$.

5.1.5 Results and discussion

In this section, we study the Rayleigh−Bénard magneto-convection under time-periodic magnetic field. A weakly nonlinear stability analysis has been performed to investigate the effect of magnetic modulation on heat transport. The effect of magnetic modulation on the Rayleigh-Bénard system has been assumed to be of order $O(\chi^2)$. This means we consider only small amplitude of magnetic field modulation. Such an assumption will help us in obtaining the amplitude equation of magneto-convection in simple and elegant manner and is much easier to obtain than in the case of the Lorenz model. The physical variables which appear in our analysis are Pr, Pm, Q, δ and Ω. The effect of various parameters has been observed keeping while fixing the others parameters. We fix the parameter values as $Pr = 1.0, Pm = 1.6, Q = 20, \delta = 0.3$ and $\Omega = 2.0$.

From the figure 5.1a, we observe that the effect of Prandtl number, which is ratio of kinematic viscosity and thermal diffusivity, is to enhance the heat transport for lower values of time τ. Similar effect is also observed for higher values of time τ. It is clear that when Pr increases, then either kinematic viscosity increases or thermal diffusivity decreases, which means in both the cases heat transfer increases. We take small values of Pr to include the time-derivative term as a coefficient in momentum Eq.(5.1.14). Though the critical value of Rayleigh number is independent of Prandtl number, the heat transport is affected due to the time derivative $\frac{1}{Pr}\frac{\partial}{\partial\tau}$. The reason for considering Pr around 1 to retain the time derivative in momentum equation. Aniss et al. (2001) considered

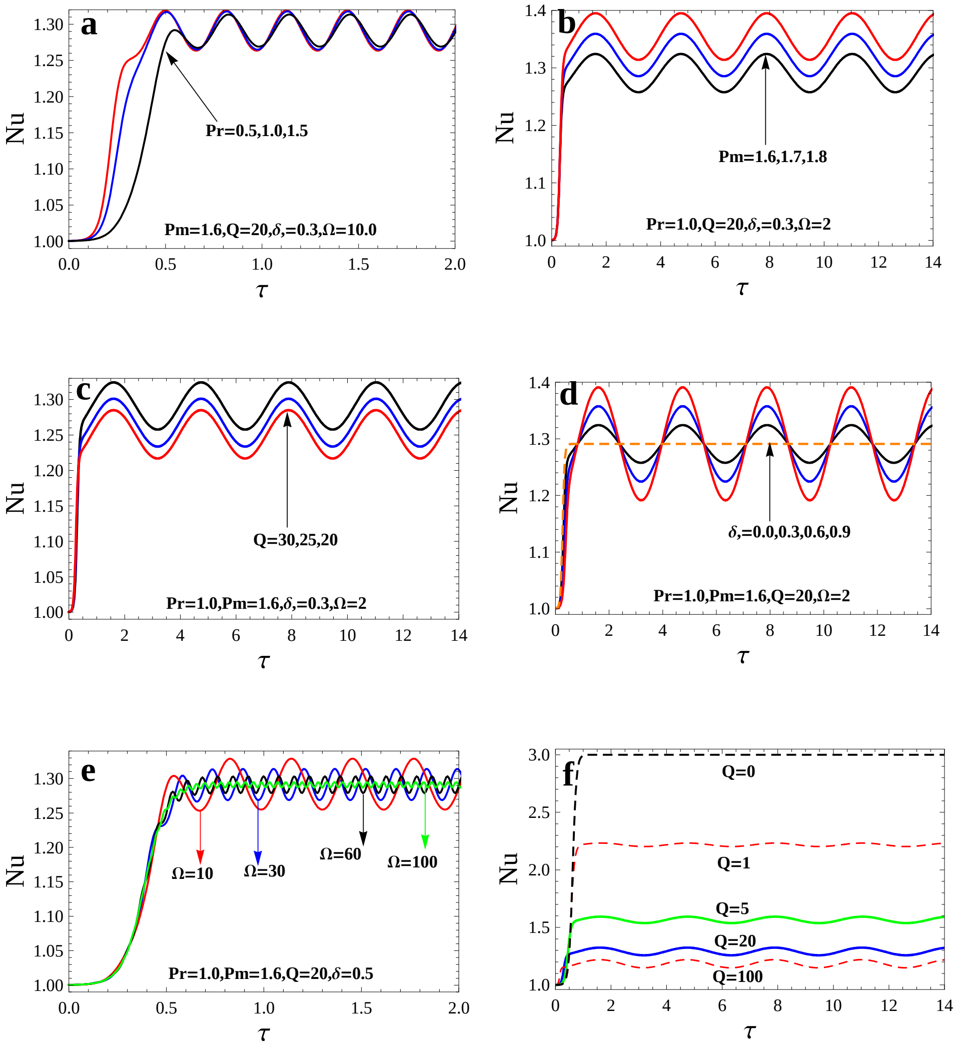


Figure 5.1: Nu versus τ for different values of system parameters

Pr=7 in the case of linear theory of same problem in which one can not see the effect of time-derivative of stream functions.

Further, from the figure 5.1b, we find that the effect of magnetic Prandtl number Pm which is the ratio of viscous diffusion rate to the magnetic diffusion rate is to increase the heat transfer. When Pm increases either viscous diffusion rate may increase or magnetic diffusion rate may decrease in both cases heat transfer increases. The Chandrasekhar number is to represent ratio of the Lorentz force to the viscous force. The Lorentz force is the combination of electric and magnetic force on a point charge due to electromagnetic fields. In figure 5.1c, we depict the effect of Chandrasekher number Q on Nu for fixed values of other parameters. We know that on increasing Q the value of critical Rayleigh number R_0 also increases, so it delays the onset of convection that means stabilize the system, and hence decreases the heat transport. It is clear from the figure 5.1c, that as Chandrasekher number Q increases, the amplitude of modulation is also increasing, so the effect of Q is also reflecting on the amplitude of modulation. From the Eq.(5.1.35), it is clear that the Chandrasekher number is multiple of amplitude of modulation, which means that an amplitude of magnetic modulation is affected by Chandrasekher number.

From figure 5.1d, we find that an increment in the amplitude of magnetic modulation increases the value of Nu, hence advances the convection, and so the heat transport. We also obtained analytically the amplitude of convection for unmodulated case, given in Eq.(5.1.37) and depicted the result in figure 5.1d. The nature of graph is found to be non oscillatory.

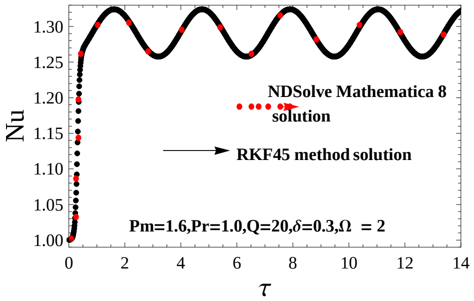


Figure 5.2: Comparison of the solution

In figure 5.1e, we have shown the effect of frequency of magnetic modulation on heat transport. We find that for small values of Ω the heat transport is more. As Ω increases, we observe that the amplitude of modulation decreases, and so the magnitude of $\mathrm{Nu}(\tau)$. As the frequency increases from 10 to 100, the magnitude of $\mathrm{Nu}(\tau)$ decreases considerable, and the effect of modulation on heat transport diminishes. On further increasing the value of Ω, the effect of modulation on magneto-convection disappears altogether. Hence the effect of Ω is to stabilize the system. From the figure 5.1f, it is observed that for small values of Q, the system is having destabilizing effect, while at large values of Q it has stabilizing effect. In order to verify the accuracy of our results, we have compared the results in figure 5.2, by solving the amplitude Eq.(5.1.35) using both RKF45 method and NDsolve Mathematica 8, which conforms our results with good approximation to RKF45 method.

In figures 5.3 and 5.4, the stream lines and the corresponding isotherms are depicted for magnetic field modulation, respectively at $\tau = 0.0, 0.1, 0.3, 0.5, 1.0$ and 2.0 for $Pr = 1.0$, $Q = 20.0, \delta = 0.3$ and $\Omega = 2.0$. From the figures, we found that initially when time is small, the magnitude of streamlines is also small figures 5.3a-b, and isotherms are straight that is the system is in conduction state figures 5.4a-b. However, as time increases, the magnitude of streamlines increases and the isotherms loses their evenness. This shows that the convection is taking place in the system. Convection becomes faster on further increasing the value of time τ. However, the system achieves the study state beyond $\tau = 1.0$ as there is no change in the streamlines and isotherms figures 5.3-5.4d-f.

5.1.6 Conclusions

1. The effect of time-periodic magnetic modulation on Rayleigh−Bénard convection has been studied by employing the non-linear stability analysis, and using the Ginzburg−Landau model.

2. The effect of increasing Prandtl number Pr is to advance the convection, hence increase the heat transfer.

3. The effect of increasing magnetic Prandtl number Pm is to advance the convection,

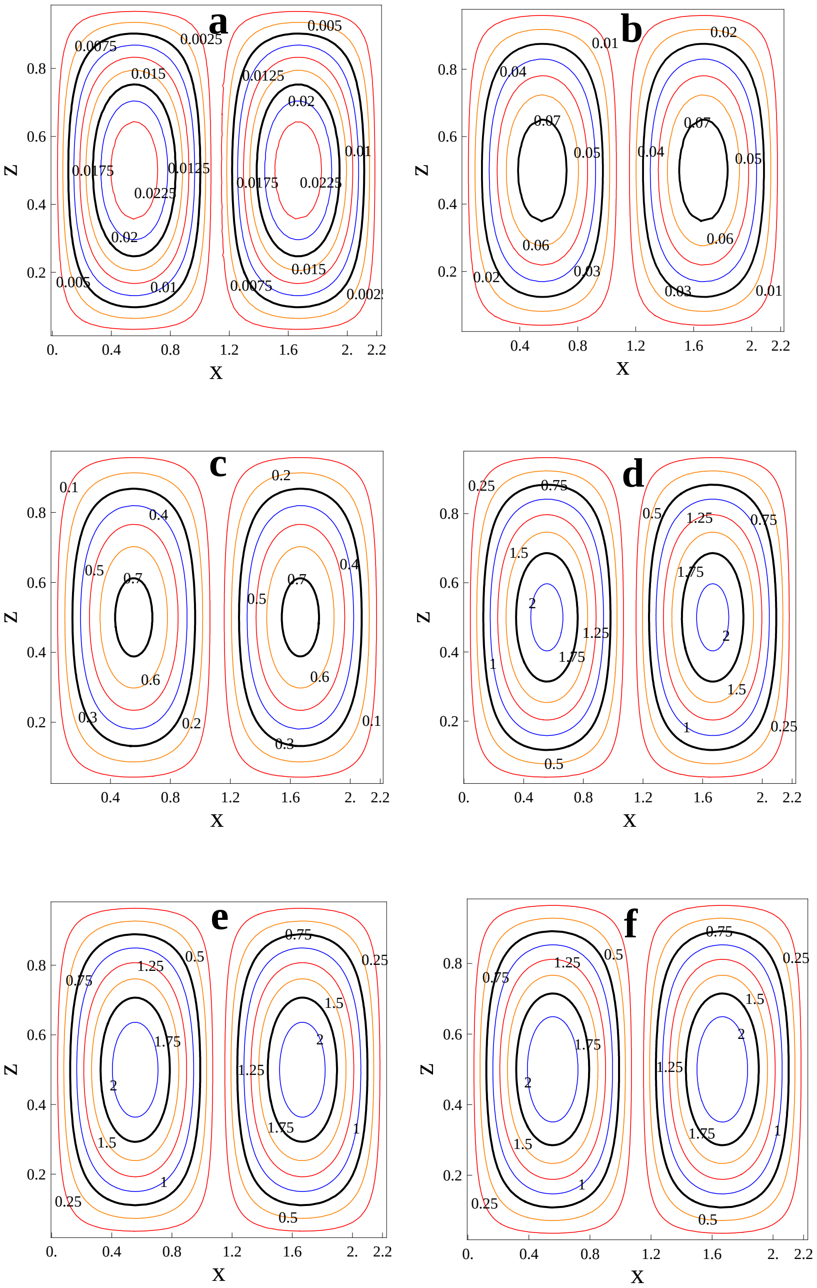


Figure 5.3: Streamlines at (a) τ= 0.0 (b) τ=0.1(c)τ=0.3(d)τ= 0.5(e)τ= 1.0(f)τ= 2.0

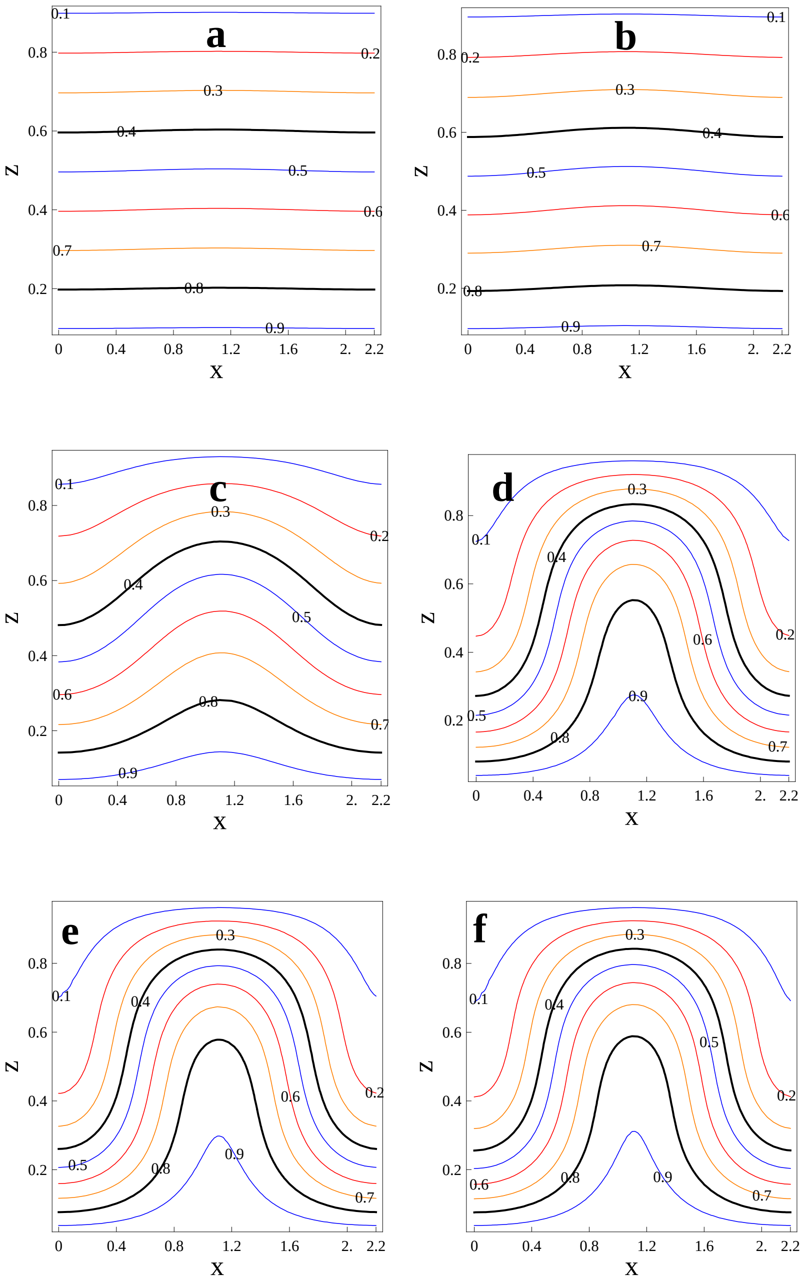


Figure 5.4: Isotherms at (a) τ= 0.0 (b) τ=0.1(c)τ=0.3(d)τ= 0.5(e)τ= 1.0(f)τ= 2.0

hence increase the heat transfer.

4. The effect of Chandrasekhar number is to delay the onset of convection and hence decrease the heat transfer.
5. The effect of increasing amplitude of modulation δ is to advance the convection hence heat transfer.
6. The effect of modulation starts vanishing at sufficiently large values of modulation frequency Ω.
7. It can be concluded that the effect of magnetic modulation is highly significant and can be used to delay the onset of convection, and hence to decrease the heat transfer.
8. It was observed that, amplitude and frequency of modulation has no effect on unmodulated system but in the case of modulated system shows sinusoidal behavior.

5.2 Effect of rotational speed modulation on the heat transport in a fluid layer with a temperature dependent viscosity and an internal heat source

5.2.1 Introduction

When we study the rotation effect then one more parameter, in form of rotation speed, exists which can affect the stability of the convective flow. Donnelly (1964) was the first who investigated the effect of rotation speed modulation on the onset of instability in fluid flow between two concentric cylinders as the speed is slowly increased beyond the point at which instability sets in. The purpose of their study is to demonstrate that under certain conditions, Couette flow can be stabilized by modulating sinusoidally the rate of rotation of the inner cylinder. It was shown that the enhancement of stability is connected with the viscous wave set up in the annulus by the modulation, and this connection is further explore by experimenting with various widths of the gap between the cylinders, as well as different frequencies and amplitudes of modulation. However, the rotation speed modulation was the originating idea of the temperature modulation (Venezian 1969), as well as, gravity modulation (Gresho and Sani 1970). But, research work in this field is scarce. Amongst the available studies, the study due to Bhattacharjee (1989) is of great importance, in which he studied the effect of rotation speed modulation on Rayleigh−Beńard convection in ordinary fluid layer. He found that the effect of modulation is stabilizing for most of the configurations. In particular, when the convection is induced by the effect of rotation only then the rotation speed modulation serves as analogues to gravity modulation applied to the natural convection. In the porous media analogue is due to Suthar et al. (2009), who investigated the effect of rotation speed modulation on the onset of centrifugal convection in a rotating vertical porous layer distant from the axis of rotation. No nonlinear study available in the literature in which the effect of rotation speed modulation has been considered where one can analyze the heat transfer in the system. Hence a weakly nonlinear study under rotational speed modulation along with internal heating and in a temperature dependent viscosity effects has been discussed in this section.

5.2.2 Problem Formulation

We consider an infinitely extended horizontal viscous-incompressible fluid layer, confined between two parallel planes which are at $z = 0$, lower plane and $z = d$, upper plane. The lower surface is heated and upper surface is cooled to maintain an adverse temperature gradient across the fluid layer. We consider the fluid layer is rotating with variable rotational speed $\vec{\Omega}_r = (0, 0, \Omega_r(t))$, about the z-axis. The effect of rotation is restricted to Coriolis term, thus we neglect the centrifugal force term. The effect of density variation is given by Boussinesq approximation. With these assumptions the basic governing equations are:

$$\begin{aligned}
\nabla \cdot \vec{q} &= 0, && (5.2.1)\\
\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} + 2\left(\vec{\Omega}_r \times \vec{q}\right) &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\vec{g} + \frac{\mu(T)}{\rho_0}\nabla^2 \vec{q}, && (5.2.2)\\
\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T &= \kappa_T \nabla^2 T + Q(T - T_0), && (5.2.3)\\
\rho &= \rho_0[1 - \alpha_T(T - T_0)], && (5.2.4)\\
\mu(T) &= \frac{\mu_0}{1 + \chi^2 \delta_0 (T - T_0)}. && (5.2.5)
\end{aligned}$$

The considered rotational speed which is varying sinusoidally with respect to time is defined as:

$$\vec{\Omega}_r(t) = \Omega_0 \left(1 + \chi^2 \ \delta \ \cos(\Omega t)\right) \hat{k}. \tag{5.2.6}$$

The constants and variables used in the above Eqs.(5.2.1)-(5.2.6) have their usual meanings and are given in the nomenclature. The thermo-rheological relationship given in Eq.(5.2.5) is guided by (Nield,1996). The considered thermal boundary conditions at the plates are:

$$T = T_0 + \Delta T \qquad \text{at} \qquad z = 0 \qquad T = T_0 \qquad \text{at} \qquad z = d. \tag{5.2.7}$$

At the steady state the fluid is at rest $\vec{q_b} = (0,0,0)$ and the heat transfer will be in the form of conduction. The other quantities of the conduction state are:

$$\rho = \rho_b(z), \qquad p = p_b(z) \qquad \text{and} \qquad T = T_b(z). \tag{5.2.8}$$

Substituting the Eq.(5.2.8) into Eqs.(5.2.1)−(5.2.4), we get the following relations which help us to define basic state pressure and temperature

$$\frac{dp_b}{dz} = -\rho_b g, \tag{5.2.9}$$

$$\kappa_T \frac{d^2(T_b - T_0)}{dz^2} + Q(T_b - T_0) = 0, \tag{5.2.10}$$

$$\rho_b = \rho_0 \left[1 - \alpha_T \left(T_b - T_0\right)\right], \tag{5.2.11}$$

where b refers the basic state. The Eq.(5.2.10) is solved for $T_b(z)$ subject to the boundary condition given in Eq.(5.2.7), we get:

$$T_b = T_0 + \Delta T \frac{\sin\sqrt{\frac{Q}{\kappa_T}}(1 - \frac{z}{d})}{\sin\sqrt{\frac{Q}{\kappa_T}}}. \tag{5.2.12}$$

The finite amplitude perturbations on the basic state are superposed in the following form:

$$\vec{q} = \vec{q_b} + \vec{q}^{'}, \quad \rho = \rho_b + \rho^{'}, \quad p = p_b + p^{'}, \quad T = T_b + T^{'}. \tag{5.2.13}$$

Substituting the Eq.(5.2.13) in Eqs.(5.2.1)-(5.2.4), and using the basic state solutions, we get:

$$\nabla.\vec{q}^{'} = 0, \tag{5.2.14}$$

$$\frac{\partial \vec{q}^{'}}{\partial t} + (\vec{q}^{'}.\nabla)\vec{q}^{'} + 2\left(\vec{\Omega}_r \times \vec{q}^{'}\right)\vec{k} = -\frac{1}{\rho_0}\nabla p + \alpha_T \vec{g} T^{'} + \frac{\mu(T)}{\rho_0}\nabla^2 \vec{q}^{'}, \tag{5.2.15}$$

$$\frac{\partial T^{'}}{\partial t} + W^{'}\frac{dT_b}{dz} + (\vec{q}^{'}.\nabla)T^{'} = \kappa_T \nabla^2 T^{'} + R_i T^{'}. \tag{5.2.16}$$

For two dimensional convection, one can introduce stream function ψ as $U^{'} = \frac{\partial \psi}{\partial z}$, $W^{'} = -\frac{\partial \psi}{\partial x}$. Non dimensionalizing the physical variables as;
$(x, y, z) = d(x^*, y^*, z^*)$, $t = \frac{d^2}{\kappa_T}t^*$, $\vec{q} = \frac{\kappa_T}{d}\vec{q}^*$, $\psi = \kappa_T \psi^*$, $T^{'} = \Delta T\, T^*$ and $\vec{\Omega}_r = \frac{\kappa_T}{d^2}\vec{\Omega}_r^*$,

then eliminating the pressure term and finally dropping the asterisk, we obtain the nondimensional governing system as

$$\frac{1}{Pr}\frac{\partial}{\partial t}(\nabla^2\psi)-\frac{1}{Pr}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}=-Ra\frac{\partial T}{\partial x}+\overline{\mu}(T)\nabla^4\psi+\sqrt{Ta}\left(1+\chi^2\ \delta\ \cos(\Omega t)\right)\frac{\partial V}{\partial z}+\frac{\partial\overline{\mu}}{\partial z}\frac{\partial\nabla^2\psi}{\partial z}, \tag{5.2.17}$$

$$-\frac{dT_b}{dz}\frac{\partial\psi}{\partial x}-(\nabla^2+R_i)T=-\frac{\partial T}{\partial t}+\frac{\partial(\psi,T)}{\partial(x,z)}. \tag{5.2.18}$$

Also from the Eq.(5.2.15), we may write the following equation for V:

$$\frac{1}{Pr}\frac{\partial V}{\partial t}-\overline{\mu}(T)\nabla^2 V=-\sqrt{Ta}\left(1+\chi^2\ \delta\ \cos(\Omega t)\right)\frac{\partial\psi}{\partial z}+\frac{1}{Pr}\frac{\partial(\psi,V)}{\partial(x,z)}. \tag{5.2.19}$$

The non-dimensionalized parameters are given in list of symbols. The non-dimensional basic temperature $T_b(z)$ which appears in the Eq.(5.2.18), can be obtained from the Eq.(5.2.12) as

$$\frac{dT_b}{dz}=-\frac{\sqrt{R_i}\cos\sqrt{R_i}(1-z)}{\sin\sqrt{R_i}}. \tag{5.2.20}$$

We assume small variations in time, and re-scaling it as $\tau=\chi^2 t$. To study the stationary convection of the system, we write the non-linear Eqs.(5.2.17)-(5.2.19) in the matrix form as given bellow

$$\begin{bmatrix} \frac{\chi^2}{Pr}\frac{\partial}{\partial\tau}\nabla^2-\overline{\mu}\nabla^4 & Ra\frac{\partial}{\partial x} & -\sqrt{Ta}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \chi^2\frac{\partial}{\partial\tau}-(\nabla^2+R_i) & 0 \\ \sqrt{Ta}\frac{\partial}{\partial z} & 0 & \frac{\chi^2}{Pr}\frac{\partial}{\partial\tau}-\overline{\mu}\nabla^2 \end{bmatrix}\begin{bmatrix}\psi\\ T\\ V\end{bmatrix}$$

$$=\begin{bmatrix} \frac{\partial(\psi,\nabla^2\psi)}{Pr\partial(x,z)}+\sqrt{Ta}\chi^2\delta\cos(\Omega t)\frac{\partial V}{\partial z}+\frac{\partial\overline{\mu}}{\partial z}\frac{\partial\nabla^2\psi}{\partial z} \\ \frac{\partial(\psi,T)}{\partial(x,z)} \\ \frac{1}{Pr}\frac{\partial(\psi,V)}{\partial(x,z)}-\sqrt{Ta}\chi^2\delta\cos(\Omega t)\frac{\partial\psi}{\partial z} \end{bmatrix} \tag{5.2.21}$$

To solve the system of Eqs.(5.2.21), we consider stress free and isothermal boundary conditions as given bellow

$$\psi=\frac{\partial^2\psi}{\partial z^2}=\frac{\partial V}{\partial z}=T=0\ \text{ at } z=0\ \text{ and }\ z=1. \tag{5.2.22}$$

5.2.3 Finite amplitude equation and heat transport

We introduce the following asymptotic expansions in Eqs.(5.2.21) as we have used in in Eqs.(2.3.1) and solve the above system Eq.(5.2.21) for different orders of χ.

