CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS AND ITS MOMENTS THROUGH ORDERED RANDOM VARIABLES

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BY

NAZIA WAHID

UNDER THE SUPERVISION OF

Dr. SURINDER KUMAR

Associate Professor

Department of Applied Statistics

(School for Physical Sciences)

BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY

(A CENTRAL UNIVERSITY, LUCKNOW-226025)

AND THE CO-SUPERVISION OF

Dr. MOHD JAHANGIR SABBIR KHAN

Assistant Professor

Department of Statistics and O.R.

ALIGARH MUSLIM UNIVERSITY

(ALIGARH)

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CONTENTS

Chapter 1

Preliminaries and Basic Concepts used in Ordered Random Variables

In this Chapter, we have introduced several models of ordered random variables and results, which are used in the subsequent Chapters.

1.1 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid) random variables (rv) of size n from a continuous population having probability density function (pdf) $f(x)$ and distribution function (df) $F(x)$. If random variables $X_1, X_2, \ldots, X_n$ are arranged in ascending order of magnitude such that

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n},$$

then $X_{r:n}$ is called the r^{th} order statistic. $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are called extremes or smallest and the largest order statistics respectively.

The subject of order statistics deals with the properties and applications of these ordered random variables and of functions involving them, (David and Nagaraja, 2003). Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos (1987). One may also refer's to Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991), Arnold *et al.* (1992) for further references. It is different from the rank statistics in which the order of the value of observation rather than its magnitude is considered. It plays an important role both in model building and in statistical inference.

For Example: extreme values are important in oceanography (waves and tides), weather conditions, material strength (strength of a chain depends on the weakest link) and meteorology (extremes of temperature, pressure etc).

Another very interesting application of the order statistics is found in reliability theory. The r^{th} order statistic $X_{r:n}$ in a sample of size n represents the life-length of a $(n-r+1)$-out-of-n-system. This system consists of n components of the same kind with independently distributed life lengths. All n components start working simultaneously and the system fails, if r or more component fails. In other words, $n-r+1$ components are necessary for the system to work. For $r=1$, the system corresponds to a series system whereas for $r=n$, it corresponds to a parallel system.

1.2 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a *pdf* $f(x)$ and the *df* $F(x)$. Then the *pdf* of the r^{th} order statistics $X_{r:n}$, $1 \le r \le n$ is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = C_{r:n} \left[F(x)\right]^{r-1} \left[1-F(x)\right]^{n-r} f(x), \qquad -\infty < x < \infty, \tag{1.2.1}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} = \left[B(r,\, n-r+1)\right]^{-1}$$

In particular, the *pdf* of the smallest and largest order statistics are

$$f_{1:n}(x) = n\left[1 - F(x)\right]^{n-1} f(x), \qquad -\infty < x < \infty \tag{1.2.2}$$

$$f_{n:n}(x) = n\left[F(x)\right]^{n-1} f(x), \qquad -\infty < x < \infty \tag{1.2.3}$$

and the *df* of $X_{r:n}$ can be obtained as follows

$$\begin{aligned}
F_{r:n}(x) &= Pr(X_{r:n} \leq x) \\
&= Pr(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x) \\
&= \sum_{i=r}^{n} Pr(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are less than or equal to } x) \\
&= \sum_{i=r}^{n} \binom{n}{i} \left[F(x)\right]^{i} \left[1 - F(x)\right]^{n-i}; \qquad -\infty < x < \infty & (1.2.4) \\
&= C_{r:n} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du & (1.2.5) \\
&= I_{F(x)}(r,\, n - r + 1) & (1.2.6)
\end{aligned}$$

where $I_p(a,b) = \dfrac{1}{B(a,b)} \displaystyle\int_0^p u^{a-1}(1-u)^{b-1} du$ is incomplete beta function and

$$B(a,b) = \frac{1}{B(a,b)} \int_0^1 u^{a-1}(1-u)^{b-1} du$$

RHS of (1.2.6) is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991).

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{i+r-1}{r-1} \left[F(x)\right]^{r} \left[1 - F(x)\right]^{i}, \quad -\infty < x < \infty \tag{1.2.7}$$

The k^{th} moment of $X_{r:n}$ is given by

$$\alpha_{r:n}^{(k)} = E\left[X_{r:n}^{k}\right] = \int_{-\infty}^{\infty} x^k f_{r:n}(x) dx \tag{1.2.8}$$

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $1 \leq r < s \leq n$, is given by

$$f_{r,s:n}(x,y) = C_{r,s:n} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} \times [1 - F(y)]^{n-s} f(x) f(y), \qquad -\infty < x < y < \infty \tag{1.2.9}$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} = [B(r,\, s-r,\, n-s+1)]^{-1}$$

The joint *df* of $X_{r:n}$ and $X_{s:n}$, $1 \leq r < s \leq n$, is

$$\begin{aligned} F_{r,s:n}(x,y) &= Pr\,(X_{r:n} \leq x,\; X_{s:n} \leq y) \\ &= Pr(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x \\ &\qquad \text{and at least } s \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \\ &= \sum_{j=s}^{n} \sum_{i=r}^{j} Pr(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x \\ &\qquad \text{and exactly } j \text{ of } X_1, X_2, X_3, \ldots, X_n \text{ are at most } y) \\ &= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} [F(x)]^{i} [F(y) - F(x)]^{j-i} [1 - F(y)]^{n-j} \end{aligned} \tag{1.2.10}$$

We can write the joint *df* of $X_{r:n}$ and $X_{s:n}$ in equation(1.2.10) equivalently as

$$\begin{aligned} F_{r,s:n} &= C_{r,s:n} \int_0^{F(x)} \int_u^{F(y)} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} du\, dv \\ &= I_{F(x)F(y)}\,(r,\, s-r,\, n-s+1)\,, \qquad -\infty < x < y < \infty \end{aligned} \tag{1.2.11}$$

which is incomplete bivariate beta function. It may be noted that for $x \geq y$, the inequality $X_{s:n} \leq y$ implies $X_{r:n} \leq x$, so that

$$F_{r,s:n}(x,\, y) = F_{s:n}(y) \tag{1.2.12}$$

The product moments of j^{th} and k^{th} order of $X_{r:n}$ and $X_{s:n}$ respectively, $1 \leq r < s \leq n$ is given by

$$\alpha_{r,s:n}^{j,k} = E\left[X_{r:n}^{j}\, X_{s:n}^{k}\right] = \iint\limits_{-\infty<x<y<\infty} x^j y^k f_{r,\,s:n}(x,\, y)dx\, dy \tag{1.2.13}$$

In general, the joint *pdf* of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n}, x_{i_2:n}, \ldots, x_{i_k:n}) = n!\left\{\prod_{j=1}^{k} f(x_{i_j})\right\}\prod_{j=0}^{k}\left\{\frac{[F(x_{i_{j+1}}) - F(x_{i_j})]^{i_{j+1}-i_j-1}}{(i_{j+1} - i_j - 1)!}\right\},$$

$$-\infty < x_{i_1} < x_{i_2} < \cdots < x_{i_k} < \infty \tag{1.2.14}$$

where $x_0 = -\infty$, $x_{k+1} = +\infty$, $i_0 = 0$, $i_{k+1} = n + 1$.

Remarks 1.1

i. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.

ii. Regarding the probability integral transformation, if $X_{r:n}$, $1 \leq r \leq n$, be the order statistic from a continuous *df* $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the r^{th} order statistic from a uniform distribution $U(0,\, 1)$.

iii. Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

iv. Let $X_1, X_2, \ldots, X_n$ be *iid* random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

v. Let X be a continuous random variable with $E\left[X_{r:n}\right] = \alpha_{r:n}$.

  (a) If $\alpha = E(X)$ exits then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of r, even though α does not exist.

  (b) $\alpha_{r:n}$ for all n determine the distribution completely.

1.3 Truncated Distribution (Khan *et al.*, 1983)

Let X be a continuous *rv* having *pdf* $f(x)$ and *df* $F(x)$ over the support $(-\infty, \infty)$. Let

$$\int_{-\infty}^{Q_1} f(x)\,dx = Q \quad \text{and} \quad \int_{-\infty}^{P_1} f(x)\,dx = P \tag{1.3.1}$$

where Q_1 and P_1 are known constants. Then doubly truncated *pdf* of X is given by

$$\frac{f(x)}{P-Q}; \quad x \in (Q_1,\, P_1)$$

and the corresponding *df* is given by

$$\frac{F(x)-Q}{P-Q}; \quad x \in (Q_1,\, P_1)$$

The lower and upper truncation points are Q_1 and P_1 respectively; the degrees of truncation are Q (from below) $1-P$ (from above). If we put $Q = 0$, the distribution will be truncated to the right. Similarly for $P = 1$, the distribution will be truncated to the left. Whereas for $Q = 0,\ P = 1$, we get non truncated distribution. Truncated distribution are useful in finding the conditional distributions of order statistics.

In the following, we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is truncated from the original population distribution $F(x)$.

Result 1.1 (David and Nagaraja, 2003):

Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with *pdf* $f(x)$ and *df* $F(x)$ and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the r^{th} order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Result 1.2 (David and Nagaraja, 2003):

Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with *pdf* $f(x)$ and *df* $F(x)$, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that$X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)^{th}$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Result 1.3 (David and Nagaraja, 2003):

Let $X_1, X_2, \ldots, X_n$ be a random sample from an absolutely continuous population with *pdf* $f(x)$ and *df* $F(x)$, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{k:n} = z$ for $1 \leq r < s < k \leq n$ is the same as the distribution of the $(s-r)^{th}$order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

1.4 Sequential Order Statistics (Kamps, 1995)

In reliability theory, if a system of n components of the same kind and without any interactions with respect to life length distributions, are working then, the system failure is modeled by an order statistics based on *iid rv*. However, the failures of some components can more are less strongly influence the remaining components. This can be thought of as damage caused by the i^{th} failure system. In this model, the life distribution of remaining components in the system may change after each failure of the components.

If we observe i^{th} failure at time x, the remaining components are now supposed to have a possibly different life length distribution. The distribution is truncated on the left of x to ensure realizations arranged in ascending order of magnitude.

Definition: Let $(Y_j^{(i)})_{1\leq j\leq n-i+1, 1\leq i\leq n}$ be independent*rv* with $(Y_j^{(i)})_{1\leq j\leq n-i+1} \backsim F_i$, $1 \leq i \leq n$ where $F_1, F_2, \ldots, F_n$ are strictly increasing and continuous distribution functions with $F_1^{-1}(1) \leqslant \cdots \leqslant F_n^{-1}(1)$, Moreover, let $X_j^{(1)} = Y_j^{(1)}$, $1 \leq j \leq n$,

$$X_*^{(1)} = \min\left\{X_1^{(1)}, \ldots, X_n^{(1)}\right\}$$

and for $2 \leq j \leq n$

$$X_j^{(i)} = F_i^{-1}\left[F_i(Y_j^{(i)})(1 - F_i(X_*^{(i-1)})) + F_i(X_*^{(i-1)})\right]$$
$$X_*^{(i)} = \min\left\{X_j^{(i)}, 1 \leq j \leq n-i+1\right\}$$

then the *rv* $X_*^{(1)}, \ldots, X_*^{(n)}$ are sequential order statistics.

If we have absolutely continuous distribution functions $F_1, \ldots, F_n$, with densities $f_1, \ldots, f_n$, respectively, then the joint *pdf* of r sequential order statistics $X_*^{(1)}, \ldots, X_*^{(r)}$ is given by

$$f_{X_*^{(1)},\ldots,X_*^{(n)}}(x_1,\ldots,x_r) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left[\frac{1-F_i(x_i)}{1-F_i(x_{i-1})}\right]^{n-i}\frac{f_i(x_i)}{[1-F_i(x_{i-1})]},\ r \le n,\ x_0 = -\infty. \tag{1.4.1}$$

Sequential order statistics from a Markov chain with transition probabilities

$$P\big(X_*^{(r)} > t \mid X_*^{(r-1)} = s\big) = \left(\frac{1-F_r(t)}{1-F_r(s)}\right)^{n-r+1}, \qquad 2 \le r \le n. \tag{1.4.2}$$

1.5 Record Values (Chandler, 1952)

Record values were defined by Chandler (1952) as a model of successive extremes in a sequence of *iid rv*. Record values are found in daily life as well as in many statistical applications. We are often interested in observing new records and in recording them, *e.g.* a list of world or Olympic records in a particular sports, the hottest day ever, the longest winning streak in professional basketball, fastest half-century and century in one day international cricket matches, the lowest stock market figure, etc. Records have been discussed extensively in books by Ahsanullah (1995) and Arnold *et al.* (1998).

It may also be helpful as a model for successively largest insurance claims in non-life insurance, for highest water levels or highest temperatures. Record values are also used in reliability theory.

Definition: Suppose that (X_i), $i \in \mathbb{N}$ be a sequence of *iid* continuous *rv* with *pdf* $f(x)$ and *df* $F(x)$. Denote upper record times by

$$u(1) = 1$$

and for

$$u(n) = \min\left\{k > u_{(n-1)} : X_k > X_{u_{(n-1)}}\right\}$$

The record value sequence is then defined by $X_{u_{(n)}}$, $(n = 1, 2, \ldots)$. Based on an *iid* sequence of *rv* (X_i), $i \in N$ with absolutely continuous *pdf* $f(x)$ and *df* $F(x)$, the joint *pdf* of first r record values $X_{u_{(1)}}, \ldots, X_{u_{(r)}}$ is given by Ahsanullah (1995)

$$f_{X_{u_{(1)}},\ldots,X_{u_{(r)}}}(x_1, \ldots, x_r) = \left(\prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)}\right) f(x_r), \qquad -\infty < x_1 < x_2 < \cdots < x_r < \infty$$

The marginal *pdf* of $X_{u_{(r)}}$ is

$$f_{X_{u_{(r)}}}(x) = \frac{1}{(r-1)!} \left[-\log(\bar{F}(x))\right]^{r-1} f(x) \tag{1.5.1}$$

and the marginal *df* of $X_{u_{(r)}}$ is

$$F_{X_{u_{(r)}}}(x) = 1 - [1 - F(x)] \sum_{j=0}^{r-1} \frac{1}{j!} \left(\log \frac{1}{1 - F(x)}\right)^j \tag{1.5.2}$$

the joint *pdf* of $X_{U(r)}$ and $X_{U(s)}$, $1 \le r < s \le n$, is given by

$$f_{r,s}(x, y) = \frac{1}{(r-1)!(s-r-1)!}[-\log \bar{F}(x)]^{r-1}[B(x, y)]^{s-r-1} \frac{f(x)}{\bar{F}(x)} f(y), \quad -\infty < x < y < \infty, \tag{1.5.3}$$

where $\bar{F}(x) = 1 - F(x)$ and $B(x, y) = [-\log \bar{F}(y) + \log \bar{F}(x)]$.

The joint *pdf* of $X_{U(r)}, X_{U(j)}$ and $X_{U(s)}$, $1 \le r < j < s \le n$, can similarly be given as

$$f_{r,j,s}(x, t, y) = \frac{1}{(r-1)!(j-r-1)!(s-j-1)!}[-\log \bar{F}(x)]^{r-1}[B(x, t)]^{j-r-1}[B(t, y)]^{s-j-1} \times \frac{f(x)}{\bar{F}(x)} \frac{f(t)}{\bar{F}(t)} f(y), \qquad -\infty < x < t < y < \infty. \tag{1.5.4}$$

Hence, the conditional *pdf* of $X_{U(j)}$ given $X_{U(r)} = x$ and $X_{U(s)} = y,\ 1 \leq r < j < s \leq n$ is given by

$$f_{j|r,s}(t|x,y) = C_{r,j,s}\frac{[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}}{[B(x,y)]^{s-r-1}}\frac{f(t)}{\bar{F}(t)}, \quad -\infty < x < t < y < \infty, \qquad (1.5.5)$$

where $C_{r,j,s} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$.

1.6 k-Record Values (Dziubdziela and Kopocinski, 1976)

In some situations, record values themselves are viewed as outlier and hence second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example.

Definition: Let (X_i), $i \in \mathbb{N}$ be a *iid rv* with a *pdf* $f(x)$ and *df* $F(x)$ and let k be a positive integer. The random variable $u^{(k)}(n)$ given by

$$u^{(k)}(1) = 1$$
$$u^{(k)}(n+1) = \min\left\{j \in N : X_{j:\,j+k-1} > X_{u^{(k)}(n):u^{(k)}(n)+k-1}\right\}, \quad n \in \mathbb{N}$$

are called k^{th} record times and the quantities $X_{u^{(k)}(n):u^{(k)}(n)+k-1}$, which we denote by $X_{u^{(k)}(n)}$, $n \in \mathbb{N}$, are termed as k^{th} record values. Obviously, we obtain ordinary record values in the case $k = 1$. Moreover, Nagaraja (1988a) points out that k-records with an underlying *df* $F(x)$ can be viewed as ordinary record values ($k = 1$) base on *df* G(minimum distribution) with

$$G(x) = 1 - (1 - F(x))^k$$

Based on a sequence (X_i), $i \in \mathbb{N}$, of *iid* rv possessing an absolutely continuous *pdf* $f(x)$ and *df* $F(x)$. The joint *pdf* of k-records $X_{u^{(k)}(1)}, \ldots, X_{u^{(k)}(r)}$ is given by

$$f_{u^{(k)}(1),\ldots,u^{(k)}(r)}(x_1, \ldots, x_r) = k^r \left(\prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)} \right) [1 - F(x_r)]^{k-1} f(x_r), \qquad (1.6.1)$$

$$x_1 < x_2 < \cdots < x_r.$$

The marginal *pdf* of $X_{u^{(k)}(r)}$

$$f_{u^{(k)}(r)}(x) = \frac{k^r}{(r-1)!} \left[-\log(\bar{F}(x)) \right]^{r-1} [1 - F(x)]^{k-1} f(x) \qquad (1.6.2)$$

and the marginal *df* of $X_{u^{(k)}(r)}$

$$F_{u^{(k)}(r)}(x) = 1 - [1 - F(x)]^k \sum_{j=0}^{r-1} \frac{1}{j!} \left(k \log \frac{1}{1 - F(x)} \right)^j \qquad (1.6.3)$$

1.7 Progressive Type-II Right Censoring (Balakrishnan and Aggrawala, 2000)

Progressive censoring is used in life testing. In this scheme units can be removed at various stages during the experiment which may lead to saving of cost and time, (Cohen, 1963; Sen, 1986). Under this scheme of censoring from a total of n units placed on a life test, only m are completely observed until failure. At the time of the first failure R_1 of the $n-1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At the time of the next failure R_2 of $n-2-R_1$ surviving units are censored and so on. Finally, at the time of the m^{th} failure, all the remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ surviving units are censored.

Suppose n *iid* units are placed on a life test with the corresponding failure times $X_1, X_2, \ldots, X_n$ being identically distributed with a continuous *pdf* $f(x)$and *df* $F(x)$. Suppose further that the prefixed number of failures to be observed is m and that the progressive Type-II right censoring scheme is $(R_1, R_2, \ldots, R_m)$. Then, we shall denote the m completely observed failure times by $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, where $X_{i:m:n}$, $i = 1, 2, \ldots, m$, denotes the i^{th} failure time in the progressive Type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$, bearing in mind that these still depend on the particular choice of $(R_1, R_2, \ldots, R_m)$ used. In a particular case, when $R_1 = R_2 = \cdots = R_m = 0$ *i.e.* no withdrawals are carried out, then this model reduces to ordinary order statistics. Above model is described as progressively Type-II right-censored order statistics from $F(x)$ arising from a sample of size n with the censoring scheme $(R_1, R_2, \ldots, R_m)$. The pdf of all m progressively Type-II right-censored order statistics is

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1, \ldots, x_m) = C \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},$$
$$-\infty < x_1 < x_2 < \cdots < x_n < \infty, \tag{1.7.1}$$

where

$$C = n(n - R_1 - 1)...(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1).$$

1.8 Generalized Order Statistics

The concept of generalized order statistics (*gos*) was introduced by Kamps (1995). Generalized order statistics provide a unified approach to a variety of models of random variable arranged in ascending order of magnitude with different interpretation and statistical applications.

