ON SOME APPLICATIONS OF STOCHASTIC ORDERS IN RELIABILITY THEORY

THESIS

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ENROLMENT NO: 229/13 **2021**

Dedicated

To

My Beloved Parents

Mrs. Urmila Devi and Mr. Girish Chandra

DECLARATION

I, **Ruby Chanchal**, Enrolment No. 229/13, hereby declare that the work which is being presented in the thesis entitled **"On some applications of stochastic orders in reliability theory"** in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy and submitted in the Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow (U.P.), India, is an authentic record of my own work carried out during a period from July, 2015 to January, 2021 under the supervision of Dr. Amit Kumar Misra, Assistant Professor, Department of Applied Statistics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow (U.P.), India.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma of this or any other Institute. I also declare that the thesis is essentially free from all kind of plagiarism.

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CERTIFICATE

This is to certify that the thesis titled **"On some applications of stochastic orders in reliability theory"** submitted by **Ruby Chanchal**, is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other university.

The thesis submitted to Babasaheb Bhimrao Ambedkar University, Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations -1999 as amended in 2013* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

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LIST OF PUBLISHED/COMMUNICATED RESEARCH PAPERS

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Chapter 1

Introduction and literature review

1.1 Introduction

In the probability theory and statistics, stochastic orders work as a tool to compare the probability distributions of random variables/vectors. They can be classified into three categories: univariate, joint, and multivariate. Univariate stochastic orders are used to compare any two random variables whereas the joint stochastic orders are used to compare two components of a random vector. Multivariate stochastic orders are extensions of univariate stochastic orders and they are used to compare two random vectors of the same dimensions. The role of stochastic ordering usually arises when the measures of central tendencies and dispersion (e.g., mean, median, and variance) of two random variables are not very informative for the comparison purposes. Because such comparisons are based on only two numbers, and they may not exist in some cases.

Stochastic orders are partial orders defined on the space of distributions. Instead of comparing merely two numbers, they mostly deal with two functions to select the random variable which is bigger than another random variable according to location, magnitude, dispersion, residual lifetimes, concentration, etc. Some important stochastic orders are defined in Section 1.2 of the thesis. For more information about various stochastic orders, we refer the reader to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), Belzunce *et al.* (2016), and references cited therein.

In many different fields of probability theory and statistics, the theory of stochastic orders has been used extensively. Such areas include reliability theory, engineering, survival analysis, biological sciences, operations research, and economics. In reliability theory, operations research, engineering, and related areas, there is a need to compare lifetimes of different systems/components. Such comparisons of the lifetimes of systems are often arises in the allocation of redundant component(s) or spare(s) to systems. In biological sciences, the lifetimes (or the residual lifetimes) of two living organisms or two classes of living organisms may be compared. For instance, comparison can be made between the class of individuals who consume a drug with the class of individuals who do not consume the drug to know the impact of that specific drug. In economics, one can compare different income distributions using stochastic orders.

In reliability engineering, one often targets to increase the system reliability by allocating some extra components (called spares or redundant components) to the system. There are generally two types of redundancies that are widely used, called active redundancy and standby redundancy. In the case of active redundancy (also known as hot redundancy), the spares are placed parallel to the system's original components and keep working with those components simultaneously. In the case of standby redundancy (also known as cold redundancy), the spares are placed on the system's original components in such a manner that a spare starts working immediately after the component to which it is attached has failed. Several researchers have studied the problem of allocating redundancies in a system to achieve optimum configurations by stochastic comparisons between the lifetimes of the resulting systems using different stochastic orders (see, for example, Boland *et al.* (1992), Shaked and Shanthikumar (1992), Singh and Misra (1994), Valdés and Zequeira (2003), Valdés and Zequeira (2006), Li and Hu (2008), Valdés *et al.* (2010), Misra *et al.* (2011a,b), Zhao *et al.* (2012), Zhao *et al.* (2016), Yan and Luo (2018), Yan *et al.* (2019), and the references cited therein).

In this thesis, we mainly present some of the applications of stochastic orders in reliability theory. We stochastically compare the lifetimes of two series and parallel systems with components having lifetimes from the Topp-Leone generated family of distributions. Also, we investigate some reliability measures or characteristics (defined in Section 1.2) for this family of distributions. Moreover, we present real data applications to compare the fits of different models of Topp-Leone generated family of distributions. We also deal with the problems related to the allocation of active and standby redundancies in series systems.

Now, we provide some useful notation, definitions, lemmas, and a review of the literature in the following sections.

1.2 Some useful notation and definitions

To compare two random variables (vectors), several notions of univariate (multivariate) stochastic orders have been established in the literature. In reliability theory (survival analysis), we want to compare lifetimes of components or systems (biological organisms), and therefore, stochastic orders are useful in these areas. For example, one may found various applications in Barlow and Proschan (1975a), Deshpande *et al.* (1990), Zhao and Jiang (1998), Müller and Stoyan (2002), Lai and Xie (2006), Shaked and Shanthikumar (2007), Misra *et al.* (2011a,b), Misra and Misra (2013), Belzunce *et al.* (2016), and Corujo *et al.* (2019). Such stochastic comparisons are based on some reliability characteristics like the survival function, the hazard rate function, the reversed hazard rate function, the mean residual life function, and the expected inactivity time. These functions are frequently used in life testing problems. Also, they are applicable in other fields, for instance, Forensic Sciences (see, for example, Kalbfleisch and Lawless (1989), Chandra and Roy (2001), and Kayid and Ahmad (2004), and references cited therein) .

To understand the stochastic orders, and to make the thesis self-contained, the following subsection describes basic definitions of those reliability characteristics which are relevant to the thesis.

1.2.1 Basic characteristics in reliability

Consider a random variable X having a distribution with support $\mathbb{R}^+$, where $\mathbb{R}^+ = [0,\infty)$. Further, let $F_X(x) = P(X \leq x)$, $x \in \mathbb{R} = (-\infty,\infty)$, $f_X(x)$, and $\bar{F}_X(x)$, respectively denote the cumulative distribution function (c.d.f.), the probability density function (p.d.f.), and the survival function of the random variable X, where $\bar{F}_X(x) = 1 - F_X(x)$, $x \in \mathbb{R}$.

1.2.1.1 Hazard rate function

The hazard rate is also known as instantaneous failure rate. It is the conditional probability of failure of an item within a small interval of time $(t, t+\varepsilon)$, given that it has survived up to time 't', per unit time. Let $r_X(\cdot)$ denote the hazard rate function of X. Then it is defined as

$$r_X(t) = \lim_{\varepsilon \to 0^+} \frac{P[X \leq t+\varepsilon | X > t]}{\varepsilon} = \frac{f_X(t)}{\bar{F}_X(t)}, \quad t \in \mathbb{R}^+.$$

1.2.1.2 Reversed hazard rate function

In Barlow *et al.* (1963), the reversed hazard rate function was first described by the name "dual of the hazard rate". It applies the concept of the hazard rate to a reverse time direction. Let $\tilde{r}_X(\cdot)$ denote the reversed hazard rate function of X. Then it is defined as

$$\tilde{r}_X(t) = \lim_{\varepsilon \to 0^+} \frac{P[X > t-\varepsilon | X \leq t]}{\varepsilon} = \frac{f_X(t)}{F_X(t)}, \quad t > 0,$$

i.e., the reversed hazard rate function is an instantaneous conditional probability that an item has survived the instant $(t-\varepsilon, t)$, given that an item fails before time t.

1.2.1.3 Mean residual life function

The mean residual life function measures the expected remaining lifetime of an item that has already survived up to time 't'. Let $\mu_X(\cdot)$ denote the mean residual life function of X. Then it is defined as

$$\mu_X(t) = E[X-t|X>t] = \frac{1}{\bar{F}_X(t)}\int_t^\infty \bar{F}_X(x)\,\mathrm{d}x, \quad t>0.$$

1.2.1.4 Expected inactivity time

The expected inactivity time is described as "dual of the mean residual life function". It defines the mean waiting time elapsed for an item that has failed in an interval $[0,t]$. Let $\tilde{\mu}_X(\cdot)$ be the expected inactivity time of X. Then it is defined as

$$\tilde{\mu}_X(t) = E[t-X|X\leq t] = \frac{1}{F_X(t)}\int_0^t F_X(x)\,\mathrm{d}x, \quad t>0.$$

For more discussion on these reliability measures, we refer the reader to Barlow and Proschan (1975b), Lai and Xie (2006), and Belzunce *et al.* (2016).

In the following subsection, we recall, from the literature, some definitions of the univariate stochastic orders which will be used throughout the thesis. Let us take another random variable Y with support $\mathbb{R}^+$ having the cumulative distribution function $F_Y(\cdot)$, the survival function $\bar{F}_Y(\cdot)$, the probability density function $f_Y(\cdot)$, the hazard rate function $r_Y(\cdot)$, and the reversed hazard rate function $\tilde{r}_Y(\cdot)$. Note that the terms increasing and decreasing are used for non-decreasing and non-increasing, respectively.

1.2.2 Definitions of some univariate stochastic orders

Definition 1.2.1. *It is said that X is smaller than Y in the*

(i) *usual stochastic order (indicated as $X \leq_{st} Y$) if $F_Y(x) \leq F_X(x)$, for all $x \in \mathbb{R}$;*

(ii) *hazard rate order (indicated as $X \leq_{hr} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}^+$, or equivalently if $r_Y(x) \leq r_X(x),$ for all $x \in \mathbb{R}^+$;*

(iii) *reversed hazard rate order (indicated as $X \leq_{rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently if $\tilde{r}_X(x) \leq \tilde{r}_Y(x),$ for all $x \in (0,\infty)$;*

(iv) *likelihood ratio order (indicated as $X \leq_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}^+$;*

(v) *dispersive order (indicated as $X \leq_{disp} Y$) if $F_X^{-1}(b) - F_X^{-1}(a) \leq F_Y^{-1}(b) - F_Y^{-1}(a)$, for all $0 < a \leq b < 1$, where, F^{-1} denotes the right continuous inverse of F;*

(vi) *star-shaped order (indicated as $X \leq_{*} Y$) if $F_X^{-1}(b)F_Y^{-1}(a) \leq F_X^{-1}(a)F_Y^{-1}(b)$, for all $0 < a \leq b < 1$, where F^{-1} denotes the right continuous inverse of F.*

The well-known relationships among the above univariate stochastic orders are as follows.

$$\begin{array}{ccccc} X \leq_{\rm lr} Y & \Rightarrow & X \leq_{\rm hr} Y & & \\ \Downarrow & & \Downarrow & & \\ X \leq_{\rm rh} Y & \Rightarrow & X \leq_{\rm st} Y & \Leftarrow & X \leq_{\rm disp} Y \end{array}$$

Note that there is no direct implication among $\leq_{\rm lr}$ $(\leq_{\rm hr}, \leq_{\rm rh})$, $\leq_{\rm disp}$, and $\leq_{*}$. For extensive details on various stochastic orders, see, Shaked and Shanthikumar (2007).