At the lowest order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -\sqrt{Ta}\frac{\partial}{\partial z} \\ -\frac{dT_b}{dz}\frac{\partial}{\partial x} & -(\nabla^2+R_i) & 0 \\ \sqrt{Ta}\frac{\partial}{\partial z} & 0 & -\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ T_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.2.23}$$

The solution of the lowest order system subject to the boundary conditions given in Eq.(5.2.22), is

$$\psi_1 = \mathbb{A}(\tau)\sin(a_c x)\sin(\pi z), \tag{5.2.24}$$

$$T_1 = -\frac{4\pi^2 a_c}{\beta_1^2(4\pi^2-R_i)}\mathbb{A}(\tau)\cos(a_c x)\sin(\pi z), \tag{5.2.25}$$

$$V_1 = -\frac{\pi\sqrt{Ta}}{\beta_1^2}A(\tau)\sin(a_c x)\cos(\pi z), \tag{5.2.26}$$

where $\beta^2 = a_c^2+\pi^2$ and $\beta_1^2 = \beta^2 - R_i$. The critical value of the Rayleigh number for the onset of stationary convection is calculated numerically, and the expression is given by

$$R_0 = \frac{\beta_1^2(4\pi^2-R_i)(\beta^6+\pi^2 Ta)}{4\beta^2\pi^2 a_c^2}. \tag{5.2.27}$$

For the system without rotation ($Ta = 0$) and internal heating ($R_i = 0$), we get:

$$R_0 = \frac{\beta^6}{a_c^2},$$

$$a_c = \frac{\pi}{\sqrt{2}},$$

which are classical results of Chandrasekhar (1961).

At the second order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -\sqrt{Ta}\frac{\partial}{\partial z} \\ -\frac{dT_b}{dz}\frac{\partial}{\partial x} & -(\nabla^2+R_i) & 0 \\ \sqrt{Ta}\frac{\partial}{\partial z} & 0 & -\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_2 \\ T_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \\ R_{23} \end{bmatrix} \tag{5.2.28}$$

where

$$\begin{aligned} R_{21} &= 0 && (5.2.29) \\ R_{22} &= \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}, && (5.2.30) \\ R_{23} &= \frac{\partial \psi_1}{\partial x}\frac{\partial V_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial V_1}{\partial x}. && (5.2.31) \end{aligned}$$

The second order solutions subjected to the boundary conditions given in Eq.(5.2.22), is obtained as:

$$\begin{aligned} \psi_2 &= 0, && (5.2.32) \\ T_2 &= -\frac{2\pi^3 a_c^2}{\beta_1^2(4\pi^2 - R_i)^2}\mathbb{A}^2(\tau)\sin(2\pi z), && (5.2.33) \\ V_2 &= \frac{\pi^2\sqrt{Ta}}{8a_c Pr\beta^2}\mathbb{A}^2(\tau)\sin(2a_c x). && (5.2.34) \end{aligned}$$

The horizontally averaged Nusselt number Nu(τ), for the stationary mode of convection is determined by substituting the expression of T_2 and $\frac{dT_b}{dz}$ in Eq.(2.3.12) and simplifying, we get the Nusselt number as:

$$\mathrm{Nu}(\tau) = 1 + \frac{4\pi^2 a_c^2 \sin\sqrt{R_i}}{\beta_1^2(4\pi^2 - R_i)^2\sqrt{R_i}\cos\sqrt{R_i}}A^2(\tau). \qquad (5.2.35)$$

At the third order, we have

$$\begin{bmatrix} -\nabla^4 & R_0\frac{\partial}{\partial x} & -\sqrt{Ta}\frac{\partial}{\partial z} \\ -\frac{dT_b}{dz}\frac{\partial}{\partial x} & -(\nabla^2 + R_i) & 0 \\ \sqrt{Ta}\frac{\partial}{\partial z} & 0 & -\nabla^2 \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \\ R_{33} \end{bmatrix} \qquad (5.2.36)$$

where

$$\begin{aligned} R_{31} &= -\frac{1}{Pr}\frac{\partial}{\partial \tau}(\nabla^2\psi_1) + \sqrt{Ta}\ \delta\ \cos(\Omega t)\frac{\partial V_1}{\partial z} - R_2\frac{\partial T_1}{\partial x} + V_T T_b \nabla^4(\psi_1) \\ &- 2R_0 V_T T_b \frac{\partial T_1}{\partial x} + 2V_T\sqrt{Ta}T_b\frac{\partial V_1}{\partial z} - V_T\frac{\partial T_b}{\partial z}\frac{\partial \nabla^2\psi}{\partial z} && (5.2.37) \\ R_{32} &= -\frac{\partial T_1}{\partial \tau} + \frac{\partial \psi_1}{\partial x}\frac{dT_2}{dz}, && (5.2.38) \\ R_{33} &= -\frac{1}{Pr}\frac{\partial V_1}{\partial \tau} - \frac{1}{Pr}\frac{\partial \psi_1}{\partial z}\frac{\partial V_2}{\partial x} - \sqrt{Ta}(V_T T_b + \delta\cos(\Omega t))\frac{\partial \psi_1}{\partial z}. && (5.2.39) \end{aligned}$$

Substituting the first and second order solutions into Eqs.(5.2.37)−(5.2.39), we can easily simplify the expressions R_{31}, R_{32} and R_{33}. Now, by applying the solvability condition for the existence of third order solutions, we get the non autonomous Ginzburg−Landau equation for stationary mode of convection, with time-periodic coefficients in the form:

$$A_1\mathbb{A}'(\tau) - A_2\mathbb{A}(\tau) + A_3\mathbb{A}(\tau)^3 = 0, \tag{5.2.40}$$

where

$$A_1 = \left[\frac{\beta^2}{Pr} + \frac{4R_0\pi^2 a_c^2}{\beta_1^4(4\pi^2 - R_i)} - \frac{Ta\pi^2}{Pr\beta^4}\right],\ A_2 = \left[\frac{4R_2\pi^2 a_c^2}{\beta_1^2(4\pi^2 - R_i)} - \frac{2Ta\pi^2}{\beta^2}\,\delta\ \cos(\Omega t) - H_1\right],$$

$$H_1 = \frac{4\pi^2 V_T(\cos\sqrt{R_i} - 1)}{(4\pi^2 - R_i)\sin\sqrt{R_i}}\left[\frac{8\pi^2 a_c^2 R_0}{\beta_1^2(4\pi^2 - R_i)\sqrt{R_i}} - \frac{\beta^4}{\sqrt{R_i}} - \frac{3\pi^2 Ta(\cos\sqrt{R_i} - 1)}{\beta^2\sqrt{R_i}} - \frac{\beta^2 R_i}{2}\right],$$

$$A_3 = \left[\frac{2R_0\pi^4 a_c^4}{\beta_1^4(4\pi^2 - R_i)^2} + \frac{Ta\pi^4}{8Pr^2\beta^4}\right].$$

The Ginzburg−Landau equation given in Eq.(5.2.40) is Bernoulli equation, and obtaining its analytical solution is difficult, due to its non-autonomous nature. Therefore, it has been solved numerically using the in-built function NDSolve of Mathematica 8, subjected to the initial condition $\mathbb{A}(0) = b_0$, where b_0 is the chosen initial amplitude of convection. In our calculations we may use $R_2 = R_0$, to keep the parameters to the minimum.

5.2.4 Analytical solution for unmodulated case

In the case of unmodulated fluid layer, the above Ginzburg−Landau equation Eq.(5.2.40) can be written as

$$A_1\mathbb{A}_u'(\tau) - A_2\mathbb{A}_u(\tau) + A_3\mathbb{A}_u(\tau)^3 = 0, \tag{5.2.41}$$

where $\mathbb{A}_u(\tau)$ is an amplitude of convection for unmodulated case. Coefficients A_1 and A_3 have the same expressions as given in the Eq.(5.2.40), while $A_2 = \left[\frac{4R_0\pi^2 a_c^2}{\beta_1^2(4\pi^2 - R_i)} - H_1\right]$, The solution of the Eq.(5.2.41) is given by

$$\mathbb{A}_u(\tau) \;=\; \frac{1}{\sqrt{\left(\frac{A_3}{2A_2} + C_1\ e^{\left(\frac{-2A_2}{A_1}\right)}\right)}}, \tag{5.2.42}$$

where C_1 is a parameter, it can be calculated for given suitable initial condition. The horizontal averaged Nusselt number in this case is obtained from the Eq.(5.2.35) by using the value of $\mathbb{A}_u(\tau)$ in the place of $\mathbb{A}(\tau)$.

5.2.5 Results and discussion

The problem addresses a nonlinear realm of Rayleigh−Bénard convection with a variable viscous liquid and internal heating effects under rotational speed modulation. Here, we have presented a weakly nonlinear stability analysis to investigate the effect of rotational speed modulation, internal heating and thermo-rheological behaviour on heat transport. The modulation of Rayleigh−Bénard system has been assumed to be of order $O(\chi^2)$, which means we consider only small amplitude of rotation speed modulation. At third order only Solvability condition exist to define an amplitude equation Eq.(5.2.40). The modulated term $(\delta\cos(\Omega\tau))$ is effective at $O(\chi^2)$ and affects the system. This assumption will help us in obtaining the amplitude equation in a simple manner, and much easier than the Lorenz model. Before writing the discussion of the results, we mention some features of the following aspects of the problem:

1. The importance and need for nonlinear stability analysis.
2. The relation of the problem to real life application.
3. The selection of all dimensionless parameters utilized in computations.
4. Consideration of numerical values for different parameters.

It is imperative to make a nonlinear study of the problem if one wants to obtain heat transport, which can not be obtained using the linear stability theory. External regulation of convection is important in the study of thermal instability in a fluid layer, therefore, in this section, we have considered rotational speed modulation and internal heating for either enhancing or inhibiting convective heat transport as is required by a real life applications in science and engineering, such as, food process industry, chemical process industry, rotating turbo machinery etc. The effect of rotational speed modulation on heat transport has been depicted in figures (5.5-5.7). The parameters that arise in the problem are R_i, Pr, V_T, Ta, δ and Ω, and these parameters influence the convective heat transport. The first two parameters are related to the fluid layer and the next three parameters concern the external mechanism of controlling convection. The fluid layer

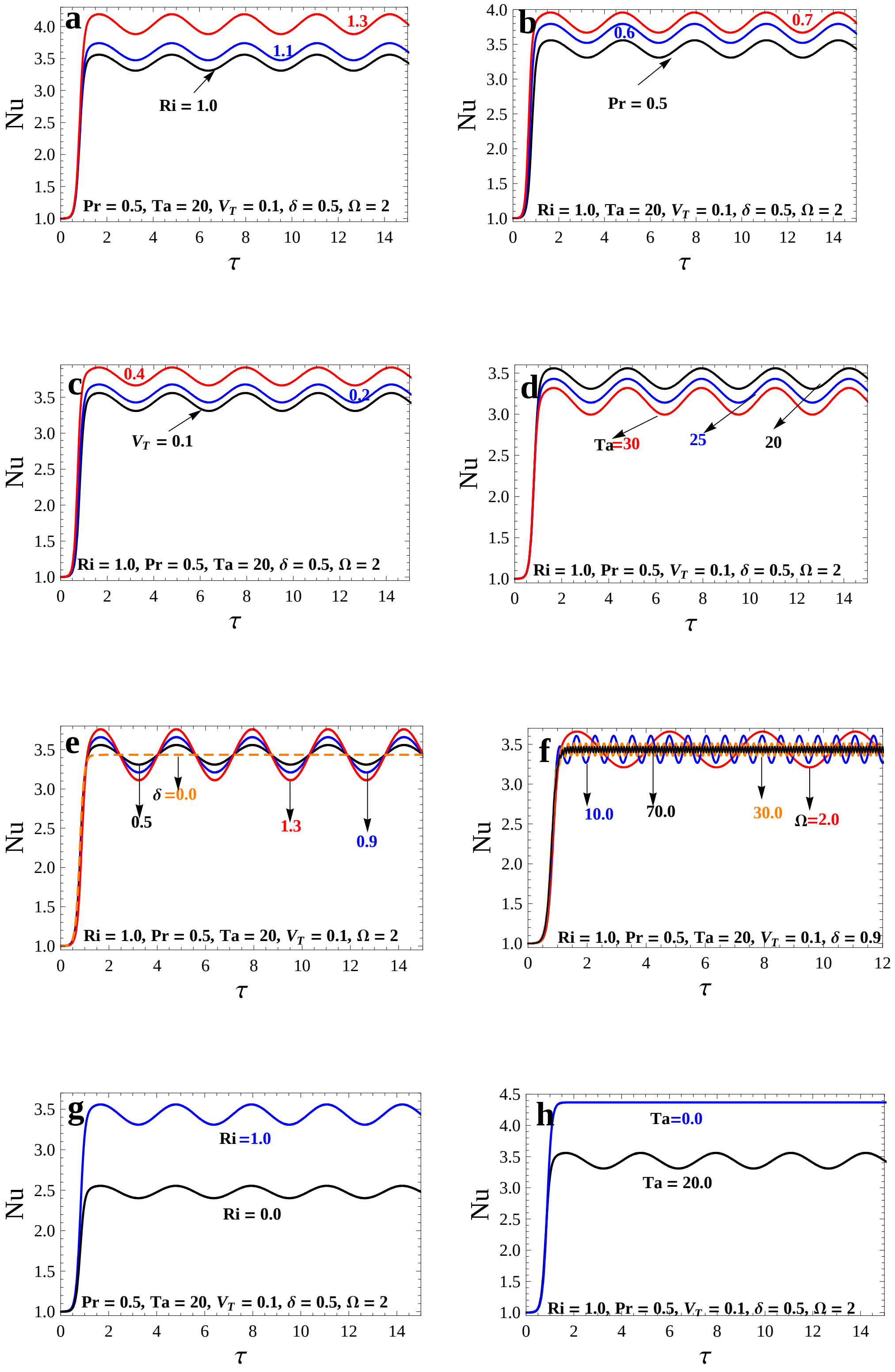


Figure 5.5: Nu versus τ for different values of system parameters

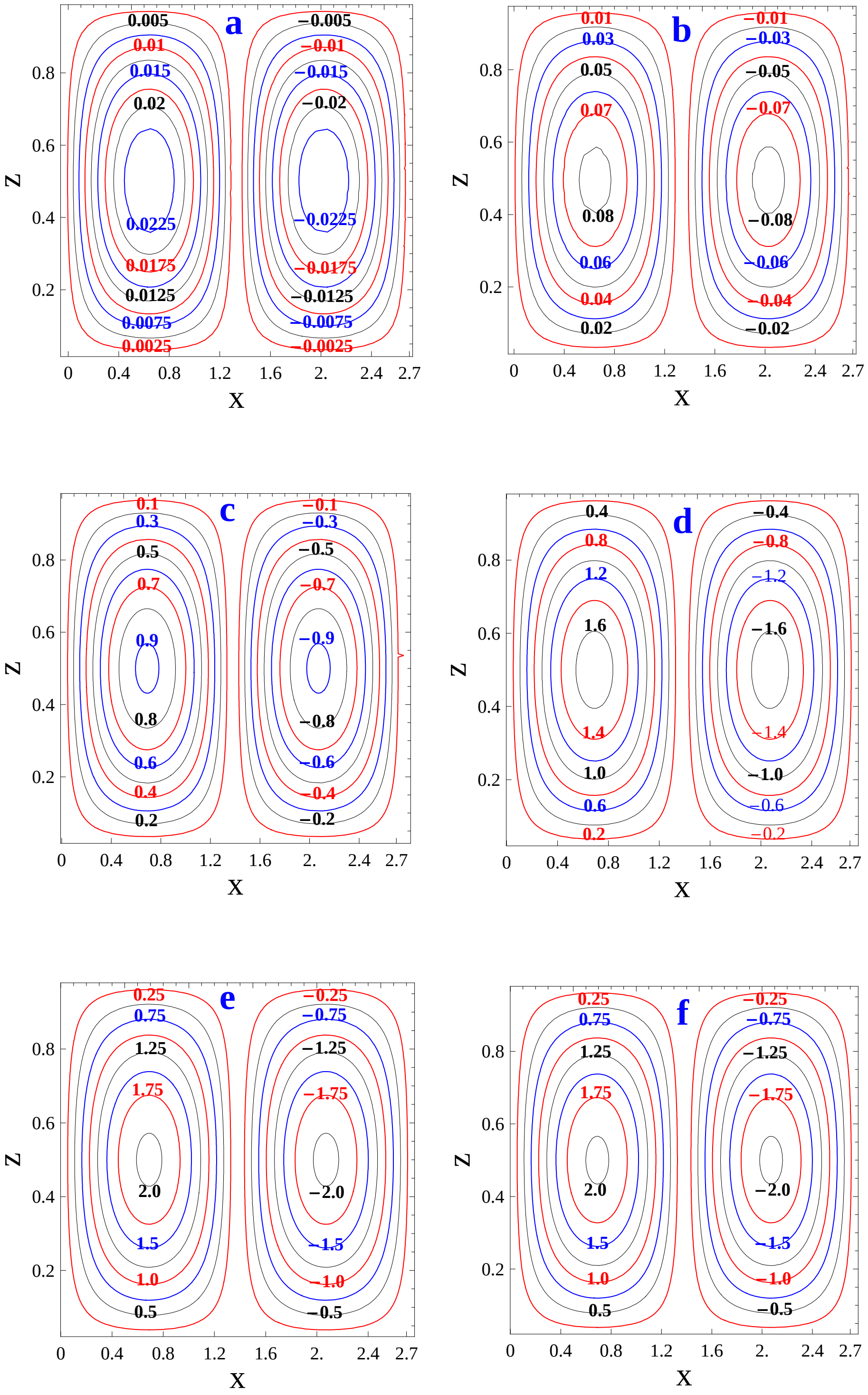


Figure 5.6: Streamlines at (a) τ= 0.0 (b) τ=0.2(c)τ=0.6(d)τ= 0.9(e)τ= 1.5(f)τ= 2.0

is not considered to be highly viscous, therefore only moderate values of Pr are taken for calculations. Because of small amplitude modulation, the values of δ are considered around 0.5. Further, the rotational speed modulation assumed to be of low frequency, as at low range of frequencies, the effect of frequencies on onset of convection as well as on heat transport is maximum. The values of R_i are considered to be moderate so that it will not affect the effect of modulation on the system by dominating it otherwise. The thermo-rheological parameter V_T has taken to be small values.