Definition: Let $X_1, X_2, \ldots, X_n$ be a sequence of *iid rv* with *pdf* $f(x)$ and the *df* $F(x)$. Let $n \in \mathbb{N}$, $n \geq 2$, $k \geq 1$, $\widetilde{m} = (m_1, m_2, m_3, \ldots, m_{n-1} \in \Re^{n-1})$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, 3, \ldots, n-1\}$. Then $X(1, n, \widetilde{m}, k), X(2, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ are said to be *gos* if their joint *pdf* is given by

$$f_{X(1,n,\widetilde{m},k),\ldots,X(n,n,\widetilde{m},k)}(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n) \tag{1.8.1}$$

on the cone $F^{-1}(0+) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1)$ of $\Re^n$.

On choosing the parameters appropriately (Kamps, 1995). Generalized order statistics includes all the models related to ordered random variables.

Table 1.1: Variants of the Generalized Order Statistics

		$\gamma_n = k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r - \gamma_{r+1} - 1)$
ii)	Ordinary order statistics	1	$(n-r+1)$	0
iii)	Record values	1	1	-1
iv)	Progressively Type-II right censored order statistics	$R_n + 1$	$n-r+1+\sum\limits_{j=r}^{n} R_j$	R_r
v)	Pfeifer's record values	β_n	β_r	$(\beta_r - \beta_{r+1} - 1)$

The joint density of the first r *gos* $X(1,\, n,\, \widetilde{m},\, k), X(2,\, n,\, \widetilde{m},\, k), \ldots, X(r,\, n,\, \widetilde{m},\, k)$ is given by

$$f_{X(1,n,\widetilde{m},k),\ldots,X(r,n,\widetilde{m},k)}(x_1,\ldots,x_r) = C_{r-1}\left(\prod_{i=1}^{r-1}\left[\bar{F}(x_i)\right]^{m_i}f(x_i)\right)\left[\bar{F}(x_r)\right]^{k+n-r+M_r-1}f(x_r) \tag{1.8.2}$$

on the cone

$$F^{-1}(0+) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1)$$

In the derivation of marginal distributions the choice of $\widetilde{m}$ is restricted

Therefore, we will consider two cases:

Case I: In this case, we shall assume that $m_1 = m_2 = \cdots = m_{n-1} = m$ and write *gos* as $X(r,n,m,k)$, $r \ge 2$. Thus, in this case, the marginal density function of the r^{th} *gos* based on an absolutely continuous *pdf* $f(x)$ with *df* $F(x)$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x))f(x), \qquad -\infty < x < \infty \tag{1.8.3}$$

where $\bar{F}(x) = 1 - F(x)$

and the joint *pdf* of the *gos* $X(r,n,m,k)$ and $X(s,n,m,k)$, the r^{th} and s^{th} *gos*, $1 \le r < s \le n$ is given by

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = C_{r,s:n}\left[\bar{F}(x)\right]^{m} g_m^{r-1}(F(x))\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1} \\ \times \left[\bar{F}(y)\right]^{\gamma_s - 1} f(x)f(y), \qquad -\infty < x < y < \infty \end{aligned} \tag{1.8.4}$$

where $x < y$ and $C_{r,s:n} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$

The joint *pdf* of the *gos* $X(r,n,m,k)$, $X(j,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < j < s \le n$, can be similarly given as

$$\begin{aligned} f_{X(r,n,m,k),X(j,n,m,k),X(s,n,m,k)}(x,t,y) = C_{r,j,s:n}\left[\bar{F}(x)\right]^{m} g_m^{r-1}(F(x))\left[h_m(F(t)) - h_m(F(x))\right]^{j-r-1} \\ \times \left[h_m(F(y)) - h_m(F(t))\right]^{s-j-1}\left[\bar{F}(t)\right]^{m}\left[\bar{F}(y)\right]^{\gamma_s - 1} \\ \times f(x)f(t)f(y), \qquad -\infty < x < t < y < \infty, \end{aligned}$$

where $C_{r,j,s:n} = \frac{C_{s-1}}{(r-1)!(j-r-1)!(s-j-1)!}$, $C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1)$ and

$$\begin{aligned} g_m(x) &= h_m(x) - h_m(0) \\ &= \int_0^x (1-t)^m dt, \qquad x \in (0,1) \end{aligned}$$

$$h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1}, & m \neq -1, \\ \log \dfrac{1}{(1-x)}, & m = -1. \end{cases}$$

Therefore, conditional distribution of $X(j,n,m,k)$ given $X(r,n,m,k) = x$, $X(s,n,m,k) = y$, $1 \leq r < j < s \leq n$, is given by

$$\begin{aligned} f_{X(j,n,m,k)|X(r,n,m,k)=x,X(s,n,m,k)=y}(t \mid x,y) &= \frac{(s-r-1)!(m+1)}{(j-r-1)!(s-j-1)!} \\ \times \frac{\left[\left\{\bar{F}(x)\right\}^{m+1} - \left\{\bar{F}(t)\right\}^{m+1}\right]^{j-r-1}\left[\left\{\bar{F}(t)\right\}^{m+1} - \left\{\bar{F}(y)\right\}^{m+1}\right]^{s-j-1}}{\left[\left\{\bar{F}(x)\right\}^{m+1} - \left\{\bar{F}(y)\right\}^{m+1}\right]^{s-r-1}} \\ \times [\bar{F}(t)]^m f(t), \qquad -\infty < x < t < y < \infty. \end{aligned} \tag{1.8.5}$$

Case II: In this case, when $\gamma_i \neq \gamma_j$, $i, j = 1, 2, 3, \ldots, n-1, \forall\, i \neq j$, the *pdf* of $X(r,n,\widetilde{m},k)$ is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\widetilde{m},k)}(x) = C_{r-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i - 1} f(x), \qquad -\infty < x < \infty \tag{1.8.6}$$

and the joint *pdf* of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,y) = C_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)}$$

$$-\infty < x < y < \infty. \tag{1.8.7}$$

The joint *pdf* of the *gos* $X(r,n,\widetilde{m},k)$, $X(j,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \le r < j < s \le n$, may similarly be given as

$$f_{X(r,n,\widetilde{m},k),X(j,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,t,y) = C_{s-1}\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\left[\sum_{i=r+1}^{j} a_i^{(r)}(j)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^{\gamma_i}\right]$$
$$\times\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(t)}\right)^{\gamma_i}\right]\frac{f(x)f(t)f(y)}{\bar{F}(x)\bar{F}(t)\bar{F}(y)},$$
$$-\infty < x < t < y < \infty. \qquad (1.8.8)$$

Therefore, conditional distribution of $X(j,n,\widetilde{m},k)$ given $X(r,n,\widetilde{m},k)=x$, $X(s,n,\widetilde{m},k)=y$, for $1 \le r < j < s \le n$ is given by

$$f_{X(j,n,\widetilde{m},k)|X(r,n,\widetilde{m},k)=x,X(s,n,\widetilde{m},k)=y}(t \mid x,y) = \frac{\left[\sum\limits_{i=r+1}^{j} a_i^{(r)}(j)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^{\gamma_i}\right]}{\left[\sum\limits_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]}$$
$$\times\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(t)}\right)^{\gamma_i}\right]\frac{f(t)}{\bar{F}(t)}$$
$$-\infty < x < t < y < \infty, \qquad (1.8.9)$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \qquad \gamma_i = k+n-i+M_i$$
$$a_i(r) = \prod_{j=1, j\neq i}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \quad \gamma_i \neq \gamma_j, \quad 1 \le i \le r \le n$$

and

$$a_i^r(s) = \prod_{j=r+1, j\neq i}^{s}\frac{1}{(\gamma_j-\gamma_i)}, \quad \gamma_i \neq \gamma_j, \quad r+1 \le i \le s \le n.$$

1.9 Lower (Dual) Generalized Order Statistics

A concept of lower (dual) generalized order statistics (*dgos*) which was introduced by Pawlas and Syznal (2001b), later Burkschat *et al.* (2003) as follows:

Let $n \in \mathbb{N}$, $k \geq 1$, $m \in \Re$, be the parameters such that $\gamma_r = k + n - r + M_r = \sum\limits_{j=r}^{n-1} m_j > 0$ for all $1 \leq r \leq n$. By the *dgos* from an absolutely continuous *pdf* $f(x)$ with *df* $F(x)$ we mean *rv* $X^*(1, n, \widetilde{m}, k)$, ..., $X^*(n, n, \widetilde{m}, k)$ having joint density function of the form

$$f_{X^*(1,n,\widetilde{m},k),\ldots,X^*(n,n,\widetilde{m},k)}(x_1, \ldots, x_n) = k\left(\prod_{r=1}^{n-1} \gamma_r\right)\left(\prod_{j=1}^{n-1} [F(x_j)]^{m_j} f(x_j)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.9.1}$$

for $\qquad F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0)$ on the cone.

Here, we consider two case:

Case I: In this case, we will assume that $m_1 = m_2 = \cdots = m_{n-1} = m$ and write *dgos* as $X^*(r, n, m, k)$. Thus, in this case the marginal density function of the r^{th} *dgos* based on an absolutely continuous *pdf* $f(x)$ with *df* $F(x)$ is given by

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x) \tag{1.9.2}$$

and the joint *pdf* of the *dgos* $X^*(r, n, m, k)$ and $X^*(s, n, m, k)$, the r^{th} and s^{th} *dgos*, $1 \leq r < s \leq n$ is given by

$$\begin{aligned} f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x, y) = C_{r,s:n}\, &[F(x)]^m\, g_m^{r-1}[F(x)]\, [h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\times [F(y)]^{\gamma_s - 1} f(x) f(y), \\ &-\infty < x < y < \infty, \end{aligned} \tag{1.9.3}$$

where $x > y$ and $C_{r,s:n} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$

The joint *pdf* of the *dgos* $X^*(r, n, m, k)$, $X^*(j, n, m, k)$ and $X^*(s, n, m, k)$,the r^{th}, j^{th} and s^{th}

dgos $1 \le r < j < s \le n$, can be similarly given as

$$f_{X^*(r,n,m,k),X^*(j,n,m,k),X^*(s,n,m,k)}(x,t,y) = C_{r,j,s:n}\,[F(x)]^m\, g_m^{r-1}[F(x)]\,[h_m(F(t)) - h_m(F(x))]^{j-r-1}$$
$$\times [h_m(F(y)) - h_m(F(t))]^{s-j-1}\,[F(t)]^m\,[F(y)]^{\gamma_s - 1}$$
$$\times f(x)f(t)f(y), \qquad -\infty < y < t < x < \infty,$$

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in (0,1)$$

$$h_m(x) = \begin{cases} -\dfrac{x^{m+1}}{m+1}, & m \neq -1, \\ \log \dfrac{1}{x}, & m = -1. \end{cases}$$

Therefore, conditional distribution of $X^*(j,n,m,k)$ given $X^*(r,n,m,k) = x$, $X*(s,n,m,k) = y$, for $1 \le r < j < s \le n$ is given by

$$f_{X^*(j,n,m,k)|X*(r,n,m,k)=x,X^*(s,n,m,k)=y}(t \mid x,y) = \frac{(s-r-1)!(m+1)}{(j-r-1)!(s-j-1)!}$$
$$\times \frac{\left[\{F(x)\}^{m+1} - \{F(t)\}^{m+1}\right]^{j-r-1}\left[\{F(t)\}^{m+1} - \{F(y)\}^{m+1}\right]^{s-j-1}}{\left[\{F(x)\}^{m+1} - \{F(y)\}^{m+1}\right]^{s-r-1}}$$
$$\times [F(t)]^m f(t), \qquad -\infty < y < t < x < \infty. \tag{1.9.4}$$

Case II: When $\gamma_i \neq \gamma_j$, i, $j = 1, 2, 3, \ldots, n-1$, $i \neq j$

$$f_{X^*(r,n,\widetilde{m},k)}(x) = C_{r-1}\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i - 1}\, f(x), \qquad -\infty < x < \infty \tag{1.9.5}$$

and the joint *pdf* of $X^*(r,n,\widetilde{m},k)$ and $X^*(s,n,\widetilde{m},k)$, $1 \le r < s \le n$, is

$$f_{X^*(r,n,\widetilde{m},k),X^*(s,n,\widetilde{m},k)}(x,y) = C_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i}\right]\frac{f(x)f(y)}{F(x)F(y)}$$

$$-\infty < y < x < \infty. \tag{1.9.6}$$

The joint *pdf* of $X^*(r,n,\widetilde{m},k)$, $X^*(j,n,\widetilde{m},k)$ and $X^*(s,n,\widetilde{m},k)$, $1 \leq r < j < s \leq n$, may similarly be given as

$$f_{X^*(r,n,\widetilde{m},k),X^*(j,n,\widetilde{m},k),X^*(s,n,\widetilde{m},k)}(x,t,y) = C_{s-1}\left[\sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i}\right]\left[\sum_{i=r+1}^{j} a_i^{(r)}(j)\left(\frac{F(t)}{F(x)}\right)^{\gamma_i}\right]$$
$$\times\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left(\frac{F(y)}{F(t)}\right)^{\gamma_i}\right]\frac{f(x)f(t)f(y)}{F(x)F(t)F(y)}$$
$$-\infty < x < t < y < \infty. \tag{1.9.7}$$

Therefore, conditional distribution of $X^*(j,n,\widetilde{m},k)$ given $X^*(r,n,\widetilde{m},k) = x$, $X^*(s,n,\widetilde{m},k) = y$, for $1 \leq r < j < s \leq n$ is given by

$$f_{X^*(j,n,\widetilde{m},k)|X^*(r,n,\widetilde{m},k)=x,X^*(s,n,\widetilde{m},k)=y}(t \mid x,y) = \frac{\left[\sum\limits_{i=r+1}^{j} a_i^{(r)}(j)\left(\frac{F(t)}{F(x)}\right)^{\gamma_i}\right]}{\left[\sum\limits_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{F(y)}{F(x)}\right)^{\gamma_i}\right]}$$
$$\times\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left(\frac{F(y)}{F(t)}\right)^{\gamma_i}\right]\frac{f(t)}{F(t)}$$
$$-\infty < x < t < y < \infty. \tag{1.9.8}$$

where C_{r-1}, $a_i(r)$, $a_i^r(s)$ are defined as above.

1.10 Generalized Order Statistics and Dual Generalized Order Statistics using Meijer's G-Function

Let $n \in \mathbb{N}$, $n \geqslant 2$, $k \geq 1$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}$, $M_r = \sum\limits_{j=r}^{n-1} m_j$, $1 \leqslant r \leqslant n-1$, be the parameters such that $\gamma_r = k + n - r + M_r \geq 0$, for all $r \in \{1, ..., n-1\}$.

Then, $U(1,n,\tilde{m},k), U(2,n,\tilde{m},k), \ldots, U(n,n,\tilde{m},k)$ are said to be uniform generalized order statistics(gos), if their joint *pdf* is given by (Kamps, 1995)

$$f_{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1,...,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1} \tag{1.10.1}$$

on the cone

$$0 < u_1 \leqslant \cdots \leqslant u_n < 1$$

Based on any arbitrary *df* $F(x)$, *gos* $X(r,n,\tilde{m},k)$ can be defined by quantile transformation $X(r,n,\tilde{m},k) = F^{-1}(U(r,n,\tilde{m},k))$, $1 \leqslant r \leqslant n$, where F^{-1} denotes the quantile function of F defined by

$$F^{-1}(u) = \sup\{x \in (\alpha,\beta) : F(x) \leqslant u\}, u \in (0,1)$$

where $\alpha = \inf\{x \in \mathbb{R} : F(x) > 0\}$ and $\beta = \sup\{x \in \mathbb{R} : F(x) < 1\}$ are the left and right end points of X.

Let P_F stands for the probability measure on $\mathbb{R}$ determined by $F(x)$, then the *pdf* of $X(r,n,\tilde{m},k)$ with respect to a measure P_F is given by (Cramer and Kamps, 2003).

$$f_r(x) = c_{r-1}G_r(\bar{F}(x) \mid \gamma_1,...,\gamma_r)I_{(\alpha,\beta)}(x) \tag{1.10.2}$$

where $\bar{F}(x) = 1 - F(x)$, $c_{r-1} = \prod_{i=1}^{r}\gamma_i$ and I_A denotes the indicator function and $G_r(x)$ is the Meijer's G-function

Meijer's G-Function

$$\begin{aligned} G_r(x) &= G_{r,r}^{r,0}(x \mid \gamma_1,...,\gamma_r) \\ &= G_{r,r}^{r,0}\left(x \,\middle|\, \begin{pmatrix} \gamma_1,...,\gamma_r \\ \gamma_1-1,...,\gamma_r-1 \end{pmatrix}\right) \end{aligned}$$

is the particular Meijer's G-Function defined by

$$G^{r,0}_{r,r}\left(s\,\middle|\,\begin{pmatrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{pmatrix}\right)=\frac{1}{2\pi i}\int_L\frac{s^z}{\prod_{j=1}^r(\gamma_j-1-z)}dz \tag{1.10.3}$$

and L is an appropriate chosen contour of integration (See Mathai, 1993, Chapter 3) for the definition of G-function and its numerous properties and applications.

For any $x\in(\alpha,\beta),\quad y\in\mathbb{R}$, denote

$$F_x(y)=\begin{cases}\frac{F(y)-F(x)}{1-F(x)}, & y\geq x,\\ 0, & y<x.\end{cases} \tag{1.10.4}$$

Here, $F_x(y)$ denotes the distribution function obtained from F by truncating on the left at x. The joint P_F density of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1\leqslant r<s\leqslant n$, is given by

$$f_{r,s}(x,y)=c_{s-1}G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},\ldots,\gamma_s\big)\frac{G_r(\bar{F}(x)\mid\gamma_1,\ldots,\gamma_r)}{\bar{F}(x)}I_{(\alpha,\beta)}(x<y), \tag{1.10.5}$$

and the joint P_F density of$X(r,n,\tilde{m},k),\ X(j,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k),\ 1\leqslant r<j<s\leqslant n$ may similarly be given as

$$\begin{aligned}f_{r,j,s}(x,t,y)=c_{s-1}\frac{1}{\bar{F}(x)}\frac{1}{\bar{F}(t)}&G_{s-j}\big(\bar{F}_t(y)\mid\gamma_{j+1},\ldots,\gamma_s\big)G_{j-r}\big(\bar{F}_x(t)\mid\gamma_{r+1},\ldots,\gamma_j\big)\\ &\times G_r(\bar{F}(x)\mid\gamma_1,\ldots,\gamma_r)I_{(\alpha,\beta)}(x<t<y)\end{aligned} \tag{1.10.6}$$

Hence, the conditional P_F density function of $X(j,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$ and $X(s,n,\tilde{m},k)=y$, $1\leqslant r<j<s\leqslant n$ is given by

$$f_{j|r,s}(t\mid x,y)=\frac{1}{\bar{F}(t)}\frac{G_{s-j}\big(\bar{F}_t(y)\mid\gamma_{j+1},\ldots,\gamma_s\big)G_{j-r}\big(\bar{F}_x(t)\mid\gamma_{r+1},\ldots,\gamma_j\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},\ldots,\gamma_s\big)}I_{(x,y)}(t) \tag{1.10.7}$$

Similarly for Dual generalized order statistics

The conditional P_F density function of $X^*(j, n, \tilde{m}, k)$ given $X^*(r, n, \tilde{m}, k) = x$ and $X^*(s, n, \tilde{m}, k) = y$, $1 \leqslant r < j < s \leqslant n$ is given by

$$f_{j|r,s}(t \mid x, y) = \frac{1}{F(t)} \frac{G_{s-j}\left(\frac{F(y)}{F(t)} \mid \gamma_{j+1}, ..., \gamma_s\right) G_{j-r}\left(\frac{F(t)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_j\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} I_{(y,x)}(t) \qquad (1.10.8)$$

Some Auxilliary Results

Various results which are used in subsequent Chapters are reproduced here:

(i) $G_1(x \mid \gamma_1) = x^{\gamma_1 - 1}$

(ii) $(\gamma_r - \gamma_1) G_r(x \mid \gamma_1, ..., \gamma_r) = G_{r-1}(x \mid \gamma_1, ..., \gamma_{r-1}) - G_{r-1}(x \mid \gamma_2, ..., \gamma_r)$

(iii) $x^\alpha G_r(x \mid \gamma_1, ..., \gamma_r) = G_r(x \mid \gamma_1 + \alpha, ..., \gamma_r + \alpha)$, $a \in R$

(iv) $\lim_{x \to 1-} G_r(x \mid \gamma_1, ..., \gamma_r) = \begin{cases} 1 & , r = 1 \\ 0 & , r \geq 2 \end{cases}$

and

$$\lim_{x \to 0+} G_r(x \mid \gamma_1, ..., \gamma_r) = \begin{cases} 0 & , \text{if} \gamma_{1:r} > 1 \\ \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_i)} & , \text{if} \gamma_{1:r} = 1 < \gamma_{2:r} \\ \infty & , \text{if} \gamma_{1:r} = \gamma_{2:r} = 1 \text{ or } \gamma_{1:r} < 1 \end{cases}$$

where $\gamma_{1:r} = \min(\gamma_1, ..., \gamma_r)$ and $l = \max(1 \leq j \leq r : \gamma_j = \gamma_{1:r})$

(v) $\frac{d}{dx} G_r(x \mid \gamma_1, ..., \gamma_r) = \frac{1}{x}[(\gamma_r - 1) G_r(x \mid \gamma_1, ..., \gamma_r) - G_{r-1}(x \mid \gamma_1, ..., \gamma_{r-1})]$

(vi) $\frac{d}{dx}G_r(x \mid \gamma_1, ..., \gamma_r) = \frac{1}{x}[(\gamma_1 - 1)G_r(x \mid \gamma_1, ..., \gamma_r) - G_{r-1}(x \mid \gamma_2, ..., \gamma_r)]$

Proof: For the property (i), see Mathai (1993, p. 130), for property (ii), see Cramer and Kamps (2003), and for the property (iii), see Mathai (1993, p. 69). Property (iv) can easily be deduced from Lemma 2.2 of Cramer et al. (2004 b), whereas (v) and (vi) can be established from (1.10.3).