For the case when X and Y are jointly distributed, the above definitions remain unchanged. Thus, univariate stochastic orders are basically based on the marginal distributions of the random variables rather than taking any dependency between these random variables. However, there may be situations in which we want to focus on the dependence structure between the two random variables, for instance, when comparing lifetimes of husband and wife, lifetimes of components working in the same environment. In such situations, the bivariate characterization of the usual stochastic order, the likelihood ratio order, and the hazard rate order, namely, the joint usual stochastic order, the joint likelihood ratio order, and the joint hazard rate order, respectively, have been defined by Shanthikumar and Yao (1991) and Shanthikumar *et al.* (1991). When the random variables to be compared are independent, such bivariate stochastic orders are equivalent to their respective univariate stochastic orders. Later on, in a number of contexts, many more bivariate characterizations were found and applied in different areas. For example, Belzunce *et al.* (2007) gave new conditions for ranking of dependent random utilities in social sciences. Belzunce *et al.* (2011) studied the problem of optimal allocation of redundancies in series and parallel systems having dependent components. Li and You (2012) provided the single-period portfolio allocation of risk assets assuming dependent random returns. Further, Pellerey and Zalzadeh (2015), Li and Li (2017), and You *et al.* (2020) have studied various problems related to the bivariate characterizations of random variables.

In this thesis, we use some of the bivariate stochastic orders given by Shanthikumar and Yao (1991) and Misra *et al.* (2020a,b). The definitions of these orders are given in the following subsection.

1.2.3 Definitions of some bivariate stochastic orders

Definition 1.2.2. *Let $f_{X,Y}(\cdot,\cdot)$ be the joint probability density function of random variables X and Y. Then, it is said that X is smaller than Y in the*

(i) joint likelihood ratio order (indicated as $X \leq_{lr:j} Y$) if, $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$, for $x \leq y$;

(ii) joint hazard rate order (indicated as $X \leq_{jhr} Y$) if, $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \geq 0$, $\forall x \geq 0$;

(iii) joint reversed hazard rate order (indicated as $X \leq_{jrh} Y$) if, $\int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \leq 0$, $\forall x \geq 0$;

(iv) residual stochastic precedence order (indicated as $X \leq_{rsp} Y$) if $P(t < X < Y) - P(t < Y < X) \geq 0$, for all $t \geq 0$;

(v) inactivity stochastic precedence order (indicated as $X \leq_{isp} Y$) if $P(Y < X \leq t) - P(X < Y \leq t) \leq 0$, for all $t > 0$.

The relationships among the above bivariate stochastic orders are as follows:

$$\begin{array}{ccccc} X \leq_{\rm jrh} Y & \Leftarrow & X \leq_{\rm lr:j} Y & \Rightarrow & X \leq_{\rm jhr} Y \\ \Downarrow & & & & \Downarrow \\ X \leq_{\rm isp} Y & & & & X \leq_{\rm rsp} Y \end{array}$$

Note that there is no implication between $\leq_{\rm jhr}$ $(\leq_{\rm rsp})$ and $\leq_{\rm jrh}$ $(\leq_{\rm isp})$.

The following subsection provides some basic idea of majorization, Schur-convexity, and Schur-concavity, used in the thesis.

1.2.4 Definitions of Majorization, Schur-convexity, and Schur-concavity

It is a problem of great interest to examine the nature of certain aging functions (e.g., survival function, hazard rate function, etc.) related to the lifetime of a system having heterogeneous components when the vector of parameters takes different values. The concept of majorization performs a good role in these type of problems.

Majorization is a pre-order defined on vectors in $\mathbb{R}^n = (-\infty, \infty)^n$. It can be said that majorization indicates how the components of a vector $\underline{x}$ are “less spread out" or “more nearly equal" than the components of a vector $\underline{y}$. It deals with the diversity of the components of the vectors in $\mathbb{R}^n$. Therefore, it has also been used as a measure of income inequality and species diversity. For a comprehensive study on this topic, we refer the reader to Marshall *et al.* (2011).

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$ be two real vectors from $\mathbb{R}^n$. Further, let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$ denote the increasing order of the components of $\underline{x}$ and $\underline{y}$, respectively. Then, some important definitions/results of majorization are presented below.

Definition 1.2.3 (Marshall *et al.* 2011, p. 8, 12). *$\underline{x}$ is said to be*

(i) majorized by $\underline{y}$ (written as $\underline{x} \overset{m}{\preceq} \underline{y}$) if

$$\sum_{k=1}^{i} x_{(k)} \geq \sum_{k=1}^{i} y_{(k)},\ i = 1, 2, \ldots, n-1,\ \textit{and}\ \sum_{k=1}^{n} x_{(k)} = \sum_{k=1}^{n} y_{(k)};$$

(ii) weakly supermajorized by $\underline{y}$ *(written as* $\underline{x} \preceq^{w} \underline{y}$*) if*

$$\sum_{k=1}^{i} x_{(k)} \geq \sum_{k=1}^{i} y_{(k)},\ i = 1, 2, \ldots, n;$$

(iii) weakly submajorized by $\underline{y}$ *(written as* $\underline{x} \preceq_{w} \underline{y}$*) if*

$$\sum_{k=i}^{n} x_{(k)} \leq \sum_{k=i}^{n} y_{(k)},\ i = 1, 2, \ldots, n.$$

From Definition 1.2.3, it is easy to verify that $\underline{x} \overset{\mathrm{m}}{\preceq} \underline{y}$ implies $\underline{x} \preceq^{\mathrm{w}} \underline{y}$ and $\underline{x} \preceq_{\mathrm{w}} \underline{y}$.

Definition 1.2.4 (Marshall *et al.* 2011, p. 80)**.** *A real valued function* ψ *defined on a set* $\mathbb{S} \subset \mathbb{R}^n$ *is said to be Schur-convex (Schur-concave) on* $\mathbb{S}$ *if*

$$\underline{x} \overset{m}{\preceq} \underline{y} \implies \psi(\underline{x}) \leq (\geq)\, \psi(\underline{y}) \quad \textit{for}\ \underline{x}, \underline{y} \in \mathbb{S}.$$

Next, in the following subsections, we provide the definition of Renyi entropy measure and present some lemmas which are beneficial in obtaining various results of the thesis.

1.2.5 Renyi entropy

The degree of unpredictability (or uncertainty) of a random variable X having p.d.f. $f_X(\cdot)$ is defined in terms of entropy. Although it is interpreted as a measure of uncertainty, it is also a measure of information. In reliability theory, it can be said that a system having low uncertainty is more reliable than a system with great uncertainty. One of the popular entropy measure is the Renyi entropy (Renyi (1961)). It has been used in various situations of science, engineering, and many other related fields. For a continuous random variable X, the Renyi entropy measure is given by

$$I_R(\delta) = \frac{1}{1-\delta}\log\left(\int f_X^{\delta}(x)\,\mathrm{d}x\right),$$

where $\delta > 0$ and $\delta \neq 1$. Also, when $\delta \to 0$ and $\delta \to 1$, it reduces to Hartley entropy (Hartley (1928)) and Shannon entropy (Shannon and Weaver (1949)), respectively.

1.3 Some important lemmas

Lemma 1.3.1 (Marshall *et al.* 2011, p. 84)**.** *Let an open interval $I \subset \mathbb{R}$ and let $\psi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for $\psi(\cdot)$ to be Schur-concave on I^n are: $\psi(\cdot)$ is symmetric on I^n, and*

$$(x_k - x_l)\left(\frac{\partial \psi(\underline{x})}{\partial x_k} - \frac{\partial \psi(\underline{x})}{\partial x_l}\right) \leq 0, \text{ for all } k \neq l.$$

Lemma 1.3.2 (Marshall *et al.* 2011, p. 87)**.** *A real valued function ψ defined on $\mathbb{S} \subset \mathbb{R}^n$ satisfies $\underline{x} \preceq_w \underline{y}$ on $\mathbb{S}$ implies $\psi(\underline{x}) \leq \psi(\underline{y})$ if, and only if, ψ is increasing and Schur-convex on $\mathbb{S}$.*

Lemma 1.3.3 (Balakrishnan *et al.* 2014)**.** *Let a function $\tau : (0,\infty) \to (0,\infty)$ be defined as*

$$\tau(\alpha) = \frac{\alpha t^{\alpha-1}}{1-t^{\alpha}}.$$

Then, $\tau(\alpha)$ is convex in α, for any $0 < t < 1$.

Lemma 1.3.4 (Marshall *et al.* 2011, Proposition C.1., p. 92)**.** *If $I \subset \mathbb{R}$ is an open interval* *and $q : I \to \mathbb{R}$ is convex, then $\psi(\underline{x}) = \sum_{i=1}^{n} q(x_i)$ is Schur-convex on I^n. Consequently, $\underline{x} \overset{m}{\preceq} \underline{y}$ on I^n implies $\psi(\underline{x}) \leq \psi(\underline{y})$.*

Lemma 1.3.5 (Saunders and Moran 1978)**.** *Let F_ν, $\nu \in \mathbb{R}$, denote a class of distribution functions such that F_ν is supported on some interval $(x_0, x_1) \subseteq (0, \infty)$ having a density function f_ν, which does not disappear on any sub-interval of (x_0, x_1). Then*

$$F_\nu \leq_{disp} F_{\nu^*}; \quad \nu, \nu^* \in \mathbb{R}, \nu \leq \nu^*, \tag{1.3.1}$$

if, and only if, $\frac{F'_\nu(x)}{f_\nu(x)}$ is decreasing in x, where F'_ν is the derivative of F_ν with respect to ν. And

$$F_\nu \leq_* F_{\nu^*}; \quad \nu, \nu^* \in \mathbb{R}, \nu \leq \nu^*, \tag{1.3.2}$$

if, and only if, $\frac{F'_\nu(x)}{x f_\nu(x)}$ is decreasing in x, where F'_ν is the derivative of F_ν with respect to ν. Note that the inequalities in (1.3.1) and (1.3.2) reverse as the quantities $\frac{F'_\nu(x)}{f_\nu(x)}$ and $\frac{F'_\nu(x)}{x f_\nu(x)}$, respectively, increase in x.