In figure 5.5, we have plotted the Nusselt number $\mathrm{Nu}(\tau)$ with respect to time τ for the case of rotational speed modulation. From the figures, we find that for lower values time τ, the value of $\mathrm{Nu}(\tau)$ does not alter and remains almost constant, then it increases on increasing τ, and finally becomes oscillatory on further increasing τ. It is clear from the figures that $\mathrm{Nu}(\tau)$ starts with one showing the conduction state. From figure 5.5a, we observe that the effect of internal heating on thermal instability is destabilizing, as heat transport increases on increasing R_i. The heat transport is more at higher values of R_i. This confirms the results obtained most recently by Bhadauria et al. (2013a,b,c). Further, we have

$$\mathrm{Nu}_{R_i=1.0} < \mathrm{Nu}_{R_i=1.1} < \mathrm{Nu}_{R_i=1.3}$$

In figure 5.5b, we find that the Nusselt number $\mathrm{Nu}(\tau)$ increases upon increasing Prandtl number Pr for fixed values of other parameters. This may happen due to the dominating role of thermal diffusivity κ_T over kinematic viscosity ν. As Prandtl number Pr increases, then for no change in kinematic viscosity, probably there is a large decrement in thermal diffusivity, and this makes sudden increase in the temperature gradient. So convection takes place early, and there is an enhancement in heat transfer. Thus, the effect of an increment in Prandtl number Pr is to advance the convection. Similar effect can be seen in the case of thermo-rheological parameter V_T in figure 5.5c. We have

$$\mathrm{Nu}_{Pr=0.5} < \mathrm{Nu}_{Pr=0.6} < \mathrm{Nu}_{Pr=0.7}$$

$$\mathrm{Nu}_{V_T=0.1} < \mathrm{Nu}_{V_T=0.2} < \mathrm{Nu}_{V_T=0.4}$$

From the figure 5.5d, we depict the effect of Taylor number Ta on $\mathrm{Nu}(\tau)$ for fixed

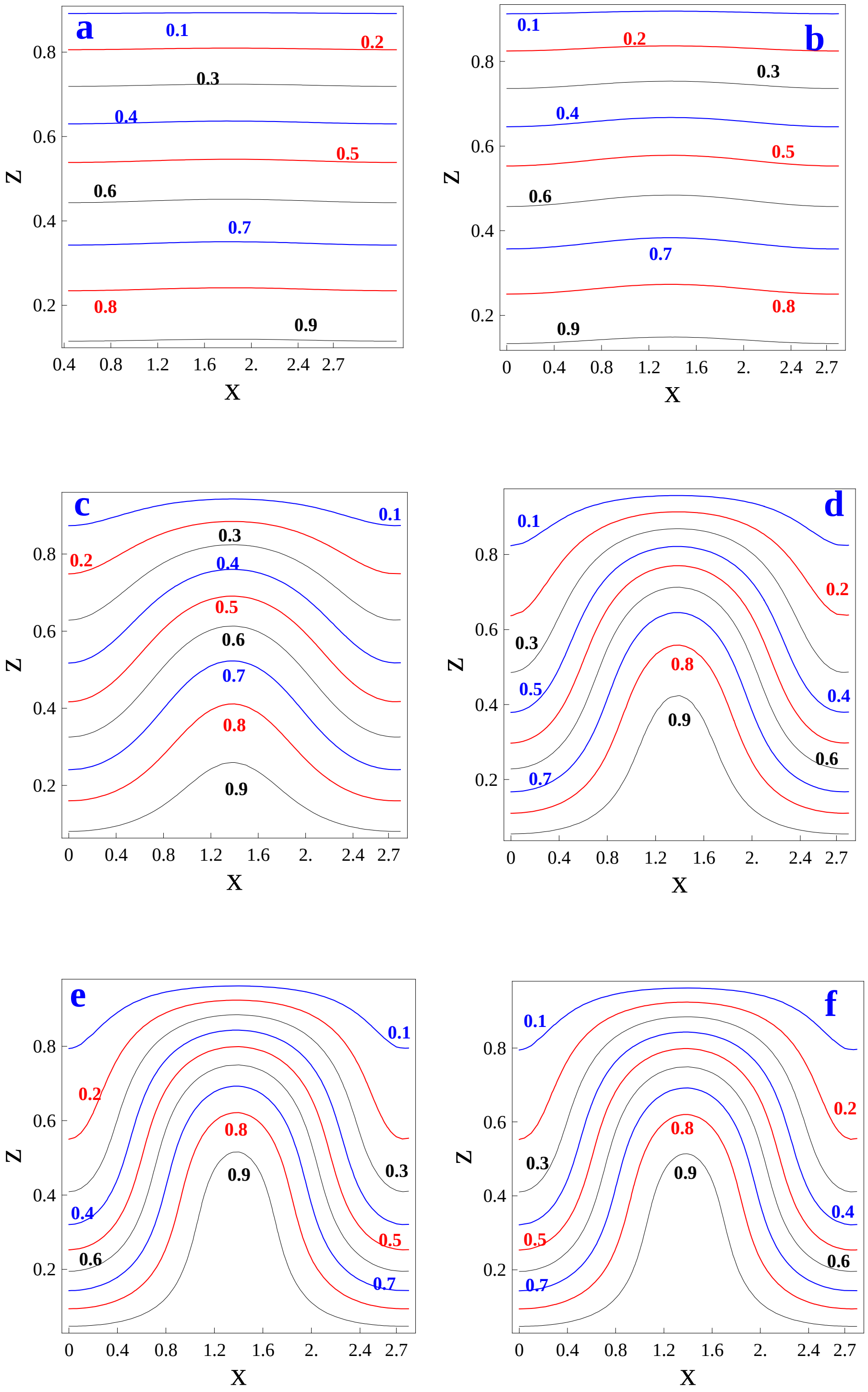


Figure 5.7: Isotherms at (a) τ= 0.0 (b) τ=0.2(c)τ=0.6(d)τ= 0.9(e)τ= 1.5(f)τ= 2.0

values of other parameters. Upon increasing Ta increases the value of critical Rayleigh number R_0, and it delays the onset of convection, hence heat transport decreases. It is clear from the figure 5.5d that, on increasing Taylor number Ta, the amplitude of modulation also increases, so the effect of Ta is reflecting on amplitude of modulation as well. From the Eq.(5.2.40), it is clear that the rotation is multiple of amplitude of rotation speed modulation. Which means that the amplitude rotation speed modulation is dependant of rotation. Generally, if there is no rotation ($Ta = 0$), it is meaningless to talk about rotation speed modulation. Further, for no rotation $Ta = 0$, the effect of frequency of modulation diminishes, so the effect of frequency of modulation can be seen when rotation is not there. Since we are studying rotation speed modulation, it is necessary to consider Ta as non-zero values, otherwise modulation effect disappears.

$$\mathrm{Nu}_{Ta=40} < \mathrm{Nu}_{Ta=30} < \mathrm{Nu}_{Ta=25} < \mathrm{Nu}_{Ta=20}$$

In figure 5.5e, we depict the effect of amplitude of modulation for moderate values of Ta and for the fixed values of other parameters. The Nusselt number Nu increases upon increasing the value of δ, hence advancing the convection. Which means that increasing upon δ increases the heat transfer. In case of unmodulated $\delta = 0$ system shows no influence on heat transport for larger values of time τ. Similar results can be obtained analytically for an unmodulated system the amplitude of convection given by Eq.(5.2.42).

$$\mathrm{Nu}_{\delta=0.0} < \mathrm{Nu}_{\delta=0.5} < \mathrm{Nu}_{\delta=0.9} < \mathrm{Nu}_{\delta=1.3}$$

From the figure 5.5f, we see the effect of frequency of modulation, for small values of Ω heat transport is more. Upon increasing the value of ω decreases the magnitude of $\mathrm{Nu}(\tau)$, and shortens the wavelength of oscillations. As the frequency increases from 2 to 100, the magnitude of $\mathrm{Nu}(\tau)$ decreases, and the effect of modulation on heat transport diminishes. On further increasing the value of Ω, the effect of modulation on thermal instability disappears altogether. Hence the effect of Ω is to stabilize the system. We have

$$\mathrm{Nu}_{\Omega=70} < \mathrm{Nu}_{\Omega=30} < \mathrm{Nu}_{\Omega=10} < \mathrm{Nu}_{\Omega=2}$$

The present result of internal heating has been compared with the results of non-internal heating in figure 5.5g. We observe that, in case of internal heating of the system, the heat transport in the system is more than that in the absence of internal heating, thus internal heating advances the onset of convection as well as heat transport.

$$\mathrm{Nu}_{R_i=0.0} < \mathrm{Nu}_{R_i=1.0}$$

The figure 5.5h show that, the heat transport is more when there is no rotation $Ta = 0$ (which means no modulation) than in the presence of rotation and modulation. Hence rotation strongly stabilize the system.

$$\mathrm{Nu}_{Ta\neq 0} < \mathrm{Nu}_{Ta=0,\Omega\neq 0}$$

In figures 5.6-5.7, the stream lines and the corresponding isotherms are depicted for rotation speed modulation, respectively at $\tau = 0.0, 0.2, 0.6, 0.9, 1.5$ and 2.0 for $Pr = 0.5.$, $Ta = 20.0, \delta = 0.5$ and $\Omega = 2.0$. From the figures, we found that initially when time is small, the magnitude of stream function is also small figures 5.6a-b, and isotherms are straight that is the system is in conduction state figures 5.7a-b. However, as time increases, the magnitude of stream function increases and the isotherms loses their evenness. This shows that the convection is taking place in the system. Convection becomes faster on further increasing the value of time τ. However, the system achieves the study state beyond $\tau = 1.0$ as there is no change in the stream function and isotherms figures 5.6-5.7d-f.

5.2.6 Conclusions

The combined effect of internal heating and rotation speed modulation on Rayleigh−Bénard convection in a rotating horizontal temperature dependent viscous fluid layer has been studied by employing nonlinear stability analysis, and using Ginzburg−Landau model. The results have been obtained in terms of the Nusselt number, and the effect of various parameters have been obtained and depicted graphically. We have the following observations

1. The effect of rotation speed modulation on Rayleigh−Bénard convection in a rotating horizontal fluid layer has been studied by employing non-linear stability analysis using Ginzburg−Landau model.

2. The effect of increasing internal Rayligh number R_i is to increase the value of Nu, thus advancing the convection, hence heat transfer.

3. The effect of increasing Prandtl number Pr is to advance the onset of convection, hence heat transfer.

4. The effect of increasing V_T is to advance the onset of convection, hence heat transfer.

5. Overall, it can be concluded that the effect of rotation speed modulation is highly significant and can be used to delay the onset of convection.

6. In modulated case, the effect of Taylor number is to delay the onset of convection, and hence heat transfer.

7. It was also observed that the effect of modulation starts vanishing at sufficiently large values of modulation frequency.

8. As time τ increases, the magnitude of streamlines increases, and isotherms loses their evenness, showing that convection is taking place. At $\tau = 1.0$ the system achieves steady state.

9. The thermo-rheological model of Nield (1996), gives physically acceptable results, namely, the destabilizing effect of variable viscosity on Bénard-Darcy convection, and thereby an enhanced heat transport.

The results of this work can be summarized as follows:

1. $\mathrm{Nu}_{R_i=1.0} < \mathrm{Nu}_{R_i=1.1} < \mathrm{Nu}_{R_i=1.3}$

2. $\mathrm{Nu}_{Pr=0.5} < \mathrm{Nu}_{Pr=0.6} < \mathrm{Nu}_{Pr=0.7}$

3. $\mathrm{Nu}_{V_T=0.1} < \mathrm{Nu}_{V_T=0.2} < \mathrm{Nu}_{V_T=0.4}$

4. $\mathrm{Nu}_{Ta=30} < \mathrm{Nu}_{Ta=25} < \mathrm{Nu}_{Ta=20}$
5. $\mathrm{Nu}_{\delta=0.0} < \mathrm{Nu}_{\delta=0.5} < \mathrm{Nu}_{\delta=0.9} < \mathrm{Nu}_{\delta=1.3}$
6. $\mathrm{Nu}_{\Omega=70} < \mathrm{Nu}_{\Omega=30} < \mathrm{Nu}_{\Omega=10} < \mathrm{Nu}_{\Omega=2}$
7. $\mathrm{Nu}_{Ta\neq0} < \mathrm{Nu}_{Ta=0}$
8. $\mathrm{Nu}_{R_i=0.0} < \mathrm{Nu}_{R_i=1.0}$.

Chapter 6

Chaotic convection in a porous medium under temperature modulation

6.1 Introduction

The concept of chaos was first introduced by Poincaré (1890, 1899), who investigated orbits in celestial mechanics, and realized that the dynamical system generated by the three body problem is quite sensitive to the initial conditions exhibiting chaotic behavior. Since the introduction of the chaotic attractors by Lorenz (1963) to study atmospheric convection, many chaotic systems have been introduced, such as the R$\ddot{o}$ssler (1976), the Chen (1999), and the L$\ddot{u}$ (2002) systems. Because of their potential applications in engineering, the study of chaotic systems has attracted the interest of many researchers. Adopting Adomian decomposition method (Adomian, 1988, 1994), Vadasz and Olek (1998) demonstrated that this method is useful in recovering the dynamics of the system as applied to centrifugally driven convection in a rotating porous medium. Vadasz and Olek (1999a) showed that for low Prandtl number convection in porous media, transition from steady to chaos is sudden and occurs via a subcritical Hopf bifurcation producing a solitary limit cycle (Vadasz 1999a), which may be associated with a homoclinic explosion. The transition from steady to chaotic convection in porous media can be recovered from a truncated Galerkin approximation which yields a system alike Lorenz system (Lorenz, 1963; Sparrow, 1982). Wang et al. (1992) and Yuen and Bau (1996) presented a summary of the sequence of transitions in the Lorenz system leading to chaos. In particular, they identified a well known experimental and numerical phenomenon of Hysteresis in which when R is increasing gradually by approaching the critical value from below, the transition to chaos occurs at ($R = R_c$), while repeating the same procedure but approaching R_c from above the transition from chaos to the stationary solution occurs at a value of ($R < R_c$).

Feki (2003) proposed a new simple adaptive controller to control chaotic systems. The constructed controller may be used for chaos control as well as for chaotic system synchronization, the main advantage of this construction is the linear structure of the controller. Yau and Chen (2007) found that the Lorenz chaos could be stabilized, even in the existence of system external distraction. Through the use of an Oldroydian-type constitutive relation, Sheu et al. (2008) have shown that stress relaxation tends to accelerate the onset of chaos. Vadasz (2010) showed that a weak non-linear solution to the problem can produce an accurate analytical expression for the transition point as long as the condition of validity and consequent accuracy of the latter solution is fulfilled. Narayana et al. (2013a) established that the applied magnetic field has a stabilizing effect, hence reducing heat and mass transport. They also derived Lorenz system to analyze a transition between steady to chaotic convection. Narayana et al. (2013b) established a binary convection in viscoelastic fluids and found the Dufour parameter enhancing both heat and mass transfer, whereas the Soret parameter reduces heat transfer and increases mass transfer. It is also found that the route to chaos in the binary viscoelastic fluid is similar to that of the single-component viscoelastic fluid due to the consideration in the study of dilute concentration of the second component.

The above last two paragraphs demonstrated the earlier work on chaotic convection with different configurations and models to control chaos. Recently Johnathan et al (2014) have investigated the effect of vertical vibrations over chaotic system. Their results show that periodic solutions and chaotic solutions alternate as the value of the scaled Rayleigh number changes, when forced vibrations are present. Due to this, the present chapter is to study chaotic motion of the system in the presence of the time-periodic heating at the boundaries.

6.2 Mathematical Formulation

We consider an infinitely extended horizontal porous layer, confined between two impermeable boundaries at $z = 0$ and $z = d$, which are heated from below and cooled from above in a time periodic manner. A Cartesian frame of reference is chosen with origin in the lower boundary and the $z-$ axis vertically upward. The gravity force is acting in vertically downward direction. It is assumed that the mechanical properties and thermal properties in x and y-directions are same. Further Darcy law and the Oberbeck-Boussinesq approximation are taken to be applicable. The equations which describe this system under above considerations are given by

$$\nabla \cdot \vec{q} = 0, \tag{6.2.1}$$

$$\frac{\rho_0}{\delta_1}\frac{\partial \vec{q}}{\partial t} = -\nabla p + \rho g - \frac{\mu}{K}\vec{q}, \tag{6.2.2}$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \kappa_T \nabla^2 T, \tag{6.2.3}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{6.2.4}$$

where the physical variables have their usual meanings as given in Nomenclature. The externally imposed thermal boundary conditions are considered as Venezian (1969):

$$\begin{aligned} T &= T_0 + \frac{\Delta T}{2}[1 + \delta\cos(\Omega t)], && \text{at } z = 0 \\ &= T_0 - \frac{\Delta T}{2}[1 - \delta\cos(\Omega t + \theta)], && \text{at } z = d \end{aligned} \tag{6.2.5}$$

where ΔT is the temperature difference across the porous medium, δ, Ω are amplitude and frequency of temperature modulation, and θ is the phase angle.

6.3 Basic state

The basic state is assumed to be quiescent and the quantities in this state are given

$$\vec{q_b} = 0, p = p_b(z,t), \quad T = T_b(z,t), \; \rho = \rho_b(z,t). \tag{6.3.1}$$

Substituting the Eq.(6.3.1) in Eqs.(6.2.1)-(6.2.4), we get the following relations which helps us to define basic state pressure and temperature:

$$\frac{\partial p_b}{\partial z} = -\rho_b \vec{g}, \tag{6.3.2}$$

$$\frac{\partial T_b}{\partial t} = \kappa_T \frac{\partial^2 T_b}{\partial z^2}, \tag{6.3.3}$$

$$\rho_b = \rho_0\left[1 - \alpha_T\left(T_b - T_0\right)\right]. \tag{6.3.4}$$

The solution of equation (6.3.3), subjected to the boundary conditions (6.2.5), is given by

$$T_b(z,t) = T_s(z) + \delta \mathrm{Re}[T_1(z,t)], \tag{6.3.5}$$

where

$$T_s(z) = T_0 + \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right), \tag{6.3.6}$$

$$T_1(z,t) = \left(\{a_1(\zeta)e^{\frac{\zeta z}{d}} + a_1(-\zeta)e^{\frac{-\zeta z}{d}}\}e^{-i\Omega t}\right), \tag{6.3.7}$$

and $a_1(\zeta) = \frac{\Delta T}{2}\frac{(e^{-i\theta}-e^{-\zeta})}{(e^{\zeta}-e^{-\zeta})}$ and $\zeta^2 = \frac{-i\Omega d^2}{\kappa_T}$. Here $T_s(z)$ is the steady part, while $T_1(z,t)$ is the oscillatory part of the basic state temperature field $T_b(z,t)$. The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q} = \vec{q}_b + q^{'},\ \rho = \rho_b + \rho^{'},\ p = p_b + p^{'},\ T = T_b + T^{'}. \tag{6.3.8}$$

Considering two dimensional fluid flow and substituting the Eq.(6.3.8), the basic state temperature field in Eqs.(6.2.1)-(6.2.4) then after simplifying obtain the following dimensionless system of coupled equations:

$$\frac{1}{Pr_D}\frac{\partial}{\partial t}(\nabla^2\psi) = -\nabla^2\psi - Ra_D\frac{\partial T}{\partial x} \tag{6.3.9}$$

$$-\frac{\partial T_b}{\partial z}\frac{\partial \psi}{\partial x} + \left(\frac{\partial}{\partial t} - \nabla^2\right)T = \frac{\partial(\psi, T)}{\partial(x,z)}. \tag{6.3.10}$$

where $Ra_D = \frac{\alpha_T g \Delta T K d}{\nu \kappa_T}$ is thermal Darcy−Rayleigh number $\nu = \frac{\mu}{\rho_0}$ is kinematic viscosity. The basic state solution which appears in Eq.(6.3.10), influences the stability problem through the factor $\frac{\partial T_b}{\partial z}$, which is given by

$$\frac{\partial T_b}{\partial z} = -1 + \delta\left(f_2(z,t)\right), \tag{6.3.11}$$

where

$$f_2(z,t) = \mathrm{Re}\left(f(z)e^{(-i\Omega t)}\right), \tag{6.3.12}$$

$$f(z) = \left(A(\zeta)e^{\zeta z} + A(-\zeta)e^{-\zeta z}\right),\ A(\zeta) = \frac{\zeta}{2}\frac{(e^{-i\theta} - e^{-\zeta})}{(e^{\zeta} - e^{-\zeta})}\ \&\ \zeta = (1-i)\sqrt{\frac{\Omega}{2}}. \tag{6.3.13}$$

To obtain the solution to the nonlinear coupled system of partial differential Eqs.(6.3.9-6.3.10), we represent the stream function and temperature in the the following Fourier expressions

$$\psi = A_{11}(\tau)\sin(\frac{\pi x}{L})\sin(\pi z) \tag{6.3.14}$$

$$T = B_{11}(\tau)\cos(\frac{\pi x}{L})\sin(\pi z) + B_{02}(\tau)\sin(2\pi z) \tag{6.3.15}$$

where the amplitudes $A_{11}(\tau), B_{11}(\tau), B_{02}(\tau)$ are functions of time. Substituting equations (6.3.14) - (6.3.15) in equations (6.3.9)-(6.3.10) taking the orthogonality condition with the eigenfunctions associated with the Eqs.(6.3.14-6.3.15) and integrating them over the domain, i.e., yields a set of three ordinary differential equations for the time evolution of the amplitudes, in the form:

$$\frac{dA_{11}(\tau)}{d\tau} = -\frac{Pr_D\beta}{\pi^2}\{A_{11} + \frac{Ra_D}{\pi\theta_1}B_{11}\}, \tag{6.3.16}$$

$$\frac{dB_{11}(\tau)}{d\tau} = -[A_{11}\frac{1}{\pi\theta_1}I_1 + \frac{1}{\theta_1}A_{11}B_{02} + B_{11}], \tag{6.3.17}$$

$$\frac{dB_{02}(\tau)}{d\tau} = \frac{1}{2\theta_1}A_{11}B_{11} - 4\beta B_{02}. \tag{6.3.18}$$

where the time was re-scaled and the following notations were introduced $\tau = \frac{(L^2+1)\pi^2}{L^2}t$, $\theta_1 = \frac{(L^2+1)}{L}$, $\beta = \frac{L}{\theta_1}$, $I_1 = \int_0^1 \sin^2(\pi z) f_2 dz$ and $\Omega = \frac{L^2}{(L^2+1)\pi^2}\Omega$. It is convenient to introduce the following further notation: $R = \frac{Ra_D}{\pi^2\theta_1^2}$, $Pr = \frac{Pr_D\beta}{\pi^2}$, $X = -\frac{A_{11}}{2\theta_1\sqrt{2\beta(R-1)}}$, $Y = \frac{\pi R B_{11}}{2\sqrt{2\beta(R-1)}}$, and $Z = -\frac{\pi R B_{02}}{(R-1)}$, to provide the following set of scaled equations which are equivalent to Eqs. (6.3.16-6.3.18)

$$\frac{dX}{d\tau} = Pr(Y - X), \tag{6.3.19}$$

$$\frac{dY}{d\tau} = R(1 + \delta I_1)X - Y - (R-1)XZ, \tag{6.3.20}$$

$$\frac{dZ}{d\tau} = 4\beta(XY - Z). \tag{6.3.21}$$

The above Eqs.Eqs.(6.3.19-6.3.21) are equivalent to Lorenz equations (Lorenz, 1963; Sparrow, 1982) although with different coefficients. The demonstration of this equivalence was provided by Vadasz and Olek (1998, 1999a). Since, the Lorenz equations are extensively analysed and solved for parameter values corresponding to convection in pure fluids (i.e. nonporous domains), Johnathan et al. (2014) studied them for porous layer case. According to Vadasz et al. (2000), the fixed points of re-scaled system for unmodulated case are $(X_1, Y_1, Z_1) = (0, 0, 0)$ corresponding to the motionless solution, and $(X_{2,3}, Y_{2,3}, Z_{2,3}) = (\pm 1, \pm 1, 1)$, corresponding to the convection solution. The critical value of R, where motionless solution loses stability and the convection solution takes over, is obtained as $R_{c1} = 1$, which corresponds to $Ra = 4\pi^2$. This pair of equilibrium points is stable only if $R < \frac{Pr(Pr+4\gamma+3)}{(Pr-4\gamma-1)}$, beyond this condition the other, periodic, quasi-periodic, or chaotic solutions take over at $R > \frac{Pr(Pr+4\gamma+3)}{(Pr-4\gamma-1)}$.

6.4 Results and Discussion

The above system of Eqs.(6.3.19-6.3.21) is solved using NDSolve mathematica 6. The initial conditions used for all numerical solutions are $\tau = 0 : X = Y = Z = 0.9$, and the parameter values for all numerical solutions are $Pr = 10$ and $\gamma = 0.5$, while the values of R, Ω and δ are varied to observe the impact of temperature modulation on the system. Although numerous cases of results can be found, however, we restricted ourself to investigated the effects only of R, Ω and δ. In the unmodulated case the critical value is $R_c = 21.4286$, where transition from steady convection to chaos takes place. For modulated system, the results are presented first at $R_c = 28.5962$ just before the transition to chaos occurs, and then just after the transition at $R_c = 29.4621$. The value of $\gamma = 0.5$ used in all computations is consistent with the critical wave number at the marginal stability in porous media convection.

The results in terms of projections of trajectories data points on the YX; ZX and ZY planes and evolution of trajectories of X, Y and Z over a time domain are presented in the figures. In fact the detailed study of the behaviour of the transition from steady to chaos due to the variation of R near its critical value has been made by Vadasz (1999a,b), Sheu et al (2008), Vadasz and Olek (2000a,b) and Johnathan et al. (2014), therefore in this article we mainly concentrate on the effects of frequency and amplitude of modulation on transition from steady to chaos solutions of the system. In Johnathan et al.(2014) one can notice that a strong forcing vibration amplitude δ does not affect the convection, however, to see the specific frequencies effect on solution δ was kept silent. Then the question arises, what happens when δ varies for lower and higher values of Ω, in the case of temperature modulation. So to see the specific effect of δ and Ω on the system, the following results have been obtained.