1.11 Characterizations through Linear Regression

Characterization results are those which shed light on modeling sequences of certain distributional assumptions and those which have potential for development of hypothesis testing for model assumptions. Continuous distributions have been characterized through conditional expectations of order statistics, record values, *gos* and *dgos* have been considered by many in the literature. Ferguson (1967) introduced the characterization of distributions based on the linearity of regression of adjacent order statistics $E(X_{r+1:n}|X_{r:n} = x)$ and its dual $E(X_{r:n}|X_{r+1:n} = x)$, where $X_{r:n}$ is the r^{th} order statistics. Shanbhag (1970) characterized exponential and geometric distributions in terms of conditional expectations for single order gap. Khan and Khan (1987) characterized Burr type XII distribution through linear regression for single order gap. Khan and Abu-Salih (1989) characterized a general class of distributions through conditional expectation of function of order statistics:

$$E\left[h(X_{r+1:n})|X_{r:n} = x\right] = a^* h(x) + b^*$$

and

$$E\left[h(X_{r:n})|X_{r+1:n} = x\right] = a_1^* h(x) + b_1^*$$

Wesolowski and Ahsanullah (1997) characterized the distributions by the regression of non-adjacent order statistics through the relation

$$E\left(X_{r+2:n}|X_{r:n}=x\right)=ax+b$$

Characterization of distributions via linearity of regression of order statistics when gap is higher is considered by Khan and Ali (1987), Franco and Ruiz (1997), Dembińska and Wesolowski (1998) and Lopez-Bláquez and Moreno-Rebollo (1997). Whereas, Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the generalized form of distributions through higher order gap. Khan and Athar (2004) also characterized some continuous distributions through linearity of regression when conditioning is done on a pair of order statistics. Using the result of Rao and Shanbhag (1994) dealing with an extended version of the integrated Cauchy functional equation Dembińska and Wesolowski (1998) and Athar *et al.* (2003) characterized the distributions by means of the regression equation

$$E\left(X_{r+i:n}|X_{r:n}=x\right)=ax+b$$

For record values Nagaraja (1977) characterized continuous distributions by using the relation

$$E\left(X_{U(r+1)}|X_{U(r)}=x\right)=ax+b$$

Nagaraja (1988b) also characterized distributions by considering the equation

$$E\left(X_{U(r)}|X_{U(r+1)}=x\right)=ax+b$$

Franco and Ruiz (1996, 1997) obtained the distribution function from the conditional expectations

$$E\left[h(X_{U(n-1)})|X_{U(n)} = x\right]$$

where $'h'$ is a real, continuous and strictly monotonic function. Ahsanullah and Wesolowski (1998) extended the result of Nagaraja (1977) and characterized the distributions for double order gap. Lopez-Bláquez and Moreno-Rebollo (1997), Dembińska and Wesolowski (2000) and Athar *et al.* (2003) extended the result of Nagaraja(1988b) and characterized the continuous distributions conditioned on non-adjacent records. Characterization of continuous distributions conditioned on a pair of adjacent records was investigated by Bairamov *et al.* (2005). They characterized the exponential distribution and continuous distributions by taking the monotone transformations.

Further, Yanev *et al.* (2008), Yanev and Ahsanullah (2009) and Khan and Khan (2009) characterized the continuous distributions conditioned on a pair of non-adjacent records. Recently Noor and Athar (2014) characterized the continuous distributions by taking the conditional expectation

$$g^p_{r,s}(x) = E[\{\Psi(X_{U(s)}) - \Psi(X_{U(r)})\}^p|X_{U(r)} = x]$$

Concept of *gos* was given by Kamps (1995). Since, many ordered variables like order statistics, record values and k -record values are special cases of *gos*, therefore, characterization through *gos* is of special interest. Keseling (1999) characterized the continuous distributions by taking the conditional expectations.

$$E[h(X(r+1, n, \tilde{m}, k))|X(r, n, \tilde{m}, k) = x]$$

where $h(.)$ is a real strictly monotonic function. They also characterized the continuous distributions by taking the conditional expectations

$$E[X(r,n,\tilde{m},k)|X(r+2,n,\tilde{m},k)=x]$$

Bieniek and Syznal(2003) investigated the characterization of the continuous distributions by considering the conditional expectations

$$E[X(r+l,n,\tilde{m},k)|X(r,n,\tilde{m},k)=x], l\geq 2.$$

Samuel (2008) characterized the conditional expectation through the relation

$$E\left[h\left(X(r+1,n,m,k)\right)|X(r,n,m,k)=x\right]=a^{*}h(x)+b^{*}$$

Khan and Alzaid (2004) characterized a general class of distribution $\bar{F}(x)=[ax+b]^{c}$ through linear regression of generalized order statistics using Rao and Shanbhag's (1994) result. They characterized the distributions by means of relation.

$$E\left[X(s,n,m,k)|X(r,n,m,k)=x\right]=a^{*}x+b^{*}$$

Khan *et al.* (2006), Beg and Ahsanullah (2006) have characterized the distribution functions through the relation

$$E\left[\xi\left\{X(s,n,m,k)\right\}|X(r,n,m,k)=x\right]=g_{s|r}(x)$$

and its dual

$$E\left[\xi\left\{X(r,n,m,k)\right\}|X(s,n,m,k)=x\right]=g_{r|s}(x)$$

Further, Ahsanullah and Beg (2008) characterized the continuous distribution functions conditioned on a pair of adjacent gos through the relation

$$E\left[\xi\left\{X(r+1,n,m,k)\right\}|X(r,n,m,k)=x,\ X(r+2,n,m,k)=x\right]=g_{r+1|r,\,r+2}(x)$$

Bieniek (2007) characterized continuous distributions based on conditional expectations, through the relation

$$E\left[g\left(X(r+1,n,m,k)\right)|X(r,n,m,k)=x\right]=h(x)$$

using Meijer's G-Function.

Characterization of continuous distributions conditioned on nonadjacent *gos* was considered by Cramer *et al.* (2004a), Raqab and Abu-Lawi (2004). In these literature, the authors are mainly concerned in finding the distribution function when the regression lines are linear. Bieniek (2009), Khan and Khan (2011) have characterized the continuous distribution functions conditioned on non-adjacent *gos* using Meijer's G-Function. Later on Khan *et al.* (2012) extended the result of Bieniek (2009), Khan and Khan (2011) and characterized the continuous distributions conditioned on a pair of non-adjacent *gos*, *i.e.* by taking the conditional expectation

$$E[h\{X(j,n,\tilde{m},k)\}|X(r,n,\tilde{m},k)=x,X(s,n,\tilde{m},k)=y],\quad 1\leq r<j<s\leq n,$$

here $h(x)$ is considered as monotonic and differentiable function of x. Recently, Noor *et al.* (2014) characterized the continuous distributions by taking the conditional expectation

$$g_{r,s,p}=E[\{\psi(X(s,n,\tilde{m},k))-\psi(X(r,n,\tilde{m},k))\}^{p}\mid X(r,n,\tilde{m},k)=x]$$

Using the concept of *gos*, Burkschat *et al.* (2003) introduced the concept of the dual generalized order statistics (*dgos*) that enables a common approach to descendingly ordered random variables like reversed ordered order statistics, lower record values etc. The various developments on *dgos* and related topic have been studied by Ahsanullah (2004), Mbah and Ahsanullah (2007), Khan *et al.* (2009), Khan *et al.* (2010 a,b), Faizan and Khan (2011), Tavangar (2011) among the others. Khan *et al.* (2009) have characterized continuous distributions through conditional expectation of *dgos*, conditioned on a pair of non-adjacent *dgos*. Recently, Khan and Khan (2012) have characterized continuous distribution functions conditioned on non-adjacent *dgos* using Meijer's G-Function.

1.12 Moments and Recurrance Relations

Order statistics and their moments have received attention from the beginning of this century. Since, Galton (1902) and Pearson (1902) studied the distribution of the difference of the successive order statistics. The moments of order statistics, assumed considerable importance in the statistics literature and have been numerically tabulated extensively for several distributions. For example one can refer to David and Nagaraja (2003), Sarhan and Greenberg (1962), Arnold and Balakrishnan (1989), Arnold *et al.* (1992) for details. There are mainly three reasons due to which recurrence relations and identities have attained importance:

i. Reduces the amount of direct computation and hence reduces the time and labour.

ii. They express the higher order moments in terms of lower order moments and hence make the evaluation of higher order moments easy.

iii. Provide some simple checks to test the accuracy of computation of moments of order statistics.

Shah (1966, 1970), Tarter (1966) have obtained moments of order statistics from logistic distribution. Malik (1967) has obtained recurrence relations for the moments of order statistics from power function distribution. Lieblien (1955), Balakrishnan and Joshi (1981) have obtained recurrence relations for moments of order statistics from Weibull distribution.

Saleh *et al.* (1975), Joshi (1978, 1979) have given recurrence relations for the moments of order statistics from exponential and truncated exponential distributions. Balakrishnan and Joshi (1983, 1984) obtained recurrence relations for single and product moments of order statistics from symmetrically truncated logistic distribution and doubly truncated exponential distributions.

Khan *et al.* (1983) developed general results for finding the k^{th} moment of order statistics without considering any particular distribution. Further, these results were utilized to obtain recurrence relations for doubly truncated and non-truncated distributions, thus unifying all the known results on recurrence relations for moments of order statistics.

Khan *et al.* (1984) obtained the inverse moments of order statistics for Weibull distribution whereas Ali and Khan (1996) obtained the ratio and inverse moments of order statistics from Weibull and exponential distribution. Unifying earlier results Khan and Athar (2000) established the relations for ratio and product moments of order statistics from doubly truncated Weibull distribution.
Khan and Khan (1987) obtained recurrence relations for single and product moments of order statistics for doubly truncated Burr distribution (Burr type XII) and utilized the relations to characterize the distribution. Athar *et al.* (2011) has obtained moments of order statistics from extended type-I generalized logistic distribution.

Kamps (1995) investigated recurrence relations for moments of generalized order statistics based on non-identically distributed random variables, which contains order statistics and record values as special cases.

Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases.

Pawlas and Szynal (2001a) derived recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Khan *et al.* (2007) obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution.

Athar and Islam (2004) established some recurrence relations between expectation of function of single and joint generalized order statistics from a general class of distribution. Further, Athar *et al.* (2009) generalized the result of Athar and Islam (2004) and established the relations for the expectation of function of *gos* for truncated distributions. Athar *et al.* (2007) obtained the ratio and inverse moments of generalized order statistics from Weibull distribution. Further, Kumar and Khan (2013) have obtained relations for generalized order statistics from doubly truncated generalized Exponential distribution and its characterization.

1.13 Some Continuous Distributions

I. Pareto Distribution

A *rv* X is said to have the Pareto distribution if its *pdf* and *df* are of the form given below:

$$f(x) = p\lambda^p x^{-(p+1)}, \qquad \lambda \leq x < \infty; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^p x^{-p}, \qquad \lambda \leq x < \infty; \quad \lambda, p > 0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

II. Power Function Distribution

A *rv* X is said to have a power function distribution if its *pdf* and *df* are of the form given below:

$$f(x) = p\lambda^{-p}x^{p-1}, \qquad 0 \leq x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p}x^{p}, \qquad 0 \leq x < \lambda; \quad \lambda, p > 0$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = \frac{1}{X}$ has a Pareto distribution.

III. Beta Distribution

Beta distribution of the First Kind

A *rv* X is said to have the beta distribution of first kind if its *pdf* is of the form

$$f(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \qquad 0 \leq x < 1; \quad p, q > 0$$

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r, n-r+1)$. The standard rectangular distribution $R(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q = 1$, the distribution reduces to power function distribution.

Beta distribution of the Second Kind

The continuous *rv* X which is distributed according to probability law:

$$f(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}}, \qquad 0 \le x < \infty;\ p,q > 0$$

is known as a beta variate of the second kind with parameters p and q. Beta distribution of second kind reduces to beta distribution of first kind if we replace $1+x$ by $\frac{1}{y}$. The beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with uncertain transition probabilities.

IV. Weibull Distribution

A *rv* X is said to have a Weibull distribution if its *pdf* is given by:

$$f(x) = \theta p x^{p-1} e^{-\theta x^p}, \qquad 0 \le x < \infty;\ \theta > 0,\ p > 0$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^p}, \qquad 0 \le x < \infty;\ \theta > 0,\ p > 0$$

Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

If we put $p = 1$ in Weibull distribution, we get the *pdf* of **Exponential distribution**.

If we put $p = 2$, it gives *pdf* of **Rayleigh distribution**.

If X has a Weibull distribution, then the *pdf* of $Y = -p \log\left(\frac{X}{\alpha}\right)$ is

$$f(y) = e^{-y} e^{-e^{-y}}$$

which is a form of an **Extreme Value Distribution**.

V. Exponential Distribution

A *rv* X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta e^{-\theta x}, \qquad 0 \leq x < \infty;\ \theta > 0$$

and the *pdf* is given by

$$F(x) = 1 - e^{-\theta x}, \qquad 0 \leq x < \infty;\ \theta > 0$$

The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s + t | X > s) = P(X > t) \ \ \forall \ \ s, t > 0.$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

VI. Burr Distribution

Let X be a continuous *rv*, then different forms of *df* of X are listed below by Johnson and Kotz (1994):

$$F(x) = x; \quad 0 < x < 1$$

$$F(x) = (1 + e^{-x})^{-k}, \quad -\infty < x < \infty$$

$$F(x) = (1 + x^{-c})^{-k}), \quad 0 \le x < \infty$$

$$F(x) = \left[1 + \left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}, \quad 0 \le x \le c$$

$$F(x) = \left[1 + ce^{-\tan x}\right]^{-k}, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$F(x) = \left[1 + ce^{-k \sinh x}\right]^{-k}, \quad -\infty < x < \infty$$

$$F(x) = 2^{-k}\left(1 + \tanh x\right)^{k}, \quad -\infty < x < \infty$$

$$F(x) = \left(\frac{2}{\pi}\tan^{-1} e^{x}\right), \quad -\infty < x < \infty$$

$$F(x) = 1 - \frac{2}{c[(1+e^{x})^{k} - 1] + 2}, \quad -\infty < x < \infty$$

$$F(x) = (1 + e^{-x^2})^{k}, \quad -\infty \le x < \infty$$

$$F(x) = \left(x - \frac{1}{2\pi}\sin 2\pi x\right)^{k}, \quad 0 \le x < 1$$

$$F(x) = 1 - (1 + x^{c})^{-k}, \quad 0 \le x < \infty$$

where k and c are positive parameters. Special attention is given to type XII, whose *pdf* is given as:

$$f(x) = kcx^{c-1}(1 + x^{c})^{-(k+1)}, \quad 0 \le x < \infty; \quad k, c > 0$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c = 1$, it is called **Lomax distribution** whereas at $k = 1$ it is known as **Log-logistic**

distribution.

VII. Extreme Value Distribution

A *rv* X is said to have Extreme value distribution of type I if its *pdf* is given by

$$f(x) = e^x \exp[-e^x], \quad -\infty < x < \infty$$

and the *df* is given by

$$F(x) = 1 - \exp[-e^x], \quad -\infty < x < \infty$$

and a *rv* X is said to have Extreme value distribution of type II if its *pdf* is given by

$$f(x) = p\theta^p x^{-(p+1)} e^{-\left(\frac{\theta}{x}\right)^p}, \quad 0 < x < \infty$$

and the *df* is given by

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^p}, \quad 0 < x < \infty$$

The Extreme value distribution is applied very much in natural phenomenon such as rain fall, floods, wind gusts, and air pollution.

VIII. Lindley Distribution

A *rv* X is said to have Lindley distribution if its *pdf* is given by

$$f(x, \theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \qquad x > 0, \quad \theta > 0 \tag{1.13.1}$$

and the *df* is given by

$$F(x) = 1 - \left[1 + \frac{\theta x}{1+\theta}\right] e^{-\theta x}, \qquad x > 0, \quad \theta > 0 \tag{1.13.2}$$

Lindley (1958) proposed Lindley distribution in the context of Bayesian statistics, as a counter example of fudicial statistics. Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distribution with shape parameter two. Lindley Distribution is widely used in Reliability Analysis and real life time data.

IX. New Weibull-Pareto Distribution

A *rv* X is said to have a New Weibull Pareto distribution (NWPD) by Nasiru and Luguterah (2015). if its *pdf* is given by

$$f(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad 0 < x < \infty;\ \beta > 0,\ \delta > 0,\ \theta > 0 \tag{1.13.3}$$

with corresponding *df*

$$F(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad 0 < x < \infty;\ \beta > 0,\ \delta > 0,\ \theta > 0 \tag{1.13.4}$$

New Weibull-Pareto distribution is used in real life data.

1.14 Content of the Thesis

Chapter 1 is an introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

In Chapter 2 of the thesis, we have characterized a families of continuous probability distributions by considering conditional expectation of difference of $p^{th},\ (p \geq 1)$ power of two records values conditioned on a pair of non-adjacent records and and the following two results are obtained:

(i) For $\Psi : \mathbb{R} \to \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$

$$\begin{aligned} g^p_{l,s}(x,y) &= E[\{\Psi(X_{U(j)}) - \Psi(X_{U(l)})\}^p | X_{U(l)} = x, X_{U(s)} = y] \\ &= [\Psi(y) - \Psi(x)]^p \frac{\Gamma(s-l)\Gamma(p+j-l)}{\Gamma(j-l)\Gamma(p+s-l)} \\ &1 \leq l < j < s \leq n,\ l = r, r+1 \end{aligned}$$

if and only if

$$F(x) = 1 - e^{-[a\Psi(x)+b]}, \qquad \alpha \leq x \leq \beta,$$

where $g^p_{r,s}(x,y)$ is a finite and differentiable function of x and $\Gamma(.)$ is a gamma function.

(ii) For $\Psi : \mathbb{R} \to \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$.

$$\begin{aligned} \xi^p_{r,l}(x,y) &= E[\{\Psi(X_{U(l)}) - \Psi(X_{U(j)})\}^p | X_{U(r)} = x, X_{U(l)} = y] \\ &= [\Psi(y) - \Psi(x)]^p \frac{\Gamma(l-r)\Gamma(p+l-j)}{\Gamma(l-j)\Gamma(p+l-r)} \\ &1 \leq r < j < l \leq n,\ l = s-1, s \end{aligned}$$

if and only if

$$F(y) = 1 - e^{-[a\Psi(y)+b]}, \qquad \alpha \le y \le \beta,$$

provided that $\xi^{p}_{r,s}(x, y)$ is a finite and differentiable function of y and there exists a $q \in (\alpha, \beta)$ such that

$$q = \inf\left[x : x \ge F^{-1}\left(\frac{e-1}{e}\right)\right]$$

Examples of various distributions are given by properly choosing parameters $\Psi(x)$, a and b.