1.4 Literature review

In this section, the literature on the Topp-Leone distribution and its generated family, related to our study, is first reviewed.

1.4.1 Some results of the Topp-Leone distribution and its generated family

In reliability and survival analysis, lifetime distributions play a significant role. Theoretically, most of these distributions have infinite support $[0,\infty)$. Topp and Leone (1955) proposed a continuous unimodal distribution, depending upon a single parameter, having bounded support $[0,1]$. Nadarajah and Kotz (2003) provided a modified version of this distribution, depending upon two parameters, having bounded support $[0,\kappa]$, $\kappa > 0$. This modified distribution, known as two-parameter Topp-Leone distribution, is useful for modeling lifetime phenomena. Nadarajah and Kotz (2003) obtained specific algebraic expressions for the hazard rate function, the first ten raw moments, and the first four central moments of the distribution. In addition, they estimated the parameters of the distribution and provided some simulation results. They identified that the two-parameter Topp-Leone distribution exhibits bathtub-shaped hazard rates. Several authors then further studied the two-parameter Topp-Leone distribution.

Ghitany *et al.* (2005) studied few reliability characteristics such as the hazard rate, the mean residual life, the reversed hazard rate, and the expected inactivity time of this distribution. Also, they compared two random variables X and Y following Topp-Leone distribution with parameters ϑ_1, κ and ϑ_2, κ, respectively. They showed that

$$\vartheta_1 < \vartheta_2 \implies X \leq_{\text{lr}} Y.$$

Zhou *et al.* (2006) considered two independent random variables X and Y of this distribution and derived the exact distributions of $X+Y$, XY, and $X/(X+Y)$.

Dorp and Kotz (2006) represented income distributions with the help of two-parameter Topp-Leone distribution. A summary on kurtosis of the distribution was discussed by Kotz and Seier (2007). Al-Zahrani (2012) and Genç (2012) discussed the goodness of fit tests and moments of order statistics for the distribution, respectively.

Although the two-parameter Topp-Leone distribution has many applications in different areas, further to make it more applicable some generalizations have also been considered. For example, Vicaria *et al.* (2008) studied two-sided generalized Topp-Leone distribution and discussed some of its properties.

Further, Pourdarvish *et al.* (2015) studied a generalization of the two-parameter Topp-Leone distribution and named it exponentiated Topp-Leone distribution. They investigated reliability characteristics such as the hazard rate and the mean residual life functions. They also studied the moments and order statistics of exponentiated Topp-Leone distribution.

Recently, Al-Shomrani *et al.* (2016) introduced a new generalization of the Topp-Leone distribution by introducing a base-line distribution $G(\cdot;\zeta)$ in the Topp-Leone distribution. They also considered a particular case of this family of distributions using the base-line distribution as exponential distribution and called it Topp-Leone exponential distribution. Moreover, they derived the hazard rate function and moments of Topp-Leone exponential distribution.

Further, Rezaei *et al.* (2017) also proposed a generalization of the Topp-Leone distribution, which is more generalized than the distribution defined by Al-Shomrani *et al.* (2016). They considered $[G(\cdot;\zeta)]^{\theta}$ as the base-line distribution and named it the Topp-Leone generated family of distributions. They also discussed a particular case of this family of distributions called Topp-Leone Gamma distribution and studied mathematical properties of this family of distributions.

The Topp-Leone generated family of distributions also exhibit the bathtub-shaped hazard rates and can be used for lifetime modeling. For some recent developments based on this family of distributions, we refer the reader to Aryal *et al.* (2017), Sebastian *et al.* (2019), and Shekhawat and Sharma (2020).

Next, we review the literature on stochastic comparisons of series and parallel systems and discuss some results used in the thesis.

1.4.2 Some stochastic comparison results for series and parallel systems

Let $X_1, X_2, \ldots, X_n$ be independent and nonnegative random variables representing the lifetimes of n-components and let $Y_1, Y_2, \ldots, Y_n$ be another set of independent and nonnegative random variables representing the lifetimes of another set of n-components. Further, let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the respective order statistics of the random variables $X_1, X_2, \ldots, X_n$. Similarly, let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ be order statistics of the random variables $Y_1, Y_2, \ldots, Y_n$. Then, in reliability theory, $X_{k:n}$ and $Y_{k:n}$ represent the lifetimes of $(n-k+1)$-out-of-n systems made up of two sets of components having lifetimes $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, X_n$, respectively. Specifically, $X_{n:n}$ (when $k = n$) and $X_{1:n}$ (when $k = 1$) denote the lifetimes of parallel (1-out-of-n) and series (n-out-of-n) systems, respectively. The problem of stochastic comparisons of order statistics from two heterogeneous distributions is widely studied by various researchers in the past several decades. Particularly, a vast literature on stochastic comparisons of the lifetimes of series and parallel systems, constructed from heterogeneous components, is available (see, for example, Pledger and Proschan (1971), Proschan and Sethuraman (1976), Kochar and Korwar (1996), Dykstra *et al.* (1997), Khaledi and Kochar (2000), Khaledi and Kochar (2006), Balakrishnan (2007), Di Crescenzo and Pellerey (2011), and Nadarajah *et al.* (2017), and references cited therein). Here, we present some results which are relevant to our work.

Let the random variables X_i and Y_i have continuous distribution functions $F(\cdot;\lambda_i)$ and $F(\cdot;\lambda_i^*)$, respectively, where, $\lambda_i, \lambda_i^* > 0$, $i = 1, 2, \ldots, n$. Moreover, let $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\underline{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$.

Dykstra *et al.* (1997) considered two-component parallel systems when X_i's and Y_i's are independent exponential random variables having hazard rates λ_i and λ_i^*, respectively. They showed that

$$(\lambda_1^*, \lambda_2^*) \overset{\text{m}}{\preceq} (\lambda_1, \lambda_2) \implies Y_{2:2} \leq_{\text{lr}} X_{2:2}. \tag{1.4.1}$$

They also considered n-component parallel systems for the case when X_i's are independent exponential random variables with hazard rate λ_i and Y_i's are independent exponential random variables with a common hazard rate $(\sum_{i=1}^{n}\lambda_i/n)$, and established that

$$Y_{n:n} \leq_{\text{disp}} X_{n:n} \quad \text{and} \quad Y_{n:n} \leq_{\text{hr}} X_{n:n}. \tag{1.4.2}$$

Khaledi and Kochar (2000) proved the result (1.4.2) for the case when the common hazard rate of Y_i's is $(\prod_{i=1}^{n}\lambda_i)^{1/n}$. Khaledi and Kochar (2007) further extended this result of Khaledi and Kochar (2000) from exponential to the proportional hazard rate model (PHR) model $\big($i.e., $F(x;\lambda) = 1-\bar{G}_0^{\lambda}(x)$, $\lambda > 0$, for some survival function $\bar{G}(\cdot)\big)$.

Moreover, Khaledi and Kochar (2006) compared two series systems with independent Weibull components $\big($i.e., $F(x;\lambda) = 1-e^{-(\lambda x)^{\alpha}}$, $\lambda > 0, \alpha > 0\big)$ having different scale parameters when the shape parameter α is fixed. They proved that

$$\underline{\lambda}^* \overset{\text{m}}{\preceq} \underline{\lambda} \implies Y_{1:n} \leq_{\text{hr}} X_{1:n}, \text{ for } 0 < \alpha \leq 1,$$

and

$$\underline{\lambda}^* \overset{\text{m}}{\preceq} \underline{\lambda} \implies X_{1:n} \leq_{\text{hr}} Y_{1:n}, \text{ for } \alpha \geq 1.$$

Further, Zhao and Balakrishnan (2011) strengthened the result presented in Equation (1.4.1). They established the following result under the condition $0 < \lambda_1 \leq \lambda_1^* \leq \lambda_2^* < \lambda_2$.

$$(\lambda_1^*, \lambda_2^*) \preceq^{\text{w}} (\lambda_1, \lambda_2) \implies Y_{2:2} \leq_{\text{lr}} X_{2:2}. \tag{1.4.3}$$

Furthermore, Fang and Tang (2014) considered two series systems with two components in each system. In the i^{th} system, two component-lifetimes are independent and follows Weibull distribution with respective scale parameters λ and λ_i, $i = 1,2$. The shape parameter is common for all the lifetimes. They proved that

$$\lambda_1 \leq \lambda_2 \leq \lambda \implies Y_{1:2} \leq_{\text{disp}} X_{1:2}.$$

Gupta *et al.* (2015) compared the lifetimes of series and parallel systems having independent and heterogeneous Fréchet components $\left(\text{i.e., } F(x;\lambda) = e^{-\left(\frac{x-\mu}{\lambda}\right)^{-\alpha}},\ x > \mu, \lambda > 0, \alpha > 0, -\infty < \mu < \infty\right)$ with different scale parameters when the shape parameter α and the location parameter μ are fixed. They proved that

$$\frac{1}{\underline{\lambda}^*} \overset{\text{m}}{\preceq} \frac{1}{\underline{\lambda}} \implies Y_{n:n} \leq_{\text{st}} X_{n:n},$$

and

$$((\lambda_1^*)^\alpha, (\lambda_2^*)^\alpha, \ldots, (\lambda_n^*)^\alpha) \overset{\text{m}}{\preceq} (\lambda_1^\alpha, \lambda_2^\alpha, \ldots, \lambda_n^\alpha) \implies X_{1:n} \leq_{\text{hr}} Y_{1:n}.$$

Fang and Wang (2017) strengthened the results of Gupta *et al.* (2015) under the assumption that $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ take a common value, say λ^*. They proved that

(i) if $(\lambda^*)^\alpha = \frac{1}{n}\sum_{i=1}^n \lambda_i^\alpha$, then $X_{1:n} \leq_{\text{lr}} Y_{1:n}$; and

(ii) if $0 < \alpha \leq 1$, $\lambda^* = \frac{1}{n}\sum_{i=1}^n \lambda_i$, then $X_{1:n} \leq_{\text{lr}} Y_{1:n}$.

Recently, Patra *et al.* (2018) presented similar results under the set up of independent and heterogeneous new Pareto type components $\left(\text{i.e., } F(x;\lambda) = \frac{x^\alpha - \lambda^\alpha}{x^\alpha + \lambda^\alpha},\ x > \lambda > 0, \alpha > 0\right)$.