We present our results in the case when both plates are in out of phase modulation. In order to see the effect of an amplitude δ and frequency Ω of modulation, we fix other

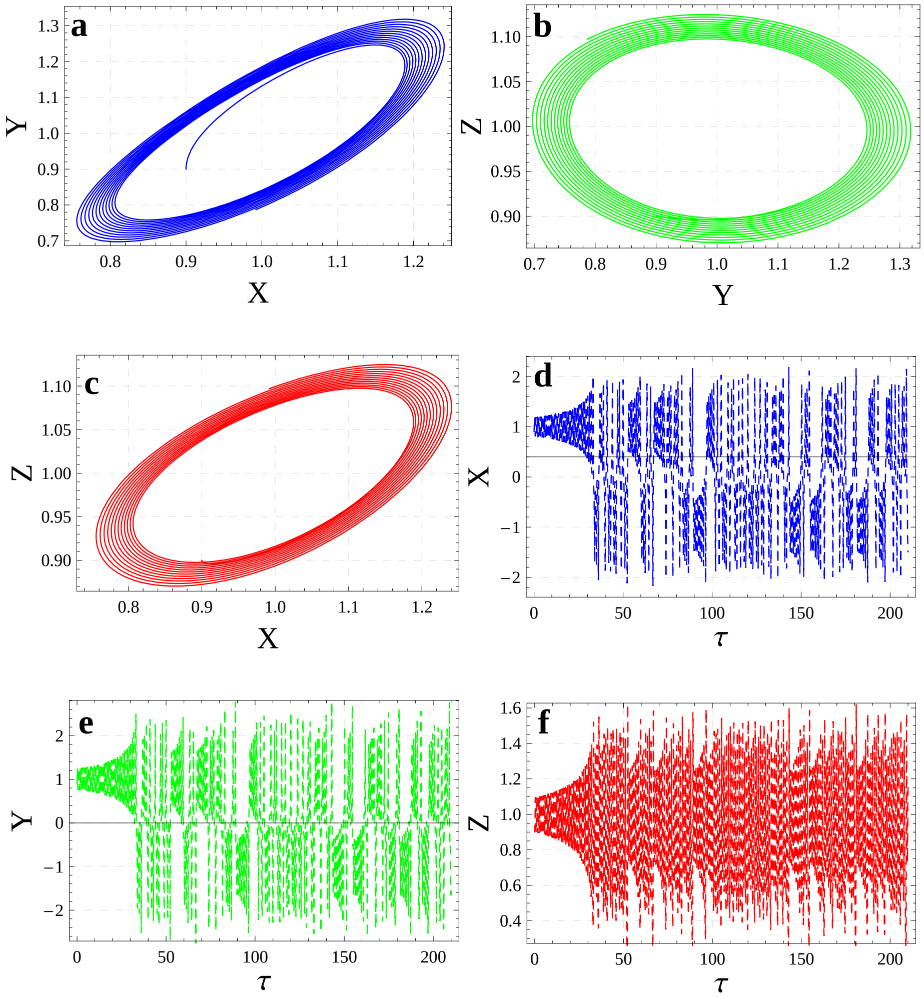


Figure 6.1: Projections and evolution of trajectories over the planes $X - Y, Y - Z, Z - X$ and a time domain of solutions for $\theta = \pi : \Omega = 10 : \delta = 0 : Pr = 10 : R = 22 : \beta = 0.5$

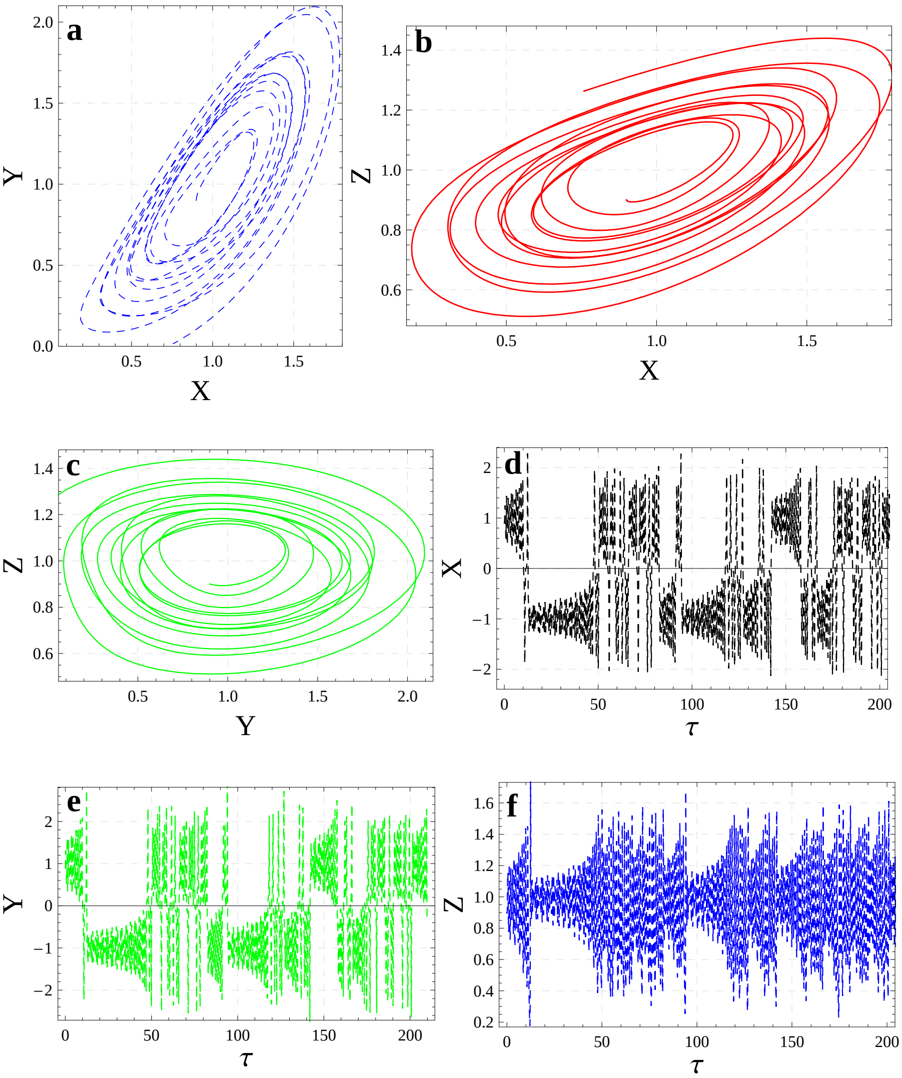


Figure 6.2: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 10 : \delta = 0.1 : Pr = 10 : R = 22 : \beta = 0.5$

system parameters and also the critical Darcy number $R < R_c$ in order to reduce the effect of R on the solution of the system. First let see the effect of δ, in figures 6.1a-c for $\delta = 0$ the solutions trajectories spiral towards one of the the fixed point resulting in steady convection. In figures 6.1d-f, it is observed for time up to $\tau = 35$ the steady behaviour can be seen further regime the solutions exhibits chaotic behaviour. For $\delta = 0.1$, in figures 6.2a-c, the solutions trajectories spiral and more vibrant towards the fixed point resulting in steady convection. In figures 6.2d-f, it is observed the solutions exhibits chaotic behaviour in time domain. For $\delta = 0.3$, in figures 6.3a-c, the solutions exhibit suddenly a typical chaotic behaviour. In figures 6.3d-f it is observed the solutions exhibits chaotic behaviour in time domain.

Secondly we see the effect of Ω while keeping $\delta = 0.3$ fixed, in figures 6.4a-c, for $\Omega = 0.01$ and figures 6.5a-c for $\Omega = 4$ the solutions show steady behaviour towards one of the the fixed point. Further values such as $\Omega = 10$ can be seen as chaotic behaviour for $\Omega = 10$ in figures 6.3a-c. In figures 6.4d-f it shows that the envelope of the function $X - Z.\tau$ converges for $\Omega = 0.01$ diverges for $\Omega = 4$ given in figures 6.5d-f. Further values of $\Omega = 10$ the same results obtained as in figure 6.3. In figure 6.6a, we can observe that for a Rayleigh number slightly above the loss of stability of the motionless solution (R=1.1) the trajectory moves to the steady convection on a straight line (except for a slight initial overshooting). In figures 6.6b-c, the trajectories are not straight lines but steady regime due to the presence of modulation. In figures 6.7 & 6.8a-c, show a transition behaviour from steady regime to chaos and in the time domain the solution converging the value for large values of time.

Figures. 6.8a-c show that at $R = 29.4621$, chaotic solution exists in the system, while from figures 6.8d-f, it is evident that the envelope of the function (X, Y, Z) does not converge nor diverge (where the data points are connected), thus demonstrates a periodic behaviour of the solution. The above results have been presented when both plates are in out of phase modulation. Now in figure 6.9, a comparison is made for three types of modulation and compare them. It is observed that for the case when the modulation is in phase the chaotic trajectories are small around negative attractor and normal with

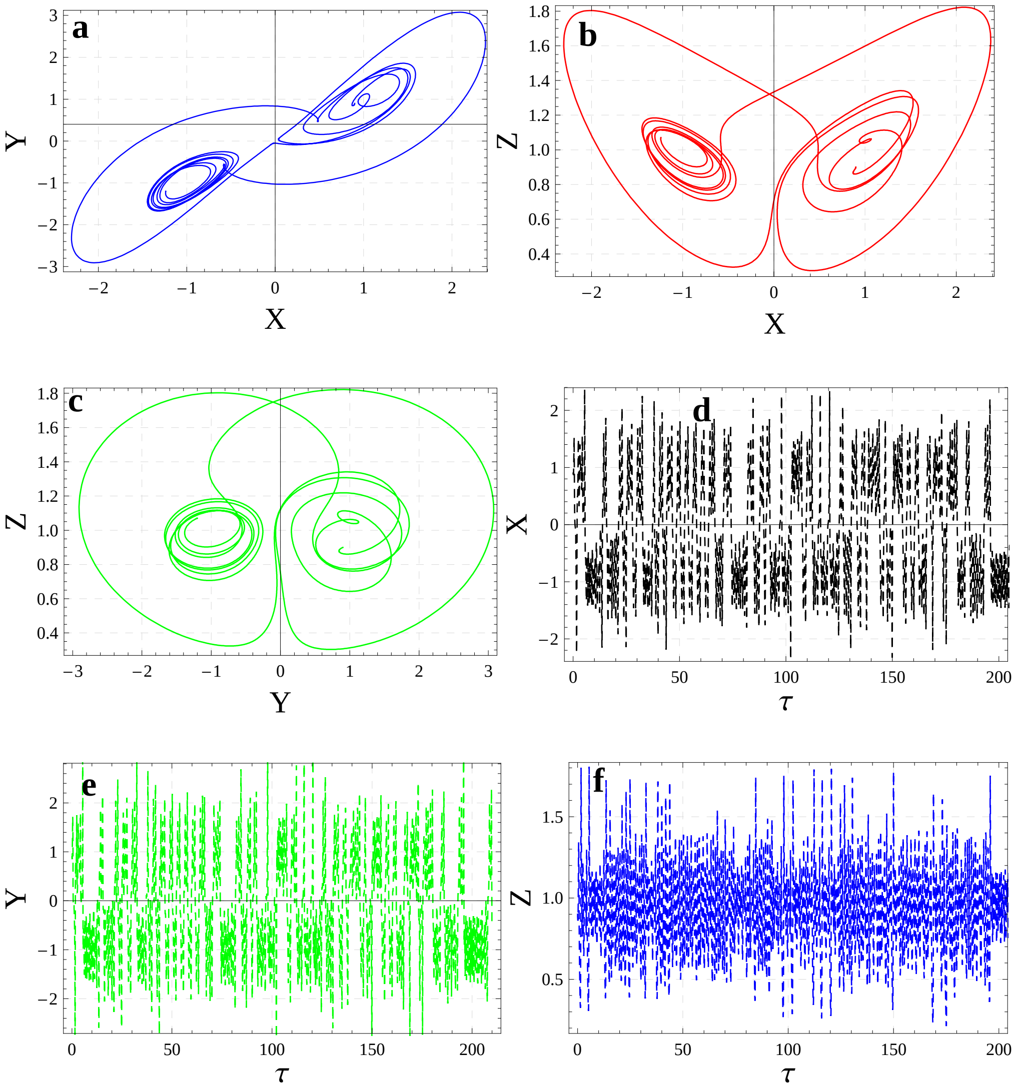


Figure 6.3: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta=\pi : \Omega=10 : \delta=0.3 : Pr=10 : R=22 : \beta=0.5$

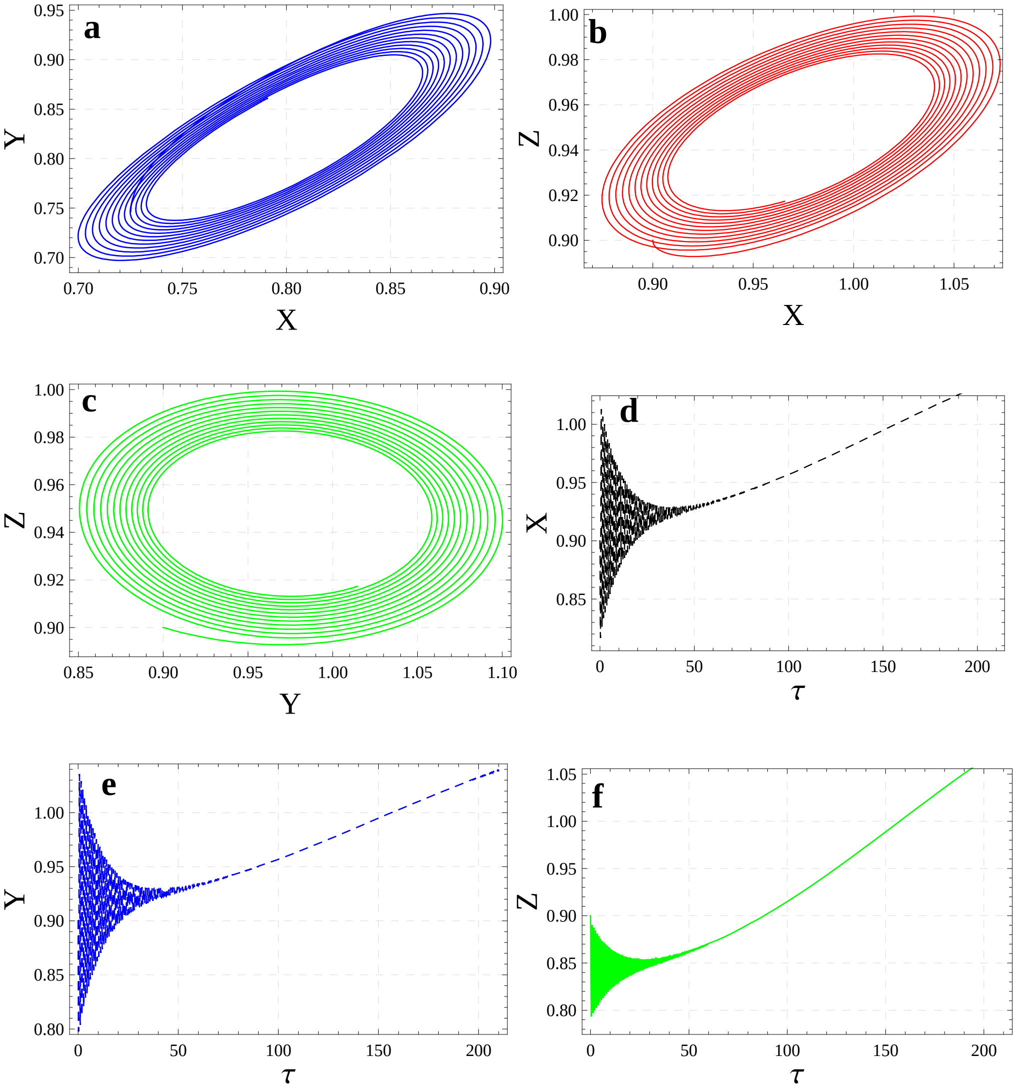


Figure 6.4: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 0.01 : \delta = 0.3 : Pr = 10 : R = 22 : \beta = 0.5$

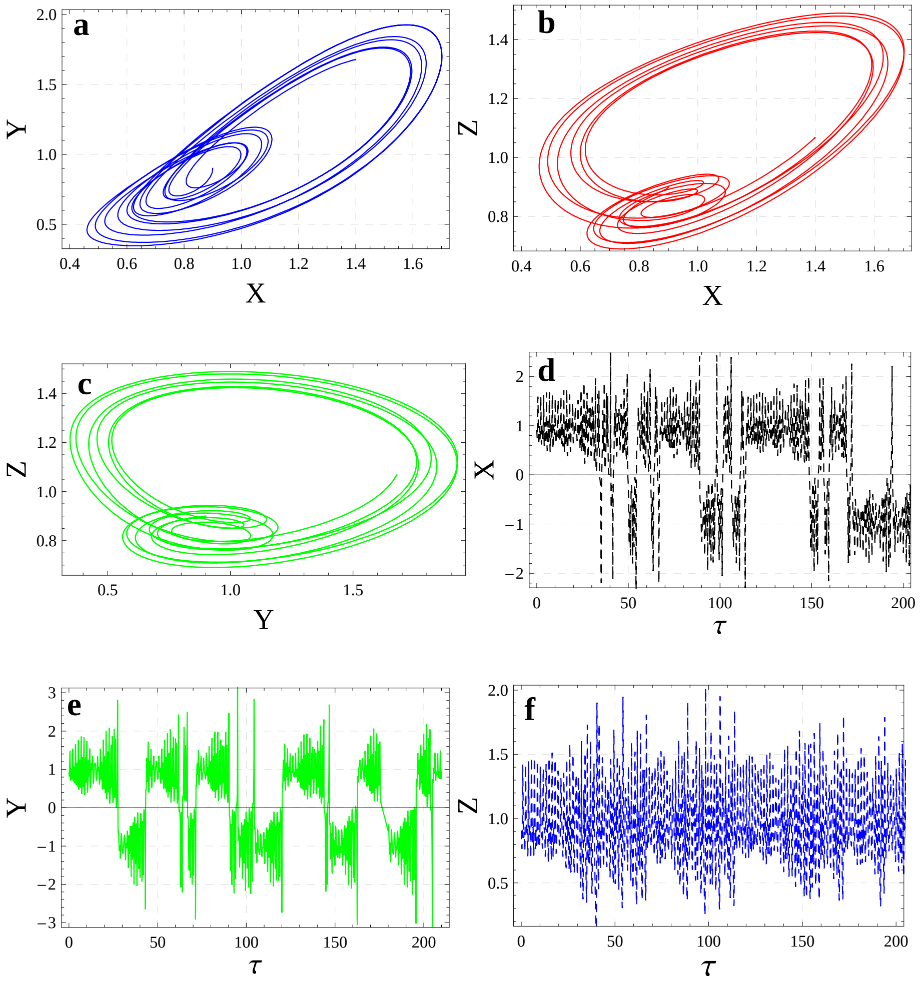


Figure 6.5: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 4 : \delta = 0.3 : Pr = 10 : R = 22 : \beta = 0.5$

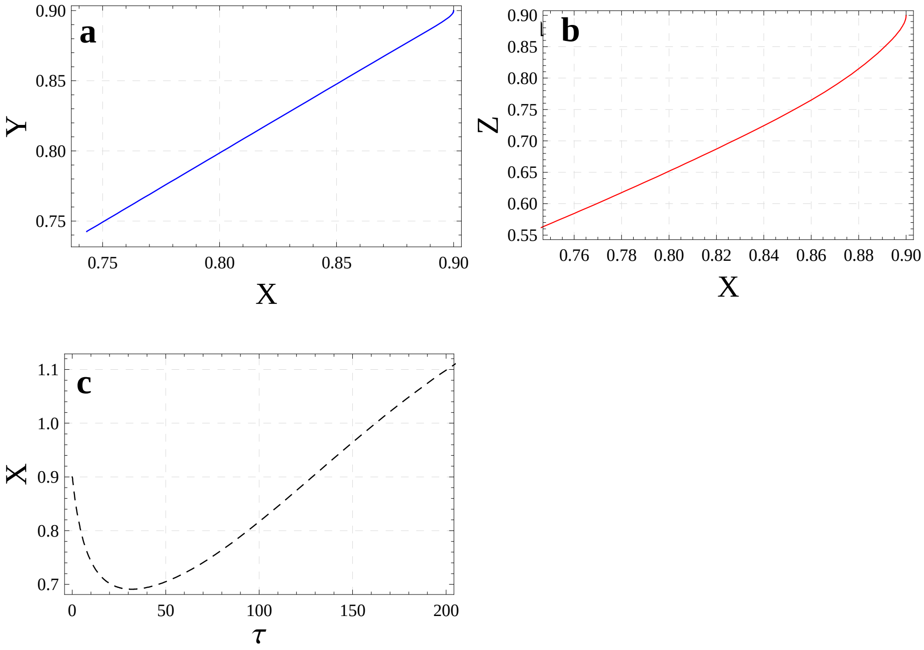


Figure 6.6: Projections and evolution of trajectories over the planes $X-Y, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 2 : \delta = 0.1 : Pr = 10 : R = 1.1 : \beta = 0.5$

positive attractor. However, for other two modulation cases, the chaotic behaviour is obvious, although it is more vigorous in out of phase modulation case.

6.5 Summary

The nonlinear convection in a porous medium for two-dimensional spatial case has been investigated. In particular, a derivation of a set of three ordinary nonlinear differential equations, which describe as a minimal model for the complex dynamic behavior in the presence of an external imposed time periodic thermal boundary conditions. Without thermal modulation, a classical Lorenz model will be recovered. The parameter regions have identified, where the stationary states or those with chaotic or regular dynamics will occur. Also a numerical simulation is performed using mathematica and found that the system has multiple transitions between regular and chaotic behavior changing the values of Ω, δ. The following conclusions are made from the above study.