In Chapter 3 of the thesis, we have obtained results based on conditional expectation of single dual generalized order statistics conditioned on a pair of non-adjacent dual generalized order statistics using Meijer's G-Function. Thus, extending the result of Khan and Khan (2012) conditioned on a non-adjacent *dgos* and the following two results are obtained:

(i) For $\psi(t)$ be a monotonic and differentiable function of t.

$$g_{j|l,s}(x, y) = E[\psi(X^*(j, n, \tilde{m}, k)) \mid X^*(l, n, \tilde{m}, k) = x, X^*(s, n, \tilde{m}, k) = y]$$

$$1 < r + 1 < j < s \le n, \quad l = r, r + 1$$

exist, then

$$(\gamma_{r+1} - 1)\frac{f(x)}{F(x)} - \frac{\frac{\partial}{\partial x} G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x, y)}{[g_{j|r+1,s}(x, y) - g_{j|r,s}(x, y)]}$$

and

$$\frac{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1} - \gamma_{r+1} + 1, ..., \gamma_s - \gamma_{r+1} + 1\right)}{G_{s-r}(F(y) \mid \gamma_{r+1} - \gamma_{r+1} + 1, ..., \gamma_s - \gamma_{r+1} + 1)} = \exp\left(-\int_x^\beta D_1(t, y)dt\right)$$

where $g()$ is a finite and differentiable function of x, and

$$D_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]}$$

(ii) For $\psi(t)$ be a monotonic and differentiable function of t.

$$\xi_{j|r,l}(x,y) = E[\psi(X^*(j,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k) = x, X^*(l,n,\tilde{m},k) = y],$$
$$1 \le r < j < s-1 < n, \quad l = s-1, s$$

exist, then

$$(\gamma_s - 1)\frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y} G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\frac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]}$$

$$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\right) = a_s^{(r)}(s) \exp\left[\int_y^\beta D_2(x,t)dt\right],$$
$$\gamma_i > \gamma_s, \quad i = r+1, ..., s-1$$

and for $\gamma_{r+1} = \cdots = \gamma_s$,

$$\frac{1 + \log\{F(y)\}}{1 + \log\{F(x)\}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_y^q D_2(x,t)dt\right]$$

where

$$p \in (\alpha, \beta) \text{ such that } \quad -\log F(p) = 1$$

and

$$D_2(x,y) = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]}$$

Further, from these results lower records values are obtained and the results obtained by Khan *et al.* (2009), Khan *et al.* (2010a) and Khan and Khan (2012) are discussed.

In Chapter 4 of the thesis, we have obtained characterization of conditional expectation of difference of p^{th}, $(p \geq 1)$ power of two generalized order statistics conditioned on a pair of non-adjacent generalized order statistics using Meijer's G-Function. Further, some of its important deductions are discussed and some examples are obtained based on the deductions.

In Chapter 5 of the thesis, we have obtained explicit expressions for single and product moment of order statistics from Lindley distribution $F(x) = 1 - \left[1 + \dfrac{\theta x}{1+\theta}\right] e^{-\theta x}$, $x > 0, \theta > 0$. Further, means and covariance of order statistics from Lindley distributions are obtained.

In Chapter 6 of the thesis, we have obtained recurrence relations for single and product moments of generalized order statistics from New Weibull Pareto distribution $F(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}$, $0 < x < \infty;\ \beta > 0,\ \delta > 0,\ \theta > 0$. Further, the distribution is characterized by a recurrence relation of single moments. Also, some deductions and particular cases are given. In the end, a comprehensive bibliography is given.

Chapter 2

On Characterization of Continuous Probability Distributions Conditioned on a Pair of Record Values

2.1 Introduction

Characterization of distributions through conditional expectation of record values was first considered by Nagaraja (1988b). They obtained the characterization result based on the linear regression of adjacent record values by considering the regression equation $g_{r+1|r}(x) = E[X_{U(r+1)}|X_{U(r)} = x] = ax + b$. Later, Franco and Ruiz (1996, 1997), Lopez-Bláquez and Rebollo (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000) and Athar *et al.* (2003) extended the result of Nagaraja (1988b) and characterized the continuous distributions conditioned on non-adjacent records. Further, Bairamov *et al.* (2005), Yanev *et al.* (2008), Yanev and Ahsanullah (2009) and Khan and Khan (2009) characterized the continuous distributions conditioned on a pair of non-adjacent records. Recently, Noor and Athar (2014) characterized the continuous distributions by taking the conditional expectation $g^{p}_{r,s}(x) = E[\{\Psi(X_{U(s)}) - \Psi(X_{U(r)})\}^{p}|X_{U(r)} = x]$. In this Chapter, we have extended the result of Noor and Athar (2014) and investigated the conditional expectation $g^p_{r,s}(x,y) = E[\{\Psi(X_{U(j)})-\Psi(X_{U(r)})\}^p | X_{U(r)} = x, X_{U(s)} = y]$, conditioned on a pair of non-adjacent records.

2.2 Characterization of Probability Distributions

Let $X_{U(r)}$ be the upper record from a continuous population with *pdf* $f(x)$ and the *df* $F(x)$ over the support (α,β). Hence, the conditional *pdf* of $X_{U(j)}$ given $X_{U(r)} = x$ and $X_{U(s)} = y$, $1 \le r < j < s \le n$ is given by

$$f_{j|r,s}(t|x,y) = C_{r,j,s}\frac{[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}}{[B(x,y)]^{s-r-1}}\frac{f(t)}{\bar{F}(t)}, \qquad -\infty < x < t < y < \infty, \qquad (2.2.1)$$

where $C_{r,j,s} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$.

Theorem 2.1: Let $X_{U(i)}, i = 1,2,\ldots,n$ be the i^{th} record from a continuous population with *pdf* $f(x)$ and *df* $F(x)$ over the support (α,β). Let $\Psi : \mathbb{R} \to \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$. Then for $1 \le l < j < s \le n,\ l = r, r+1$

$$\begin{aligned} g^p_{l,s}(x,y) &= E[\{\Psi(X_{U(j)}) - \Psi(X_{U(l)})\}^p | X_{U(l)} = x, X_{U(s)} = y] \\ &= [\Psi(y) - \Psi(x)]^p \frac{\Gamma(s-l)\Gamma(p+j-l)}{\Gamma(j-l)\Gamma(p+s-l)} \end{aligned} \qquad (2.2.2)$$

if and only if

$$F(x) = 1 - e^{-[a\Psi(x)+b]}, \qquad \alpha \le x \le \beta, \qquad (2.2.3)$$

where $g^p_{r,s}(x,y)$ is a finite and differentiable function of x and $\Gamma(.)$ is a gamma function.

Proof: Here, first we shall prove the necessary condition, equation (2.2.3) implies (2.2.2), then we have

$$g_{r,s}^{p}(x,y) = E[\{\Psi(X_{U(j)}) - \Psi(X_{U(l)})\}^{p} | X_{U(l)} = x, X_{U(s)} = y] \tag{2.2.4}$$

Therefore, in view of equation (2.2.1),

$$g_{r,s}^{p}(x,y)[B(x,y)]^{s-r-1} = C_{r,j,s} \int_{x}^{y} [\Psi(t) - \Psi(x)]^{p} [B(x,t)]^{j-r-1} [B(t,y)]^{s-j-1} \frac{dF(t)}{\bar{F}(t)}$$

$$g_{r,s}^{p}(x,y) = \frac{C_{r,j,s}}{[B(x,y)]} \int_{x}^{y} [\Psi(t) - \Psi(x)]^{p} \left[\frac{B(x,t)}{B(x,y)}\right]^{j-r-1} \left[1 - \frac{B(x,t)}{B(x,y)}\right]^{s-j-1} \times \frac{dF(t)}{\bar{F}(t)}$$

$$g_{r,s}^{p}(x,y) = \frac{C_{r,j,s}}{[-\log \bar{F}(y) + \log \bar{F}(x)]} \times \int_{x}^{y} [\Psi(t) - \Psi(x)]^{p} \left[\frac{[-\log \bar{F}(t) + \log \bar{F}(x)]}{[-\log \bar{F}(y) + \log \bar{F}(x)]}\right]^{j-r-1} \times \left[1 - \frac{[-\log \bar{F}(t) + \log \bar{F}(x)]}{[-\log \bar{F}(y) + \log \bar{F}(x)]}\right]^{s-j-1} \frac{dF(t)}{\bar{F}(t)}$$

$$= \frac{C_{r,j,s}}{[\Psi(y) - \Psi(x)]} \int_{x}^{y} [\Psi(t) - \Psi(x)]^{p} \left[\frac{\Psi(t) - \Psi(x)}{\Psi(y) - \Psi(x)}\right]^{j-r-1} \times \left[1 - \frac{\Psi(t) - \Psi(x)}{\Psi(y) - \Psi(x)}\right]^{s-j-1} d\Psi(t)$$

Set $u = \frac{\Psi(t) - \Psi(x)}{\Psi(y) - \Psi(x)}$, we get

$$g_{r,s}^{p}(x,y) = C_{r,j,s} [\Psi(y) - \Psi(x)]^{p} \int_{0}^{1} u^{j-r+p-1} (1-u)^{s-j-1} du$$

$$= [\Psi(y) - \Psi(x)]^{p} \frac{\Gamma(s-r)}{\Gamma(j-r)\Gamma(s-j)} \times \frac{\Gamma(j-r+p)\Gamma(s-j)}{\Gamma(j-r+p+s-j)}$$

$$g^p_{r,s}(x,y) = [\Psi(y)-\Psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}$$

To prove the sufficient condition, on using the equation (2.2.2) which implies (2.2.3), we have

$$g^p_{r,s}(x,y)[B(x,y)]^{s-r-1} = C_{r,j,s}\int_x^y [\Psi(t)-\Psi(x)]^p [B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\frac{dF(t)}{\bar{F}(t)}$$

Differentiating both the sides *w.r.t.* x, we get

$$\begin{aligned}
&\frac{\partial}{\partial x} g^p_{r,s}(x,y)[B(x,y)]^{s-r-1} - g^p_{r,s}(x,y)(s-r-1)[B(x,y)]^{s-r-2}\frac{f(x)}{\bar{F}(x)} \\
&\quad = C_{r,j,s}\int_x^y \left[-p\Psi^{'}(x)[\Psi(t)-\Psi(x)]^{p-1}[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\right]\frac{dF(t)}{\bar{F}(t)} \\
&\qquad - C_{r,j,s}\int_x^y \left[[\Psi(t)-\Psi(x)]^p (j-r-1)[B(x,t)]^{j-r-2}[B(t,y)]^{s-j-1}\frac{f(x)}{\bar{F}(x)}\right]\frac{dF(t)}{\bar{F}(t)}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial x} g^p_{r,s}(x,y)[B(x,y)]^{s-r-1} + C_{r,j,s}\int_x^y \Big[p\Psi^{'}(x)[\Psi(t)-\Psi(x)]^{p-1}[B(x,t)]^{j-r-1} \\
&\quad \times [B(t,y)]^{s-j-1}\Big]\frac{dF(t)}{\bar{F}(t)} = g^p_{r,s}(x,y)(s-r-1)[B(x,y)]^{s-r-2}\frac{f(x)}{\bar{F}(x)} \\
&\qquad - C_{r,j,s}\int_x^y \left[[\Psi(t)-\Psi(x)]^p (j-r-1)[B(x,t)]^{j-r-2}\,[B(t,y)]^{s-j-1}\frac{f(x)}{\bar{F}(x)}\right]\frac{dF(t)}{\bar{F}(t)}
\end{aligned}$$

This implies that

$$\frac{f(x)}{\bar{F}(x)B(x,y)} = \frac{p\Psi^{'}(x)g^{p-1}_{r,s}(x,y) + \frac{\partial}{\partial x}g^p_{r,s}(x,y)}{(s-r-1)[g^p_{r,s}(x,y) - g^p_{r+1,s}(x,y)]} \tag{2.2.5}$$

Now consider,

$$p\Psi^{'}(x)g^{p-1}_{r,s}(x,y) + \frac{\partial}{\partial x}g^p_{r,s}(x,y)$$

$$
\begin{aligned}
&= p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r-1)} \\
&\quad - p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}
\end{aligned}
$$

$$
= p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\left[\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r-1)} - \frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}\right]
$$

$$
= p(s-j)\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}
$$

and

$$
\begin{aligned}
& g^{p}_{r,s}(x,y) - g^{p}_{r+1,s}(x,y) \\
&= [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)} \\
&\quad - [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r-1)\Gamma(p+s-r-1)}
\end{aligned}
$$

$$
= [\Psi(y)-\Psi(x)]^{p}\left[\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)} - \frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r-1)\Gamma(p+s-r-1)}\right]
$$

$$
= p(s-j)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}
$$

Therefore, in view of equation (2.2.5), we have

$$
\frac{f(x)}{\bar{F}(x)B(x,y)} = \frac{p(s-j)\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}}{(s-r-1)p(s-j)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}}
$$

$$
\frac{f(x)}{\bar{F}(x)B(x,y)} = \frac{\Psi^{'}(x)}{[\Psi(y)-\Psi(x)]}
$$

Integrating both the sides *w.r.t.* x, over (α, x), we get

$$\int_{\alpha}^{x} \frac{f(x)}{\bar{F}(x)B(x,y)}dx = \int_{\alpha}^{x} \frac{\Psi^{'}(x)}{[\Psi(y)-\Psi(x)]}dx$$

$$\log B(x,y) - \log B(\alpha,y) = -\log[\Psi(y)-\Psi(t)]|_{\alpha}^{x}$$

or,

$$\log\left[1-\frac{\log \bar{F}(x)}{\log \bar{F}(y)}\right] = \log\left[\frac{\Psi(y)-\Psi(\alpha)}{\Psi(y)-\Psi(x)}\right]$$

or,

$$-\frac{\log \bar{F}(x)}{\log \bar{F}(y)} = \left[\frac{\Psi(x)-\Psi(\alpha)}{\Psi(y)-\Psi(x)}\right]$$

This implies that

$$-\log \bar{F}(x) = a[\Psi(x)-\Psi(\alpha)] = a\Psi(x)+b,$$

where $b = -a\Psi(\alpha)$. This proves the theorem.

Theorem 2.2: Let $X_{U(i)}, i = 1, 2, \ldots, n$ be the i^{th} record from a continuous population with *pdf* $f(x)$ and *df* $F(x)$ over the support (α, β). Let $\Psi : \mathbb{R} \to \mathbb{R}$ is a monotonic and differentiable function and $p \in \mathbb{N}$. Then for $1 \leq r < j < l \leq n,\ l = s-1, s$

$$\begin{aligned}\xi_{r,l}^{p}(x,y) &= E[\{\Psi(X_{U(l)})-\Psi(X_{U(j)})\}^{p}|X_{U(r)}=x, X_{U(l)}=y] \\ &= [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(l-r)\Gamma(p+l-j)}{\Gamma(l-j)\Gamma(p+l-r)} \qquad (2.2.6)\end{aligned}$$

if and only if

$$F(y) = 1 - e^{-[a\Psi(y)+b]}, \qquad \alpha \le y \le \beta, \tag{2.2.7}$$

provided that $\xi^p_{r,s}(x,y)$ is a finite and differentiable function of y and there exists a $q \in (\alpha, \beta)$ such that

$$q = \inf\left[x : x \ge F^{-1}\left(\frac{e-1}{e}\right)\right] \tag{2.2.8}$$

Proof: Here, first we shall prove that the necessary condition, equation (2.2.7) implies (2.2.6), then we have

$$\xi^p_{r,s}(x,y) = E[\{\Psi(X_{U(l)}) - \Psi(X_{U(j)})\}^p | X_{U(r)} = x, X_{U(l)} = y]$$

Therefore, in view of equation (2.2.1),

$$\xi^p_{r,s}(x,y)[B(x,y)]^{s-r-1} = C_{r,j,s}\int_x^y [\Psi(y) - \Psi(t)]^p [B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\frac{dF(t)}{\bar{F}(t)}$$

$$\begin{aligned}\xi^p_{r,s}(x,y) = {} & \frac{C_{r,j,s}}{[B(x,y)]}\int_x^y [\Psi(y) - \Psi(t)]^p \left[\frac{B(t,y)}{B(x,y)}\right]^{s-j-1}\left[1 - \frac{B(t,y)}{B(x,y)}\right]^{j-r-1} \\ & \times \frac{dF(t)}{\bar{F}(t)}\end{aligned}$$

$$\begin{aligned}\xi^p_{r,s}(x,y) = {} & \frac{C_{r,j,s}}{[-\log \bar{F}(y) + \log \bar{F}(x)]} \\ & \times \int_x^y [\Psi(y) - \Psi(t)]^p \left[\frac{[-\log \bar{F}(y) + \log \bar{F}(t)]}{[-\log \bar{F}(y) + \log \bar{F}(x)]}\right]^{j-r-1} \\ & \times \left[1 - \frac{[-\log \bar{F}(y) + \log \bar{F}(t)]}{[-\log \bar{F}(y) + \log \bar{F}(x)]}\right]^{s-j-1}\frac{dF(t)}{\bar{F}(t)}\end{aligned}$$

$$= \frac{C_{r,j,s}}{[\Psi(y)-\Psi(x)]}\int_x^y [\Psi(y)-\Psi(t)]^p \left[\frac{\Psi(y)-\Psi(t)}{\Psi(y)-\Psi(x)}\right]^{j-r-1} \times \left[1-\frac{\Psi(y)-\Psi(t)}{\Psi(y)-\Psi(x)}\right]^{s-j-1} d\Psi(t)$$

Set $v=\frac{\Psi(y)-\Psi(t)}{\Psi(y)-\Psi(x)}$, we get

$$\begin{aligned}
\xi^p_{r,s}(x,y) &= C_{r,j,s}[\Psi(y)-\Psi(x)]^p \int_0^1 v^{p+s-j-1}(1-v)^{j-r-1}dv \\
&= [\Psi(y)-\Psi(x)]^p \frac{\Gamma(s-r)}{\Gamma(j-r)\Gamma(s-j)} \times \frac{\Gamma(p+s-j)\Gamma(j-r)}{\Gamma(p+s-j+j-r)} \\
\xi^p_{r,s}(x,y) &= [\Psi(y)-\Psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)}
\end{aligned}$$

To prove the sufficient condition, on using equation (2.2.6) which implies (2.2.7), we have

$$\xi^p_{r,s}(x,y)[B(x,y)]^{s-r-1} = C_{r,j,s}\int_x^y [\Psi(y)-\Psi(t)]^p[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\frac{dF(t)}{\bar{F}(t)}$$

Differentiating both the sides *w.r.t.* y, we get

$$\begin{aligned}
&\frac{\partial}{\partial y}\xi^p_{r,s}(x,y)[B(x,y)]^{s-r-1} + \xi^p_{r,s}(x,y)(s-r-1)[B(x,y)]^{s-r-2}\frac{f(y)}{\bar{F}(y)} \\
&\quad = C_{r,j,s}\int_x^y \left[p\Psi'(y)[\Psi(y)-\Psi(t)]^{p-1}[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\right]\frac{dF(t)}{\bar{F}(t)} \\
&\qquad + C_{r,j,s}\int_x^y \left[[\Psi(y)-\Psi(t)]^p(s-j-1)[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-2}\frac{f(y)}{\bar{F}(y)}\right]\frac{dF(t)}{\bar{F}(t)}
\end{aligned}$$

$$\begin{aligned}
&-\frac{\partial}{\partial y}\xi^p_{r,s}(x,y)[B(x,y)]^{s-r-1} + C_{r,j,s}\int_x^y \Big[p\Psi'(y)[\Psi(y)-\Psi(t)]^{p-1}[B(x,t)]^{j-r-1} \\
&\times [B(t,y)]^{s-j-1}\Big]\frac{dF(t)}{\bar{F}(t)} = \xi^p_{r,s}(x,y)(s-r-1)[B(x,y)]^{s-r-2}\frac{f(y)}{\bar{F}(y)} \\
&\quad - C_{r,j,s}\int_x^y \left[[\Psi(y)-\Psi(t)]^p(s-j-1)[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-2}\frac{f(y)}{\bar{F}(y)}\right]\frac{dF(t)}{\bar{F}(t)}
\end{aligned}$$

This implies that

$$\frac{f(y)}{\bar{F}(y)B(x,y)} = \frac{p\Psi^{'}(y)\xi_{r,s}^{p-1}(x,y) - \frac{\partial}{\partial y}\xi_{r,s}^{p}(x,y)}{(s-r-1)[\xi_{r,s}^{p}(x,y) - \xi_{r,s-1}^{p}(x,y)]} \tag{2.2.9}$$

Now consider,

$$\begin{aligned}
&p\Psi^{'}(y)\xi_{r,s}^{p-1}(x,y) - \frac{\partial}{\partial y}\xi_{r,s}^{p}(x,y) \\
&= p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r-1)} \\
&\quad - p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)}
\end{aligned}$$

$$= p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\left[\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r-1)} - \frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)}\right]$$

$$= p(j-r)\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}$$

and

$$\begin{aligned}
&\xi_{r,s}^{p}(x,y) - \xi_{r,s-1}^{p}(x,y) \\
&= [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} \\
&\quad - [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j-1)\Gamma(p+s-r-1)}
\end{aligned}$$

$$= [\Psi(y)-\Psi(x)]^{p}\left[\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} - \frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j-1)\Gamma(p+s-r-1)}\right]$$

$$= p(j-r)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}$$

Therefore, in view of equation (2.2.9), we have

$$\frac{f(y)}{\bar{F}(y)B(x,y)} = \frac{p(j-r)\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}}{(s-r-1)p(j-r)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}}$$
$$\frac{f(y)}{\bar{F}(y)B(x,y)} = \frac{\Psi^{'}(y)}{[\Psi(y)-\Psi(x)]}$$

Integrating both the sides *w.r.t.* y, over (y,q), we get

$$\int_y^q \frac{f(y)}{\bar{F}(y)B(x,y)}dy = \int_y^q \frac{\Psi^{'}(y)}{[\Psi(y)-\Psi(x)]}$$

or,

$$\frac{1+\log \bar{F}(y)}{1+\log \bar{F}(x)} = 1 - e^{\left[-\int_y^q \frac{[\Psi(t)}{\Psi(t)-\Psi(x)}dt\right]}$$

$$\frac{1+\log \bar{F}(y)}{1+\log \bar{F}(x)} = \frac{\Psi(q)-\Psi(y)}{\Psi(q)-\Psi(x)}$$

Thus, we have

$$1+\log \bar{F}(y) = a_1[\Psi(q)-\Psi(y)] = a\Psi(y)+b,$$

where $a = -a_1$ and $b = a_1\Psi(q)$. This proves the theorem.

