NUMERICAL STUDY OF (NON)LINEAR PARTIAL DIFFERENTIAL EQUATIONS VIA COLLOCATION TECHNIQUES

ABSTRACT

of

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ABSTRACT

Most of the phenomena, whether it is physical, chemical, biological or economical, can be modeled in the form of partial differential equations (PDEs); so the solution of modeled PDEs is required in order to study these various phenomena accurately. Various methods are available in the literature to study these PDEs but each method is not applicable for all type of PDEs, so there is always a need to develop new techniques for the complete study phenomena modeled in the form of PDEs. Methods to solve a PDE can be broadly classified in three categories, (i) analytical methods, (ii) semi-analytical methods and (iii) numerical methods. These methods has their special features in solving a PDE, but the our main focus is on numerical methods as they are easy to apply, have low computation time, accuracy of the solution can be controlled as per the requirement. With the advent of modern computer, numerical methods are growing very fast and for the qualitative study of a phenomena demand of numerical methods is also increasing. Collocation method is also one of the numerical method, which is easily programable and effective to solve the PDEs accurately. The aim of this thesis is to proposed some new collocation techniques for classical as well as fractional PDEs and using them to study the various PDEs arising in various fields of science and technology. For classical order PDEs we have proposed some collocation techniques with fourth order accuracy and utilizing them we have evaluated the numeric solution of Burgers' equation, Fisher's equation, sine-Gordon equation, Klein-Gordon equation. A collocation scheme based on trigonometric tension spline for fractional order PDEs is also proposed and we tested the scheme for fractional Burgers' equation. The proposed techniques are found to be accurate, efficient and stable for the considered problems.

In **chapter 1**, we give a brief motivation behind the work. The brief introduction about partial differential equations (PDEs) and some important PDEs arising in various fields of science and technology along with the discussion about the methods to solve PDEs is done. Categorization of methods with some examples is presented

there. Apart from these some basic definition and preliminaries, which will be used in forthcoming chapters is also discussed there.

In **chapter 2**, a new fourth order modified cubic B-spline (mCB) based upon collocation technique (mCBCT4) has been developed to evaluate the numeric results of the nonlinear Burgers' equation,

$$\frac{\partial}{\partial \tau}\wp(\sigma,\tau) + \alpha\wp(\sigma,\tau)\frac{\partial}{\partial \sigma}\wp(\sigma,\tau) - \nu\frac{\partial^2}{\partial \sigma^2}\wp(\sigma,\tau) = 0, \quad \sigma \in \Omega_\ell^\kappa,\ \tau \geq 0,$$

where $\wp(\sigma,\tau)$ is the fluid velocity at mentioned position vector σ and temporal co-ordinate τ in the computational domain $\Omega_\ell^\kappa = [\ell,\kappa]$ for the referred equation, and $\nu = \dfrac{1}{Re} > 0$ be the kinetic viscosity / diffusion coefficient, Re is the Reynolds number. This equation appears in rigorous real-world physical phenomena like - sound & shock waves in viscous medium, waves in fluid filled viscous elastic tubes, magneto-hydrodynamic-waves in medium with finite electrical-conductivity, in modeling of turbulent fluid, and in continuous stochastic processes. At first, the Burgers' equation is remodeled into a set of 1st order ordinary differential equations (ODE), in which fourth order accurate approximation of the unknown functions, and its spatial derivatives obtained via mCBCT4. In this way a set of first-order ODE is obtained, which we solve via SSP-RK$(\ell+1,\ell)$ scheme $(\ell = 3,4)$. The accuracy, efficiency and effectiveness of the developed technique mCBCT4 is demonstrated in terms of six different test examples of nonlinear Burgers' equation by computing the error norms: L_2 and L_∞ errors. The proposed mCBCT4 scheme is also tested for nonlinear Burgers' equation with very small kinematic viscosities. The proposed mCBCT4 is shown unconditionally stable scheme. The numerical findings demonstrate that the developed mCBCT4 performs better than some recently developed good techniques and enables to produces comparably more accurate solutions than some recently developed reliable techniques.

In **chapter 3**, we implemented a fourth order collocation scheme based upon the modified form of cubic B-splines, developed in chapter 1, on sine-Gordon (sG) equation

$$\left(\frac{\partial^2}{\partial\tau^2}+\alpha\frac{\partial^2}{\partial\sigma^2}\right)\wp(\sigma,\tau)+\sin(\wp)=0,\ \sigma\in\Omega_\ell^\kappa,\ \tau\geq 0,$$

This equation is used as model to study ferromagnetic-waves transmission, in crystals as motion of dislocations, oliton dynamics of DNA, fluid dynamics and nonlinear optics and many more phenomena of science and engineering. At first, the sG equation is remodeled into a coupled system of first-order ordinary differential equations (ODE), in which fourth order accurate mCB as base functions have been utilized for spatial coordinates. This yields a system of 1st-order ODE which is evaluated via a variant of strong stability preserving(SSP) Runge-Kutta(RK) scheme (SSP-RK43/RK54 technique). The accuracy/efficiency of the developed technique has been measured in terms of L_2, L_∞ and root mean square (RMS) errors for three different problems of S-G equation. The numerical findings confirm that the computed results from this technique are acceptable and better than some existing results.