They derived, for $0 < \alpha \le 1$, that

$$\underline{\lambda}^* \overset{\text{m}}{\preceq} \underline{\lambda} \implies Y_{1:n} \le_{\text{hr}} X_{1:n} \text{ and } Y_{1:n} \le_{\text{disp}} X_{1:n}. \tag{1.4.4}$$

Apart from these types of comparisons, the stochastic comparisons between two generated families of distributions have also been considered in some recent research articles. Let $F(\cdot;\vartheta_i,\theta_i,\zeta)$ and $F(\cdot;\vartheta_i^*,\theta_i^*,\zeta)$ be the distribution functions of generated families corresponding to X_i and Y_i, respectively, where, $\vartheta_i,\vartheta_i^*,\theta_i,\theta_i^* > 0$, $i = 1,2,\ldots,n.$. Also, let $G(\cdot;\zeta)$ be the base-line distribution function of the generated family of distributions, and ζ contains the parameters which specify the base-line distribution. Kundu and Chowdhury (2018) proved the stochastic comparison results for series and parallel systems with components having Kumaraswamy generalized (Kumaraswamy-G) family of distributions $\left(\text{i.e., } F(x;\vartheta,\theta,\zeta) = 1-(1-(G(x;\zeta))^{\theta})^{\vartheta}\right)$. They proved the following result when this family of distributions has a fixed base-line distribution and fixed ϑ_i's.

$$\underline{\theta}^* \preceq^{\text{w}} \underline{\theta} \implies X_{1:n} \le_{\text{hr}} Y_{1:n}. \tag{1.4.5}$$

They also derived the results for different base-line distributions. Further, Kayal (2018) proved that

$$\underline{\theta}^* \preceq^{\text{w}} \underline{\theta} \implies X_{1:n} \le_{\text{st}} Y_{1:n}. \tag{1.4.6}$$

Note that (1.4.6) directly follows from (1.4.5) as $X_{1:n} \le_{\text{hr}} Y_{1:n}$ implies $X_{1:n} \le_{\text{st}} Y_{1:n}$. Kayal (2018) also studied the problems related to series systems on taking a common value

ϑ for all ϑ_i's. They showed that

$$\underline{\theta}^* \preceq_{\mathrm{w}} \underline{\theta} \implies Y_{n:n} \leq_{\mathrm{st}} X_{n:n}, \text{ when } \vartheta \geq 1,$$

and

$$\underline{\theta}^* \preceq^{\mathrm{w}} \underline{\theta} \implies X_{n:n} \leq_{\mathrm{st}} Y_{n:n}, \text{ when } 0 < \vartheta \leq 1.$$

Apart from the above results, they also proved the results when base-line distributions are different.

Furthermore, Das and Kayal (2019) proved similar results using the Marshall and Olkin's family of distributions. Recently, Kayal and Nanda (2020) considered the generalized form of Kumaraswamy-G family of distributions and proved the results on comparisons of parallel systems with this family of distributions.

Now, in the following subsection, we review the problems on the allocations of redundancies in series systems with one spare.

1.4.3 Allocation of redundancies in series systems with one spare

Consider a series (n-out-of-n) system having components $C_1, C_2, \ldots, C_n$ with lifetimes $X_1, X_2, \ldots, X_n$, respectively. Suppose that we have a spare R having the lifetime X. Let us consider a model in which the spare R is available for active or standby redundancy. Assume that the spare R is assigned either to component C_1 or to component C_2. Then, in the case of active redundancy, the lifetimes of two resulting systems are defined by

$$S_1 = \wedge\{\vee(X_1, X), X_2, \ldots, X_n\} \tag{1.4.7}$$

and

$$S_2 = \wedge\{X_1, \vee(X_2, X), \ldots, X_n\}, \tag{1.4.8}$$

respectively. The symbols $\wedge$ and $\vee$ denote the *minimum* and *maximum*, respectively. In the case of standby redundancy, the lifetimes of two resulting systems are given by

$$T_1 = \wedge\{X_1 + X, X_2, \ldots, X_n\} \tag{1.4.9}$$

and

$$T_2 = \wedge\{X_1, X_2 + X, \ldots, X_n\}, \tag{1.4.10}$$

respectively. Let $\leq_{\text{icx}}, \leq_{\text{icv}}$, and $\leq_{\text{sp}}$ denote, respectively, the increasing convex order, the increasing concave order, and the stochastic precedence order (see for definitions, Boland *et al.* (2004) and Shaked and Shanthikumar (2007)). To compare the performances of both the systems, Boland *et al.* (1992) established that

$$\text{if } X_1 \leq_{\text{st}} X_2, \quad \text{then} \quad S_2 \leq_{\text{st}} S_1, \tag{1.4.11}$$

and

$$\text{if } X_1 \leq_{\text{hr}} X_2, \quad \text{then} \quad T_2 \leq_{\text{st}} T_1. \tag{1.4.12}$$

Singh and Misra (1994) proved that (a) if $X_1 \leq_{\text{st}} X_2$ then $S_2 \leq_{\text{sp}} S_1$; (b) if $X_1 \leq_{\text{st}} X_2$ then $T_2 \leq_{\text{sp}} T_1$. For two-component systems ($n = 2$), they also proved that if X_1, X_2, and X have exponential distributions with means $1/\lambda_1, 1/\lambda_2$, and $1/\lambda$, respectively, then

$$\lambda_1 \geq \vee\{\lambda_2, \lambda\} \implies S_2 \leq_{\text{hr}} S_1. \tag{1.4.13}$$

Li and Hu (2008) proved the result for two-component series systems. They showed that (a) if $X_1 \leq_{\text{icv}} X_2$, then $S_2 \leq_{\text{icv}} S_1$; and (b) if $S_2 \leq_{\text{icx}} S_1$, then $X_1 \leq_{\text{icx}} X_2$. Further, they have provided a counterexample to demonstrate that the converse of the result given in (b) may not hold. Also, they derived that if $X_1 \leq_{\text{icv}} X_2$ and if X and X_1 (or X_2) have convex survival functions, then $S_2 \leq_{\text{sp}} S_1$. Moreover, they established that if $X_1 \leq_{\text{icv}} X_2$ and if $X_1, X_3, \ldots, X_n$ have convex survival functions, then $T_2 \leq_{\text{sp}} T_1$.

Li *et al.* (2011) established the following result for two-component systems.

$$X_1 \leq_{\text{sp}} X_2 \iff T_2 \leq_{\text{sp}} T_1 \tag{1.4.14}$$

Zhao *et al.* (2012) strengthened the result in (1.4.13) in terms of the likelihood ratio order as

$$\lambda_1 \geq \vee\{\lambda_2, \lambda\} \implies S_2 \leq_{\text{lr}} S_1. \tag{1.4.15}$$

Under the assumption that X_1, X_2, and X have exponential distributions with means $1/\lambda_1$, $1/\lambda_2$, and $1/\lambda$, respectively, Zhao *et al.* (2012), also improved the result in (1.4.12) from the usual stochastic order to the likelihood ratio order as

$$\lambda_1 \geq \lambda_2 \implies T_2 \leq_{\text{lr}} T_1. \tag{1.4.16}$$

You and Li (2014) extended the result (1.4.15) from the exponential distributions to PHR models. Also, they pointed out, through a counterexample, that the result (1.4.16) does not hold for the PHR models.

Recently, Zhao *et al.* (2016) strengthened the results (1.4.15) and (1.4.16) for the n-component series systems.

Readers can refer to Bhattacharya and Samaniego (2008), Belzunce *et al.* (2011), Misra *et al.* (2011a,b), Yan *et al.* (2013), Da and Ding (2016), Yan and Luo (2018), and Yan *et al.* (2019) for numerous other studies related to the problems of redundancy allocations.

1.5 Outline of the thesis

One of the main results of Chapter 2 provides the condition under which the reversed hazard rate function of any random variable from Topp-Leone generated family of distributions, is decreasing. Chapter 2 also includes the stochastic comparison results of two random variables from the Topp-Leone generated family of distributions. In this chapter, we also consider a particular case of this family of distributions, namely, Topp-Leone exponential distribution. We study few reliability characteristics of this distribution, such as the hazard rate function, the reversed hazard rate function, the mean residual life function, and the expected inactivity time. Renyi entropy measure for the Topp-Leone exponential distribution has also been discussed. Moreover, we define the Topp-Leone generated log-logistic distribution, and the Topp-Leone generated Lomax distribution using the base-line distributions as log-logistic and Lomax distributions, respectively.

Chapter 3 presents real data applications to discuss the importance of the Topp-Leone generated family of distributions. In this chapter, we compare the fits of the Topp-Leone generated Weibull, Topp-Leone generated log logistic, and Topp-Leone generated Lomax distributions using three real data sets. Also, we provide the applications of the Topp-Leone exponential distribution with two real data sets, and compare the fit of this distribution with the Lomax and Burr-XII distributions.

In Chapter 4, we stochastically compare the lifetimes of two series and parallel systems with components having lifetimes from the Topp-Leone generated family of distributions. We present the comparison results with heterogeneity in one parameter while another is fixed. We compare the lifetimes of two series systems with respect to the hazard rate order, and with the help of a counterexample, we show that the hazard rate order cannot be extended to the likelihood ratio order for this comparison. Also, we provide the comparison results for parallel systems with respect to the usual stochastic order and the likelihood ratio order. We show that the usual stochastic order cannot be extended to the hazard rate order using a counterexample. Moreover, we derive the results when this family of distributions has different base-line distributions. All the comparison results we study with the help of vector majorization technique.

In Chapter 5, we consider the models defined in Subsection 1.4.3. We study the problem of allocating one active/standby redundant spare in the *n*-component series system using the residual stochastic precedence and the inactivity stochastic precedence orders. The comparison results based on these orders have special concern as they take care of the dependence structure between the residual lifetimes (inactivity times) of the random variables. We also compare two parallel systems in terms of the inactivity stochastic precedence order.

Chapter 6 summarizes the work done in the thesis.