2.3 Examples

Table 2.1 shows that for particular choices of a, b and $\Psi(x)$ the following distributions can be characterized using Theorem 2.1 and Theorem 2.2.

Table 2.1: Examples based on $F(x) = 1 - e^{-[a\Psi(x)+b]}$

Distribution	$F(x)$	a	b	$\Psi(x)$
Power function	$a^{-p}x^{p}$	1	$p \log a$	$-\log(a^{p} - x^{p})$
Pareto	$1 - a^{p}x^{-p}$	p	$-p \log a$	$\log x$
Beta of I kind	$1 - (1 - x)^{p}$	p	0	$-\log(1 - x)$
Exponential	$1 - e^{-\theta x}$	θ	0	x
Rayleigh	$1 - e^{-\theta x^{2}}$	θ	0	x^{2}
Weibull	$1 - e^{-\theta x^{p}}$	θ	0	x^{p}
Extreme value II	$1 - e^{e^{\theta x}}$	1	0	$e^{\theta x}$
Burr Type XII	$1 - (1 + \theta x^{p})^{-m}$	m	0	$\log(1 + \theta x^{p})$

2.4 Conclusion

A probability distribution can be characterized in many ways and the method under study here is one of them. We have used here the conditional expectation of record statistics conditioned on a pair of non-adjacent records to characterize the probability distribution. That is, we have characterized the probability distribution if the regression equation truncated from both sides is given, i.e. the data are truncated from left side at x and truncated from right side at y. In real practice, several times we get the data of which observations are missing either in beginning or in the end. In such type of data we can use the result of this Chapter, *i.e.* when the data are in form of records and when the data are missing at both.

Keeping this in view, we have characterized probability distributions through conditional expectation conditioned on a pair of non-adjacent records.

Chapter 3

Characterization of Continuous Distributions Conditioned on a pair of Non-Adjacent Dual Generalized Order Statistics using Meijer's G-Function

3.1 Introduction

In this Chapter, a generalized family of continuous distributions has been characterized through conditional expectation of dual generalized order statistics (*dgos*) conditioned on a pair of non-adjacent *dgos* using Meijer's G-Function. Various developments on dual generalized order statistics and related topic have been studied by Ahsanullah (2004), Mbah and Ahsanulah (2007), Khan *et al.* (2009), Khan *et al.* (2010 a,b), Faizan and Khan (2011), Tavangar (2011) amongst others. Khan *et al.* (2009) have characterized continuous distributions through conditional expectation of *dgos*, conditioned on a pair of non-adjacent *dgos*. Recently, Khan and Khan (2012) have characterized continuous distribution functions conditioned on non-adjacent *dgos* using Meijers G-Function. In this Chapter, we have extended the result of Khan and Khan (2012) when the regression is based on two non-adjacent *dgos*. Also the result is deduced for known results on *dgos*.

3.2 Characterization of Probability Distributions

Let $X^*(i, n, \tilde{m}, k),\ i = 1, 2, ..., n$ be the *dgos* from a continuous population with *pdf* $f(x)$ and *df* $F(x)$ over the support (α, β). Hence, the conditional P_F density function of $X^*(j, n, \tilde{m}, k)$ given $X^*(r, n, \tilde{m}, k) = x$ and $X^*(s, n, \tilde{m}, k) = y,\ 1 \leqslant r < j < s \leqslant n$ is given by

$$f_{j|r,s}(t \mid x, y) = \frac{1}{F(t)} \frac{G_{s-j}\left(\dfrac{F(y)}{F(t)} \mid \gamma_{j+1}, ..., \gamma_s\right) G_{j-r}\left(\dfrac{F(t)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_j\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} I_{(y,x)}(t), \qquad (3.2.1)$$

where $C_{r-1} = \left(\prod_{i=1}^{r} \gamma_i\right)$

Theorem 3.1: Let $X^*(i, n, \tilde{m}, k),\ i = 1, ..., n$ be the i^{th} *dgos* from a continuous population with *pdf* $f(x)$ and the *df* $F(x)$ over the support (α, β), and $\psi(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $(r+1),\ 1 < r+1 < j < s \leq n$,

$$g_{j|l,s}(x, y) = E[\psi(X^*(j, n, \tilde{m}, k)) \mid X^*(l, n, \tilde{m}, k) = x, X^*(s, n, \tilde{m}, k) = y],$$
$$l = r, r+1, \qquad (3.2.2)$$

exist, then

$$(\gamma_{r+1} - 1)\frac{f(x)}{F(x)} - \frac{\dfrac{\partial}{\partial x} G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\dfrac{\partial}{\partial x} g_{j|r,s}(x, y)}{\left[g_{j|r+1,s}(x, y) - g_{j|r,s}(x, y)\right]} \qquad (3.2.3)$$

and

$$\frac{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}-\gamma_{r+1}+1, ..., \gamma_s-\gamma_{r+1}+1\right)}{G_{s-r}(F(y) \mid \gamma_{r+1}-\gamma_{r+1}+1, ..., \gamma_s-\gamma_{r+1}+1)} = \exp\left(-\int_x^\beta D_1(t,y)dt\right), \tag{3.2.4}$$

where $g()$ is a finite and differentiable function of x, and

$$D_1(x,y) = \frac{\dfrac{\partial}{\partial x} g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]} \tag{3.2.5}$$

Proof: We have

$$\begin{aligned} g_{j|r,s}(x,y) G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right) &= \int_y^x \frac{\psi(t)}{F(t)} G_{s-j}\left(\frac{F(y)}{F(t)} \mid \gamma_{j+1}, ..., \gamma_s\right) \\ &\quad \times G_{j-r}\left(\frac{F(t)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_j\right) f(t) dt \end{aligned} \tag{3.2.6}$$

Differentiating both the sides *w.r.t.* x and using the property (vi) from Auxilliary Results (Chapter 1), we have

$$\begin{aligned} &\frac{\partial}{\partial x} g_{j|r,s}(x,y) G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right) - g_{j|r,s}(x,y)(\gamma_{r+1}-1) G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right) \\ &\times \frac{f(x)}{F(x)} + g_{j|r,s}(x,y) G_{s-r-1}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+2}, ..., \gamma_s\right) \frac{f(x)}{F(x)} = -g_{j|r,s}(x,y)(\gamma_{r+1}-1) \\ &\times G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right) \frac{f(x)}{F(x)} + g_{j|r+1,s}(x,y) G_{s-r-1}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+2}, ..., \gamma_s\right) \frac{f(x)}{F(x)} \end{aligned}$$

After rearranging the terms, we get

$$\frac{\dfrac{f(x)}{F(x)} G_{s-r-1}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+2}, ..., \gamma_s\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\dfrac{\partial}{\partial x} g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]}$$

$$\Rightarrow \quad \frac{\frac{f(x)}{F(x)}(\gamma_{r+1}-1)G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right) - \frac{\partial}{\partial x}G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)} = \frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]}$$

Implies that

$$(\gamma_{r+1}-1)\frac{f(x)}{F(x)} - \frac{\frac{\partial}{\partial x}G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)} = \frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{\left[g_{j|r+1,s}(x,y) - g_{j|r,s}(x,y)\right]},$$

and hence the theorem follows.

Corollary 3.1

Using residue theorem, it can be proved that

$$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right) = \left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{F(y)}{F(x)}\right]^{\gamma_i-1}\right) \tag{3.2.7}$$

where $a_i^{(r)}(s) = \prod_{j=r+1, j\neq i}^{s} \frac{1}{(\gamma_j-\gamma_i)}, \quad \gamma_j \neq \gamma_i, \quad r+1 \leq i \leq s \leq n.$

Therefore, in this case, equation (3.2.4) reduces to

$$\frac{[F(x)]^{\gamma_{r+1}} B_r^s(x,y)}{B_r^s(\alpha,y)} = \exp\left(-\int_x^{\beta} D_1(t,y)dt\right), \tag{3.2.8}$$

where

$$B_r^s(x,y) = \sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{F(y)}{F(x)}\right]^{\gamma_i}$$

as obtained by Khan *et al.* (2009).

Also since for $m_1 = \cdots = m_{n-1} = m \neq -1$,

$$a_i^{(r)}(s) = \frac{1}{\prod\limits_{j=r+1, i\neq j}^{s} (\gamma_j - \gamma_i)} = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}$$

Thus, for $m_1 = \cdots = m_{n-1} = m \neq -1$, equation (3.2.4) reduces to

$$\frac{1-\{F(x)\}^{m+1}}{1-\{F(y)\}^{m+1}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_x^\beta D_1(t,y)dt\right], \quad m \neq -1 \tag{3.2.9}$$

and

$$\frac{\log\{F(x)\}}{\log\{F(y)\}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_x^\beta D_1(t,y)dt\right], \quad m = -1 \tag{3.2.10}$$

as obtained by Khan *et al.* (2009)

Remark 3.1

At $\gamma_s = 0$ *i.e.* $s = k+n+M$, by convention $X^*(s,n,\tilde{m},k) = y = \alpha$, and hence $F(\alpha) = 0$. Therefore,

$$g_{j|r}(x) = E[\psi(X^*(j,n,\tilde{m},k)) \mid X^*(r,n,\tilde{m},k) = x],$$

and

$$F(x) = \exp\left(\frac{1}{\gamma_{r+1}}\int_x^\beta \frac{g'_{j|r}(t)}{\left[g_{j|r+1}(t) - g_{j|r}(t)\right]}\right)dt \tag{3.2.11}$$

as given by Khan *et al.* (2010a), Khan and Khan (2012). The result for lower record is given by

$$F(x) = \exp\left(-\int_x^\beta \frac{g'_{j|r}(t)}{[g_{j|r+1}(t) - g_{j|r}(t)]} \right) dt$$

Theorem 3.2: Let $X^*(i, n, \tilde{m}, k),\ i = 1, ..., n$ be the i^{th} *dgos* from a continuous population with *pdf* $f(x)$ and the *df* $F(x)$ over the support (α, β), and $\psi(t)$ be a monotonic and differentiable function of t. If for two consecutive values $(s-1)$ and s, $1 \le r < j < s-1 < n$,

$$\xi_{j|r,l}(x, y) = E[\psi(X^*(j, n, \tilde{m}, k)) \mid X^*(r, n, \tilde{m}, k) = x, X^*(l, n, \tilde{m}, k) = y], \quad l = s-1, s \qquad (3.2.12)$$

exist, then

$$(\gamma_s - 1)\frac{f(y)}{F(y)} - \frac{\frac{\partial}{\partial y} G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)}{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}, ..., \gamma_s\right)} = \frac{\frac{\partial}{\partial y}\xi_{j|r,s}(x, y)}{[\xi_{j|r,s}(x, y) - \xi_{j|r,s-1}(x, y)]} \qquad (3.2.13)$$

$$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\right) = a_s^{(r)}(s) \exp\left[\int_y^\beta D_2(x, t)dt\right], \quad \forall \gamma_i > \gamma_s,\ i = r+1, ..., s-1 \qquad (3.2.14)$$

and for $\gamma_{r+1} = \cdots = \gamma_s$,

$$\frac{1 + \log\{F(y)\}}{1 + \log\{F(x)\}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_y^q D_2(x, t)dt\right] \qquad (3.2.15)$$

where

$$p \in (\alpha, \beta) \text{ such that } -\log F(p) = 1 \qquad (3.2.16)$$

and

$$D_2(x,y) = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]} \tag{3.2.17}$$

Proof: We have

$$\xi_{j|r,s}(x,y)G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right) = \int_y^x \frac{\psi(t)}{F(t)}G_{s-j}\left(\frac{F(y)}{F(t)} \mid \gamma_{j+1},...,\gamma_s\right) \times G_{j-r}\left(\frac{F(t)}{F(x)} \mid \gamma_{r+1},...,\gamma_j\right)f(t)dt \tag{3.2.18}$$

Differentiating both the sides *w.r.t.* y and using the property (v) from Auxilliary Results (Chapter 1), we have

$$\frac{\partial}{\partial y}\xi_{j|r,s}(x,y)G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right) + \xi_{j|r,s}(x,y)(\gamma_s-1)G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)\frac{f(y)}{F(y)}$$
$$- \xi_{j|r,s}(x,y)G_{s-r-1}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_{s-1}\right)\frac{f(y)}{F(y)} = \xi_{j|r,s}(x,y)(\gamma_s-1)$$
$$\times G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)\frac{f(y)}{F(y)} - \xi_{j|r,s-1}(x,y)G_{s-r-1}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_{s-1}\right)\frac{f(y)}{F(y)}$$

After rearranging the terms, we get

$$\frac{\dfrac{f(y)}{F(y)}G_{s-r-1}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_{s-1}\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)} = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]}$$

Thus,

$$(\gamma_s-1)\frac{f(y)}{F(y)} - \frac{\dfrac{\partial}{\partial y}G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)}{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right)} = \frac{\dfrac{\partial}{\partial y}\xi_{j|r,s}(x,y)}{\left[\xi_{j|r,s}(x,y) - \xi_{j|r,s-1}(x,y)\right]} \tag{3.2.19}$$

$$\frac{G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}-\gamma_s+1,...,\gamma_s-\gamma_s+1\right)}{G_{s-r}\left(\frac{F(\alpha)}{F(x)} \mid \gamma_{r+1}-\gamma_s+1,...,\gamma_s-\gamma_s+1\right)} = \exp\left(-\int_y^\alpha D_2(x,t)dt\right) \tag{3.2.20}$$

It can be seen that when $\gamma_i > \gamma_s,\ i = r+1,...,s-1$,

$G_{s-r}(x \mid \gamma_{r+1}-\gamma_s+1,...,\gamma_s-\gamma_s+1) \to a_s^{(r)}(s)$ as $x \to 0$,

and therefore

$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1}-\gamma_s+1,...,\gamma_s-\gamma_s+1\right) = a_s^{(r)}(s)\ \exp\left[-\int_\alpha^y D_2(x,t)dt\right], \quad \forall\ \gamma_i > \gamma_s,\ i = r+1,...,s-1$

also, for $\gamma_{r+1} = \cdots = \gamma_s$, see Cramer (2002, p.35)

$$G_{s-r}\left(\frac{F(y)}{F(x)} \mid \gamma_{r+1},...,\gamma_s\right) = \frac{1}{(s-r-1)!}[-\log F(y) + \log F(x)]^{s-r-1} \times \left[\frac{F(y)}{F(x)}\right]^{\gamma_{r+1}-1}.$$

Thus, in the case of lower record statistic,

$$\frac{1+\log\{F(y)\}}{1+\log\{F(x)\}} = 1 - \exp\left[-\frac{1}{(s-r-1)}\int_p^y D_2(x,t)dt\right], \tag{3.2.21}$$

where p is defined in equation (3.2.16)

Corollary 3.2

It may be noted that at $\gamma_i \neq \gamma_j$ but $m_1 = \cdots = m_{n-1} = m > -1$

$$\frac{G_{s-r}\left(\dfrac{F(y)}{F(x)} \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\right)}{a_s^{(r)}(s)} = \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i - \gamma_s}$$
$$= \left[1 - \left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}.$$

Therefore, equation (3.2.9), it reduces to

$$\left[\frac{F(y)}{F(x)}\right]^{m+1} = 1 - \exp\left[-\frac{1}{(s-r-1)} \int_\alpha^y D_2(x,t)dt\right], \ m > -1, \tag{3.2.22}$$

as obtained by Khan *et al.* (2010a).

Remark 3.2

With the convention $X^*(0, n, \tilde{m}, k) = x = \beta$, at $r = 0$, Theorem 3.2 reduces to

$$G_s(F(y) \mid \gamma_1 - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1) = \sum_{i=1}^{s} a_i(s)[F(y)]^{\gamma_i - \gamma_s}$$
$$= a_s(s) \exp\left[-\int_\alpha^y D(t)dt\right], \quad \text{if} \ \ \gamma_{1:s-1} > \gamma_s \tag{3.2.23}$$

and for $m_1 = \cdots = m_{n-1} = m > -1$

$$F(y)^{m+1} = 1 - \exp\left[-\frac{1}{s-1} \int_\alpha^y D(t)dt\right] \tag{3.2.24}$$

whereas for $\gamma_{r+1} = \cdots = \gamma_s$

$$-\log F(y) = \exp\left[-\frac{1}{s-1} \int_p^y D(t)dt\right] \tag{3.2.25}$$

where p is as defined in equation (3.2.16), and

$$D(y) = \frac{\xi'_{j|s}(y)}{\left[\xi_{j|s}(y) - \xi_{j|s-1}(y)\right]} = D_2(\alpha, y) \tag{3.2.26}$$

as obtained by Khan *et al.* (2010a), Khan and Khan (2012).

Corollary 3.3

Under the assumptions given in Corollary 3.1 and Corollary 3.2,

$$F(x) = \left[\frac{e^{I_1}}{e^{I_1} + e^{I_2} - 1}\right]^{\frac{1}{m+1}}, \quad m > -1 \tag{3.2.27}$$

and

$$F(y) = \left[\frac{e^{I_1} - 1}{e^{I_1} + e^{I_2} - 1}\right]^{\frac{1}{m+1}}, \quad m > -1 \tag{3.2.28}$$

where $I_1 = \int_x^\beta A_1(t, y)dt, \quad I_2 = \int_\alpha^y A_2(x, t)dt$ and

$$A_1(x, y) = \frac{D_1(x, y)}{(s - r - 1)}, \quad A_2(x, y) = \frac{D_2(x, y)}{(s - r - 1)}.$$

Similar, result for lower records can be obtained.

3.3 Conclusion

We can characterized the probability distributions by using different methods. In this Chapter, we have obtained conditional expectation of single dual generalized order statistics conditioned on a pair of non-adjacent dual generalized order statistics using Meijer's G-Function. Further, we can use these results for lower record values too.

Chapter 4

Characterization of Continuous Distributions Conditioned on a pair of Non-Adjacent Generalized Order Statistics using Meijer's G-Function

4.1 Introduction

Kamps (1995) was the first who introduced the concept of the *gos*. Since, then several characterizing result have been appeared in literature using conditional expectation. Keseling (1999), Bienik and Szynal (2003), Cramer *et al.* (2004a), Khan and Alzaid (2004), Raqab and Abu-Lawi (2004), Ahsanullah and Raqab (2004) and Khan *et al.* (2006) have characterized the distributions based on conditional expectation conditioned on adjacent and non-adjacent *gos*. Ahsanullah and Beg (2008) and Ahsanullah *et al.* (2009) have characterized continuous distributions through conditional expectation of *gos* conditioned on pair of adjacent and non-adjacent *gos*. Further, Bieniek (2009), Khan and Khan (2011) have characterized the continuous distribution functions conditioned on non-adjacent *gos* using Meijer's G-Function.