In **chapter 4**, we have successfully implemented 4th order collocation technique based upon the modified cubic B-splines to evaluate the new numerical solutions of the Fishers' equation

$$\frac{\partial\wp}{\partial\tau}=\nu\frac{\partial^2\wp}{\partial\sigma^2}+\rho\mathrm{F}(\wp(\sigma,\tau)),\quad \sigma\in\mathcal{R},\ \tau\geq 0,$$

where σ and τ are spatial and temporal coordinate respectively. ν is diffusion coefficient, ρ is reaction coefficient and F is reaction term. At first, the Fishers' equation is considered at different grid points of the computational region and then the 4th order spatial approximation of the unknown is implemented on it, which yields a system of first order ODE that we solved using strong stability preserving(SSP) Runge-Kutta(RK)43 method. The accuracy/effectiveness/efficiency of the technique has been discussed in terms of L_2 and L_∞ errors for some test problems of Fishers' equation. The numerical observations show that the computed results from this technique are acceptable and better than some existing results.

In **chapter 5**, a fourth order numerical scheme for Klein-Gordon (K-G) equation with cubic nonlinearity is proposed.

$$\frac{\partial^2 \wp}{\partial \tau^2} + \alpha \frac{\partial^2 \wp}{\partial \sigma^2} + P'(\wp) = 0$$

At first, we transform the K-G equation into a coupled system of differential equation and then discretize the time derivative using finite difference approximation and spatial derivatives using Crank-Nicolson. After that fourth-order cubic B-spline approximation is used to convert the coupled system of differential equations into a linear system of simultaneous equations. We implement the proposed scheme on the K-G equation and found good results in terms of L_2 and L_∞ error norms. Fourth order spatial convergence of the technique is also shown numerically in considered test problems. The proposed technique is found to be economical, powerful, easy and efficient for the study of various kinds of linear and nonlinear physical models as compared to the existing techniques.

In **chapter 6**, two new efficient redefined quartic B-spline (mQB) based collocation methods have been proposed to investigate and simulate the dynamical behavior of the most celebrated turbulent fluid motion model equation termed as viscous Burgers' equation. First collocation scheme (mQBCM-I): mQB collocation technique is utilized for spatial-descretization in the referred model equation, and so, a system of time dependent ordinary differential equations (ODE) of first order is achieved. The achieved system of ODEs is evaluated via SSP-RK43. In the second collocation scheme (mQBCM-II), a system of linear difference equations in different time-levels is achieved via fully descretization of the referred model equation via Crank-Nicoloson scheme followed quasi-linearization for the diffusion term and mQB collocation method for the spatial descretization, and after that Gauss elimination method is preferred to get the desired approximation. The dynamical behavior of Burgers equation has been investigated via both schemes: mQBCM-I and mQBCM-II to validate their efficiency/accuracy and effectiveness as compared to some recently developed rigorous techniques. Numerical findings concludes that both techniques are capable of in investigating the dynamics of the viscous Burgers' equation for large Reynolds numbers and have high-resolution shock-capturing ability. As compare to mQBCM-I, mQBCM-II has advantage in terms of computational efficiency and convergence, and additionally mQBCM-II is unconditionally stable while mQBCM-I is stable.

In **chapter 7**, our aim is to form and examine the numerical simulation of Caputo-time fractional nonlinear Burgers' equation via collocation approach with trigonometric tension B-splines as base functions.

$$\frac{\partial^\beta \wp}{\partial \tau^\beta} + \alpha \wp \frac{\partial \wp}{\partial \sigma} = \nu \frac{\partial^2 \wp}{\partial \sigma^2} + f(\sigma, \tau),$$

First, $L1$ discretization formula is utilized for the time fractional derivative and after linearizing the nonlinear term, the trigonometric tension B-spline interpolants are utilized to get a system of simultaneous linear equations that are solved via Gauss elimination method. Thus, numerical approximation at the desired time level is obtained. It is demonstrated via von-Neumann approach that proposed scheme produces stable solutions. The results of six different test examples having their analytical solutions are compared with the results in the literature to validate the accuracy and efficiency of the scheme.

At the end future scope of presented work is suggested.

List of Publications

A. Published/Accepted

1. B.K. Singh, **M. Gupta**, "A new efficient fourth order collocation scheme for solving Burgers' equation" *Applied Mathematics and Computation* **(2021) 399: 126011. (Impact Factor-4.397)**

2. B.K. Singh, **M. Gupta**, "A new efficient fourth order collocation scheme for solving sine–Gordon equation" *Int. J. Appl. Comput. Math* **(2021) 7(4): 138.**

3. B.K. Singh, **M. Gupta**, "Trigonometric tension B-spline collocation approximations for time fractional Burgers' equation" *Journal of Ocean Engineering and Science* (2022). **https://doi.org/10.1016/j.joes.2022.03.023 (Impact Factor-4.803)**

B. Book Chapter

1. Singh B.K., **Gupta M.**, "Numerical solution of Fisher's equation by using fourth-order collocation scheme based on modified cubic B-splines" in the book *Computing and Simulation for Engineers* by CRC Press(Taylor & Francis), **(2022): 209-220.** DOI:10.1201/9781003222255-14

C. Communicated

1. **M. Gupta**, B.K. Singh,"Dynamical behavior of Burgers' equation via two efficient higher order collocation techniques".

2. **M. Gupta**, B.K. Singh,"New fourth-order efficient numerical solutions of Klein-Gordon equation".

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