Chapter 2

Stochastic properties of Topp–Leone generated family of distributions

2.1 Introduction

Topp-Leone (TL) distribution was first discussed by Topp and Leone (1955), which is a simple bounded J-shaped distribution. Its p.d.f. and c.d.f. are given by

$$f_{\mathrm{TL}}(x) = 2\vartheta(1-x)[x(2-x)]^{\vartheta-1}, \quad 0 \le x \le 1, 0 < \vartheta < 1,$$

and

$$F_{\mathrm{TL}}(x) = [x(2-x)]^{\vartheta}, \quad 0 \le x \le 1, 0 < \vartheta < 1,$$

respectively. We can see that it is not very versatile due to having only one parameter, and its support is limited to $[0,1]$. Also, this distribution had not given due consideration until Nadarajah and Kotz (2003) studied different aspects of this distribution. Let the continuous random variable X follows the TL distribution with two parameters ϑ and κ. Then, according to Nadarajah and Kotz (2003), the p.d.f. of X is written as

$$f(x) = \frac{2\vartheta}{\kappa}\left(\frac{x}{\kappa}\right)^{\vartheta-1}\left(1-\frac{x}{\kappa}\right)\left(2-\frac{x}{\kappa}\right)^{\vartheta-1}, \quad 0 < x < \kappa,\, 0 < \vartheta < 1,\ \kappa > 0,$$

and the c.d.f. of X is

$$F(x) = \left(\frac{x}{\kappa}\right)^{\vartheta}\left(2-\frac{x}{\kappa}\right)^{\vartheta}, \quad 0 \leq x < \kappa,\, 0 < \vartheta < 1,\ \kappa > 0.$$

They identified that the hazard rate function of the TL distribution has a bathtub shape for all $\vartheta \in (0,1)$. Several authors then further studied TL distribution. One may see Chapter 1 of the thesis for a review of the literature.

To make the TL distribution more applicable, some generalizations of it have also been considered in the literature. For example, Vicaria *et al.* (2008) presented two-sided generalized TL distribution. The authors talked about some properties of this family of distributions and defined a procedure for estimating the parameters using the method of maximum likelihood. Pourdarvish *et al.* (2015) proposed the exponentiated TL distribution and studied its hazard rate function, moments, and order statistics. Recently, Al-Shomrani *et al.* 2016 introduced a generalization of the TL distribution using $G(\cdot;\zeta)$ as the base-line c.d.f. in the TL distribution and derived the hazard rate function and its moments. Rezaei *et al.* (2017) also proposed a generalization of the TL distribution. They used $[G(\cdot;\zeta)]^{\theta}$ as the base-line distribution and named it Topp-Leone generated (TL-G) family of distributions. The distributions of this family also have the hazard rate functions with bathtub shape for all $\vartheta \in (0,1)$, and maybe utilized for modeling lifetime events. The p.d.f. of the TL-G family of distributions having parameters ϑ and θ is

$$f(x;\vartheta,\theta,\zeta) = 2\vartheta\theta g(x;\zeta)G(x;\zeta)^{\theta\vartheta-1}(1-G(x;\zeta)^{\theta})(2-G(x;\zeta)^{\theta})^{\vartheta-1}, \quad x \in \mathbb{R},\ \theta,\vartheta > 0, \tag{2.1.1}$$

and the corresponding c.d.f. is

$$F(x;\vartheta,\theta,\zeta) = (G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta}, \quad x \in \mathbb{R},\ \theta,\vartheta > 0, \tag{2.1.2}$$

where $G(x;\zeta)$ and $g(x;\zeta)$ are the c.d.f. and the p.d.f. of the base-line distribution, respectively, and ζ contains the parameters which specify the base-line distribution. For ease of notation, we write $X \sim$TL-$G(\vartheta,\theta,\zeta)$ for a random variable X having p.d.f. written as (2.1.1). If we take the base-line distribution as $U(0,1)$ along with $\theta = 1$, then the TL-G family of distributions reduces to the Topp-Leone's distribution. Recently, some authors introduced and studied new distributions by choosing different $G(\cdot;\zeta)$, see, for example, Aryal *et al.* (2017), Brito *et al.* (2017), Sharma (2018), Korkmaz *et al.* (2019), and Shekhawat and Sharma (2020).

Motivated from the TL-G family of distributions proposed by Rezaei *et al.* (2017), our aim is to compare two random variables from this family of distributions in terms of stochastic orders. We also consider one of its special cases, namely, the TL-exponential distribution, and study some well-known reliability characteristics such as the hazard rate function, the mean residual life function, the reversed hazard rate function, and the expected inactivity time for this distribution. Moreover, we define two other special cases of this family of distributions, namely, the Topp-Leone generated log-logistic distribution (TL-log logistic) and the Topp-Leone generated Lomax distribution (TLGLo).

Throughout the chapter, we consider two nonnegative random variables X and Y. Let $f_X(\cdot)$ and $f_Y(\cdot)$ be the p.d.f.'s, and $F_X(\cdot)$ and $F_Y(\cdot)$ be the c.d.f.'s of X and Y, respectively.

This chapter is presented in the following scenario. In Section 2.2, we show that the reversed hazard rate function of the TL-G family of distributions is decreasing (and hence, the expected inactivity time is increasing) if the reversed hazard rate function of the base-line distribution is decreasing. Also, we make stochastic comparisons between two random variables from the TL-G family of distributions in terms of the dispersive and the star-shaped orders. With the help of an example, we show that the likelihood ratio order may not be taken in some situations. In Section 2.3, we discuss a particular case of this family of distributions called the TL-exponential distribution, and examine few reliability characteristics of this distribution, and derive some results. Also, we define the Topp-Leone generated log logistic (TL-log logistic) distribution, and the Topp-Leone generated Lomax (TLGLo) distribution using the genesis of the TL-G family of distributions. Moreover, we derive the expression of the Renyi entropy measure for the TL-exponential distribution.

2.2 Main results

In this section, we first discuss the condition under which the reversed hazard rate function (the expected inactivity time) of the TL-$G(\vartheta, \theta, \zeta)$ distribution is decreasing (increasing).

Then, we provide some comparison results based on the dispersive and the star-shaped orders.

2.2.1 The reversed hazard rate function

Let $\tilde{r}(x;\zeta)$, $x > 0$, denotes the reversed hazard rate function of the base-line distribution of the TL-$G(\vartheta,\theta,\zeta)$ family of distributions, i.e.,

$$\tilde{r}(x;\zeta) = \frac{g(x;\zeta)}{G(x;\zeta)}, \quad x > 0.$$

The theorem below shows that if $\tilde{r}(x;\zeta)$ is decreasing in $x \in (0,\infty)$, then the reversed hazard rate function of the TL-$G(\vartheta,\theta,\zeta)$ distribution is decreasing in $x \in (0,\infty)$.

Theorem 2.2.1. *Let $X \sim$ TL-$G(\vartheta,\theta,\zeta)$ and let $\tilde{r}(x;\zeta)$ be the reversed hazard rate function of the base-line distribution. Then, for fixed θ, $\vartheta > 0$ and for any fixed ζ, the reversed hazard rate function of X is decreasing in $x \in (0,\infty)$ if $\tilde{r}(x;\zeta)$ is decreasing in $x \in (0,\infty)$.*

Proof. Let $X \sim$ TL-$G(\vartheta,\theta,\zeta)$ with p.d.f. and c.d.f. given by Equations 2.1.1 and 2.1.2, respectively. Further, let $\tilde{r}_X(x;\vartheta,\theta,\zeta)$ denotes the reversed hazard rate function of X. Then,

$$\tilde{r}_X(x;\vartheta,\theta,\zeta) = \frac{f_X(x;\vartheta,\theta,\zeta)}{F_X(x;\vartheta,\theta,\zeta)} = \frac{\mathrm{d}}{\mathrm{d}x}\log(F(x;\vartheta,\theta,\zeta)).$$

Using Equation (2.1.2), we obtain that

$$\begin{aligned}\tilde{r}_X(x;\vartheta,\theta,\zeta) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\vartheta\theta\,\log(G(x;\zeta)) + \vartheta\,\log(2 - G(x;\zeta)^{\theta})\Big)\\ &= \vartheta\theta\frac{g(x;\zeta)}{G(x;\zeta)} - \frac{\vartheta\theta g(x;\zeta)G(x;\zeta)^{\theta-1}}{2 - G(x;\zeta)^{\theta}}\end{aligned}$$

$$= \frac{2\vartheta\theta g(x;\zeta)(1-G(x;\zeta)^{\theta})}{G(x;\zeta)(2-G(x;\zeta)^{\theta})}$$
$$= 2\vartheta\theta\tilde{r}(x;\zeta)\left(1-\frac{1}{2-G(x;\zeta)^{\theta}}\right), \quad x>0.$$

Now, it is straightforward to look that $\tilde{r}_X(x;\vartheta,\theta,\zeta)$ is decreasing in $x \in (0,\infty)$ whenever $\tilde{r}(x;\zeta)$ is decreasing in $x \in (0,\infty)$. □

Remark 2.2.1. It is popularly recognized that the decreasing reversed hazard rate implies increasing expected inactivity time (see, Chandra and Roy (2001)). Then, using Theorem 2.2.1, it follows that the expected inactivity time of the TL-$G(\vartheta,\theta,\zeta)$ random variable is increasing in $x \in (0,\infty)$ if $\tilde{r}(x;\zeta)$ is decreasing in $x \in (0,\infty)$.

2.2.2 Stochastic comparisons of the TL-G family of distributions

In this subsection, we compare two random variables from the TL-G family of distributions in terms of the dispersive and the star-shaped orders. The following theorem provides the stochastic comparisons of the TL-G family of distributions when the parameter θ varies.