1. The IPM case is not much effective on chaotic regime of the system whereas OPM

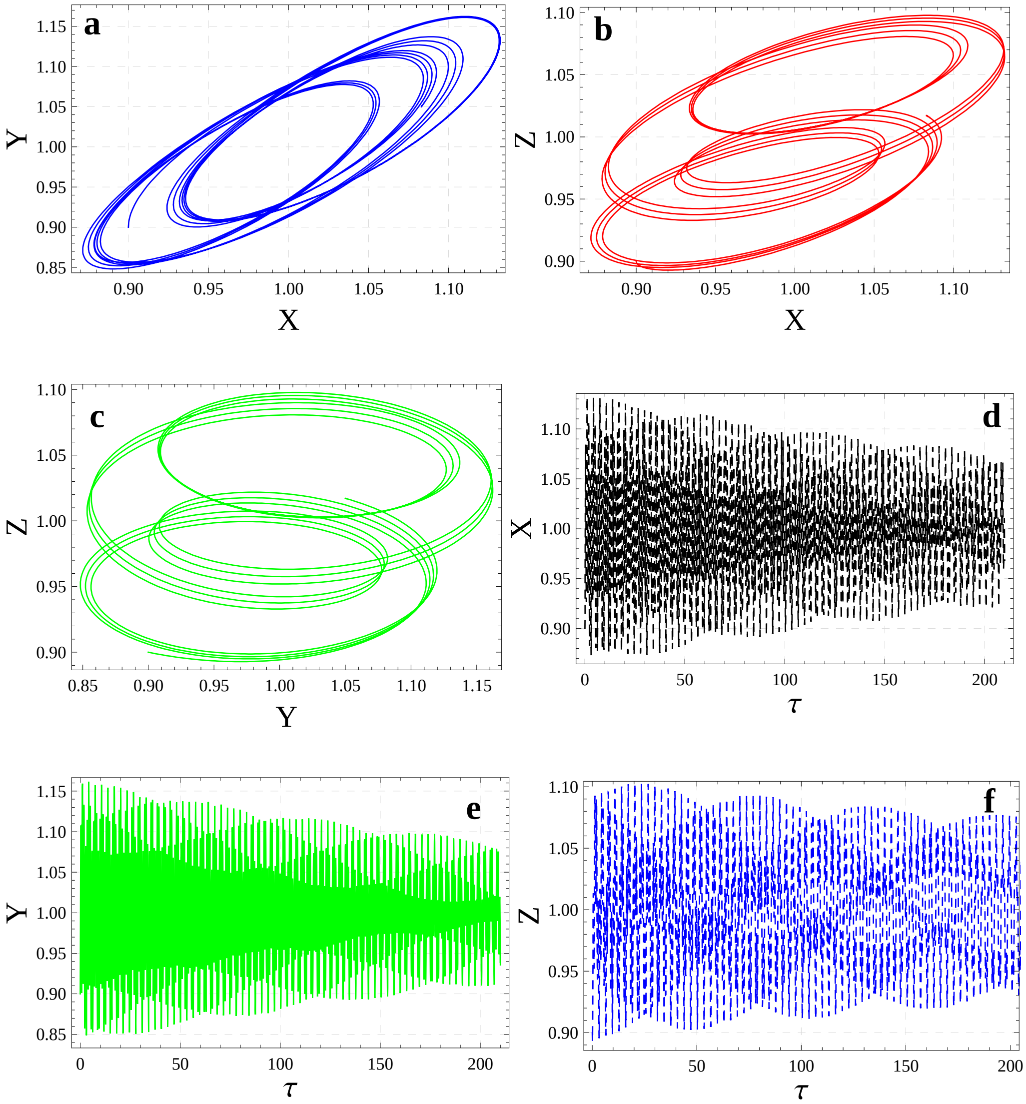


Figure 6.7: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 2 : \delta = 0.1 : Pr = 10 : R = 21.2172 : \beta = 0.5$

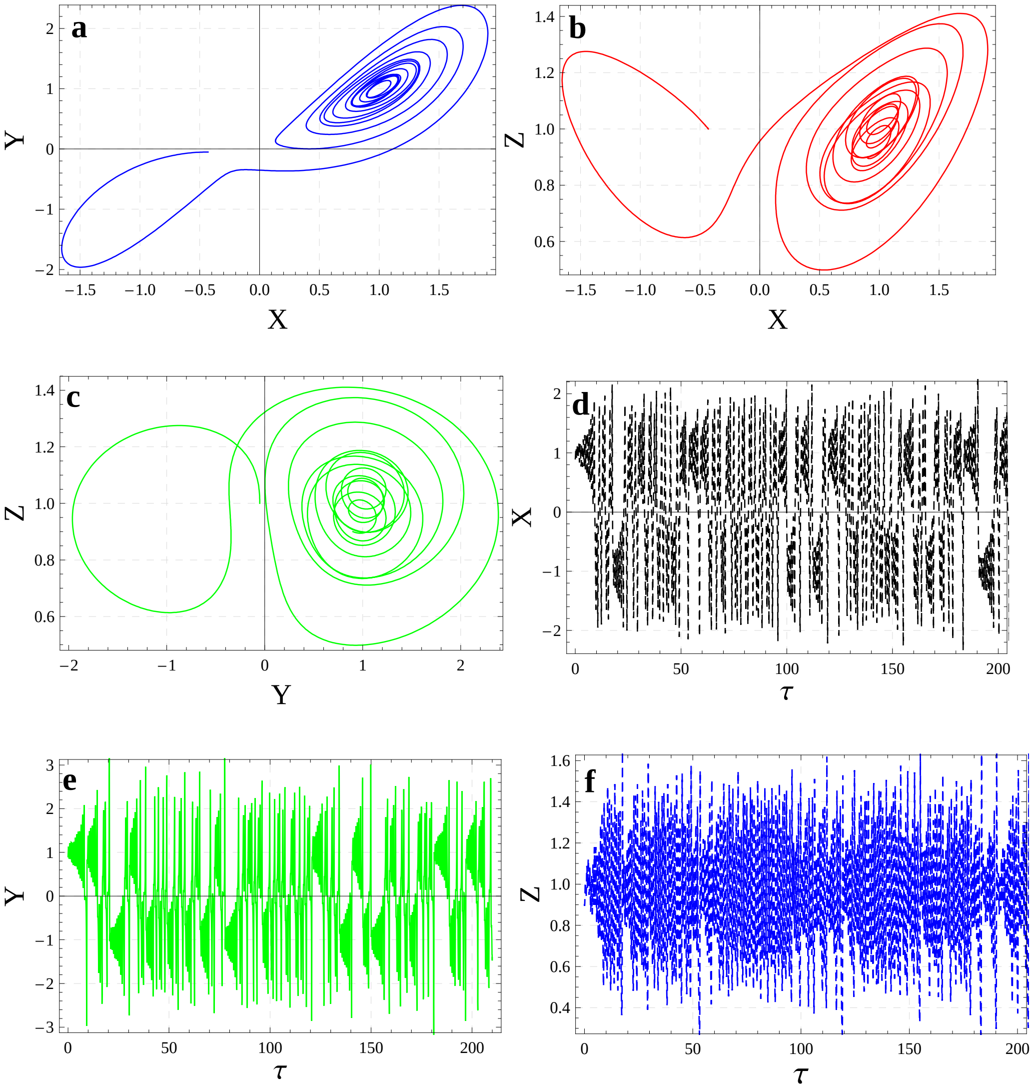
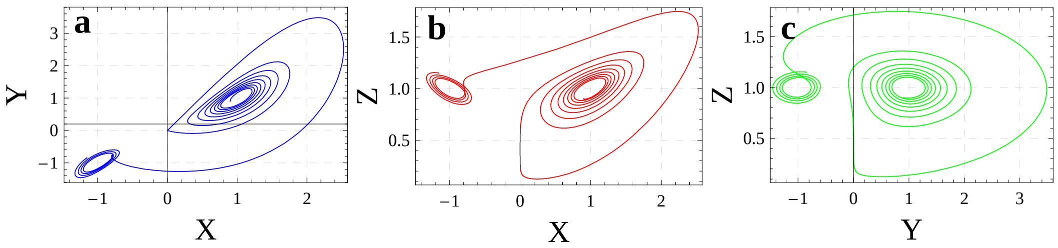
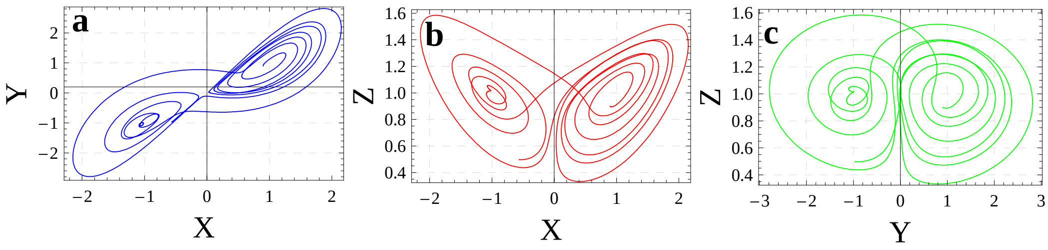


Figure 6.8: Projections and evolution of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 2 : \delta = 0.1 : Pr = 10 : R = 29.4621 : \beta = 0.5$

In phase modulation ($\phi = 0$)



Out of phase modulation ($\phi = \pi$)



Only lower boundary modulation ($\phi = -i\,\infty$)

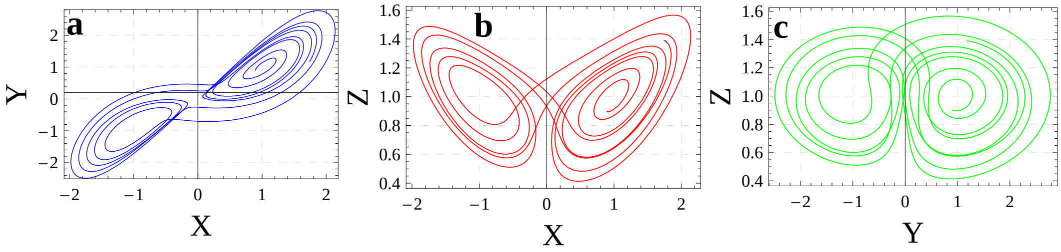


Figure 6.9: Effect of three types of modulations on the projections of trajectories over the planes $X-Y, Y-Z, Z-X$ and a time domain of solutions for $\theta = \pi : \Omega = 2 : \delta = 0.1 : Pr = 10 : R = 29.4621 : \beta = 0.5$

and LBMO cases are.

2. By suitable combination of the values of Ω, δ and R, the chaotic behaviour of the solution of the system can be controlled.
3. Thermal modulation of the plates is to inhabit the chaotic motions in the system.

Chapter 7

Nonlinear thermal instability in Non-Newtonian fluids

7.1 Nonlinear thermal Darcy convection in a nanofluid saturated porous medium under gravity modulation

7.1.1 Introduction

Common fluids have limited heat transfer capabilities while some of the metals have very high thermal conductivity in comparison to these fluids. Therefore, the basic idea behind nanofluids was to make a substance by combining these two, which would behave like a fluid and have thermal conductivity of a metal, thus nanofluids were made by suspending the nanoparticles in the common fluids, called base fluids. Presence of these nanoparticles in the base fluids may increase the thermal conductivity of the fluids by 15-40%. A large number of studies are available in the literature in which thermal instability in nanofluids have been investigated.

A significant feature of nanofluids is thermal conductivity enhancement which was first reported by Masuda et al.(1993). Nanofluids are mixtures of base fluid such as water or ethylene-glycol along with small amount of nanoparticles such as metallic or metallic oxide particles (Cu, Cuo, $Al_2\ O_3$), having dimensions from 1 to 100nm. Choi (1995) was the first, who proposed this term ‘nanofluid’. Natural convection or buoyancy driven convection, is the heat removal strategy adopted in a wide variety of industries ranging from transportation (heating, ventilation, and air conditioning), energy production and supply to electronics, textiles and paper production, geophysical problems, nuclear reactors to name a few (Choi 1999). The ballistic nature of heat transport within nanoparticles was analyzed by Chen (2001). Eastman et al. (2001) reported an increase of 40% in the effective thermal conductivity of ethylene-glycol with 0.3% volume of copper nanoparticles of 10 nm diameter. Further 10-30% increase of the effective thermal conductivity in alumina/water nanofluids with 1-4% of alumina was reported by Das et al. (2003). These reports led Buongiorno and Hu (2005) to suggest the possibility of using nanofluids in advanced nuclear systems. Another application of the nanofluid flow is in the delivery of nano-drug as suggested by Kleinstreuer et al. (2008).

Khanafer et al. (2003) reported an increase in concentration of suspended nanoparticles, is to increase in heat transfer in Cu-water nanofluids in a two-dimensional rectangular enclosures while Putra et al. (2003) reported that in natural convection, using Al_2O_3 and CuO nanofluids, the heat transfer coefficient was smaller than that in a clear fluid. Various studies have been conducted to determine the governing mechanisms in nanoscale, including a modified Maxwell model accounting for the ordered nanolayer near the particle fluid interface by Yu and Choi (2003), Brownian motion of nanoparticles in fluids by Jang and Choi (2004), ballistic nature of heat transport within nanoparticles by Keblinski and Cahill (2005) and thermal lagging in nanoparticles with a large surface area to volume ratio by Vadasz (2006). A comprehensive review of heat transport in nanofluids is due to Eastman et al. (2004). In spite of several reported studies, it is a fact that no satisfactory explanation could be found so far, for abnormal enhancement in the thermal conductivity and viscosity of the fluid due to the presence of nano-particles. Wen and Ding (2006) reported a reduction in heat transfer after changing a clear fluid to a nanofluid while Abu-Nada et al. (2008) showed the enhancement of heat transfer in nanofluids at higher values of the Rayleigh number. Buongiorno (2006) has given an extensive study to account for the unusual behavior of nanofluids based on inertia, Brownian diffusion thermophoresis, diffusiophoresis, Magnus effects, fluid drainage and gravity settling, and proposed a model incorporating the effects of Brownian diffusion and the thermophoresis. With the help of these equations, studies were conducted by Tzou (2008a,b) and Nield and Kuznetsov (2010).

Employing Darcy model, the Horton-Rogers-Lapwood problem was investigated by Nield and Kuznetsov (2009). The effect of local thermal non-equilibrium among the nanoparticle, fluid, and solid-matrix phases was investigated by Kuznetsov and Nield (2010a) using a three-temperature model. They conclude that in some circumstances, the effect of LTNE can be significant, but for large Lewis number, the effect was small. The onset of double-diffusive convection in a layer of nanofluid saturated porous medium was analyzed by Kuznetsov and Nield (2010b). They found that when Soret and Dufour parameters are negligible, the non-oscillatory mode was expected when concentration Rayleigh number was positive, a situation which physically corresponds to the fact that for oscillations to occur, two of the buoyancy forces have to be in opposite directions. Using Brinkman model, Kuznetsov and Nield (2010c) studied the onset of thermal instability in a porous medium saturated by a nanofluid, incorporating the effects of Brownian motion and thermophoresis of nanoparticles. They found that the critical thermal Rayleigh number can be reduced or increased depending on whether the basic nanoplarticle distribution is bottom heavy or top heavy nanoparticles suspension. The effect of LTNE on the onset of convection in a porous medium saturated by a nanofluid using Brinkman model was investigated by Kuznetsov and Nield (2011). Nield and Kuznetsov (2011) investigated analytically the effect of vertical throughflow on the onset of convection in a horizontal porous medium saturated by a nanofluid. Nield and Kuznetsov (2012) investigated the linear theory for the Horton-Rogers-Lapwood problem of porous medium saturated by nanofluid with thermal conductivity and viscosity dependent on the nanoparticle volume fraction. They found that the critical value of Rayleigh number is increased by these parameters when compared to constant viscosity and thermal conductivity results.

Bhaduria and Agarwal (2011a,b) studied the effect of local thermal non equilibrium on linear and nonlinear thermal instability in a horizontal porous layer saturated by a nanofluid. Agarwal et al. (2011a) and Agarwal and Bhaduria (2011b) studied thermal instability in a rotating porous layer saturated by a nanofluid for top and bottom-heavy suspension for Darcy model. Agarwal et al. (2012) studied double diffusive convection in a horizontal porous layer saturated by a nanofluid, for the case where the base fluid of the nanofluid is itself a binary fluid such as salty water. Chand and Rana (2012) studied the onset of thermal convection in rotating nanofluid layer saturated porous medium. Boundary and internal heat source effects on the onset of Darcy-Brinkman convection in a porous layer saturated by nanofluid was studied by Yadav et al. (2012).

Recently, Umavathi (2013) studied both temperature and gravity modulation of convection in a porous medium saturated by a nanofluid by using a linear stability analysis, where one can see only onset criteria fails at heat and concentration transports in the system. Hence in this section a nonlinear analysis of thermal instability in a nanofluid saturated porous medium under gravity modulation is discussed as till date no nonlinear study is available on this aspect.

7.1.2 Governing Equations

We consider a porous layer saturated by a nanofluid, confined between two horizontal boundaries, respectively at $z = 0$ and $z = d$, heated from below and cooled from above. The boundaries are impermeable and perfectly thermally conducting. The porous layer is extended infinitely in x and y-directions, and z-axis is taken vertically upward with the origin at the lower boundary. In addition, the local thermal equilibrium between the fluid and solid has been considered, thus the heat flow has been described using one equation model. T_h and T_c are the temperatures at the lower and upper walls respectively such that $T_h > T_c$. Employing the Oberbeck Boussinesq approximation, the governing equations to study the thermal instability in a nanofluid saturated porous medium are (Buongiorno 2006, Kuznetsov and Nield 2010a,b,c)

$$\begin{aligned}
\nabla \cdot \mathbf{v_D} &= 0, && (7.1.1)\\
\frac{\rho_f}{\delta_1}\frac{\partial \mathbf{v_D}}{\partial t} + \nabla p &= -\frac{\mu}{K}\mathbf{v}_D + [\phi\rho_p + (1-\phi)\{\rho(1-\beta(T-T_c))\}]\vec{g}, && (7.1.2)\\
(\rho c)_m\frac{\partial T}{\partial t} + (\rho c)_f \mathbf{v}_D \cdot \nabla T &= k_m \nabla^2 T + \delta(\rho c)_p[D_B \nabla\phi \cdot \nabla T + \frac{D_T}{T_c}\nabla T \cdot \nabla T], && (7.1.3)\\
\frac{\partial \phi}{\partial t} + \frac{1}{\delta_1}\mathbf{v}_D \cdot \nabla\phi &= D_B \nabla^2\phi + \frac{D_T}{T_c}\nabla^2 T, && (7.1.4)\\
\vec{g} &= g_0\,(1 + \delta\cos(\Omega t))\,\vec{k}, && (7.1.5)
\end{aligned}$$

where $\mathbf{v}_D = \delta_1 v$ is the Darcy velocity, δ is the amplitude and Ω frequency of modulation. Assuming temperature and volumetric fraction of the nanoparticles to be constant at the stress-free boundaries, we may take the boundary conditions on T and ϕ as

$$\mathbf{v} = 0, \quad T = T_h, \ \phi = \phi_0 \text{ at } z = 0, \qquad (7.1.6)$$

$$\mathbf{v} = 0, \quad T = T_c, \ \phi = \phi_1 \text{ at } z = d, \qquad (7.1.7)$$

where ϕ_1 is greater than ϕ_0. The dimensionless variables are considered as given below: $(x^*, y^*, z^*) = (x, y, z)/d$, $\tau^* = \tau k_T/\gamma d^2$, $(u^*, v^*, w^*) = (u, v, w)d/k_T$, $p^* = pK/\mu k_T, \phi^* =$

$\dfrac{\phi-\phi_0}{\phi_1-\phi_0}$ and $T^* = \dfrac{T-T_c}{T_h-T_c}$, where $k_T = \dfrac{k_m}{(\rho c)_f}$, $\gamma = \dfrac{(\rho c_p)_m}{(\rho c_p)_f}$. The non-dimensionalized governing equations are (after dropping the asterisk for simplicity)

$$\begin{aligned} \nabla\cdot\mathbf{v} &= 0, && (7.1.8)\\ \frac{1}{Pr_D}\frac{\partial\mathbf{v}}{\partial\tau} &= -\nabla p-\mathbf{v}-\mathrm{g}_m\left(Rm-Ra\;T+Rn\phi\right)\hat{e_z}, && (7.1.9)\\ \gamma\frac{\partial T}{\partial\tau}+\mathbf{v}\cdot\nabla T &= \nabla^2T+\frac{N_B}{Le}\nabla\phi\cdot\nabla T+\frac{N_AN_B}{Le}\nabla T\cdot\nabla T, && (7.1.10)\\ \frac{\partial\phi}{\partial\tau}+\mathbf{v}\cdot\nabla\phi &= \frac{1}{Le}\nabla^2\phi+\frac{N_A}{Le}\nabla^2T, && (7.1.11) \end{aligned}$$

$$\mathbf{v}=0,\;\; T=1,\; \phi=0 \text{ at } z=0, \text{ and } \mathbf{v}=0,\;\; T=0,\; \phi=1 \text{ at } z=1, \qquad (7.1.12)$$

where $g_m = (1+\delta\cos(\Omega t))$. The non-dimensionlized parameters in the above equations have their usual meanings given in nomenclature, N_A is the modified diffusivity ratio, which is similar to the Soret parameter that arises in cross diffusion in thermal instability. At the basic state, the nanofluid is assumed to be at rest, therefore the quantities at the basic state will vary only in z-direction, and are given by:

$$\mathbf{v}=0, p=p_b(z),\;\; T=T_b(z),\; \phi=\phi_b(z). \qquad (7.1.13)$$

Substituting the Eq.(7.1.13) in Eq.(7.1.10) and Eq.(7.1.11), we get:

$$\frac{d^2T_b}{dz^2}+\frac{N_B}{Le}\frac{d\phi_b}{dz}\frac{dT_b}{dz}+\frac{N_AN_B}{Le}(\frac{dT_b}{dz})^2=0. \qquad (7.1.14)$$

According to Buongiorno (2006), for most nanofluids investigated so far $Le/(\phi_1-\phi_0)$ is large of order 10^5-10^5, since the nanoparticle fraction decrement $(\phi_1-\phi_0)$ is typically no smaller than 10^{-3} this means that Le is large of order 10^2-10^3, while N_A is no greater then about 10. Using the above analysis, Kuznetsov and Nield (2010a), Tzou (2008a,b) showed that the second and third terms in equation Eq.(7.1.14) are small and hence we have

$$\frac{d^2T_b}{dz^2}=0,\;\; \frac{d^2\phi_b}{dz^2}=0. \qquad (7.1.15)$$

The boundary conditions for solving Eq.(7.1.15) can be obtained from Eq.(7.1.12) as

$$T_b=1,\; \phi_b=0 \text{ at } z=0, \qquad (7.1.16)$$

$$T_b=0,\; \phi_b=1 \text{ at } z=1. \qquad (7.1.17)$$

Solving the Eq.(7.1.15), subject to the above conditions Eq.(7.1.16) and Eq.(7.1.17), we obtain

$$T_b = 1 - z, \tag{7.1.18}$$

$$\phi_b = z. \tag{7.1.19}$$

7.1.3 Nonlinear stability analysis

We now superimpose perturbations on the basic state as given below

$$\mathbf{v} = \mathbf{v}', \ p = p_b + p', \ T = T_b + T', \ \phi = \phi_b + \phi'. \tag{7.1.20}$$

Substituting the above expression Eq.(7.1.20) in Eqs.(7.1.8)-(7.1.11), and using the expressions Eqs.(7.1.18-7.1.19), eliminating the pressure term and introducing the stream function, we obtain

$$\frac{1}{Pr_D}\frac{\partial}{\partial \tau}(\nabla^2\psi) + \nabla^2\psi = \left(Rn\frac{\partial \phi}{\partial x} - Ra\frac{\partial T}{\partial x}\right)\mathrm{g}_m\vec{k}, \tag{7.1.21}$$

$$\frac{\partial \psi}{\partial x} - \nabla^2 T = -\frac{\partial T}{\partial \tau} + \frac{\partial(\psi, T)}{\partial(x,z)}, \tag{7.1.22}$$

$$-\frac{\partial \psi}{\partial x} - \frac{N_A}{Le}\nabla^2 T = \frac{1}{Le}\nabla^2\phi - \frac{1}{\gamma}\frac{\partial \phi}{\partial \tau} + \frac{\partial(\psi, \phi)}{\partial(x,z)}. \tag{7.1.23}$$

A local nonlinear stability analysis shall be performed and hence we will take the following Fourier expressions

$$\psi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}(\tau)\sin(m\alpha x)\sin(n\pi z), \tag{7.1.24}$$

$$T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}(\tau)\cos(m\alpha x)\sin(n\pi z), \tag{7.1.25}$$

$$\phi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}(\tau)\cos(m\alpha x)\sin(n\pi z). \tag{7.1.26}$$

In what follows we take the modes $(1,1)$ for stream function, and $(0,2)$ and $(1,1)$ for temperature and nanoparticle concentration

$$\psi = A_{11}(\tau)\sin(\alpha x)\sin(\pi z), \tag{7.1.27}$$

$$T = B_{11}(\tau)\cos(\alpha x)\sin(\pi z) + B_{02}(\tau)\sin(2\pi z), \tag{7.1.28}$$

$$\phi = C_{11}(\tau)\cos(\alpha x)\sin(\pi z) + C_{02}(\tau)\sin(2\pi z), \tag{7.1.29}$$

where the amplitudes $A_{11}(\tau), B_{11}(\tau), B_{02}(\tau), C_{11}(\tau)$ and $C_{02}(\tau)$ are functions of time and are to be determined. Substituting the equations (7.1.27-7.1.29) in equations (7.1.21-7.1.23) taking the orthogonality condition with the eigenfunctions associated with the considered minimal mode, we get

$$\begin{aligned}
\frac{dA_{11}(\tau)}{d\tau} &= \frac{Pr_D}{\delta^2}\{\alpha[Rn\ \ (1+\delta\cos(\Omega t))\,C_{11}(\tau) - Ra\,(1+\delta\cos(\Omega t))\,B_{11}(\tau)] - \ \delta^2 A_{11}(\tau)\}, && (7.1.30)\\
\frac{dB_{11}(\tau)}{d\tau} &= -[\alpha A_{11}(\tau) + \delta^2 B_{11}(\tau) + \pi\alpha A_{11}(\tau)B_{02}(\tau)], && (7.1.31)\\
\frac{dB_{02}(\tau)}{d\tau} &= \frac{\pi\alpha}{2}A_{11}(\tau)B_{11}(\tau) - 4\pi^2 B_{02}(\tau), && (7.1.32)\\
\frac{1}{\gamma}\frac{dC_{11}(\tau)}{d\tau} &= -[\alpha A_{11}(\tau) + \frac{1}{Le}\delta^2 C_{11}(\tau) + \pi\alpha A_{11}(\tau)C_{02}(\tau) + \frac{N_A}{Le}\delta^2 B_{11}(\tau)], && (7.1.33)\\
\frac{1}{\gamma}\frac{dC_{02}(\tau)}{d\tau} &= \frac{\pi\alpha}{2}A_{11}(\tau)C_{11}(\tau) - \frac{4\pi^2}{Le}[C_{02}(\tau) + N_A B_{02}(\tau)]. && (7.1.34)
\end{aligned}$$

The above system of simultaneous autonomous ordinary differential equations can be subsequently solved numerically using Mathematic 8 NDSolve.