Later, Khan *et al.* (2012) extended the result of Bieniek (2009), Khan and Khan (2011) and characterized the continuous distributions conditioned on a pair of non-adjacent *gos*. Recently Noor *et al.* (2014) characterized the continuous distributions by taking the conditional expectation

$$g_{r,s,p} = E[\{\psi(X(s,n,\tilde{m},k)) - \psi(X(r,n,\tilde{m},k))\}^p \mid X(r,n,\tilde{m},k) = x]$$

In this Chapter, motivated by the work of Noor *et al.* (2014), we have investigated the characterization of the continuous distribution functions by considering the conditional expectation

$$g^p_{r,s}(x,y) = E[\{\psi(X(j,n,\tilde{m},k)) - \psi(X(r,n,\tilde{m},k))\}^p \mid X(r,n,\tilde{m},k) = x, X(s,n,\tilde{m},k) = y] \tag{4.1.1}$$

and

$$\xi^p_{r,s}(x,y) = E[\{\psi(X(s,n,\tilde{m},k)) - \psi(X(j,n,\tilde{m},k))\}^p \mid X(r,n,\tilde{m},k) = x, X(s,n,\tilde{m},k) = y], \tag{4.1.2}$$

where $1 \leq r < j < s \leq n, \;\; p \geq 1$, using Meijer's G-Function. Further, we have assumed that $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing function. Where $g^p_{r,s}(x,y)$ and $\xi^p_{r,s}(x,y)$ are finite and differentiable function of x and y respectively and we have derived the characterization result based on $g^p_{r,s}(x,y)$ and $\xi^p_{r,s}(x,y)$ respectively and deduced some examples based on the results

4.2 Characterization of Probability Distributions

Let $X(i,n,\tilde{m},k)$, $i = 1,2,...,n$ be the *gos* from a continuous population with the *pdf* $f(x)$ and *df* $F(x)$ over the support (α,β).

Hence the conditional P_F density function of $X(j,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k) = x$ and

$X(s,n,\tilde{m},k)=y$, $1\leqslant r<j<s\leqslant n$ is given by

$$f_{j|r,s}(t\mid x,y)=\frac{1}{\bar{F}(t)}\frac{G_{s-j}\big(\bar{F}_t(y)\mid\gamma_{j+1},...,\gamma_s\big)G_{j-r}\big(\bar{F}_x(t)\mid\gamma_{r+1},...,\gamma_j\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}I_{(x,y)}(t) \tag{4.2.1}$$

Theorem 4.1: Let $X(i,n,\tilde{m},k)$, $i=1,...,n$ be the i^{th} *gos* from a continuous population with *pdf* $f(x)$ and the *df* $F(x)$ over the support(α,β). Let $\psi:\mathbb{R}\to\mathbb{R}$ is strictly increasing function and $E|[\psi(X(s,n,\tilde{m},k))-\psi(X(l,n,\tilde{m},k)]|^p<\infty$, for $l=r,\ r+1$. If for two consecutive values r and $(r+1)$, $1<r+1<j<s\leq n$,

$$g^p_{l,s}(x,y)=E[\{\psi(X(j,n,\tilde{m},k))-\psi(X(l,n,\tilde{m},k))\}^p\mid X(l,n,\tilde{m},k)=x,X(s,n,\tilde{m},k)=y],\quad l=r,r+1, \tag{4.2.2}$$

exist, then

$$(\gamma_{r+1}-1)\frac{f(x)}{\bar{F}(x)}-\frac{\frac{\partial}{\partial x}G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}=\frac{p\psi'(x)g^{p-1}_{r,s}(x,y)+\frac{\partial}{\partial x}g^p_{r,s}(x,y)}{\big[g^p_{r,s}(x,y)-g^p_{r+1,s}(x,y)\big]} \tag{4.2.3}$$

and

$$\frac{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1}-\gamma_{r+1}+1,...,\gamma_s-\gamma_{r+1}+1\big)}{G_{s-r}\big(\bar{F}(y)\mid\gamma_{r+1}-\gamma_{r+1}+1,...,\gamma_s-\gamma_{r+1}+1\big)}=\exp\left(-\int_\alpha^x D_1(t,y)dt\right) \tag{4.2.4}$$

where $g^p_{r,s}$ is a finite and differentiable function of x and

$$D_1(x,y)=\frac{p\psi'(x)g^{p-1}_{r,s}(x,y)+\frac{\partial}{\partial x}g^p_{r,s}(x,y)}{\big[g^p_{r,s}(x,y)-g^p_{r+1,s}(x,y)\big]} \tag{4.2.5}$$

Proof: We have

$$g^p_{r,s}(x,y)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)=\int_x^y\frac{[\psi(t)-\psi(x)]^p}{\bar{F}(t)}G_{s-j}\big(\bar{F}_t(y)\mid\gamma_{j+1},...,\gamma_s\big)\times G_{j-r}\big(\bar{F}_x(t)\mid\gamma_{r+1},...,\gamma_j\big)f(t)dt$$

Differentiating both the sides *w.r.t.* x and using the property (vi) from Auxilliary Results (Chapter 1), we have

$$\frac{\partial}{\partial x}g^p_{r,s}(x,y)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)+g^p_{r,s}(x,y)(\gamma_{r+1}-1)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)\frac{f(x)}{\bar{F}(x)}$$
$$-g^p_{r,s}(x,y)G_{s-r-1}\big(\bar{F}_x(y)\mid\gamma_{r+2},...,\gamma_s\big)\frac{f(x)}{\bar{F}(x)}=-p\psi'(x)g^{p-1}_{r,s}(x,y)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)$$
$$+g^p_{r,s}(x,y)(\gamma_{r+1}-1)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)\frac{f(x)}{\bar{F}(x)}-g^p_{r+1,s}(x,y)G_{s-r-1}\big(\bar{F}_x(y)\mid\gamma_{r+2},...,\gamma_s\big)$$
$$\times\frac{f(x)}{\bar{F}(x)}$$

After rearranging the terms, we get

$$\frac{\frac{f(x)}{\bar{F}(x)}G_{s-r-1}\big(\bar{F}_x(y)\mid\gamma_{r+2},...,\gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}=\frac{p\psi'(x)g^{p-1}_{r,s}(x,y)+\frac{\partial}{\partial x}g^p_{r,s}(x,y)}{\big[g^p_{r,s}(x,y)-g^p_{r+1,s}(x,y)\big]}$$

$$\Rightarrow\quad\frac{\frac{f(x)}{\bar{F}(x)}(\gamma_{r+1}-1)G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)-\frac{\partial}{\partial x}G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}$$
$$=\frac{p\psi'(x)g^{p-1}_{r,s}(x,y)+\frac{\partial}{\partial x}g^p_{r,s}(x,y)}{\big[g^p_{r,s}(x,y)-g^p_{r+1,s}(x,y)\big]}$$

Implying that

$$(\gamma_{r+1}-1)\frac{f(x)}{\bar{F}(x)}-\frac{\frac{\partial}{\partial x}G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)}=\frac{p\psi'(x)g_{r,s}^{p-1}(x,y)+\frac{\partial}{\partial x}g_{r,s}^{p}(x,y)}{\big[g_{r,s}^{p}(x,y)-g_{r+1,s}^{p}(x,y)\big]},$$

and hence the theorem follows.

Corollary 4.1

Using the theorem of residue's under the condition that $\gamma_i\neq\gamma_j,\ \forall\ \ i,j=r+1,\ldots,$ s For $i\neq j$, it can be proved that Cramer *et al.* (2004b, p. 36)

$$G_{s-r}\big(\bar{F}_x(y)\mid\gamma_{r+1},...,\gamma_s\big)=\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\bar{F}_x(y)^{\gamma_i-1}\right), \tag{4.2.6}$$

where $a_i^{(r)}(s)=\prod_{j=r+1,j\neq i}^{s}\frac{1}{(\gamma_j-\gamma_i)}$, $\gamma_j\neq\gamma_i$, $r+1\leq i\leq s\leq n$.

Therefore, in this case, equation (4.2.4) reduces to

$$\frac{\big[\bar{F}(x)\big]^{\gamma_{r+1}}B_r^s(x,y)}{B_r^s(\alpha,y)}=\exp\left(-\int_{\alpha}^{x}D_1(t,y)dt\right), \tag{4.2.7}$$

where

$$B_r^s(x,y)=\sum_{i=r+1}^{s}a_i^{(r)}(s)\big[\bar{F}_x(y)\big]^{\gamma_i}$$

Also since, for $m_1=\cdots=m_{n-1}=m\neq-1$,

$$a_i^{(r)}(s)=\frac{1}{\prod\limits_{j=r+1,j\neq i}^{s}(\gamma_j-\gamma_i)}=\frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}$$

Thus, for $m_1 = \cdots = m_{n-1} = m \neq -1$, equation (4.2.4) reduces to

$$\frac{1-\{\bar{F}(x)\}^{m+1}}{1-\{\bar{F}(y)\}^{m+1}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)dt\right], \quad m \neq -1 \tag{4.2.8}$$

and when $\gamma_{r+1} = \gamma_{r+2} = \cdots = \gamma_s$ *i.e.* in case of record statistics see Cramer (p. 35)(2002)

$$G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big) = \frac{1}{(s-r-1)!}\big[-\log\bar{F}(y)+\log\bar{F}(x)\big]^{s-r-1}\big[\bar{F}_x(y)\big]^{\gamma_{r+1}-1} \tag{4.2.9}$$

equation (4.2.4) reduces to

$$\frac{\log\{\bar{F}(x)\}}{\log\{\bar{F}(y)\}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)dt\right], \quad m = -1. \tag{4.2.10}$$

Theorem 4.2: Let $X(i,n,\tilde{m},k), i = 1,2,\ldots,n$ be the i^{th} *gos* from a continuous population with *pdf* $f(x)$ and the *df* $F(x)$ over the support (α,β). Suppose $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing function and $E|[\psi(X(l,n,\tilde{m},k)) - \psi(X(j,n,\tilde{m},k)]|^p < \infty$, for $l = s-1,\ s$. If for two consecutive values $s-1$ and s, $1 \leq r+1 < j < s-1 < n$,

$$\xi^p_{r,l}(x,y) = E[\{\psi(X(l,n,\tilde{m},k)) - \psi(X(j,n,\tilde{m},k))\}^p \mid X(r,n,\tilde{m},k) = x, X(l,n,\tilde{m},k) = y],$$
$$l = s-1, s \tag{4.2.11}$$

then

$$(\gamma_s - 1)\frac{f(y)}{\bar{F}(y)} - \frac{\frac{\partial}{\partial y}G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)} = \frac{\frac{\partial}{\partial y}\xi^p_{r,s}(x,y) - p\psi'(y)\xi^{p-1}_{r,s}(x,y)}{\big[\xi^p_{r,s}(x,y) - \xi^p_{r,s-1}(x,y)\big]} \tag{4.2.12}$$

The following two cases will arise:

Case (i): when $\min(\gamma_{r+1}, \gamma_{r+2}, ..., \gamma_{s-1}) > \gamma_s$, then

$$G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\big) = a_s^{(r)}(s) \exp\left[-\int_y^\beta D_2(x,t)dt\right] \tag{4.2.13}$$

Case (ii): when $\min(\gamma_{r+1}, \gamma_{r+2}, ..., \gamma_{s-1}) \leq \gamma_s$, then the characterizing result is

$$G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\big) = \exp\left[-\int_y^q D_2(x,t)dt\right], \tag{4.2.14}$$

where q is defined as

$$q = \inf\{z \in (\alpha, \beta) : \ G_{s-r}\left(\bar{F}_x(z) \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\right) \geq 1\} \tag{4.2.15}$$

and

$$D_2(x,y) = \frac{\dfrac{\partial}{\partial y}\xi_{r,s}^p(x,y) - p\psi'(y)\xi_{r,s}^{p-1}(x,y)}{\big[\xi_{r,s}^p(x,y) - \xi_{r,s-1}^p(x,y)\big]} \tag{4.2.16}$$

Proof: We have

$$\xi_{r,s}^p(x,y)G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big) = \int_x^y \frac{[\psi(y) - \psi(t)]^p}{\bar{F}(t)} G_{s-j}\big(\bar{F}_t(y) \mid \gamma_{j+1}, ..., \gamma_s\big)$$
$$\times G_{j-r}\big(\bar{F}_x(t) \mid \gamma_{r+1}, ..., \gamma_j\big) f(t)dt$$

On Differentiating both the sides *w.r.t.* y and using the property (v) of Auxilliary Results (Chapter 1), we get

$$\frac{\partial}{\partial y}\xi_{r,s}^p(x,y)G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big) - \xi_{r,s}^p(x,y)(\gamma_s - 1)G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)\frac{f(y)}{\bar{F}(y)}$$
$$+ \xi_{r,s}^p(x,y)G_{s-r-1}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_{s-1}\big)\frac{f(y)}{\bar{F}(y)} = p\psi'(y)\xi_{r,s}^{p-1}(x,y)G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)$$
$$- \xi_{r,s}^p(x,y)(\gamma_s - 1)G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)\frac{f(y)}{\bar{F}(y)} + \xi_{r,s-1}^p(x,y)G_{s-r-1}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_{s-1}\big)$$

$$\times \frac{f(y)}{\bar{F}(y)}$$

After rearranging the terms, we get

$$\frac{\frac{f(y)}{\bar{F}(y)} G_{s-r-1}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_{s-1}\big)}{G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)} = \frac{\frac{\partial}{\partial y}\xi^p_{r,s}(x,y) - p\psi'(y)\xi^{p-1}_{r,s}(x,y)}{\big[\xi^p_{r,s}(x,y) - \xi^p_{r,s-1}(x,y)\big]}$$

$$\Rightarrow \quad \frac{\frac{f(y)}{\bar{F}(y)}(\gamma_s - 1) G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big) - \frac{\partial}{\partial y} G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)}$$
$$= \frac{\frac{\partial}{\partial y}\xi^p_{r,s}(x,y) - p\psi'(y)\xi^{p-1}_{r,s}(x,y)}{\big[\xi^p_{r,s}(x,y) - \xi^p_{r,s-1}(x,y)\big]}$$

Implying that

$$(\gamma_s - 1)\frac{f(y)}{\bar{F}(y)} - \frac{\frac{\partial}{\partial y} G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)}{G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}, ..., \gamma_s\big)} = \frac{\frac{\partial}{\partial y}\xi^p_{r,s}(x,y) - p\psi'(y)\xi^{p-1}_{r,s}(x,y)}{\big[\xi^p_{r,s-1}(x,y) - \xi^p_{r,s}(x,y)\big]}$$

It can be seen that when $\min(\gamma_{r+1}, \gamma_{r+2}, ..., \gamma_{s-1}) > \gamma_s$,
$G_{s-r}(x \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1) \to a^{(r)}_s(s)$ as $x \to 0$.

Therefore,
$G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\big) = a^{(r)}_s(s) \exp\Big[-\int_y^\beta D_2(x,t)dt\Big]$.

Further, when $\min(\gamma_{r+1}, \gamma_{r+2}, ..., \gamma_{s-1}) \leq \gamma_s$
In this case, $G_{s-r}(x \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1) \to \infty$ as $x \to 0$ (see Lemma 2.2: Cramer *et. al.*(2004b).
Thus, in this case, the characterization result is $G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1\big) =$

$\exp\left[-\int_y^q D_2(x,t)dt\right]$, where q is defined in equation (4.2.15).

Corollary 4.2

It may be noted that at $\gamma_i \neq \gamma_j$ but $m_1 = \cdots = m_{n-1} = m > -1$

$$\begin{aligned}\frac{G_{s-r}\big(\bar{F}_x(y) \mid \gamma_{r+1}-\gamma_s+1, ..., \gamma_s-\gamma_s+1\big)}{a_s^{(r)}(s)} &= \sum_{i=r+1}^{s} a_i^{(r)}(s)(\bar{F}_x(y))^{\gamma_i-\gamma_s} \\ &= \left[1-(\bar{F}_x(y))^{m+1}\right]^{s-r-1}\end{aligned}$$

Therefore, it reduces to

$$[\bar{F}_x(y)]^{m+1} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_y^\beta D_2(x,t)dt\right],\ m > -1. \tag{4.2.17}$$

Also for $\gamma_{r+1} = \cdots = \gamma_s$, *i.e.* in case of records and using equations (4.2.10), (4.2.14) reduces to

$$\frac{\{1+\log \bar{F}(y)\}}{\{1+\log \bar{F}(x)\}} = 1-\exp\left[-\frac{1}{(s-r-1)}\int_y^q D_2(x,t)dt\right] \tag{4.2.18}$$

where q is as defined as $q = \inf\left\{z \in (\alpha,\beta) :\ z \geq F^{-1}\left(\frac{e-1}{e}\right)\right\}$.

Corollary 4.3

Under the assumptions given in Corollary 4.1 and Corollary 4.2,

$$\bar{F}(x) = \left[\frac{e^{I_1}}{e^{I_1}+e^{I_2}-1}\right]^{\frac{1}{m+1}}, \qquad m > -1 \tag{4.2.19}$$

and

$$\bar{F}(y) = \left[\frac{e^{I_2} - 1}{e^{I_1} + e^{I_2} - 1}\right]^{\frac{1}{m+1}}, \qquad m > -1 \tag{4.2.20}$$

where $I_1 = \int_\alpha^x A_1(t,y)dt$, $I_2 = \int_y^\beta A_2(x,t)dt$ and $A_1(x,y) = \dfrac{D_1(x,y)}{(s-r-1)}$, $A_2(x,y) = \dfrac{D_2(x,y)}{(s-r-1)}$

Similarly, for records, we have

$$\bar{F}(x) = \exp\left[-\frac{e^{I_1} - 1}{e^{I_1} + e^{I_2} - 1}\right], \quad m = -1 \tag{4.2.21}$$

and

$$\bar{F}(y) = \exp\left[-\frac{e^{I_1}}{e^{I_1} + e^{I_2} - 1}\right], \quad m = -1. \tag{4.2.22}$$

4.3 Examples

In this Section, continuous distribution is characterized under the condition stated in Corollary 4.1 and Corollary 4.2 respectively. Based on $g^p_{r,s}(x,y)$ and $\psi^p_{r,s}(x,y)$, these result can be utilized to get the answer that from which distribution the sample is being obtained. Also any intermediate *gos* can be predicted if we have information about $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $r<s$ for a family of distribution.

(i) For $m_1 = ... = m_{n-1} = m \geq -1$

$$g^p_{r,s}(x,y) = [\psi(y)-\psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)} \tag{4.3.1}$$

If and only if

$$1-\{\bar{F}(x)\}^{m+1} = a\psi(x)+b, \quad m>-1 \tag{4.3.2}$$

where $F(x)$ is so chosen that $a\psi(\beta)+b=1$. And for record values

$$F(x) = 1-e^{-[a\psi(x)+b]}, \quad m=-1 \tag{4.3.3}$$

Provided that there exist a $q \in (\alpha,\beta)$ such that $a\psi(q)+b=1$.