Theorem 2.2.2. *Let $X \sim$TL-$G(\vartheta,\theta_1,\zeta)$ and $Y \sim$TL-$G(\vartheta,\theta_2,\zeta)$ be two random variables having base-line distribution $G(x;\zeta)$ with support $(0,\infty)$. Then, for a fixed $\vartheta>0$ and for any fixed ζ, $Y \le_{disp} X$ ($Y \le_{*} X$) whenever $\theta_1 \le \theta_2$ and $\frac{log(G(x;\zeta))}{\tilde{r}(x;\zeta)}$ $\left(\frac{log(G(x;\zeta))}{x\tilde{r}(x;\zeta)}\right)$ is increasing in $x \in (0,\infty)$, where $\tilde{r}(x;\zeta)$ is the reversed hazard rate function of base-line distribution.*

Proof. For a fixed $\vartheta > 0$ and for any fixed ζ, let the p.d.f. of X be written as

$$f_{\theta_1}(x) = 2\vartheta\theta_1 g(x;\zeta)G(x;\zeta)^{\theta_1\vartheta-1}(1-G(x;\zeta)^{\theta_1})(2-G(x;\zeta)^{\theta_1})^{\vartheta-1},\ x>0,\ \theta_1>0.$$

Clearly, the c.d.f. of X is

$$F_{\theta_1}(x) = (G(x;\zeta)^{\theta_1}(2-G(x;\zeta)^{\theta_1}))^{\vartheta},\ x>0,\ \theta_1>0.$$

Then,

$$\begin{aligned}
F'_{\theta_1}(x) &\equiv \frac{\partial}{\partial\theta_1}F_{\theta_1}(x)\\
&= -\vartheta G(x;\zeta)^{\theta_1\vartheta+\theta_1}(2-G(x;\zeta)^{\theta_1})^{\vartheta-1}\log(G(x;\zeta)) + \vartheta\log(G(x;\zeta))G(x;\zeta)^{\theta_1\vartheta}\\
&\quad\times(2-G(x;\zeta)^{\theta_1})^{\vartheta}\\
&= \vartheta G(x;\zeta)^{\theta_1\vartheta}(2-G(x;\zeta)^{\theta_1})^{\vartheta}\log(G(x;\zeta))\left(1-\frac{G(x;\zeta)^{\theta_1}}{2-G(x;\zeta)^{\theta_1}}\right)\\
&= 2\vartheta G(x;\zeta)^{\theta_1\vartheta}(2-G(x;\zeta)^{\theta_1})^{\vartheta}\log(G(x;\zeta))\left(\frac{1-G(x;\zeta)^{\theta_1}}{2-G(x;\zeta)^{\theta_1}}\right).
\end{aligned}$$

Therefore,

$$\frac{F'_{\theta_1}(x)}{f_{\theta_1}(x)} = \frac{1}{\theta_1}\frac{\log(G(x;\zeta))}{\tilde{r}(x;\zeta)},$$

which is increasing in $x\in(0,\infty)$ as $\frac{\log(G(x;\zeta))}{\tilde{r}(x;\zeta)}$ is increasing in $x\in(0,\infty)$. Hence, on adopting Lemma 1.3.5, we achieve that $Y \le_{\text{disp}} X$ whenever $\theta_1 \le \theta_2$. Further, we have

$$\frac{F'_{\theta_1}(x)}{xf_{\theta_1}(x)} = \frac{1}{x\theta_1}\frac{\log(G(x;\zeta))}{\tilde{r}(x;\zeta)},$$

which is also increasing in $x\in(0,\infty)$ as $\frac{\log(G(x;\zeta))}{x\tilde{r}(x;\zeta)}$ is increasing in $x\in(0,\infty)$. Hence, on adopting Lemma 1.3.5, we have $Y \le_* X$ whenever $\theta_1 \le \theta_2$. □

The following example supports the existence of the assumptions made in Theorem 2.2.2.

Example 2.2.1. Let $X \sim$TL-$G(\vartheta,\theta_1,\zeta)$ and $Y \sim$TL-$G(\vartheta,\theta_2,\zeta)$ be two random variables with base-line distribution $G(x;\tau,\rho) = 1-e^{-\tau x^{\rho}}$, $x>0$, $\tau>0$, $\rho>1$. Here $\zeta = (\tau,\rho)$. Then, $g(x;\tau,\rho) = \tau\rho x^{\rho-1}e^{-\tau x^{\rho}}$, $x>0$, $\tau>0$, $\rho>1$, and $\tilde{r}(x;\tau,\rho) = \frac{\tau\rho x^{\rho-1}}{e^{\tau x^{\rho}}-1}$, $x>0$, $\tau>0$, $\rho>1$. Now, let

$$\omega_1(x) = \frac{\log(G(x;\tau,\rho))}{\tilde{r}(x;\tau,\rho)} = \frac{(e^{\tau x^{\rho}}-1)\log(1-e^{-\tau x^{\rho}})}{\tau\rho x^{\rho-1}},\ x>0,$$

and

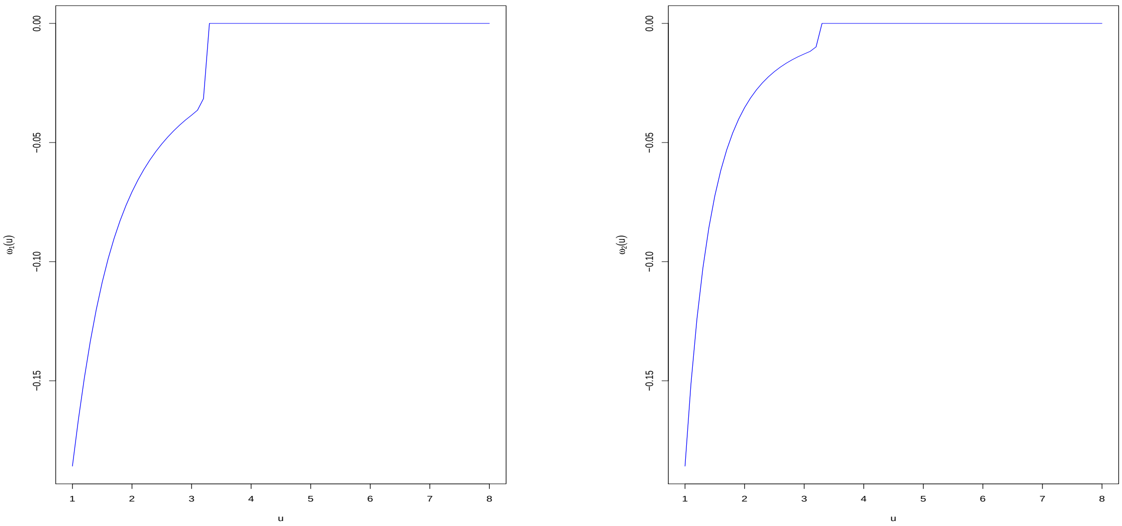
$$\omega_2(x) = \frac{\log(G(x;\tau,\rho))}{x\tilde{r}(x;\tau,\rho)} = \frac{(e^{\tau x^{\rho}}-1)\log(1-e^{-\tau x^{\rho}})}{\tau\rho x^{\rho}},\ x>0.$$

Now, for $\tau = 2$ and $\rho = 2.5$, with the help of R-software, we plot $\omega_1(x)$ and $\omega_2(x)$ as given in Figure 2.1. Clearly, $\omega_1(x)$ and $\omega_2(x)$ both are increasing in $x \in (0,\infty)$ and hence, on adopting Theorem 2.2.2, we achieve that $Y \leq_{\text{disp}} X$ and $Y \leq_{*} X$. □

The following theorem directly follows from Theorem 5.1 of Sharma (2018).

Theorem 2.2.3. *Let $X \sim$TL-$G(\vartheta_1,\theta_1,\zeta)$ and $Y \sim$TL-$G(\vartheta_2,\theta_2,\zeta)$ be two random variables. Then, for $\theta_1 = \theta_2\,(>0)$ and for any fixed ζ, $X \leq_{lr} Y$ whenever $\vartheta_1 < \vartheta_2$.*

One may be interested in comparing the random variables X and Y in terms of the likelihood ratio order when $\theta_1 \neq \theta_2$. The upcoming counterexample demonstrates that the likelihood ratio order may not hold when $\theta_1 \neq \theta_2$.



(a) Plot of $\omega_1(x)$ **(b)** Plot of $\omega_2(x)$

Figure 2.1: Plots of $\omega_1(x)$ and $\omega_2(x)$

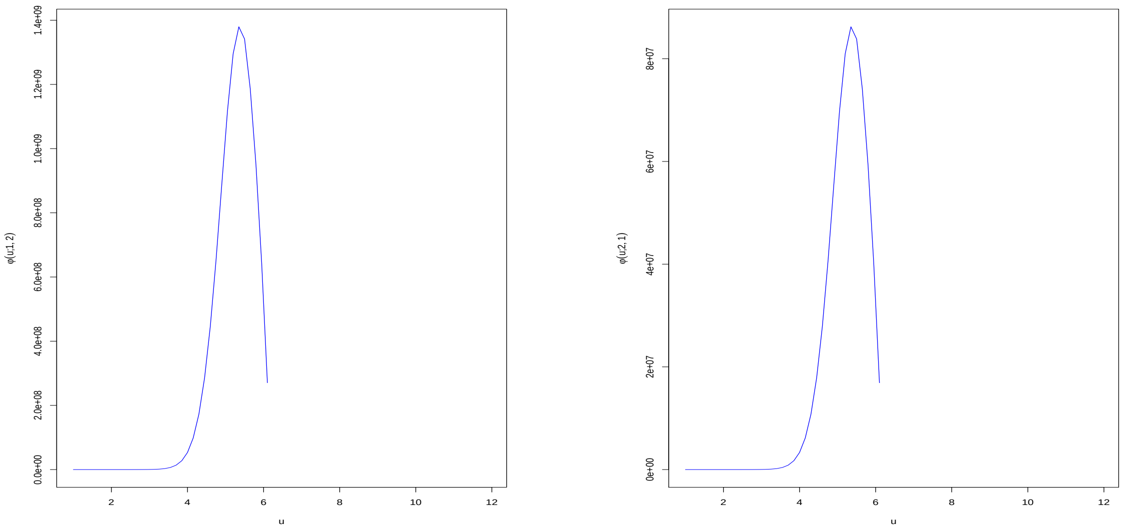
Counterexample 2.2.1. Let $X \sim$TL-$G(\vartheta_1, \theta_1, (h_1, k_1))$ and $Y \sim$TL-$G(\vartheta_2, \theta_2, (h_2, k_2))$ with base-line c.d.f. $G(x;h,k) = 1 - e^{-(x/h)^k}$, $x > 0, h > 0, k > 0$, i.e., X and Y follows the Topp-Leone Generated Weibull (TLGW) distributions proposed by Aryal *et al.* (2017). Then, this can be easily get that

$$\frac{f_Y(x)}{f_X(x)} = \frac{\vartheta_2 \theta_2}{\vartheta_1 \theta_1} \frac{k_2}{k_1} \left(\frac{h_1^{k_1}}{h_2^{k_2}} \right) x^{k_2 - k_1} e^{-((x/h_2)^{k_2} - (x/h_1)^{k_1})} \frac{(1 - e^{-(x/h_2)^{k_2}})^{\theta_2 \vartheta_2 - 1}}{(1 - e^{-(x/h_1)^{k_1}})^{\theta_1 \vartheta_1 - 1}}$$
$$\times \frac{(1 - (1 - e^{-(x/h_2)^{k_2}})^{\theta_2})}{(1 - (1 - e^{-(x/h_1)^{k_1}})^{\theta_1})} \frac{(2 - (1 - e^{-(x/h_2)^{k_2}})^{\theta_2})^{\vartheta_2 - 1}}{(2 - (1 - e^{-(x/h_1)^{k_1}})^{\theta_1})^{\vartheta_1 - 1}},$$