7.1.4 Heat and Nanoparticle Concentration Transport

The horizontal averaged thermal Nusselt number, Nu(τ) is defined as

$$\begin{aligned}
\mathrm{Nu}(\tau) &= \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}}\\
&= 1 + \left[\frac{\int\limits_0^{2\pi/\alpha_c}\left(\frac{\partial T}{\partial z}\right)dx}{\int\limits_0^{2\pi/\alpha_c}\left(\frac{\partial T_b}{\partial z}\right)dx}\right]_{z=0}. \qquad (7.1.35)
\end{aligned}$$

Substituting the Eq.(7.1.18) and Eq.(7.1.28) in Eq.(7.1.35), we get

$$\mathrm{Nu}(\tau) = 1 - 2\pi B_{02}(\tau). \qquad (7.1.36)$$

The nanoparticle concentration Nusselt number, Nu$_\phi(\tau)$ is defined similar to the thermal Nusselt number. Following the procedure adopted for arriving at Nu(τ), one can obtain the expression for Nu$_\phi(\tau)$ in the form

$$\mathrm{Nu}_\phi(\tau) = (1 - 2\pi C_{02}(\tau)) + N_A(1 - 2\pi B_{02}(\tau)). \qquad (7.1.37)$$

7.1.5 Results and Discussion

The nanofluids have an enhanced magnitude of thermal conductivity than normal fluids or metals, being attributed to the presence of nanoparticles in them. Nanofluids being combinations of base fluids with some metallic nano-scale sized particles or fibers, suspended in them, combine the thermal properties of both, leading to an enhanced rate of thermal conductivity. This has been experimentally verified by many researchers in the past one decade. Here we have investigated the effect of gravity modulation in a horizontal porous layer saturated by a nanofluid. Using Brinkman model and considering top heavy porus layer, we performed a nonlinear stability analysis to study the heat transport. A linear theory has been investigated by Umavathi (2013) concerning temperature and gravity modulations. It is well known that, we make a nonlinear theory to analyze heat transport, which is not possible by linear stability theory. Moreover external regulations of convection is important in the study of thermal instability, therefore, in this paper we have considered gravity modulation for either enhancing or inhibiting the convective heat transport as is required by a real life application. Here note that according to Buongiorno (2006), for most nanofluids investigated so far Le is large, but as per Bhaduria and Agarwal (2011b), we have considered $Le = 10$ in the case of nanopartical concentration Rayleigh number. The effect of gravity modulation on heat transport has been depicted in figures 7.1-7.3. The following parameters $Pr_D, Rn, N_A, Le, \gamma, \delta$ and Ω occurring in the present study, influence the convective heat transport. The first five parameters are related to the porous layer and the next two parameters concern the external mechanism of controlling convection. Because of small amplitude of modulation, the value of δ is considered to be small. Further, the gravity modulation assumed to be of low frequency, as at low range of frequencies, the effect of frequency on onset of convection as well as on heat transport is maximum. The coefficient of heat transport, i.e. thermal Nusselt number and the coefficient of nanoparticle concentration transport, i.e. concentration Nusselt number are calculated as functions of time and other parameters of the system. The obtained results are depicted in the figures 7.1-7.3 for $\mathrm{Nu}(\tau)$ and $\mathrm{Nu}_\phi(\tau)$ versus time τ. In the figures the values of $\mathrm{Nu}(\tau)$ and $\mathrm{Nu}_\phi(\tau)$ start with 1 and 2 respectively, and

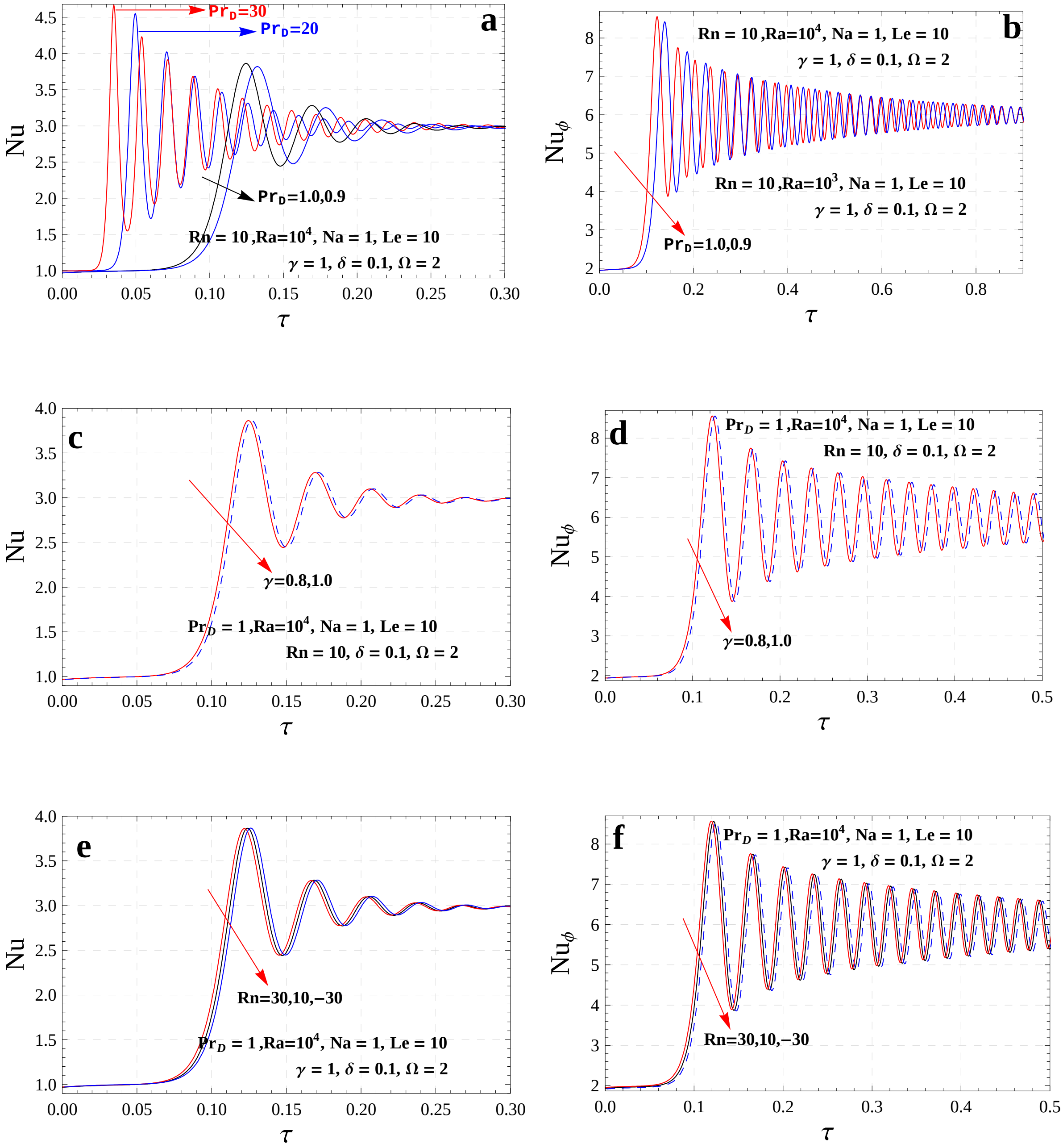


Figure 7.1: Nu and Nu_ϕ versus τ for different values of system parameters

remains constant for a quite some time, showing the conduction state. Then the values of Nu(τ) and Nu$_\phi$(τ) increase as time passes, thus showing that the convection is taking place. These values oscillate and then approach constant values thus showing that the steady state has been achieved.

From the figures 7.1a-b, we find that initially when time τ is small the vibrations become of high amplitudes as the value of Prantdl-Darcy number Pr_D increases, and so Nusselt and concentration Nusselt numbers increase, thus increasing the rate of heat and nanoparticle concentration transport. But at large values of time τ, the vibrations become smaller and subsequently the values of Nu and Nu$_\phi$ approach steady state values. We also can take Pr_D more than one, in that case the effect of local acceleration term which appear in momentum equation will be disappeared. Taking $Pr_D = 10, 30$ one can see there is increment in heat transfer, the same effect can be seen in the case of concentration Nusselt number. However, from the figures 7.1c-d, we find that as the value of thermal capacity ratio γ increases the values of Nusselt and concentration Nusselt numbers decrease, thus decreasing the rate of heat and concentration transport. Further, the influence of concentration Rayleigh number Rn on both thermal Nusselt number as well as on concentration Nusselt number is found to be similar that is to enhance the heat and concentration transport as given in figures 7.1e-f, which is due to the fact that the nanoparticle concentration is more at the top.

In figure 7.2a, shows that N_A and Le do not have significant effect on the thermal Nusselt number as reported earlier by Bhadauria and Agarwal (2011a). On the contrary in the case of concentration Nusselt number both Le and N_A have increasing effect, given in the figure 7.2b, and figure 7.2c, and so the heat and concentration transport. One can see in Fig. figure 7.2b, that when we consider $Le = 50$, there is an increment in nanoparticle concentration. Also the frequency and magnitude of oscillations increases. Figures 7.2d-e, show that the effect of increase in the amplitude of gravity modulation on heat and nanoparticle concentration transport is to increase the values of Nu and Nu$_\phi$ and hence transport phenomena in both the cases. In figure 7.2f, we compare the results between modulated and unmodulated system, in the absence of modulation we obtain the results

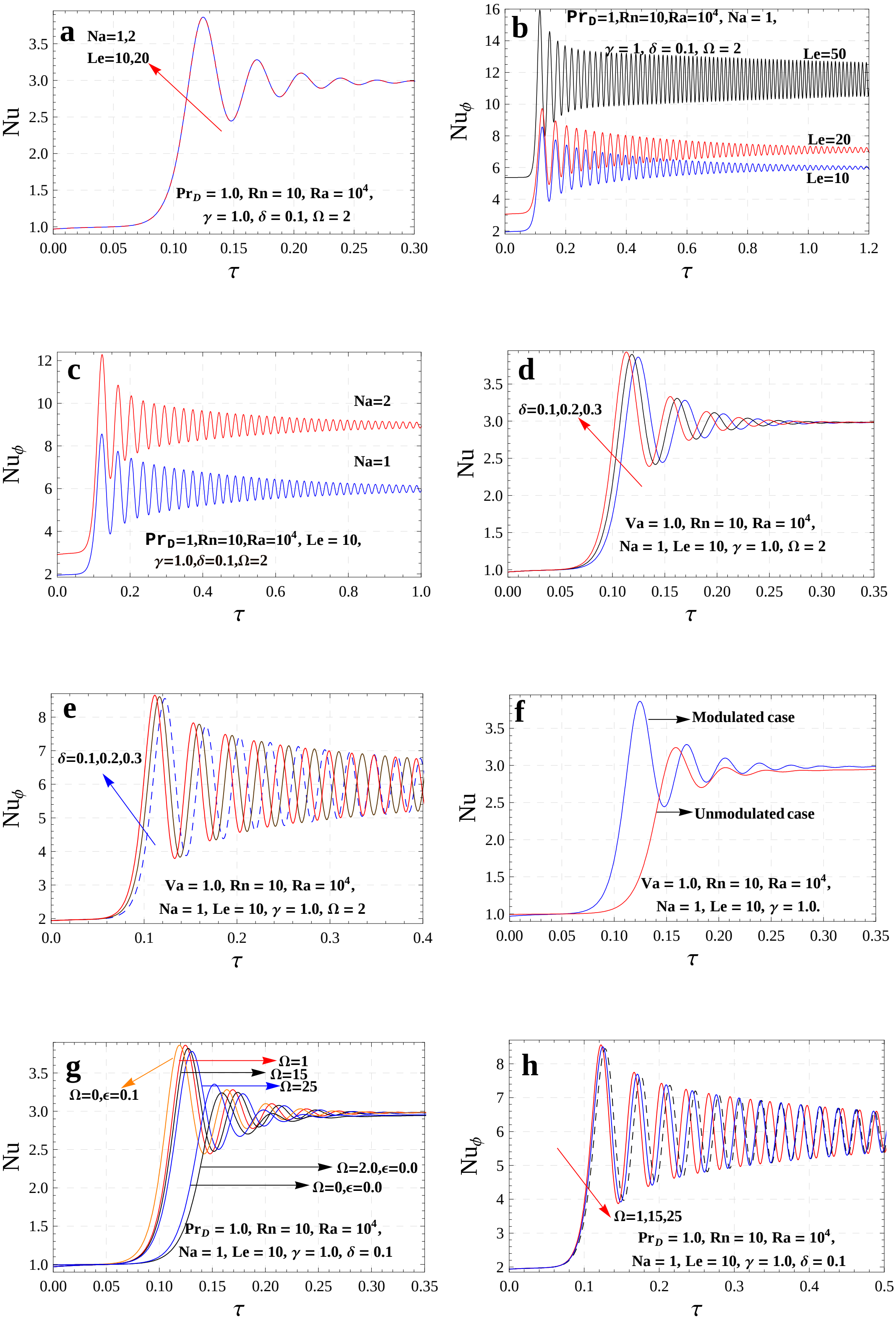


Figure 7.2: The effect of system parameters on heat and concentration transport

obtained by Agarwal et al. (2012). Figures 7.2g-h, show that the effect of increasing frequency of gravity modulation on heat and nanoparticle concentration transport is to decrease the values of Nu and Nu_ϕ, and hence stabilize the system. Thus we obtain the classical results as per Gresho and Sani (1970) and Malashetty and Padmavathi (1997). It is observed in most of the cases that there is much effect of parameters on Nu and Nu_ϕ at low values of time, but less effect at large time, since vibrations become smaller in magnitude, and disappears as Nu, Nu_ϕ reach steady state value. Finally, the parameters Pr_D, Rn, δ have destabilizing effects on the system, while γ, Ω have stabilizing effects. The parameters N_A, Le does not show any effect on Nu, but increase as value of Nu_ϕ.

In figures 7.3a-c show the time-dependent fields for different values of ψ (streamlines), T (isotherms), ϕ (isohalines), at different times. It is clear that with increasing in time, magnitudes of stream function, isotherms and isohalines increases. In all the figures 7.3a-c, for ψ, the sense of motion in the subsequent cells is alternately identical with and opposite to that of the adjoining cell. In case of isotherms, at the starting time, conduction occurs which approaches to convection stage very soon, with the magnitude of isotherms increasing with time. In the intermediate time range, uniform convection cells are observed which change to strong convection with the passage of time.

7.1.6 Conclusions

We have investigated nonlinear stability of a vertically vibrating horizontal porous layer saturated by a nanofluid, which is heated from below and cooled from above. Further, we incorporate the effect of Brownian motion along with thermophoresis. The top heavy suspension of nanoparticles has been considered. The results have been obtained in terms of the concentration and thermal Nusselt numbers with the help of the finite amplitude equation. The effect of various parameters have been obtained and depicted graphically. We have the following observations.

1. Gravity modulation can be used to regulate the heat transport effectively.

2. Increase in concentration Rayleigh number Rn, Modified diffusivity ratio N_A and Lewis number Le increases the effect of gravity modulation.

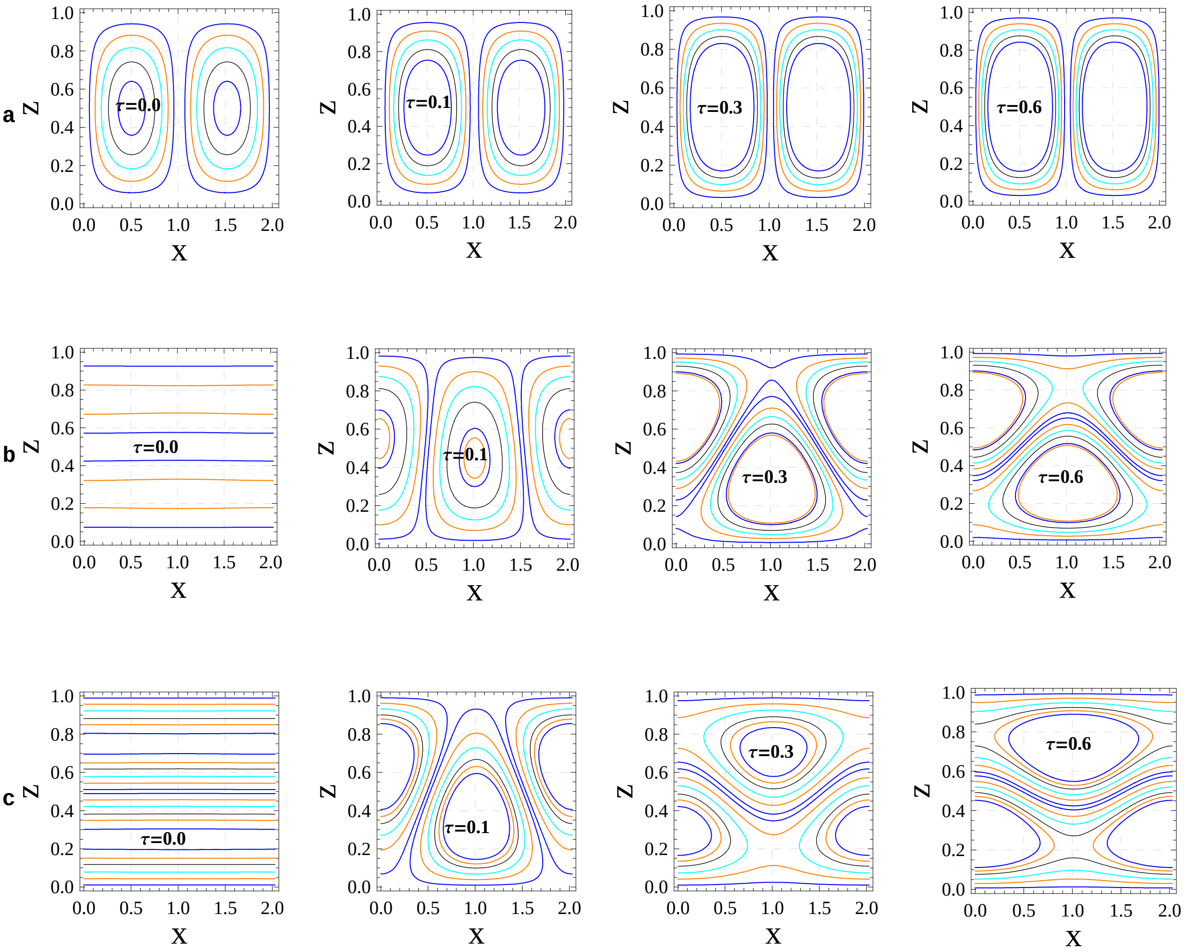


Figure 7.3: (a) Streamlines (b) Isohalines (c) Isotherms

3. An increment in Prantdl-Darcy number Pr_D is to increase the values of Nu(τ) and Nu$_\phi$(τ) at small values of time τ but no effect at large time τ.

4. The effect of increased nanoparticle concentration Rn is to enhance the heat and concentration transport.

5. There is no significant effect of N_A and Le on thermal Nusselt number.

6. Increasing Le, δ, N_A and Rn is to increase the concentration Nusselt number, whereas an increase in Ω decreases the same, and so the concentration transport.

7.2 Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under throughflow effects

7.2.1 Introduction

In several technological and geophysical applications involve flow of fluids through porous media, called throughflow. This kind of flow alters the conduction state temperature from linear to nonlinear with layer height, which in turn affects the stability of the system significantly. The effect of throughflow on the onset of convection in a horizontal porous medium has been investigated by Wooding (1960), Homsy and Sherwood (1976), Jones and Persichetti (1986). Nield (1987), Shivakumara (1997) have shown that a small amount of throughflow can have a destabilizing effect if the boundaries are of different types, and a physical explanation for the same has been given by them. Khalili and Shivakumara (1998) have investigated the effect of throughflow and internal heat generation on the onset of convection in a porous medium. They have shown that throughflow destabilizes the system even if the boundaries are of the same type; a result which is not true in the absence of an internal heat source. The non-Darcian effects on convective instability in a porous medium with throughflow has been investigated in order to account for inertia and boundary effects by Shivakumara (1999), Khalili and Shivakumara (2003). Later on many researchers have investigated throughflow effects considering different physical models some of them are: Shivakumara and Nanjundappa (2006), Shivakumara and Sureshkumar (2007), Barletta et al. (2010), Nield and Kuznetsov (2011), Reza and Gupta (2012) and Nield and Kuznetsov (2013).

Most of the above studies on throughflow have been done using linear stability analysis. However, if one wants to calculate the heat transfer across the boundaries, one needs to do a nonlinear stability analysis of the problem. Further, to the best of authors' knowledge, not even a single study is available in which oscillatory convection along with throughflow has been investigated. Therefore, the purpose of the present section is to

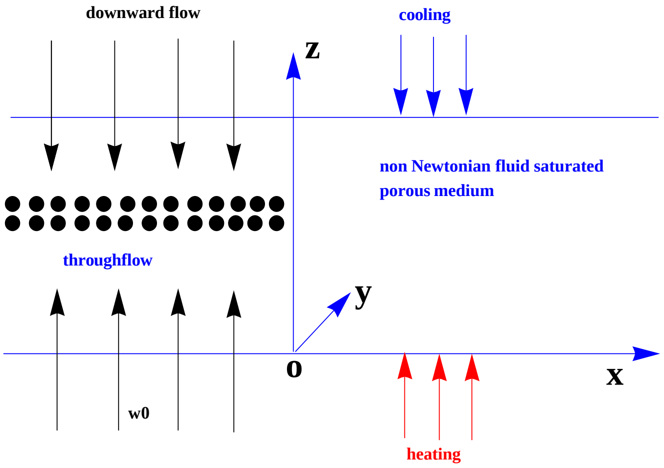


Figure 7.4: A sketch of the physical problem

investigate the oscillatory mode of convection in an horizontal porous layer saturated by a viscoelastic fluid, under the effect of vertical throughflow. A weak nonlinear stability analysis is done to quantify the heat transfer using a complex non−autonomous Ginzburg−Landau equation derived for oscillatory mode of convection.

7.2.2 Problem Formulation

We consider an infinitely extended horizontal fluid saturated porous layer of depth 'd' as given in figure 7.4. The porous layer is homogeneous and isotropic, and saturated with viscoelastic fluid. The porous medium is heated slowly from below. Using modified Darcy's model (Alishaev 1975) and employing the Boussinesq approximation, the governing equations of flow are given by

$$\nabla \cdot \vec{q} = 0, \tag{7.2.1}$$

$$\left(\overline{\lambda_1}\frac{\partial}{\partial t}+1\right)\left(\frac{\rho_0}{\delta_1}\frac{\partial \vec{q}}{\partial t}-\nabla P+\rho\vec{g}\right)-\frac{\mu}{K}\left(\overline{\lambda_2}\frac{\partial}{\partial t}+1\right)\vec{q}=0, \tag{7.2.2}$$

$$\frac{\partial T}{\partial t}+(\vec{q}\cdot\nabla)T=\kappa_T\nabla^2 T, \tag{7.2.3}$$

$$\rho = \rho_0[1 - \alpha_T(T - T_0)], \tag{7.2.4}$$

where $\vec{q}$ is the fluid velocity, $\vec{g}$ is acceleration due to gravity, ρ is density, K is the permeability of porous material, κ_T is the effective thermal diffusivity, $\overline{\lambda_1}$ is stress relaxation time, $\overline{\lambda_2}$ is strain retardation time and α_T is the coefficient of thermal expansion. The externally imposed thermal boundary conditions are given by

$$\begin{aligned} T &= T_0 + \Delta T && \text{at } z = 0 \\ &= T_0 && \text{at } z = d, \end{aligned} \tag{7.2.5}$$

where ΔT is the temperature difference across the porous medium. The basic state is assumed to be quiescent, and the quantities in this state are given by

$$q_b = (0, 0, w_0),\ \rho = \rho_b(z, t),\ p = p_b(z, t),\ T = T_b(z, t). \tag{7.2.6}$$

Substituting the Eq.(7.2.6) in Eqs.(7.2.1-7.2.4), we get the following equations which help us to define hydrostatic pressure and temperature

$$\frac{\partial p_b}{\partial z} = \frac{\mu}{K} w_0 - \rho_b g, \tag{7.2.7}$$

$$w_0 \frac{\partial T_b}{\partial z} = \kappa_T \frac{\partial^2 T_b}{\partial z^2}, \tag{7.2.8}$$

$$\rho_b = \rho_0 \left[1 - \alpha_T \left(T_b - T_0\right)\right]. \tag{7.2.9}$$

The solution of Eq.(7.2.8) subject to the thermal boundary conditions Eq.(7.2.5), is given by:

$$T_b = T_0 + \Delta T \frac{e^{Qz} - e^{Q}}{1 - e^{Q}}. \tag{7.2.10}$$

The finite amplitude perturbations on the basic state are superposed in the form:

$$\vec{q} = \vec{q_b} + \vec{q}',\ \rho = \rho_b + \rho',\ p = p_b + p',\ T = T_b + T'. \tag{7.2.11}$$

We introduce the Eq.(7.2.11) and the basic state temperature field, Eq.(7.2.10) in Eqs.(7.2.1-7.2.4) and considering two dimensional flow the non-dimensionalized (chapters 2 and 3.1) system of equations

$$\left(\lambda_2 \frac{\partial}{\partial t} + 1\right) \nabla^2 \psi + \left(\lambda_1 \frac{\partial}{\partial t} + 1\right) \left(\frac{1}{Pr_D} \frac{\partial \nabla^2 \psi}{\partial t} + Ra \frac{\partial T}{\partial x}\right) = 0, \tag{7.2.12}$$

$$-\frac{dT_b}{dz}\frac{\partial\psi}{\partial x}-\left(\nabla^2-Q\frac{\partial}{\partial z}\right)T=-\frac{\partial T}{\partial t}+\frac{\partial(\psi,T)}{\partial(x,z)}. \tag{7.2.13}$$

where $Q=\frac{w_0d^2}{\kappa_T}$ is Peclet number, $Pr_D=\frac{\nu\delta_1d^2}{K\kappa_T}$ is Prandtl−Darcy number. The Eq.(7.2.13) shows that the basic state solution influences the stability problem through the factor $\frac{dT_b}{dz}$, which is given by Eq.(7.2.10). The above system of coupled Eqs.(7.2.12-7.2.13) will be solved for stress free and isothermal boundary conditions as given in Eq.(2.2.26).