Proof: For $m_1 = \cdots = m_{n-1} = m \neq -1$, the value of $g^p_{r,s}(x,y)$ is given by

$$g^p_{r,s}(x,y) = C_{r,j,s}(m+1)\int_x^y \frac{[\psi(t)-\psi(x)]^p}{[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]}\left[1-\frac{[\bar{F}(x)^{m+1}-\bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]}\right]^{s-j-1}$$
$$\times\left[\frac{[\bar{F}(x)^{m+1}-\bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]}\right]^{j-r-1}[\bar{F}(t)]^m f(t)dt,$$

where

$$C_{r,j,s} = \frac{\Gamma(s-r)}{\Gamma(j-r)\Gamma(s-j)}$$

Now, To prove the necessary part *i.e.* equation (4.3.2) implies (4.3.1), we have

$$g^p_{r,s}(x,y) = C_{r,j,s} \int_x^y \frac{[\psi(t)-\psi(x)]^p}{[\psi(y)-\psi(x)]} \left[1 - \frac{[\psi(t)-\psi(x)]}{[\psi(y)-\psi(x)]}\right]^{s-j-1} \left[\frac{[\psi(t)-\psi(x)]}{[\psi(y)-\psi(x)]}\right]^{j-r-1} \times \psi'(t)dt$$

Set $u = \left[\frac{[\psi(t)-\psi(x)]}{[\psi(y)-\psi(x)]}\right]$

Therefore,

$$\begin{aligned} g^p_{r,s}(x,y) &= C_{r,j,s}[\psi(y)-\psi(x)]^p \int_0^1 u^{p+j-r-1}(1-u)^{s-j-1}du \\ &= [\psi(y)-\psi(x)]^p \frac{\Gamma(s-r)}{\Gamma(j-r)\Gamma(s-j)} \frac{\Gamma(s-j)\Gamma(p+j-r)}{\Gamma(p+j-r+s-j)} \\ g^p_{r,s}(x,y) &= [\psi(y)-\psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)} \end{aligned}$$

Now, To prove the sufficiency part *i.e.* equation (4.3.1) implies (4.3.2), we have

$$g^p_{r,s}(x,y)[B(x,y)]^{s-r-1} = C_{r,j,s}(m+1) \int_x^y [\Psi(t)-\Psi(x)]^p [B(x,t)]^{j-r-1} [B(t,y)]^{s-j-1} \times [\bar{F}(t)]^m f(t)dt$$

On Differentiating both the sides *w.r.t.* x, and rearranging we get

$$A_1(x,y) = \frac{(m+1)f(x)[\bar{F}(x)]^m}{B(x,y)} = \frac{p\psi'(x)g^{p-1}_{r,s}(x,y) + \frac{\partial}{\partial x} g^p_{r,s}(x,y)}{(s-r-1)\left[g^p_{r,s}(x,y) - g^p_{r+1,s}(x,y)\right]},$$

Now consider,

$$
\begin{aligned}
&p\Psi^{'}(x)g_{r,s}^{p-1}(x,y)+\frac{\partial}{\partial x}g_{r,s}^{p}(x,y)\\
&=p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r-1)}\\
&\qquad -p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}
\end{aligned}
$$

$$
=p\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\left[\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r-1)}-\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}\right]
$$

$$
=p(s-j)\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}
$$

and

$$
\begin{aligned}
&g_{r,s}^{p}(x,y)-g_{r+1,s}^{p}(x,y)\\
&=[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}\\
&\qquad -[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r-1)\Gamma(p+s-r-1)}
\end{aligned}
$$

$$
=[\Psi(y)-\Psi(x)]^{p}\left[\frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)}-\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r-1)\Gamma(p+s-r-1)}\right]
$$

$$
=p(s-j)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}
$$

Therefore, we have

$$
\begin{aligned}
A_1(x,y)&=\frac{p(s-j)\Psi^{'}(x)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}}{(s-r-1)p(s-j)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+j-r-1)}{\Gamma(j-r)\Gamma(p+s-r)}}\\
A_1(x,y)&=\frac{\Psi^{'}(x)}{[\Psi(y)-\Psi(x)]}
\end{aligned}
$$

Now, we get from equation (4.2.8)

$$\frac{1-\{\bar{F}(x)\}^{m+1}}{1-\{\bar{F}(y)\}^{m+1}} = \frac{a\psi(x)+b}{a\psi(y)+b}, \quad m > -1 \tag{4.3.4}$$

The solution of equation (4.3.4) is

$$1-\{\bar{F}(x)\}^{m+1} = K[a\psi(x)+b],$$

where K is constant of integration. Thus, as $x \to \beta,\ a\psi(x)+b \to 1$, and the value of K is one and hence the sufficient part. Similarly we can prove the result for records.

(ii) For $m_1 = \cdots = m_{n-1} = m \geq -1$,

$$\xi^p_{r,s}(x,y) = [\psi(y)-\psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} \tag{4.3.5}$$

If and only if

$$1-\{\bar{F}(y)\}^{m+1} = a\psi(y)+b, \quad m > -1 \tag{4.3.6}$$

here, $F(y)$ is so chosen that $a\psi(\alpha)+b = 1$.

And for records

$$F(y) = 1 - e^{-[a\psi(y)+b]}, \quad m = -1 \tag{4.3.7}$$

Provided that there exists a $q \in (\alpha, \beta)$ such that $a\psi(q) + b = 1$.

Proof: For $m_1 = \cdots = m_{n-1} = m \neq -1$, the value of $\psi^p_{r,s}(x, y)$ is given by

$$\xi^p_{r,s}(x, y) = C_{r,j,s}(m+1) \int_x^y \frac{[\psi(y) - \psi(t)]^p}{[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]} \left[1 - \frac{[\bar{F}(x)^{m+1} - \bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]}\right]^{s-j-1}$$
$$\times \left[\frac{[\bar{F}(x)^{m+1} - \bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]}\right]^{j-r-1} [\bar{F}(t)]^m f(t)dt$$

Now to prove the necessary part *i.e.* equation (4.3.6) implies (4.3.5), we have

$$\xi^p_{r,s}(x, y) = \int_x^y C_{r,j,s} \frac{[\psi(y) - \psi(t)]^p}{[\psi(y) - \psi(x)]} \left[1 - \frac{[\psi(y) - \psi(t)]}{[\psi(y) - \psi(x)]}\right]^{j-r-1} \left[\frac{[\psi(y) - \psi(t)]}{[\psi(y) - \psi(x)]}\right]^{s-j-1}$$
$$\times \psi'(t)dt$$

Set $v = \left[\frac{[\psi(y) - \psi(t)]}{[\psi(y) - \psi(x)]}\right]$ Therefore,

$$\begin{aligned}
\xi^p_{r,s}(x, y) &= C_{r,j,s}[\psi(y) - \psi(x)]^p \int_0^1 v^{p+s-j-1}(1-v)^{j-r-1} dv \\
&= [\psi(y) - \psi(x)]^p \frac{\Gamma(s-r)}{\Gamma(j-r)\Gamma(s-j)} \frac{\Gamma(j-r)\Gamma(p+s-j)}{\Gamma(p+s-j+j-r)} \\
\xi^p_{r,s}(x, y) &= [\psi(y) - \psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)}
\end{aligned}$$

Now to prove the sufficient part *i.e.* equation(4.3.5) implies (4.3.6), we have

$$\xi^p_{r,s}(x, y)[B(x, y)]^{s-r-1} = C_{r,j,s}(m+1) \int_x^y [\Psi(t) - \Psi(x)]^p [B(x, t)]^{j-r-1} [B(t, y)]^{s-j-1}$$
$$\times [\bar{F}(t)]^m f(t)dt$$

On Differentiating both the sides *w.r.t.* y, and rearranging we get

$$A_2(x, y) = \frac{f(y)}{\bar{F}(y)B(x, y)} = \frac{p\Psi'(y)\xi^{p-1}_{r,s}(x, y) - \frac{\partial}{\partial y}\xi^p_{r,s}(x, y)}{(s-r-1)[\xi^p_{r,s}(x, y) - \xi^p_{r,s-1}(x, y)]} \tag{4.3.8}$$

Now consider,

$$\begin{aligned}
&p\Psi^{'}(y)\xi_{r,s}^{p-1}(x,y) - \frac{\partial}{\partial y}\xi_{r,s}^{p}(x,y) \\
&= p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r-1)} \\
&\quad - p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} \\
&= p\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\left[\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r-1)} - \frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)}\right] \\
&= p(j-r)\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j1)}{\Gamma(s-j)\Gamma(p+s-r)}
\end{aligned}$$

and

$$\begin{aligned}
&\xi_{r,s}^{p}(x,y) - \xi_{r,s-1}^{p}(x,y) \\
&= [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} \\
&\quad - [\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j-1)\Gamma(p+s-r-1)} \\
&= [\Psi(y)-\Psi(x)]^{p}\left[\frac{\Gamma(s-r)\Gamma(p+s-j)}{\Gamma(s-j)\Gamma(p+s-r)} - \frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j-1)\Gamma(p+s-r-1)}\right] \\
&= p(j-r)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}
\end{aligned}$$

Therefore, in view of equation (4.3.8), we have

$$\begin{aligned}
A_2(x,y) &= \frac{p(j-r)\Psi^{'}(y)[\Psi(y)-\Psi(x)]^{p-1}\frac{\Gamma(s-r)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}}{(s-r-1)p(j-r)[\Psi(y)-\Psi(x)]^{p}\frac{\Gamma(s-r-1)\Gamma(p+s-j-1)}{\Gamma(s-j)\Gamma(p+s-r)}} \\
A_2(x,y) &= \frac{\Psi^{'}(y)}{[\Psi(y)-\Psi(x)]}
\end{aligned}$$

Now, in view of equation (4.2.17), we get

$$\frac{\{\bar{F}(y)\}^{m+1}}{\{\bar{F}(x)\}^{m+1}} = \frac{a\psi(y)+b}{a\psi(x)+b}, \quad m>-1 \tag{4.3.9}$$

The solution of equation (4.3.9) is

$$\{\bar{F}(y)\}^{m+1} = K[a\psi(y) + b],$$

where K is constant of integration. Thus, as $y \to \alpha$, $a\psi(y) + b \to 1$, and the value of K is one and hence the sufficient part. Similarly we can prove the result for records.

4.4 Conclusion

In real problem, a statistician is often interested in guessing the distribution from which the true data is obtained. Characterization problem is theoretical approach to obtain the distribution function if certain condition is fulfilled. A probability distribution can be characterized in many ways and the method under study here is one of them. We have used here the conditional expectation of generalized order statistics conditioned on a pair of non-adjacent generalized order statistics, *i.e.* We have used the regression equation which is truncated at both sides, left side at point x and right side at point y. In many real problems, we find the data which is truncated from both ends. In those cases, we can use the result obtained in this Chapter to get the answer that from which population the sample is drawn. Keeping this in view, we have characterized the probability distributions through the difference of p^{th}, $(p \geq 1)$ power of two generalized order statistics (gos) conditioned on a pair of two non-adjacent gos using Meijer's G-Function. Further, this result can be utilized in case of ordinary order statistics, record values, sequential order statistics, order statistics with non integral sample size, Pfeifer record values and progressive type II right censored order statistics.

Chapter 5

On Exact Moments of Order Statistics from Lindley Distribution

5.1 Introduction

A random variable X is said to follow Lindley distribution if its (pdf) is of the form

$$f(x,\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \qquad x > 0,\ \theta > 0 \tag{5.1.1}$$

and the corresponding (df) is

$$F(x) = 1 - \left[1 + \frac{\theta x}{1+\theta}\right]e^{-\theta x}, \qquad x > 0,\ \theta > 0 \tag{5.1.2}$$

The basic concept of order statistics is given in Chapter 1 and the moments of order statistics for some specific distributions are investigated by several authors such as Malik (1966) derived the expression for moments of order statistics from Pareto distribution. Malik (1967) obtained the explicit expression for moments of order statistics of power function distribution. Further, Khan and Khan (1987) obtained the moments of order statistics from Burr distribution. Recently Athar *et al.* (2011) has obtained moments of order statistics from extended type-I generalized logistic distribution. In this Chapter, we have obtained the explicit expressions for the moments of order statistics from Lindley distribution. Further, the mean and covariance of the order statistics from Lindley distribution has been computed.

5.2 Moment of Order Statistics from Lindley Distribution

In this Section, explicit expression for single and product moment of order statistics are obtained from Lindley distribution.

Theorem 5.1: The single moment of order statistic from Lindley distribution is given by

$$\alpha_{r:n}^{p} = C_{r:n} \sum_{j=0}^{r-1} \sum_{k=0}^{n-r+j} \binom{r-1}{j} \binom{n-r+j}{k} (-1)^{j} \frac{\theta^{2+k}}{(1+\theta)^{1+k}} \times \left[\frac{\Gamma(p+k+1)}{\{\theta(n-r+j+1)\}^{p+k+1}} + \frac{\Gamma(p+k+2)}{\{\theta(n-r+j+1)\}^{p+k+2}} \right], \quad p \in \mathbb{N} \tag{5.2.1}$$

Proof: In view of equations (1.2.1), (5.1.1) and (5.1.2), we have

$$\begin{aligned} \alpha_{r:n}^{p} &= C_{r:n} \int_{0}^{\infty} x^{p} \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} f(x) dx \\ &= C_{r:n} \int_{0}^{\infty} x^{p} \left[1 - \left(1 + \frac{\theta x}{1+\theta}\right) e^{-\theta x}\right]^{r-1} \left[\left(1 + \frac{\theta x}{1+\theta}\right) e^{-\theta x}\right]^{n-r} \frac{\theta^{2}}{1+\theta} (1+x) e^{-\theta x} dx \end{aligned}$$

Expanding the terms inside the brackets Binomially, we get

$$= C_{r:n} \frac{\theta^{2}}{1+\theta} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{j} \int_{0}^{\infty} x^{p} \left[\left(1 + \frac{\theta x}{1+\theta}\right)\right]^{n-r+j} e^{-\theta x(n-r+j+1)} (1+x) dx$$

$$= C_{r:n}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^j\sum_{k=0}^{n-r+j}\binom{n-r+j}{k}\frac{\theta^{2+k}}{(1+\theta)^{1+k}}\int_0^\infty x^{p+k}(1+x)$$
$$\times\, e^{-\theta x(n-r+j+1)}dx$$

$$= C_{r:n}\sum_{j=0}^{r-1}\sum_{k=0}^{n-r+j}\binom{r-1}{j}\binom{n-r+j}{k}(-1)^j\frac{\theta^{2+k}}{(1+\theta)^{1+k}}$$
$$\times\left[\int_0^\infty x^{p+k+1-1}e^{-\theta x(n-r+j+1)}dx + \int_0^\infty x^{p+k+2-1}e^{-\theta x(n-r+j+1)}\right]dx$$

$$\alpha_{r:n}^p = C_{r:n}\sum_{j=0}^{r-1}\sum_{k=0}^{n-r+j}\binom{r-1}{j}\binom{n-r+j}{k}(-1)^j\frac{\theta^{2+k}}{(1+\theta)^{1+k}}$$
$$\times\left[\frac{\Gamma(p+k+1)}{\{\theta(n-r+j+1)\}^{p+k+1}}+\frac{\Gamma(p+k+2)}{\{\theta(n-r+j+1)\}^{p+k+2}}\right]$$

Hence the theorem follows.

Theorem 5.2: The product moment of order statistic from Lindley distribution is given by

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s+i}\sum_{l=0}^{k+s-r-i-1}\binom{s-r-1}{i}\binom{r-1}{k}\binom{k+s-r-i-1}{l}\binom{n-s+i}{j}$$
$$\times(-1)^{i+k}\frac{\theta^{4+j+l}}{(1+\theta)^{2+j+l}}\frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j}}\left[\sum_{m=0}^{q+j}\frac{[\theta(n-s+i+1)]^m}{m!}\right.$$
$$+\sum_{m=0}^{q+j+1}\frac{(q+j+1)}{[\theta(n-s+i+1)]^{1-m}\,m!}\left(\frac{\Gamma(l+p+m+1)}{[\theta(k-r+n+1)]^{l+p+m+1}}\right.$$
$$\left.\left.+\frac{\Gamma(l+p+m+2)}{[\theta(k-r+n+1)]^{l+p+m+2}}\right)\right],\qquad 1\le r<s\le n,\quad p,q\in\mathbb{N} \tag{5.2.2}$$

Proof: In view of equation (1.2.9), we have

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n} \int_0^\infty \int_x^\infty x^p y^q [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dx dy,$$

$$= C_{r,s:n} \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \int_0^\infty x^p [F(x)]^{r-1} \left[\bar{F}(x)\right]^{s-r-1-i} f(x) I(x) dx$$

where

$$I(x) = \left[\int_x^\infty y^q \left[\bar{F}(y)\right]^{n-s+i} f(y) dy\right]$$

$$= \frac{\theta^2}{(1+\theta)} \int_x^\infty y^q \left[\left(1 + \frac{\theta y}{1+\theta}\right) e^{-\theta y}\right]^{n-s+i} e^{-\theta y}(1+y) dy$$

Therefore,

$$I(x) = \frac{\theta^2}{(1+\theta)} \left[\int_x^\infty y^q \left(1 + \frac{\theta y}{1+\theta}\right)^{n-s+i} e^{-\theta y(n-s+i+1)} dy \right.$$

$$\left. + \int_x^\infty y^{q+1} \left(1 + \frac{\theta y}{1+\theta}\right)^{n-s+i} e^{-\theta y(n-s+i+1)} dy\right]$$

Proceeding similarly as in Theorem 5.1 and after some simple algebraic computations, it can be shown that

$$I(x) = \frac{\theta^2}{(1+\theta)} (I_1 + I_2)$$

$$I_1 = \int_x^\infty y^q \left(1 + \frac{\theta y}{1+\theta}\right)^{n-s+i} e^{-\theta y(n-s+i+1)} dy$$

$$= \sum_{j=0}^{n-s+i} \binom{n-s+i}{j} \left(\frac{\theta}{1+\theta}\right)^j \int_x^\infty y^{q+j} e^{-\theta y(n-s+i+1)} dy$$

Let us substitute $\theta y(n-s+i+1)=u$,

$$I_1=\sum_{j=0}^{n-s+i}\binom{n-s+i}{j}\left(\frac{\theta}{1+\theta}\right)^j\frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j+1}}\frac{1}{\Gamma(q+j+1)}\int_{\theta x(n-s+i+1)}^{\infty}u^{q+j+1-1}e^{-u}du$$

where $\Gamma[a,x]=\int_x^\infty u^{a-1}e^{-u}du$ is incomplete gamma function, On using the relationship between incomplete gamma function and poisson distribution from [Saleh and Rohatgi (2000) p.218, 15], we get

$$I_1=\sum_{j=0}^{n-s+i}\binom{n-s+i}{j}\left(\frac{\theta}{1+\theta}\right)^j\frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j+1}}\sum_{m=0}^{q+j}\frac{e^{-\theta x(n-s+i+1)}\left[\theta x(n-s+i+1)\right]^m}{m!}$$

Similarly,

$$\begin{aligned}I_2=&\sum_{j=0}^{n-s+i}\binom{n-s+i}{j}\left(\frac{\theta}{1+\theta}\right)^j\frac{\Gamma(q+j+2)}{[\theta(n-s+i+1)]^{q+j+2}}e^{-\theta x(n-s+i+1)}\\&\times\sum_{m=0}^{q+j+1}\frac{[\theta x(n-s+i+1)]^m}{m!}\end{aligned}$$

$$\begin{aligned}I(x)=&\sum_{j=0}^{n-s+i}\binom{n-s+i}{j}\frac{\theta^{2+j}}{(1+\theta)^{1+j}}\frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j+1}}e^{-\theta x(n-s+i+1)}\\&\times\left[\sum_{m=0}^{q+j}\frac{[\theta x(n-s+i+1)]^m}{m!}+\frac{(q+j+1)}{[\theta(n-s+i+1)]}\sum_{m=0}^{q+j+1}\frac{[\theta x(n-s+i+1)]^m}{m!}\right]\end{aligned}$$

Thus,

$$\alpha_{r,s:n}^{p,q}=C_{r,s:n}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\int_0^\infty x^p\left[F(x)\right]^{r-1}\left[\bar{F}(x)\right]^{s-r-i-1}f(x)I(x)dx$$

$$=C_{r,s:n}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\frac{\theta^2}{1+\theta}\int_0^\infty x^p\left[1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{r-1}$$

$$\times \left[\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{s-r-i-1}(1+x)e^{-\theta x}I(x)dx$$

$$= C_{r,s:n}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}\sum_{k=0}^{r-1}\binom{r-1}{k}(-1)^{i+k}\frac{\theta^2}{1+\theta}\int_0^\infty x^p(1+x)e^{-\theta x}$$
$$\times \left[\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}\right]^{k+s-r-i-1} I(x)dx$$

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1}\sum_{l=0}^{k+s-r-i-1}\frac{\theta^{l+2}}{(1+\theta)^{l+1}}(-1)^{i+k}\binom{s-r-1}{i}\binom{r-1}{k}$$
$$\times \binom{k+s-r-i-1}{l}\left[\int_0^\infty x^{l+p}e^{-\theta x(k+s-r-i)}+\int_0^\infty x^{l+p+1}e^{-\theta x(k+s-r-i)}\right]I(x)dx$$

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s+i}\sum_{l=0}^{k+s-r-i-1}\frac{\theta^{4+j+l}}{(1+\theta)^{2+j+l}}(-1)^{i+k}\binom{s-r-1}{i}\binom{r-1}{k}\binom{n-s+i}{j}$$
$$\times \binom{k+s-r-i-1}{l}\left[\int_0^\infty x^{l+p}e^{-\theta x(k+s-r-i)}+\int_0^\infty x^{l+p+1}e^{-\theta x(k+s-r-i)}\right]$$
$$\times \frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j+1}}e^{-\theta x(n-s+i+1)}\left[\sum_{m=0}^{q+j}\frac{[\theta x(n-s+i+1)]^m}{m!}\right.$$
$$\left.+\frac{(q+j+1)}{[\theta(n-s+i+1)]}\sum_{m=0}^{q+j+1}\frac{[\theta x(n-s+i+1)]^m}{m!}\right]dx$$

Implying that

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s+i}\sum_{l=0}^{k+s-r-i-1}\binom{s-r-1}{i}\binom{r-1}{k}\binom{k+s-r-i-1}{l}\binom{n-s+i}{j}$$
$$\times (-1)^{i+k}\frac{\theta^{4+j+l}}{(1+\theta)^{2+j+l}}\frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j}}\left[\sum_{m=0}^{q+j}\frac{[\theta(n-s+i+1)]^m}{m!}\right.$$
$$+\sum_{m=0}^{q+j+1}\frac{(q+j+1)}{[\theta(n-s+i+1)]^{1-m}m!}\left(\int_0^\infty x^{l+p+m}e^{-\theta x(k-r+n+1)}dx\right.$$

$$+\int_{0}^{\infty} x^{l+p+m+1} e^{-\theta x(k-r+n+1)} dx \Bigg)\Bigg]$$

$$\alpha_{r,s:n}^{p,q} = C_{r,s:n} \sum_{k=0}^{r-1} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s+i} \sum_{l=0}^{k+s-r-i-1} \binom{s-r-1}{i} \binom{r-1}{k} \binom{k+s-r-i-1}{l} \binom{n-s+i}{j}$$
$$\times (-1)^{i+k} \frac{\theta^{4+j+l}}{(1+\theta)^{2+j+l}} \frac{\Gamma(q+j+1)}{[\theta(n-s+i+1)]^{q+j}} \Bigg[\sum_{m=0}^{q+j} \frac{[\theta(n-s+i+1)]^{m}}{m!}$$
$$+ \sum_{m=0}^{q+j+1} \frac{(q+j+1)}{[\theta(n-s+i+1)]^{1-m}\, m!} \Bigg(\frac{\Gamma(l+p+m+1)}{[\theta(k-r+n+1)]^{l+p+m+1}}$$
$$+ \frac{\Gamma(l+p+m+2)}{[\theta(k-r+n+1)]^{l+p+m+2}} \Bigg)\Bigg]$$

Hence the theorem follows.