$x > 0,\ \theta_i, \vartheta_i, h_i, k_i > 0,\ i = 1, 2$. On taking $\vartheta_1 = 0.1$, $\vartheta_2 = 0.2$, $h_1 = 1$, $h_2 = 2$, $k_1 = 2$, and $k_2 = 3$ in the above equation, we get

$$\frac{f_Y(x)}{f_X(x)} = \frac{3}{8} \frac{\theta_2}{\theta_1} x e^{-((x/2)^3 - x^2)} \frac{(1 - e^{-(x/2)^3})^{(\theta_2 \times 0.2) - 1}}{(1 - e^{-x^2})^{(\theta_1 \times 0.1) - 1}} \frac{(1 - (1 - e^{-(x/2)^3})^{\theta_2})}{(1 - (1 - e^{-x^2})^{\theta_1})}$$
$$\times \frac{(2 - (1 - e^{-(x/2)^3})^{\theta_2})^{-0.8}}{(2 - (1 - e^{-x^2})^{\theta_1})^{-0.9}}. \tag{2.2.1}$$
$$= \varphi(x; \theta_1, \theta_2), \text{ say.}$$

Now, we plot $\varphi(x;1,2)$ and $\varphi(x;2,1)$ (see, Figure 2.2). Clearly, in both the cases, $\varphi(x)$ is not monotone, which means that neither $X \leq_{\mathrm{lr}} Y$ nor $Y \leq_{\mathrm{lr}} X$. This shows that the likelihood ratio order may not exist between X and Y if $\theta_1 \neq \theta_2$. □



(a) Plot of $\varphi(x;1,2)$ **(b)** Plot of $\varphi(x;2,1)$

Figure 2.2: Plots of $\varphi(x;1,2)$ and $\varphi(x;2,1)$

2.3 Some special cases of the TL-G family of distributions

This section of the chapter is devoted to some particular cases of the TL-G family of distributions. First, we consider the TL-exponential distribution by using the exponential distribution as the base-line distribution. In addition, we study some reliability characteristics and the Renyi entropy measure for the TL-exponential distribution. Second, we define the TL-log logistic, and the TLGLo distributions using the log-logistic distribution and the Lomax distribution as the base-line distributions, respectively. Also, we show some graphical representations of the density functions and the hazard rate functions for both distributions.

2.3.1 The TL-exponential distribution

In this subsection, we use the exponential distribution as the base-line distribution with $G(x;\mu) = 1 - e^{-\mu x}$ and $\theta = 1$, then the p.d.f. of the TL-exponential distribution with parameters ϑ and μ is written as

$$f_X(x) = 2\vartheta\mu e^{-2\mu x}(1 - e^{-2\mu x})^{\vartheta - 1}, \qquad x > 0,\, \mu > 0,\, \vartheta > 0, \tag{2.3.1}$$

and the corresponding c.d.f. is

$$F_X(x) = (1 - e^{-2\mu x})^{\vartheta}, \qquad x > 0,\, \mu > 0,\, \vartheta > 0.$$

For a random variable X with p.d.f. written as (2.3.1), we write $X \sim$ TL-Exp(ϑ, μ). Figure 2.3 indicates the shapes of p.d.f. of the TL-exponential distribution for $\vartheta > 1$ and for distinct values of μ.

Few authors have studied the shapes of the hazard rate functions and the p.d.f. of the TL-exponential distribution for different values of ϑ and μ (see, Al-Shomrani *et al.* (2016) and Sebastian *et al.* (2019)).

Now, we discuss some reliability characteristics of the TL-exponential distribution.

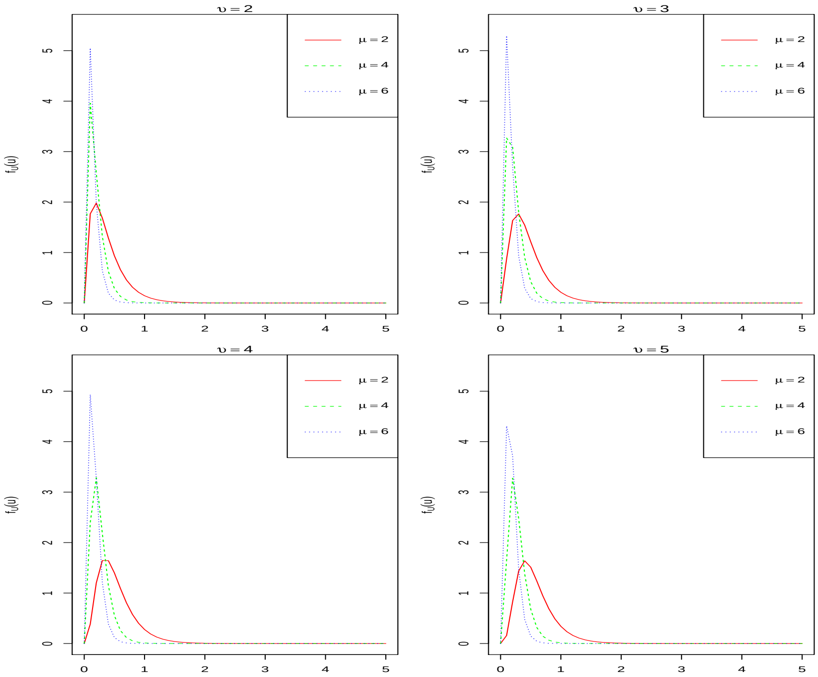


Figure 2.3: Shapes of the p.d.f. of the TL-exponential distribution for $\vartheta > 1$ and for distinct values of μ

2.3.1.1 Hazard rate and mean residual life functions of the TL-exponential distribution

The hazard rate and the mean residual life functions are very useful and important functions in survival analysis and reliability theory. Moreover, they defined earlier in terms of the time to failure of a device. However, the mean residual life function becomes a more appropriate reliability measure than the hazard rate function because it compiles the whole remaining life, whereas the hazard rate function gives instantaneous failure rate at time $x > 0$. For the definitions and other details on the hazard rate and the mean residual life functions, we refer the readers to Chapter 1 of the thesis, Ramos-Romero and Sordo-Díaz (2001), Belzunce *et al.* (2002), and Lai and Xie (2006).

The hazard rate function for the TL-exponential distribution is written as

$$\begin{aligned} r_X(x) &= \frac{f_X(x)}{1-F_X(x)} \\ &= \frac{2\vartheta}{\mu}\frac{e^{-2\mu x}(1-e^{-2\mu x})^{\vartheta-1}}{1-(1-e^{-2\mu x})^{\vartheta}}, \quad x>0,\ \vartheta>0, \mu>0. \end{aligned}$$

Al-Shomrani *et al.* (2016) and Sebastian *et al.* (2019) discussed the shape of the hazard rate function for various values of ϑ, and for $\mu = 3$ and $\mu = 0.3$, respectively. They graphically showed that the TL-exponential distribution has a strictly increasing hazard rate function for $\vartheta > 1$ and has a strictly decreasing hazard rate function for $\vartheta < 1$. Now, the following theorem provides the proof of the above-mentioned graphical observations for any $\mu > 0$, i.e., we show that the hazard rate function of the TL-Exp(ϑ, μ) distribution is strictly increasing for $\vartheta > 1$, strictly decreasing for $\vartheta < 1$, and constant for $\vartheta = 1$ for any $\mu > 0$.

Theorem 2.3.1. *Let $X \sim$ TL-Exp(ϑ, μ) and let $r_X(\cdot)$ be the hazard rate function of X. Then, $r_X(x)$ is strictly increasing (strictly decreasing, constant) in $x \in (0,\infty)$ for $\vartheta > 1$ ($\vartheta < 1$, $\vartheta = 1$) for any $\mu > 0$.*

Proof. The p.d.f., $f_X(x)$, of the random variable X is written as (2.3.1). Define

$$\eta(x) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}f_X(x)}{f_X(x)}, \quad x > 0.$$

Then,

$$\eta(x) = \frac{2\mu(1-\vartheta e^{-2\mu x})}{1-e^{-2\mu x}}, \quad x > 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta(x) = \frac{4\mu^2(\vartheta-1)e^{-2\mu x}}{(1-e^{-2\mu x})^2}, \quad x>0.$$

Clearly, for any fixed $\mu > 0$, we have $\frac{\mathrm{d}}{\mathrm{d}x}\eta(x) > 0\,(<0, =0)$, for every $x > 0$, if $\vartheta > 1$ ($\vartheta < 1, \vartheta = 1$), which means that $\eta(x)$ is strictly increasing (strictly decreasing, constant) in $x \in (0,\infty)$ for $\vartheta > 1$ ($\vartheta < 1, \vartheta = 1$). Now, on adopting Theorem 2.1 of Lai and Xie (2006, p. 13), we conclude that the hazard rate function, $r_X(x)$, is strictly increasing (strictly decreasing, constant) in $x \in (0,\infty)$ for $\vartheta > 1$ ($\vartheta < 1, \vartheta = 1$) for any $\mu > 0$. □

Now, the mean residual life function for the TL-exponential distribution is written as

$$\begin{aligned}\mu_X(x) &= \frac{\int_x^\infty (1-F_X(t))\,\mathrm{d}t}{1-F_X(x)}\\ &= \frac{\int_x^\infty (1-(1-e^{-2\mu t})^\vartheta)\,\mathrm{d}t}{1-(1-e^{-2\mu x})^\vartheta}, \quad x>0,\ \vartheta>0, \mu>0.\end{aligned}$$

On using binomial expansion (any index), we obtain the expression for the mean residual life function of TL-exponential distribution as follows.

$$\begin{aligned}\mu_X(x) &= \frac{1}{1-(1-e^{-2\mu x})^\vartheta}\int_x^\infty \sum_{m=1}^\infty \binom{\vartheta}{m}(-1)^{m-1}(e^{-2\mu t})^m\,\mathrm{d}t\\ &\qquad\qquad \left[\text{where } \binom{\vartheta}{m} = \frac{\vartheta(\vartheta-1)\cdots(\vartheta-m+1)}{m!}\right]\\ &= \frac{1}{1-(1-e^{-2\mu x})^\vartheta}\sum_{m=1}^\infty \binom{\vartheta}{m}(-1)^{m-1}\int_x^\infty e^{-2\mu tm}\,\mathrm{d}t \qquad \text{[by Fubini's theorem]}\\ &= \frac{1}{1-(1-e^{-2\mu x})^\vartheta}\sum_{m=1}^\infty \binom{\vartheta}{m}(-1)^{m-1}\frac{1}{2\mu m}e^{-2\mu um}, \quad x>0.\end{aligned}$$

2.3.1.2 Reversed hazard rate function and expected inactivity time of the TL-exponential distribution

Despite being similar to the hazard rate and the mean residual life functions, the reversed hazard rate and the expected inactivity time functions are frequently used in life testing problems. They played a significant role in censored data research. Also, they are applicable in the fields like Forensic Sciences (see, for example, Kalbfleisch and Lawless (1989), Chandra and Roy (2001), and Kayid and Ahmad (2004), and references cited therein).