7.2.3 Amplitude equation for oscillatory convection

Using asymptotic expansion Eq.(2.3.1) for χ on physical quantities of the system Eqs.(7.2.12-7.2.13) is solved for different orders of the perturbation parameter χ. In order to allow for anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of time τ and the slow time scale of time s. Therefore, the scaling of time variable is such that $\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}+\chi^2\frac{\partial}{\partial s}$. In the first order problem, the nonlinear term in energy equation vanishes, therefore, the first order problem reduces to the linear stability problem for overstability.

At the lowest order, we have

$$\begin{bmatrix}\left(\lambda_2\frac{\partial}{\partial\tau}+1\right)\nabla^2 & R_0\left(\lambda_1\frac{\partial}{\partial\tau}+1\right)\frac{\partial}{\partial x}\\ -\frac{dT_b}{dz}\frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2+Q\frac{\partial}{\partial z}\right)\end{bmatrix}\begin{bmatrix}\psi_1\\ T_1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{7.2.14}$$

The solutions of the above system is assumed to be of the form

$$\psi_1 = \left(\mathbb{A}_1(s)e^{i\omega\tau}+\overline{\mathbb{A}_1}(s)e^{-i\omega\tau}\right)\cos ax\sin\pi z, \tag{7.2.15}$$

$$T_1 = \left(\mathbb{B}_1(s)e^{i\omega\tau}+\overline{\mathbb{B}_1}(s)e^{-i\omega\tau}\right)\sin ax\sin\pi z. \tag{7.2.16}$$

The undetermined amplitudes are functions of slow time scale, and are related by the following relation

$$\mathbb{B}_1(s)=-\frac{4\pi^2a}{(4\pi^2+Q^2)(c+i\omega)}\mathbb{A}_1(s), \tag{7.2.17}$$

where $c=a^2+\pi^2$. The thermal Darcy-Rayleigh number for stationary mode of convection is obtained as

$$Ra=\frac{c^2(4\pi^2+Q^2)}{4a^2\pi^2}. \tag{7.2.18}$$

For $Q = 0$, we get the classical results of Horton and Rogers (1945) and Lapwood (1948). We find Darcy−Rayleigh number for oscillatory convection as given below

$$R_0 = \frac{(4\pi^2 + Q^2)}{4a^2\pi^2}\left(\frac{c^2 + \omega^2(\lambda_1\lambda_2 c^2 + \lambda_1 c - \lambda_2 c)}{1 + \omega^2\lambda_1^2} - \frac{c\omega^2}{Pr_D}\right). \tag{7.2.19}$$

The critical wave number is defined as that value which minimizes Ra. The critical wave number does not depend on (λ_1, λ_2) in stationary mode but in oscillatory mode. Also, we see that the overstability can occur for a particular wave number a only, if the following inequality holds.

$$\lambda_1 > \frac{Pr_D + c}{cPr_D} + \lambda_2. \tag{7.2.20}$$

The dimensionless frequency of the neutral oscillatory mode is

$$\omega^2 = \frac{(cPr_D(\lambda_1 - \lambda_2) - c - Pr_D)}{(Pr_D\lambda_1\lambda_2 + c\lambda_1)}. \tag{7.2.21}$$

In the second order, we get

$$\frac{\partial(\psi_1, T_1)}{(x, z)} = \frac{\pi a}{2}\{\mathbb{A}_1(s)\mathbb{B}_1(s)e^{2i\omega\tau} + \overline{\mathbb{A}_1}(s)\overline{\mathbb{B}_1}(s)e^{-2i\omega\tau} + \mathbb{A}_1(s)\overline{\mathbb{B}_1}(s) + \overline{\mathbb{A}_1}(s)\mathbb{B}_1(s)\}\sin 2\pi z. \tag{7.2.22}$$

From the above relation, we can deduce that the velocity and temperature fields have the terms having frequency 2ω and independent of past time scale. Thus, we write the second order temperature term as follows

$$T_2 = \{T_{20} + T_{22}e^{2i\omega\tau} + \overline{T}_{22}e^{-2i\omega\tau}\}\sin 2\pi z \tag{7.2.23}$$

where T_{22} and T_{20} are temperature fields having the terms having the frequency 2ω and independent of fast time scale, respectively. The solutions of the second order problem is of the following form

$$\psi_2 = 0, \quad T_{20} = \frac{a}{8\pi}\{\mathbb{A}_1(s)\overline{\mathbb{B}_1}(s) + \overline{\mathbb{A}_1}(s)\mathbb{B}_1(s)\}, \tag{7.2.24}$$

and

$$T_{22} = \frac{\pi a}{8\pi^2 + 4i\omega}\mathbb{A}_1(s)\mathbb{B}_1(s). \tag{7.2.25}$$

The horizontally averaged Nusselt number, Nu, for the oscillatory mode of convection is defined while using the expression of T_2 in Eq.(3.1.28) as

$$\mathrm{Nu}(s) = 1 + \left(\frac{2a^2c\pi^2}{(4\pi^2 + Q^2)(c^2 + \omega^2)} + \frac{2a^2\pi^4}{(4\pi^2 + Q^2)\sqrt{4\pi^4 + \omega^2}\sqrt{c^2 + \omega^2}}\right)|\mathbb{A}_1(s)|^2. \tag{7.2.26}$$

Although, in this problem we concentrate on oscillatory mode of convection only, however, for the sake of comparison of results, we also define the Nusselt number for stationary (following chapter 2) mode of convection as

$$\mathrm{Nu}(\tau) = 1 + \frac{4\pi^4 a^2 (e^Q - 1)}{\delta^2 Q (4\pi^2 + Q^2)^2} \mathbb{A}^2(\tau). \tag{7.2.27}$$

At the third order, we have

$$\begin{bmatrix} \left(\lambda_2 \frac{\partial}{\partial \tau} + 1\right) \nabla^2 & R_0 \left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial}{\partial x} \\ -\frac{dT_b}{dz} \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial \tau} - \nabla^2 + Q \frac{\partial}{\partial z}\right) \end{bmatrix} \begin{bmatrix} \psi_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix} \tag{7.2.28}$$

where

$$R_{31} = -\frac{1}{Pr_D} \frac{\partial \nabla^2 \psi_1}{\partial s} - \lambda_2 \frac{\partial \nabla^2 \psi_1}{\partial s} - R_0 \lambda_1 \frac{\partial}{\partial s} \frac{\partial T_1}{\partial x} - R_2 \left(\lambda_1 \frac{\partial}{\partial \tau} + 1\right) \frac{\partial T_1}{\partial x}, \tag{7.2.29}$$

$$R_{32} = \frac{\partial \psi_1}{\partial x} \frac{\partial T_2}{\partial z} - \frac{\partial T_1}{\partial s}. \tag{7.2.30}$$

Now, applying the solvability condition for the existence of third order solution, we obtain the following Ginzburg–Landau equation that describes the temporal variation of the amplitude $\mathbb{A}_1(s)$ of the convection cell

$$\frac{\partial \mathbb{A}_1(s)}{\partial s} - \gamma_1^{-1} F \mathbb{A}_1(s) + \gamma_1^{-1} k |\mathbb{A}_1(s)|^2 \mathbb{A}_1(s) = 0, \tag{7.2.31}$$

where

$$\gamma_1 = \left[\frac{c}{Pr_D} + \lambda_2 c + \frac{4\pi^2 a^2 R_0 (1 + i\omega\lambda_1)}{(c + i\omega)^2 (4\pi^2 + Q^2)} - \frac{4\pi^2 a^2 R_0 \lambda_1}{(c + i\omega)(4\pi^2 + Q^2)} \right],$$

$$k = \left[\frac{\pi^2 a^4 c R_0 (1 + i\omega\lambda_1)}{(c + i\omega)(c^2 + \omega^2)(4\pi^2 + Q^2)} + \frac{\pi^4 a^4 R_0 (1 + i\omega\lambda_1)}{(c + i\omega)^2 (2\pi^2 + i\omega)(4\pi^2 + Q^2)} \right],$$

$F = \left[\frac{4\pi^2 a^2 R_2 (1 + i\omega\lambda_1)}{(c + i\omega)(4\pi^2 + Q^2)} \right]$. Writing $\mathbb{A}_1(s)$ in the phase-amplitude form as

$$\mathbb{A}_1(s) = |\mathbb{A}_1(s)| e^{i\phi} \tag{7.2.32}$$

We obtain the following equation for the amplitude $|\mathbb{A}_1(s)|$ as

$$\frac{\partial |\mathbb{A}_1(s)|^2}{\partial s} = 2p_r |\mathbb{A}_1(s)|^2 - 2l_r |\mathbb{A}_1(s)|^4 \tag{7.2.33}$$

$$\frac{\partial (ph(\mathbb{A}_1(s)))}{\partial s} = p_i - l_i |\mathbb{A}_1(s)|^2, \tag{7.2.34}$$

where $\gamma_1^{-1}F(s) = p_r + ip_i$, $\gamma_1^{-1}k = l_r + il_i$ and $ph(.)$ represents the phase shift. One can observe here from Eq.(7.2.33) for the case $l_r > 0$ and $Ra > Ra_c$ i.e. $p_r > 0$, the solution gives as $\mathbb{A} \sim \mathbb{A}_0 e^{p_r s}$ as $s \longrightarrow -\infty$, and $\mathbb{A} \longrightarrow 0$ is unstable solution, and a new stable solution develops, $\mathbb{A} = \sqrt{\frac{p_r}{l_r}}$ as $s \longrightarrow \infty$, whatever be the value of $\mathbb{A}_0$. This is called supercritical pitch fork bifurcation, the base system being linearly unstable for $Ra > Ra_c$ but settling down as a new laminar flow. The steady state amplitude exists when Ra_c takes positive values. For better understanding the results on heat transport, we calculate the mean value of Nusselt number MNu. For this a representative time interval that allows a clear comprehension needs to be chosen. The interval $(0, 2\pi)$ seemed to be an appropriate interval to calculate MNu. The time-averaged Nusselt number is defined as

$$\mathrm{MNu} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Nu}\, ds. \tag{7.2.35}$$

A discussion of the results now follows culminating in a listing of conclusions.

7.2.4 Results and discussions

A weak nonlinear oscillatory convective instabilities in a viscoelastic fluid saturated porous layer has been studied in the presence of vertical throughflow. The viscoelastic model is considered for the existence of oscillatory mode of convection. It is to be noted that in Eq.(2.2.26), the only stress free and isothermal boundaries are considered to simplify the weak nonlinear theory. However, one can consider different boundaries for linear stability analysis, as done by Shivakumara (1999), Shivakumara and Khalili (2001) and Shivakumara et al. (2007). Amplitude of convection is obtained by deriving a non-autonomous complex form of Ginzburg−Landau Eq.(7.2.31) in terms of the system parameters. Ginzburg−Landau Eq.(7.2.31) has been solved for the amplitude using the in built function NDsolve Mathematica 8. The Nusselt numbers for both stationary and oscillatory mode of convection have been obtained. For good approximation and understanding, the average value of Nu is calculated as MNu. Since the porous medium is assumed to be closely packed, the Darcy−model is considered. In order to illustrate

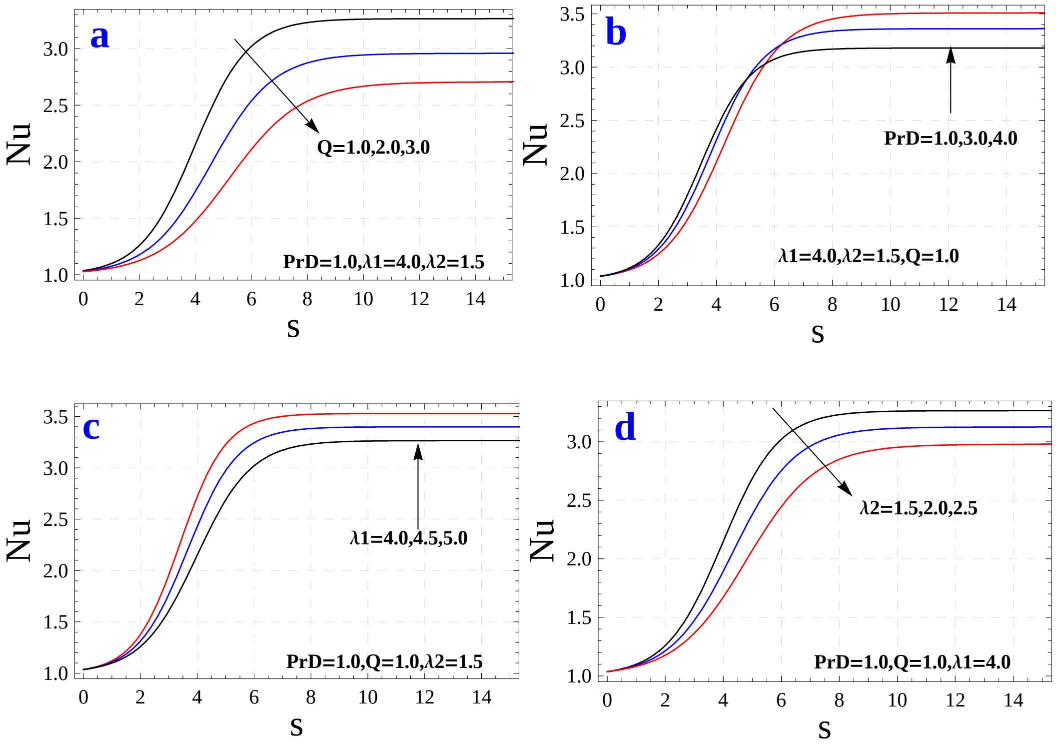


Figure 7.5: Effect of various parameters on heat transport (a) Pr_D (b) Q (c) λ_1 (d) λ_2

the effects of relaxational parameters, λ_1, λ_2, the Peclet number Q, on heat transport, we plot the curves for both the Nusselt and mean Nusselt numbers, with respect to time and Q. It is observed that the relation Eq.(7.2.20) leads to an interesting result; that for a horizontal porous layer heated from bellow; the oscillatory type of instability is possible only when the relaxation parameter λ_1 is greater than the retardation parameter λ_2.

A small amount of throughflow in a particular direction either is to destabilizes or stabilize the system, hence, we consider Q values around one. The numerical results for Nu obtained from the expression, Eq.(7.2.26) by solving the amplitude Eq. (7.2.33) have been presented in the figures 7.5−7.9. It is clear from the expressions Eqs.(7.2.26-7.2.35) in conjunction with Eq.(7.2.33) that Nu and MNu are functions of the system parameters. The effect of each parameter on heat transport is shown in the figures, wherein the plots of Nusselt number Nu versus s and MNu versus Q are presented. It is found from the figures that the value of Nu starts with 1 and remains constant for quite some time, thus showing the conduction state initially. Then the value of Nu increases with time, showing the convection state, and finally becomes constant upon further increasing

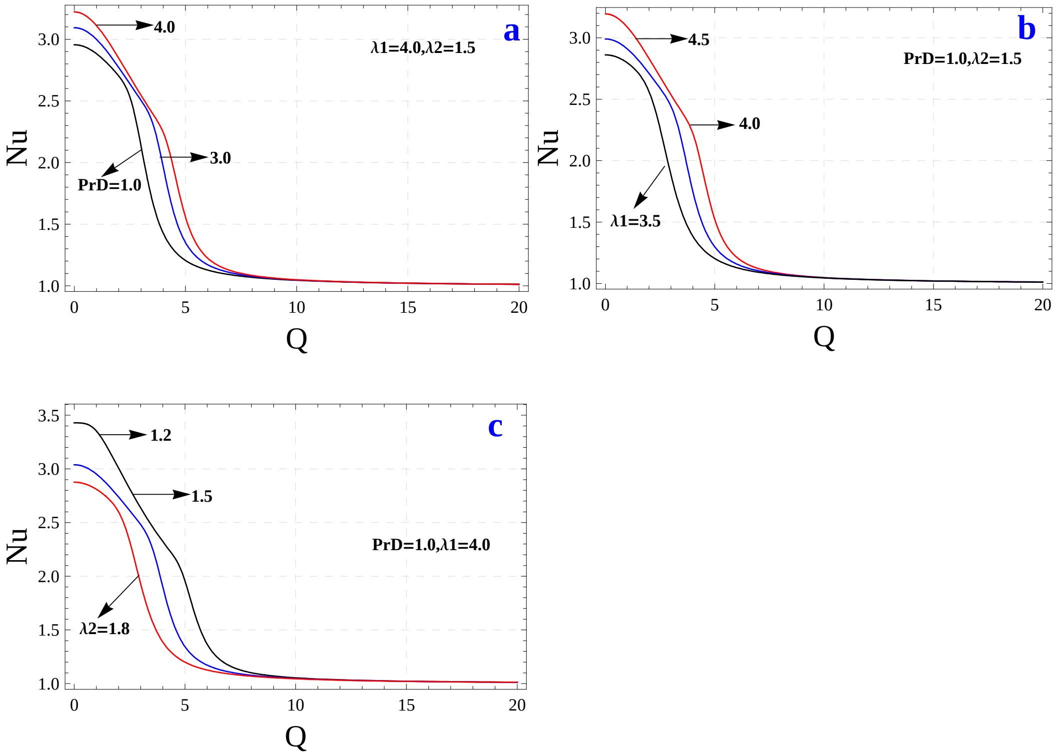


Figure 7.6: Effect of Q on Nu for various values of (a) Pr_D (b) λ_1 (c) λ_2

s, thus achieving the steady state. For the case of MNu versus Q, it is found that as Q increases MNu approaches 1.

The effect of Q on heat transfer, figure 7.5a is found to be stabilize the system. The oscillatory Rayleigh number increases with an increment in Q, and is independent from the throughflow direction. This may be due to the fact that the throughflow is to confine significant thermal gradients to a thermal boundary layer at the boundary towards which the throughflow is directed. The effective length scale is thus smaller than the thickness of the porous layer. Hence the Rayleigh number will be much less than the actual value of Rayleigh number. Therefore, large values of Rayleigh number are needed for the onset of convection when the throughflow strength increases, which are the results obtained by Khalili and Shivakumar (1998) in the case of free-free boundaries. The opposite results were obtained by Nield (1987) in the case of a fluid layer for small amount of throughflows. According to Shivakumara and Sureshkumar (2007), the opposite effect may be due to the distortion of steady-state basic temperature distribution from linear to nonlinear by the throughflow. A measure of this is given by the basic state temperature and this can be interpreted as a rate of energy transfer into the disturbance by interaction of the perturbation convective motion with basic temperature gradient. The maximum temperature occurs at a place where the perturbed vertical velocity is high, and this leads to an increase in energy supply for destabilization. The effect of Prandtl−Darcy number Pr_D on Nu is given in figure 7.5b. The effect is to destabilize the system hence the heat transfer, which are results earlier obtained by Bhadauria et al. (2013a,b,c). The effects of relaxational parameters λ_1 and λ_2 on Nu for the onset of oscillatory convection can be seen in figures 7.5c and 7.5d, respectively. It is clear that the Nu decreases upon decreasing λ_1 and increasing λ_2. This is due to the stabilizing effects of low λ_1 and high λ_2, which are related with viscosity and elasticity of the viscoelastic fluids respectively. On the other hand, the amplitude of oscillation will decrease and the convective flow becomes stable with increasing λ_2 but decreasing λ_1. Opposite effect can be seen in the case of time retardation parameter λ_2 given in figure 7.5d. Thus, it is confirmed that the elastic behavior of the non-Newtonian fluids leads to the oscillatory motions. The critical value of dimensionless frequency for marginally oscillatory modes is obtained from the relation, Eq.(7.2.21). One can notice here that the oscillatory frequency is dependent of viscoelastic parameters and independent of throughflow parameter Q. The effect of λ_1, λ_2 on ω^2, the critical value of the frequency increases with increasing relaxation time, but with decreasing retardation time. These are the common results for most of the viscoelastic fluids, compatible with the results of Kim et al. (2003) and Rajib and Layek (2012), therefore not depicted graphically.

In figures 7.6a-c, we have calculated Nu as a function of Q at time $s = 6.0$ and depicted the variation of Nu with respect to Q. From the figure 7.5, it is noted that at time $s = 6.0$ the heat transfer in the system is maximum for lower values of Q. The results presented in figures 7.6a-c have maximum heat transfer at lower values of Q. Further Nu approaches 1 as Q increases, showing that throughflow has a strongly stabilizing effects. In figure 7.6a, we found that the effect of Pr_D is to enhance the heat transfer. In figure 7.6b, the effect of time relaxation parameter λ_1 on Nu is found to increase the amount of

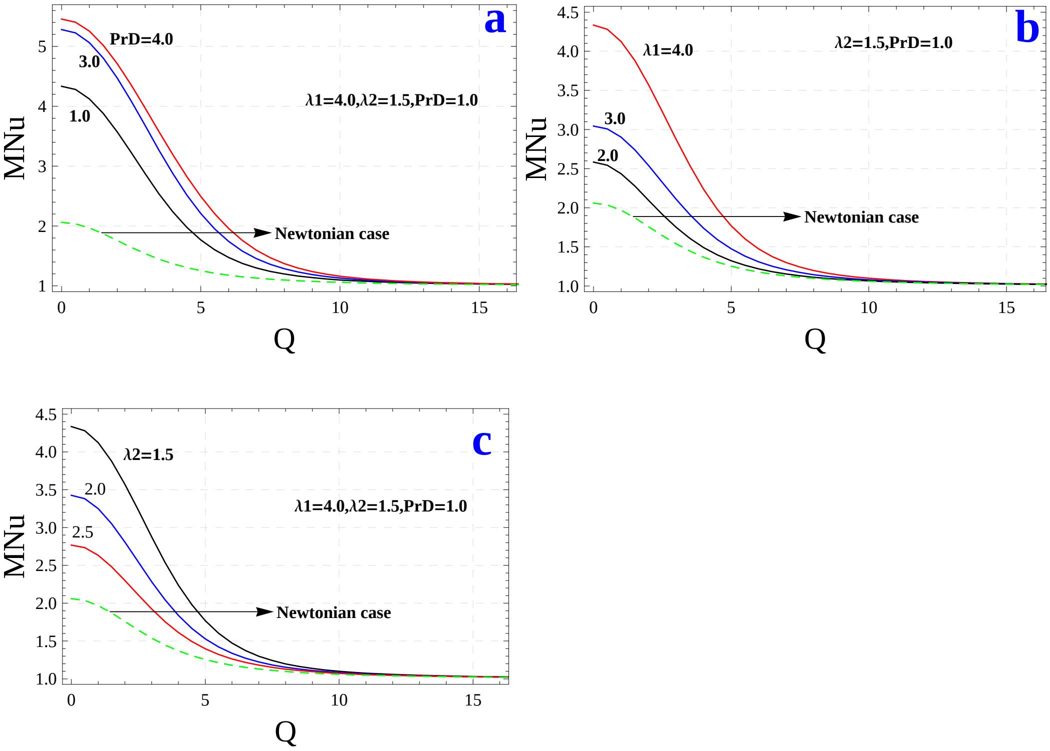


Figure 7.7: Effect of Q on MNu for various values of (a) Pr_D (b) λ_1 (c) λ_2

heat transfer, but opposite effect was found for time retardation parameter λ_2 as shown in figure 7.6c. In figures 7.7a-c, we have calculated MNu as a function of Q. It is found that MNu is maximum at lower values of Q, and approaches 1 as Q increases, thus throughflow has a strongly stabilizing effect. The effects of Pr_D, λ_1 and λ_2 on are found similar to those given in figures 7.6a-c.