5.3 Numerical Computations of Mean and Covariance of Order Statistics from Lindley Distribution

In this Section, we have obtained mean and covariance of order statistics from Lindley distribution for n=1:1:5 and a fix value of the parameter θ by using MATLAB. Here, it can be seen that as the sample size increases, for fixed values of r, mean keep on decreasing.

Table 5.1: Mean of Order Statistics from Lindley Distribution for fixed $\theta = 2$

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$r=1$	0.6667	0.3472	0.2359	0.179	0.1443
$r=2$		0.9861	0.5698	0.4069	0.3178
$r=3$			1.1943	0.7327	0.5405
$r=4$				1.3481	0.8609
$r=5$					1.4699

Table 5.2: Covariance of Order Statistics from Lindley Distribution for fixed $\theta = 2$

r	s	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	1	0.38881	0.108652	0.051151	0.029759	0.019478
1	2		0.102126	0.225484	0.028665	0.018841
1	3			0.13884	0.027447	0.018206
1	4				0.02619	0.017672
1	5					0.016793
2	2		0.465107	0.149328	0.076032	0.046703
2	3			0.141588	0.073164	0.045129
2	4				0.069658	0.043606
2	5					0.041766
3	3			0.492948	0.169451	0.09046
3	4				0.161747	0.087284
3	5					0.083619
4	4				0.506226	0.180951
4	5					0.173463
5	5					0.513394

5.4 Conclusion

The explicit expression obtained in Theorems 5.1 and Theorem 5.2. enables us to compute the means and covariance of order statistics from Lindley distribution. Further, these means and covariance of order statistics (given in the Table 5.1 and Table 5.2) can be utilized to obtain the Best Linear Unbiased Estimator (BLUE) of location and scale parameters of Lindley distribution(see Kaminsky and Nelson (1975)).

Chapter 6

Recurrence Relation for Single and Product Moments of Generalized Order Statistics from New Weibull-Pareto Distribution and its Characterization

6.1 Introduction

A random variable X is said to have a New Weibull Pareto distribution (NWPD) if its (pdf) is of the form

$$f(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad 0 < x < \infty;\ \beta > 0,\ \delta > 0,\ \theta > 0 \tag{6.1.1}$$

with corresponding (df)

$$F(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad 0 < x < \infty;\ \beta > 0,\ \delta > 0,\ \theta > 0 \tag{6.1.2}$$

Therefore, in view of equations (6.1.1) and (6.1.2)

$$\bar{F}(x) = \frac{\theta^{\beta} x^{1-\beta} f(x)}{\beta\delta} \tag{6.1.3}$$

where $\bar{F}(x) = 1 - F(x),\ 0 < x < \infty,\ \beta > 0,\ \theta > 0$, and $\delta > 0$.

Recurrence relations for moments of generalized order statistics has obtained by several authors like Pawlas and Szynal (2001a), Khan *et al.* (2007) and Kumar and Khan (2013). In this Chapter, we have obtained recurrence relations for single and product moments of generalized order statistics from NWPD. Further, the distribution is characterized by a recurrence relation for single moments. Also some deductions and particular cases are given. The concept of generalized order statistics is as introduced in Chapter 1.

6.2 Recurrence Relation for Single and Product Moments of Generalized Order Statistics for New Weibull-Pareto Distribution

Theorem 6.1: Let X be a non-negative continuous random variable and follows NWPD. Suppose that for any $j > 0$ and $1 \leq r \leq n,\ \ E\left|[\phi\left(X(r,n,m,k)\right)]\right|$ is finite, then

$$E\left[X^{j}(r,n,m,k)\right] - E\left[X^{j}(r-1,n,m,k)\right] = \frac{j\theta^{\beta}}{\beta\delta\gamma_r} E\left[\phi(X(r,n,m,k))\right], \tag{6.2.1}$$

where $\phi(x) = x^{j-\beta}$

Proof:In view of equation (1.8.3), we have from (Athar and Islam, 2004)

$$E\left[X^j(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right] f(x)dx$$

Integrating by parts taking $\left[\bar{F}(x)\right]^{\gamma_r-1} f(x)$ as the part of the integrated, we get

$$\begin{aligned}E\left[X^j(r,n,m,k)\right] &= \frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] dx \\ &+ \frac{\gamma_r C_{r-2}}{(r-2)!\gamma_r}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1} g_m^{r-2}\left[F(x)\right] f(x)dx\end{aligned}$$

which implies that

$$\begin{aligned}E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right] &= \frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} \\ &\times g_m^{r-1}\left[F(x)\right] dx\end{aligned}$$

Now, we have from equation (6.1.3)

$$\begin{aligned}E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right] &= \frac{j\theta^\beta C_{r-1}}{(r-1)!\gamma_r\beta\delta}\int_0^\infty x^{j-\beta}\left[\bar{F}(x)\right]^{\gamma_r-1} \\ &\times g_m^{r-1}\left[F(x)\right] f(x)dx\end{aligned}$$

Thus, we get,

$$E\left[X^j(r,n,m,k)\right] - E\left[X^j(r-1,n,m,k)\right] = \frac{j\theta^\beta}{\gamma_r\beta\delta}E\left[\phi(X(r,n,m,k))\right]$$

and hence the theorem follows.

Remark 6.1

Putting $m = 0$ and $k = 1$ in equation (6.2.1), we have obtain a recurrence relation for single moments of order statistics of the NWPD as

$$E\left[X_{r:n}^{j}\right] - E\left[X_{r-1:n}^{j}\right] = \frac{j\theta^{\beta}}{\beta\delta(n-r+1)} E\left[\phi(X_{r:n})\right]$$

Remark 6.2

Putting $m = -1$ and $k \geqslant 1$ in equation (6.2.1), we have obtain a recurrence relation for single moments of k^{th} upper record values from NWPD as

$$E\left[X^{j}(r,n,-1,k)\right] - E\left[X^{j}(r-1,n,-1,k)\right] = \frac{j\theta^{\beta}}{\beta\delta k} E\left[\phi(X(r,n,-1,k))\right]$$

Theorem 6.2: Let X be a non-negative continuous random variable and follow NWPD. Suppose $E\left|\left[\Psi\left\{X(r,n,m,k)X(s,n,m,k)\right\}\right]\right|$ is finite for any $i, j > 0$ and $1 \leq r < s \leq n$, then the recurrence relation for product moment is

$$\begin{aligned} E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right] - E\left[X^{i}(r,n,m,k)X^{j}(s-1,n,m,k)\right] \\ = \frac{j\theta^{\beta}}{\beta\delta\gamma_{s}} E\left[\Psi\left\{X(r,n,m,k)X(s,n,m,k)\right\}\right], \end{aligned} \tag{6.2.2}$$

where $\Psi(x,y) = x^{i}y^{j-\beta}$.

Proof: In view of equation (1.8.4), we have (Athar and Islam, 2004)

$$E\left[X^{i}(r,n,m,k)X^{j}(s,n,m,k)\right] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{0}^{\infty} x^{i}\left[\bar{F}(x)\right]^{m} f(x) g_{m}^{r-1}\left[F(x)\right] I(x)dx \tag{6.2.3}$$

where $I(x) = \int_{x}^{\infty} y^{j}\left[h_{m}\left(F(y)\right) - h_{m}\left(F(x)\right)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1} f(y)dy$

On solving the integral $I(x)$ and substituting the resulting expression in equation (6.2.3), we get

$$
\begin{aligned}
&E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right] \\
&= \frac{C_{s-1}}{(r-1)!(s-r-2)!\gamma_s}\int_0^\infty\int_x^\infty x^i\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right] \\
&\times y^j\left[h_m\left(F(y)\right)-h_m\left[F(x)\right)\right]^{s-r-2}\left[\bar{F}(y)\right]^{\gamma_{s-1}-1}f(y)dydx \\
&+ \frac{jC_{s-1}}{(r-1)!(s-r-1)!\gamma_s}\int_0^\infty\int_x^\infty x^i\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right] \\
&\times y^{j-1}\left[h_m\left(F(y)\right)-h_m\left[F(x)\right)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s}dydx
\end{aligned}
$$

$$
\begin{aligned}
&E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k)X^j(s-1,n,m,k)\right] \\
&= \frac{jC_{s-1}}{(r-1)!(s-r-1)!\gamma_s}\int_0^\infty\int_x^\infty x^i\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right] \\
&\times y^{j-1}\left[h_m\left(F(y)\right)-h_m\left[F(x)\right)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s}dydx
\end{aligned}
$$

$$
\begin{aligned}
&E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k)X^j(s-1,n,m,k)\right] \\
&= \frac{jC_{s-1}}{(r-1)!(s-r-1)!\gamma_s}\int_0^\infty\int_x^\infty x^i\left[\bar{F}(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]y^{j-1} \\
&\times \left[h_m\left(F(y)\right)-h_m\left[F(x)\right)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}\frac{y^{1-\beta}\theta^\beta}{\beta\delta}f(y)dydx
\end{aligned}
$$

this implies that

$$
\begin{aligned}
&E\left[X^i(r,n,m,k)X^j(s,n,m,k)\right]-E\left[X^i(r,n,m,k)X^j(s-1,n,m,k)\right] \\
&\qquad= \frac{j\theta^\beta}{\gamma_s\beta\delta}E\left[\Psi\left\{X(r,n,m,k)X(s,n,m,k)\right\}\right]
\end{aligned}
$$

Remark 6.3

Putting $m = 0$ and $k = 1$ in equation (6.2.2), we obtain a recurrence relations for product moments of order statistics of the NWPD as

$$E\left[X_{r:n}^{i} X_{s:n}^{j}\right] - E\left[X_{r:n}^{i} X_{s-1:n}^{j}\right] = \frac{j\theta^{\beta}}{\beta\delta(n-s+1)} E\left[\Psi(X_{r:n} X_{s:n})\right]$$

Remark 6.4

Putting $m = -1$ and $k \geq 1$ in equation (6.2.2), we get the recurrence relations for product moments of upper k^{th} record values from NWPD in the form

$$\begin{aligned} E\left[X^{i}(r,n,-1,k)X^{j}(s,n,-1,k)\right] - E\left[X^{i}(r,n,-1,k)X^{j}(s-1,n,-1,k)\right] \\ = \frac{j\theta^{\beta}}{\beta\delta k} E\left[\Psi\left\{X(r,n,-1,k)X(s,n,-1,k)\right\}\right] \end{aligned}$$

6.3 Characterization

Theorem 6.3: For $m > -1$, the necessary and sufficient condition for a random variable X to be distributed with *pdf* given in (Theorem 6.1) is that

$$\frac{j\theta^{\beta}}{\gamma_r \beta\delta} E\left[\phi(X(r,n,m,k))\right] = E\left[X^{j}(r,n,m,k)\right] - E\left[X^{j}(r-1,n,m,k)\right] \tag{6.3.1}$$

if and only if

$$F(x) = 1 - e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad x, \delta, \beta > 0$$

Proof: A necessary part follows immediately from equation (6.3.1). On the other hand if the recurrence relation from equation (6.3.1) satisfied, then we have

$$\begin{aligned}\frac{j\theta^{\beta}C_{r-1}}{(r-1)!\beta\delta\gamma_r}\int_0^{\infty}\phi(x)\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx\\ =\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx\\ -\frac{C_{r-1}}{(r-1)!}\frac{(r-1)}{\gamma_r}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r+m}g_m^{r-2}\left[F(x)\right]f(x)dx\end{aligned}$$

$$=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-2}\left[F(x)\right]f(x)\left[\frac{g_m[F(x)]}{[\bar{F}(x)]}-\frac{(r-1)[\bar{F}(x)]^m}{\gamma_r}\right]dx$$

Let

$$v(x)=-\frac{[\bar{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]}{\gamma_r},$$

then

$$v'(x)=\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-2}\left[F(x)\right]f(x)\left[\frac{g_m[F(x)]}{[\bar{F}(x)]}-\frac{(r-1)[\bar{F}(x)]^m}{\gamma_r}\right]\tag{6.3.2}$$

Thus

$$\frac{j\theta^{\beta}C_{r-1}}{(r-1)!\beta\delta\gamma_r}\int_0^{\infty}\phi(x)\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^jv'(x)dx\tag{6.3.3}$$

Now integrating RHS of equation (6.3.3) by parts and using the value of $v(x)$, we get

$$\begin{aligned}\frac{j\theta^{\beta}C_{r-1}}{(r-1)!\beta\delta\gamma_r}\int_0^{\infty}\phi(x)\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]f(x)dx\\ =\frac{C_{r-1}}{(r-1)!\gamma_r}\int_0^{\infty}jx^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx\end{aligned}\tag{6.3.4}$$

which reduces to

$$\frac{jC_{r-1}}{(r-1)!\gamma_r}\int_0^{\infty}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]\left[x^{j-1}-\frac{\theta^{\beta}\phi(x)f(x)}{\beta\delta\bar{F}(x)}\right]dx=0 \tag{6.3.5}$$

Applying a generalization of the M*ü*ntz-Sz*á*sz Theorem from Hwang and Lin (1984) to equation (6.3.5), which states that on a space $L(a,b)$ of all summable functions defined on the interval (a,b), a sequence of functions $f_n(x)$ is complete on (a,b) if for any $g\in L(a,b)$ the equalities

$$\int_a^b f_n(x)g(x)dx=0, \qquad n=1,2,\ldots$$

Imply that $g(x)=0$ almost everywhere on (a,b), then we get

$$\frac{\bar{F}(x)}{f(x)}=\frac{x^{1-\beta}\theta^{\beta}}{\beta\delta}$$

which implies that

$$F(x)=1-e^{\delta\left(\frac{x}{\theta}\right)^{\beta}}, \qquad x,\delta,\beta>0$$

Hence the theorem follows.

6.4 Conclusion

The result obtained in present Chapter can be used to compute the moments of ordered random variables if the parent population follows New Weibull-Pareto distribution.

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List of Publications

i. M.J.S Khan, M.I. Khan and N.Wahid (2013): Characterization of Continuous Distributions Conditioned on a pair of Non-Adjacent Dual Generalized Order Statistics using Meijer's G function. *Asian Journal of Current Engineering and Maths*, **Vol 2(5)**,329-332.

ii. M.J.S Khan and N.Wahid (2015): On Exact Moments of Order Statistics From Lindley Distribution. *Mathematical Sciences International Research Journal*, **Vol 4(1)**, 276-280.

iii. M.J.S Khan, S.Kumar, Z.Akhtar and N.Wahid (2016): On Characterization of Continuous Probability Distributions Conditioned on a Pair of Record Values, *Prob.Stat.Forum*, **Vol 9(12)**, 109-114.

iv. M.J.S Khan, M.A.R. Khan and N.Wahid (2017): Characterization of Continuous Distributions Conditioned on a pair of Non-Adjacent Generalized Order Statistics using Meijer's G Function. *Applied Mathematical Sciences*, **Vol 11(16)**, 759-771.

Conference Attended

i. Conference on Applied Statistics and its Applications 2013(CASA-2013)(From 16th-17th March 2013) Organized by Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University(A Central University), Rae Bareli Road, Lucknow, INDIA. (Attended)

ii. National Conference on Recent Advances in Mathematics and Application (NCRAMA-2014) (From 30th-31st October 2014) Organized by Department of Applied Mathematics, School for Physical Science, Babasaheb Bhimrao Ambedkar University, (A Central University), Vidya Vihar, Rae Bareli Road, Lucknow, INDIA. (Paper Presented)

    Title:Characterization of Continuous Distributions by Conditional Expectation of Record Values Conditioned on Pair of Non-Adjacent Records.

iii. International Conference on Advances in Mathematical Sciences 2015 (ICAMS-2015) (From 19th-21st March, 2015) Organized by GSSDGS Khalsa College Patiala, INDIA and International Multidisciplinary Research Foundation.(Paper Presented)

    Title: On Exact Moments of Order Statistics from Lindley Distribution.

iv. International Conference on Statistics and Related Areas for Equity, Sustainability and Development (SRAESD-2015) In Conjunction with XXXV Annual Convention of Indian Society for Probability and Statistics (ISPS) (28th -30th November 2015) Organized by

Department of Statistics, University of Lucknow, Lucknow, INDIA.(Paper Presented)

Title: Characterization of continuous distributions conditioned on a pair of non- adjacent generalized order statistics using Meijers G-function.

v. International Multidisciplinary Academic Conference Thailand- 2016 (From 26th-30th September, 2016) Organized by International Multidisciplinary Research Foundation (IMRF) in colloboration with Center for Scientific Research Education, Institute for Foreign Affairs and Diplomacy, IMRF Thailand Chapter, Chonburi, Thailand. (Paper Presented)

   Title: Recurrence Relation for Single and Product Moments of Generalized Order Statistics from New Weibull-Pareto Distribution and its Characterization

Workshop and Seminar Attended

i. Workshop on Statistical Modelling using Softwares 2013 (WSMS 2013) (From 14th-15th March, 2013), organized by Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University(A Central University), Rae Bareli Road, Lucknow, INDIA.

ii. One-Day Seminar on Awareness of Official Statistical System in India and Career Prospects (Sponsored by Ministry of Statistics and Programme Implementation, Government of India), (On 23rd November 2013), organized by Department of Statistics, University of Lucknow, Lucknow, INDIA

iii. Workshop on MATLAB and LATEX: (Simulation with Documentation) (SIMDOC-2014) (From 30th June to 5th July 2014), organized by Department of Computer Science and Engineeringn, Motilal Nehru National Institute of Technology, Allahabad, INDIA.

iv. National Workshop on Scientific Writing using Latex (From 14th-20th February, 2017), organised by Dr.Shakuntala Misra National Rehabilition University, Lucknow, INDIA.