In figures 7.8-7.9, the stream lines and the corresponding isotherms are depicted, respectively at slow time $s = 0.0, 1.0, 2.0, 3.0, 6.0, 8.0$, and for fixed values of the parameters as $\lambda_1 = 4.0, \lambda_2 = 1.5, Q = 1.0$ and $\chi = 0.5$. From the figures, we found that initially when the time is small, the magnitude of stream function is also small as given in figures 7.8a-b, and isotherms are straight lines that is the system is in conduction state, figures 7.9a-b. However, as time increases, the magnitude of stream functions increase and the isotherms lose their evenness. This shows that onset of convection is taking place in the system, and becomes faster on further increasing the time s. However, the system achieves steady state beyond $s = 6.0$ as there is no change in the streamlines and isotherms given in

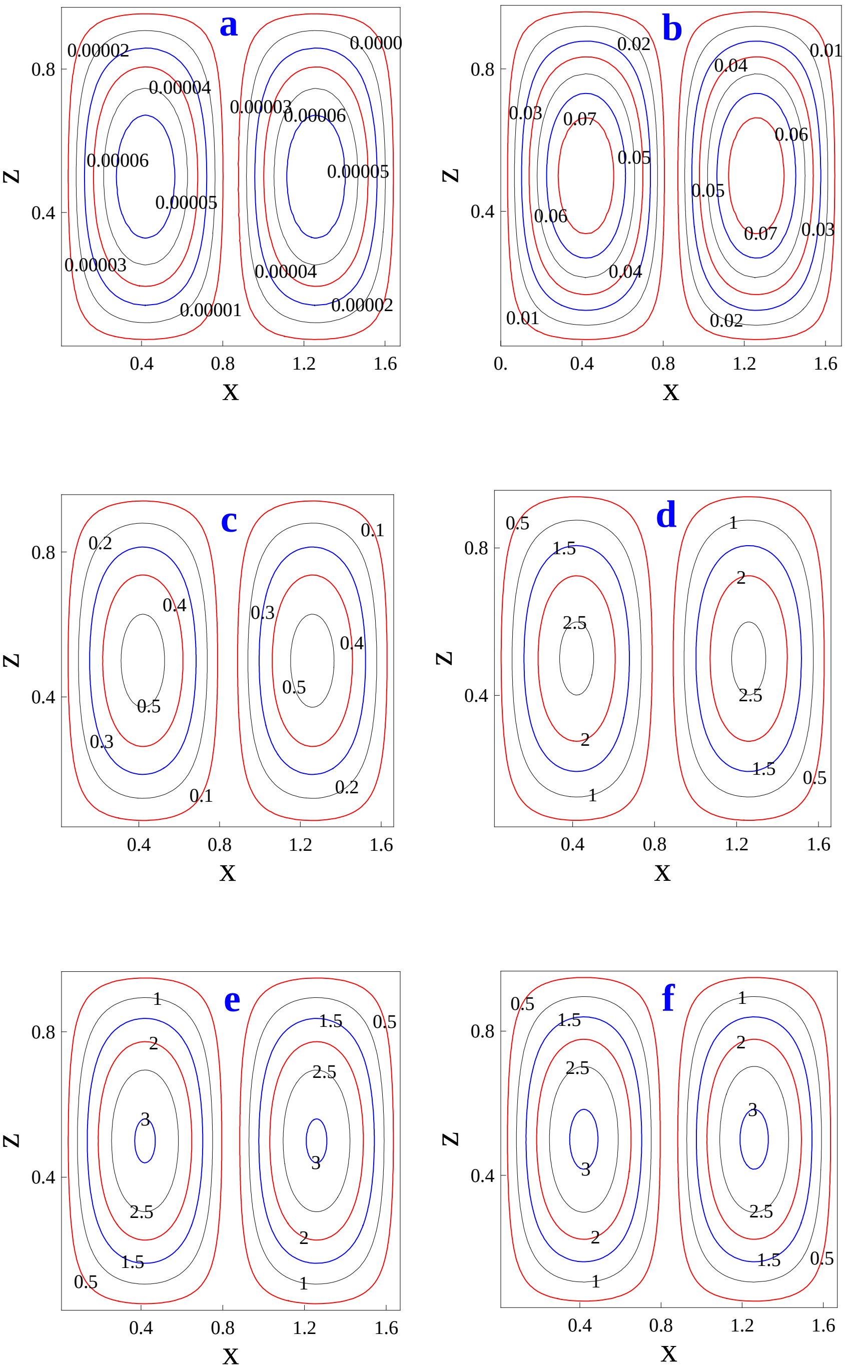


Figure 7.8: Streamlines for various values of time s (a) s = 0 (b) s = 1 (c) s = 2(d) s = 3 (e) s = 6 (f) s = 8

figures 7.8-7.9d-f. These results can also be confirmed from the figure 7.5.

7.2.5 Conclusions

We have studied the effect of throughflow on overstability of Bénard-Darcy convection by performing a weakly nonlinear stability analysis resulting in the complex Ginzburg−Landau amplitude equation. The following conclusions are made upon the pervious analysis

1. Upon increasing Pr_D, Nu and MNu increase, hence advances the onset of convection.
2. Upon increasing λ_1, Nu and MNu increase, hence advances the onset of convection.
3. Upon increasing λ_2, Nu and MNu decreases, hence delays the onset of convection.
4. Critical Rayleigh-Darcy number depends on λ_1, λ_2 for oscillatory case, but independent in stationary case.
5. Oscillatory mode exists only when the values of λ_1, λ_2 chosen according to the Eq.(7.2.20).
6. Supercritical pitch fork bifurcation exits for Eq.(7.2.31).
7. Throughflow Q has strongly stabilizing effect on the system for oscillatory case, irrespective of the direction of the flow.
8. Throughflow Q has both stabilizing and destabilizing effects, corresponding to downward and upward directions, for stationary case.

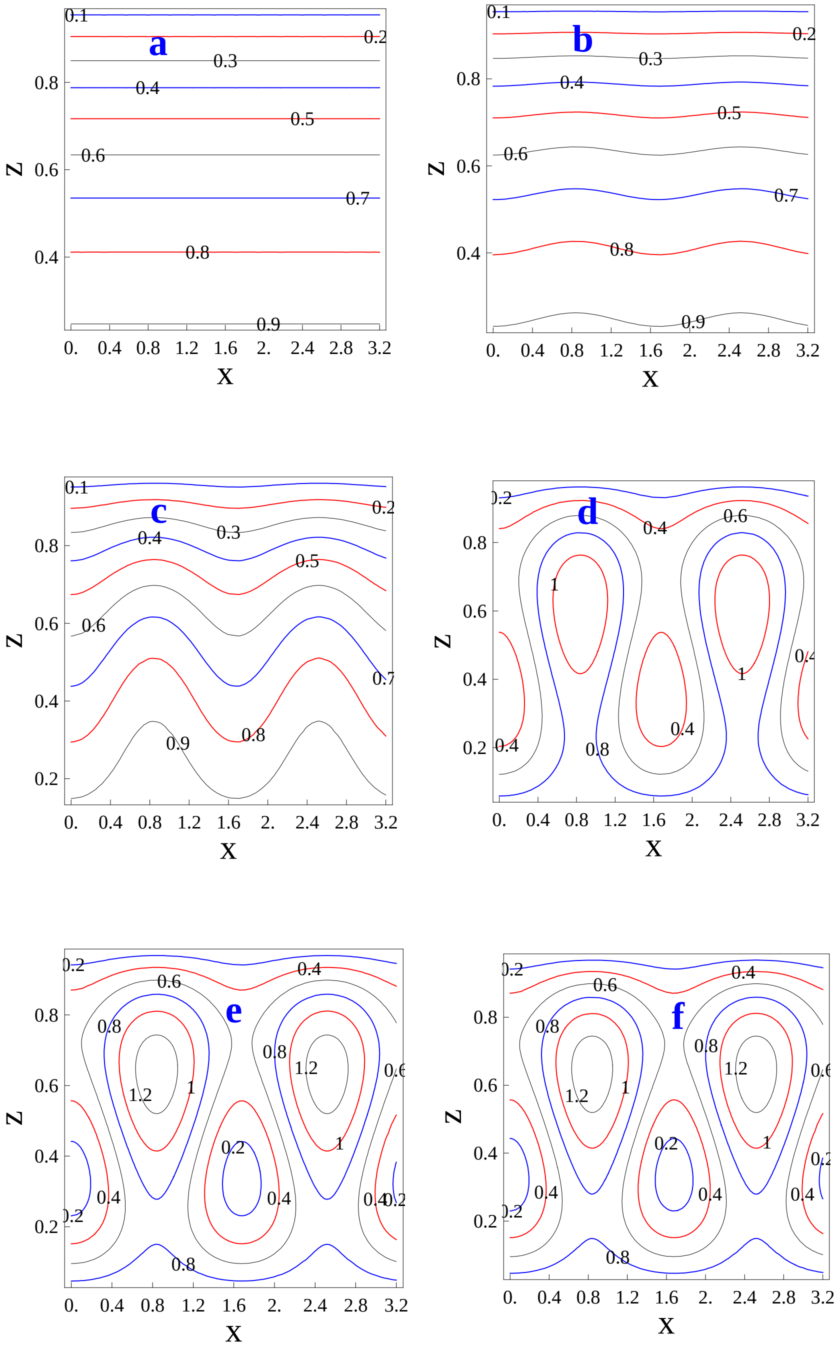


Figure 7.9: Isotherms for various values of time s (a) s = 0 (b) s = 1 (c) s = 2 (d) s = 3 (e) s = 6 (f) s = 8

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List of Publications

Publications in International Journals

1. Bhadauria, B.S., Kiran, P.: Heat transport in an anisotropic porous medium saturated with variable viscosity liquid under temperature modulation. Transp Porous Media 100, 279−295 (2013a) DOI 10.1007/s11242-013-0216-0 **Chapter-2 Springer IF 1.55**
2. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid layer under gravity modulation. Int. J Non-Linear Mech. 65, 133−140 (2014a) DOI: 10.1016/j.ijnonlinmec.2014.05.002 **Chapter-3.1 Elsevier IF 1.463**
3. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation. Transp Porous Media. 104, 451−467 (2014b) DOI 10.1007/s11242-014-0343-2 **Chapter-3.2 Springer IF 1.55**
4. Bhadauria, B.S., Kiran, P.: Weakly nonlinear oscillatory convection in a viscoelastic fluid saturating porous medium under temperature modulation. Int. J Heat Mass Transf. 77, 843−851 (2014c) DOI: 10.1016/j.ijheatmasstransfer.2014.05.037 **Chapter-4.1 Elsevier IF 2.522**
5. Bhadauria, B.S., Kiran, P.: Heat transfer for oscillatory convection in a binary viscoelastic fluid layer subjected to temperature modulation at the boundaries. Int. Commu in Heat & Mass Transf. **Accepted** (2014d) **Chapter-4.2 Elsevier IF 2.124**

6. Bhadauria, B.S., Kiran, P.: Weak nonlinear thermal instability under magnetic field and rotational speed modulation. Physica Scripta. 89, 095209 (2014e) doi:10.1088/0031-8949/89/9/095209**Chapter-5.1 IOP IF 1.296**

7. Bhadauria, B.S., Kiran, P.: Effect of rotational speed modulation on heat transport in a fluid layer with temperature dependent viscosity and internal heat source. Ain Shams Eng J. (2014f) DOI: 10.1016/j.asej.2014.05.005 **Chapter-5.2 Elsevier**

8. Bhadauria, B.S., Kiran, P.: Weak nonlinear double diffusive magnetoconvection in a Newtonian liquid under temperature modulation. Int. J Eng Math 2014, 01−14 (2014g) http://dx.doi.org/10.1155/2014/296216 (**Hindawi publication**)

9. Bhadauria, B.S., Kiran, P.: Study of heat and mass transport in temperature dependent viscous fluid under gravity modulation. Int. J Comp Sci with Appli S(1) 33−38 (2013b) **MJM publishers IF 0.3**

10. Bhadauria, B.S., Kiran, P.: Study of heat and mass transport in a temperature dependent viscosity fluid layer under temperature modulation, Int. J Sci Eng Res, 5, 1954−1963 (2014h)

11. Bhadauria, B.S., Kiran, P.: Weak nonlinear thermal instability under vertical magnetic field, temperature modulation and heat source. Int. J Eng Res Appli. 4, 200−208 (2014i)

12. Bhadauria, B.S., Kiran, P.: Study of heat transport by stationary magnetoconvection in a Newtonian liquid under gravity modulation with internal heating effects. Int Research J. Math Sci 3, 01−07 (2014j)

13. Bhadauria, B.S., Kiran, P.: Weakly nonlinear convection in a variable viscosity fluid saturated porous medium under internal heating and temperature modulation. Int Research J. Eng Sci 2, 01−07 (2014k)

Publications in Proceedings

1. Bhadauria, B.S., Kiran, P.: Weakly nonlinear Bénard−Darcy convection under rotation speed modulation and internal heating effects. ISTAM 01−19 (2013)

2. Bhadauria, B.S., Kiran, P.: Weak nonlinear thermal convection in a fluid layer under rotation speed modulation. ISBN-(13):978-93-392-0316-0, ISBN(10):93-392-0316-x (2014) BY McGraw Hill Education

3. Bhadauria, B.S., Kiran, P.: Weakly nonlinear double diffusive convection in a temperature dependent viscosity fluid saturated porous medium under temperature modulation. Int J. Eng Trends & Tech 146−153 (2014) ISSN: 2231−5381 NCETMS

Papers presented in /International and National Conferences

International Conferences

1. Study of heat transport by stationary magneto-convection in a Newtonian liquid under temperature modulation with internal heating effects. Int Conf on Mathematical modeling and Numerical Simulation 2013 July 01−03
2. Study of heat and mass transport in a temperature dependent viscosity fluid layer under temperature modulation. 18th Annual cum 3rd Int Conf on Gwalior Academy of Mathematical Science 2013 Sept 22−26
3. Weakly nonlinear Darcy convection in Nanofluids under gravity modulation. Int Conf on Nanoscience and Nano-Technology 2013 Nov 18−20
4. Weakly nonlinear Bénard−Darcy convection under rotation speed modulation and internal heating effects. 58th Congress of the Indian Society Of Theoretical and Applied Mechanics (An Internal Meet) 2013 Dec 18−21
5. Study of heat transport in a porous medium saturated by Nanofluid under thermal modulation. Int Conf on Recent Advances in Mathematical Sci and Applications 2013 Dec 24−26
6. Study of heat transport by stationary magnetoconvection in a Newtonian liquid under gravity modulation with internal heating effects. Int Conf on Mathematics

and Engineering Sci 2014 March 20−22

7. Weakly nonlinear convection in a variable viscosity fluid saturated porous medium under internal heating and temperature modulation. Int Conf on Mathematics and Engineering Sci 2014 March 20−22

8. Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation. Int Conf on Emerging Trends in Compu and Appli Mathematics 2014 June 2−4

9. A Mathematical model for chaotic convection in a porous medium with time periodic heating at the boundaries. Int Conf on Modeling and Computing 2014 July 10−11

10. Oscillatory and chaotic magnetoconvection in a binary viscoelastic fluid with time periodic heating. Int Symposium on Advances in Materials Characterization 2014 July 14 Poster Presentation

National Conferences

1. Heat transport in an anisotropic porous medium saturated with variable viscosity liquid under temperature modulation. 1^{st} Lucknow Science Congress 2013 March 20−21

2. Thermal convection in a temperature dependent viscosity fluid layer under modulation. Frontier areas of research in mathematics. 2013 Aug 24

3. Study of heat and mass transport in a temperature dependent viscous fluid under gravity modulation. National Conf Recent Advances in Mathematical Analysis and Applications 2013 Sept 6−7

4. Nonlinear thermal convection in a porous medium saturated by Nanofluid under g−jitter and internal heating. 8^{th} National Conf on Thermodynamics of Chemical, Biological and Environmental System 2013 Nov 25−26

5. Thermo-rheological effect on weakly nonlinear Rayleigh−Bénard convection under rotation speed modulation. National Conf on Mathematics 2013 Nov29−Dec01

6. Weak nonlinear thermal convection in a fluid layer under rotation speed modulation. First National Conf on Emerging Trends in Eng and Sci 2014 Jan 30−31
7. Weakly nonlinear double diffusive convection in a temperature dependent viscosity fluid saturated porous medium under temperature modulation. NCETMS 2014 Feb 06−07
8. Weakly nonlinear double diffusive oscillatory convection in a viscoelastic fluid layer under gravity modulation. 2^{st} Lucknow Science Congress 2014 March 27−28

Conferences Attended

1. Patent workshop. 2013 March 18
2. Conference on mathematics. Bharata Ganita Parisad Lucknow University 2013 March 24

Papers Communicated

1. Bhadauria, B.S., Kiran, P.: Chaotic convection in a porous medium under temperature modulation. Transp Porous Media (2014) **Chapter-6**
2. Bhadauria, B.S., Kiran, P.: Nonlinear thermal Darcy convection in a nanofluid saturated porous medium under gravity modulation. Advanced Sci Letters (2014) **Revised Chapter-7.1 American Scientific Publishers IF 1.253**
3. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under throughflow effects. J of Porous Media (2014) **Chapter-7.2**
4. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid layer under thermal modulation. J Heat Transfer ASME (2014)
5. Bhadauria, B.S., Kiran, P.: Weak nonlinear double diffusive magneto-convection in a Newtonian liquid under gravity modulation. J of Applied Fluid Mech **Revised** (2014)
6. Bhadauria, B.S., Kiran, P.: Nonlinear thermal instability in a Dielectric liquid under temperature modulation. Physica Scripta. (2014)
7. Bhadauria, B.S., Kiran, P.: Throughflow effect on weakly nonlinear oscillatory convection in a viscoelastic fluid saturating porous medium under temperature modulation. Transp Porous Media (2014)

8. Bhadauria, B.S., Kiran, P.: Effect of K*ü*ppers Lortz instability in a rotating fluid layer for an oscillatory convection under gravity modulation. Heat Mass Transf. (2014)
9. Bhadauria, B.S., Kiran, P.: Oscillatory and chaotic magneto−convection in a binary viscoelastic fluid with time periodic heating. J of The Franklin Inst (2014)
10. Kiran P and Bhadauria, B.S.: Weak nonlinear oscillatory magnetoconvection under gravity modulation. Physica Scripta. (2014)
11. Bhadauria, B.S., Kiran, P.: Nonlinear throughflow effects on thermally modulated porous medium. Ain Shams Eng J. (2014)
12. Bhadauria, B.S., Kiran, P., Chamkha A.J.: Study of Heat Transport in a Temperature-Dependent Viscosity Liquid Under g-Jitter and Internal Heating Effects. Appl Math. Mech (2014)

RESUME

Bio-Brief of Mr. Palle Kiran
DOB: June 4, 1986.

Palle Kiran
Department of Applied Mathematics,
Babasheb Bimrao Ambedkar Central University,
Lucknow-UP, India. 226025.
E-mail: kiran40p@gmail.com

EDUCATION

Institute Name	**Degree Name**	**Year of passing**	**Percentage & CGPA**
Babasaheb Bhimrao Ambedkar University	Ph.D.(2014..,)	2012(Aug)-2014(Aug)	Course work 9.0/10.0
Sri Ganesh College of Eng & Tech (worked as Lecturer)	Lecturer	2010-2012	
Pondicherry University (Puducherry).	M.Phil (Mathematics)	2008-2010	7.95(72.5)
Pondicherry University (Puducherry).	M.Sc (Mathematics)	2006-2008	6.68(58.40)
Silver Jubilee Degree College Boys (Kurnool) AP.	B.Sc (Computers)	2003-2006	70.2
APSWR Boys College (Kurnool) AP.	Intermediate (12th)	2000-2003	84.6
APR Boys School (Kurnool) AP.	SSC (10th)	1999-2000	77

AREA OF INTEREST

- Fluid Mechanics (Research)
- Differential Equation.
- Linear Algebra.
- Complex Analysis.

DISSERTATIONS UNDER TAKEN

- Pythagorean Theory (Number Theory). MPhil (Pondicherry University)
- Nonlinear Thermal Instability Under Various Physical Configurations (Ph.D.BBAU)

TEACHING EXPERIENCE

Sno	Name of the institute	Designations	Duration
1	Sri Ganesh College of Engineering and Technology	Lecturer	17-08-2010 to 31-07-2012

INTERESTS

- Playing Chess.
- Listening Music.

EXTRA CURRICULAR ACTIVITIES (PERIOD AUG 2012 TO AUG 2014)

- GATE-2013 Qualified
- Conferences Attended 22

  National 12: International 10
- Papers Presented

  National 8: International 11
- Papers Published

  International 15

  Proceedings Publications 4 (International)
- Thought C-Language for MSc (Mathematics) 2 semesters 2012/2013(Aug-Dec)

LIST OF SELECTED PUBLICATIONS

1. Bhadauria, B.S., Kiran, P.: Heat transport in an anisotropic porous medium saturated with variable viscosity liquid under temperature modulation. Transp Porous Media 100, 279-295 (2013) Springer **IF 1.55**.
2. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid layer under gravity modulation. Int. J Non-Linear Mech. 65, 133-140 (2014) Elsevier IF **1.463.**

3. Bhadauria, B.S., Kiran, P.: Weak nonlinear oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation. Transp Porous Media. 104, 451-467 (2014). Springer **IF 1.460**
4. Bhadauria, B.S., Kiran, P.: Weakly nonlinear oscillatory convection in a viscoelastic fluid saturating porous medium under temperature modulation. Int. J Heat Mass Transf. 77, 843-851 (2014) Elsevier **IF 2.55**
5. Bhadauria, B.S., Kiran, P.: Heat transfer for oscillatory convection in a binary viscoelastic fluid layer subjected to temperature modulation at the boundaries. Int. Commu in Heat Mass Transf Accepted (2014) Elsevier **IF 2.124**
6. Bhadauria, B.S., Kiran, P.: Weak nonlinear thermal instability under magnetic field and rotational speed modulation. Physica Scripta 89, 095209 (2014) **IF 1.296**
7. Bhadauria, B.S., Kiran, P.: Effect of rotational speed modulation on heat transport in a fluid layer with temperature dependent viscosity and internal heat source. Ain Shams Eng J. (2014) Elsevier

LANGUAGES KNOWN

- Telugu
- English
- Hindi
- Tamil (Speak).

PERSONAL DETAILS

Father's Name: Palle Thikkanna,

Address:
Hno: 17-912-178 Maruti Nagar., Near water tank
Nandikotkur (po/vi)
Kurnool (District),
Andhra Pradesh -518401.

Declaration

I hereby declare that all the above-furnished information is true and correct to the best of my knowledge and belief.

Place: Lucknow

Date: 24-08-2014

(Palle Kiran)