For the TL-Exp(ϑ,μ) distribution, the reversed hazard rate function and the expected inactivity time are written as

$$\tilde{r}_X(x)=\frac{f_X(x)}{F_X(x)}=\frac{2\vartheta}{\mu}\frac{1}{(e^{2\mu x}-1)},\quad x>0,\ \vartheta>0,\mu>0,$$

and

$$\tilde{\mu}_X(x)=\frac{\int_0^x F_X(t)\,\mathrm{d}t}{F_X(x)}=\frac{1}{(1-e^{-2\mu x})^{\vartheta}}\int_0^x(1-e^{-2\mu t})^{\vartheta}\,\mathrm{d}t,\quad x>0,\ \vartheta>0,\mu>0,$$

respectively. On adopting binomial expansion (any index), we obtain that

$$\begin{aligned}\tilde{\mu}_X(x)&=\frac{1}{(1-e^{-2\mu x})^{\vartheta}}\int_0^x\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m(e^{-2\mu t})^m\,\mathrm{d}t\\ &\qquad\left[\text{where }\binom{\vartheta}{0}=1\text{ and }\binom{\vartheta}{m}=\frac{\vartheta(\vartheta-1)\cdots(\vartheta-m+1)}{m!},m\geq 1\right]\\ &=\frac{1}{(1-e^{-2\mu x})^{\vartheta}}\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m\int_0^x e^{-2\mu tm}\,\mathrm{d}t\qquad\text{[by Fubini's theorem]}\\ &=\frac{1}{(1-e^{-2\mu x})^{\vartheta}}\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m\left(1-\frac{1}{2\mu m}e^{-2\mu um}\right),\quad x>0.\end{aligned}$$

Now, we state the following corollary which comes directly under the observation that $\tilde{r}_X(x)$ is decreasing in $x \in (0,\infty)$ and Remark 2.2.1.

Corollary 2.3.1. *If X ~TL-Exp(ϑ,μ), then X has a decreasing reversed hazard rate function and increasing expected inactivity time.*

2.3.1.3 Renyi entropy

The degree of unpredictability of a random variable X is defined in terms of entropy, and the one which is well-suited in this case is the Renyi entropy (Renyi (1961)). For a continuous random variable X having p.d.f. $f_X(\cdot)$, the Renyi entropy is written as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left(\int f_X^{\delta}(x)\,\mathrm{d}x\right),$$

where $\delta > 0$ and $\delta \neq 1$.

For the TL-Exp(ϑ,μ), the expression for the Renyi entropy is written as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}e^{-2\mu\delta x}(1-e^{-2\mu x})^{\delta(\vartheta-1)}\,\mathrm{d}x\right).$$

Now, on using binomial expansion (any index), we obtain that

$$\begin{aligned}
I_R(\delta) &= \frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}e^{-2\mu\delta x}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m e^{-2\mu mu}\,\mathrm{d}x\right)\\
&\left[\text{where } \binom{\delta(\vartheta-1)}{m} = \frac{\delta(\vartheta-1)(\delta(\vartheta-1)-1)\cdots(\delta(\vartheta-1)-m+1)}{m!}, m\geq 1\right]\\
&= \frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m e^{-2\mu x(\delta+m)}\,\mathrm{d}x\right)\\
&= \frac{1}{1-\delta}\log\left((2\vartheta\mu)^{\delta}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m\int_0^{\infty}e^{-2\mu x(\delta+m)}\,\mathrm{d}x\right)
\end{aligned}$$

[by Fubini's theorem]

$$= \frac{1}{1-\delta} \log\left((2\vartheta\mu)^{\delta} \sum_{m=0}^{\infty} \binom{\delta(\vartheta-1)}{m} (-1)^m \frac{1}{2\mu(\delta+m)} \right).$$

2.3.2 The TL-log logistic distribution and the TLGLo distribution

In this subsection, we take the log-logistic and the Lomax distributions (see, Lomax (1954)) as the base-line distributions with

$$G(x;\beta,\eta) = \frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}} \quad \text{and} \quad G^{*}(x;\beta,\eta) = 1-(1+\eta x)^{-\beta},$$

respectively. Then, the p.d.f.'s of the TL-log logistic distribution and the TLGLo distribution are given by

$$f_X(x) = 2\vartheta\theta\beta\eta^{\beta\theta\vartheta} \frac{x^{\beta\theta\vartheta-1}}{[1+(\eta x)^{\beta}]^{1+\theta\vartheta}} \left[1-\left(\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right)^{\theta}\right] \left[2-\left(\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right)^{\theta}\right]^{\vartheta-1},$$
$$x>0, \vartheta>0, \theta>0, \beta>0, \eta>0, \quad (2.3.2)$$

and

$$f_X^{*}(x) = 2\vartheta\theta\beta\eta(1+\eta x)^{-(\beta+1)} \left[1-(1+\eta x)^{-\beta}\right]^{\theta\vartheta-1} \left[1-(1-(1+\eta x)^{-\beta})^{\theta}\right]$$
$$\times \left[2-(1-(1+\eta x)^{-\beta})^{\theta}\right]^{\vartheta-1}, \qquad x>0, \vartheta>0, \theta>0, \beta>0, \eta>0, \quad (2.3.3)$$

respectively, and their corresponding c.d.f.'s are

$$F_X(x) = \left[\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right]^{\theta\vartheta} \left[2-\left(\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right)^{\theta}\right]^{\vartheta}, \quad x>0, \vartheta>0, \theta>0, \beta>0, \eta>0,$$

and

$$F_X^*(x) = \left[1-(1+\eta x)^{-\beta}\right]^{\theta\vartheta}\left[2-\left(1-(1+\eta x)^{-\beta}\right)^{\theta}\right]^{\vartheta},$$

$$x > 0, \vartheta > 0, \theta > 0, \beta > 0, \eta > 0.$$

Figures 2.4 and 2.5 represent the possible shapes of the p.d.f.'s of the TL-log logistic distribution and the TLGLo distribution, respectively. Figures 2.6 and 2.7 illustrate the possible shapes of the hazard rate functions of the TL-log logistic distribution and the TL-GLo distribution, respectively.

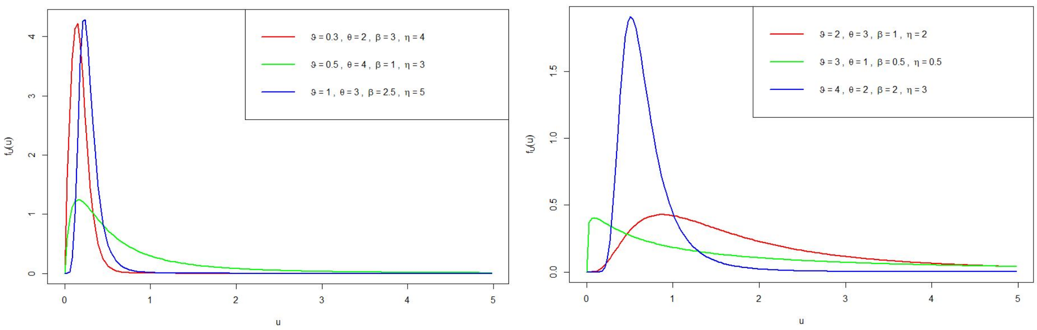


Figure 2.4: Shapes of the p.d.f.'s of the TL-log logistic distribution

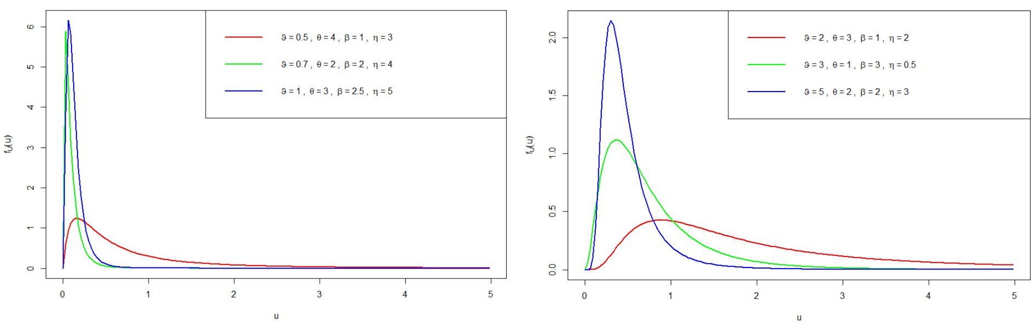


Figure 2.5: Shapes of the p.d.f.'s of the TLGLo distribution

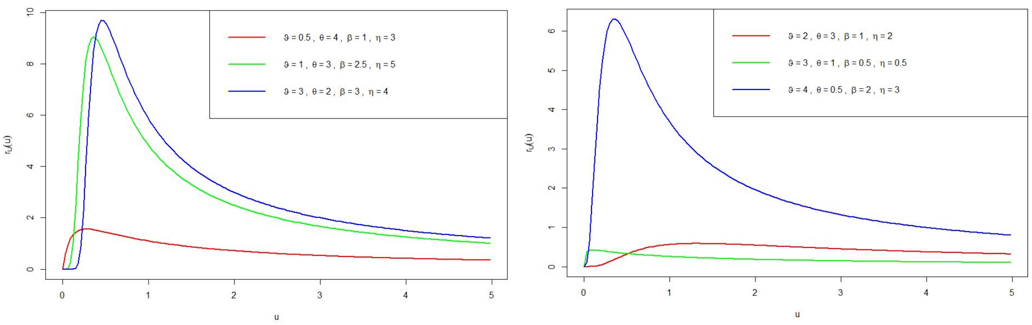


Figure 2.6: Shapes of the hazard rate functions of the TL-log logistic distribution

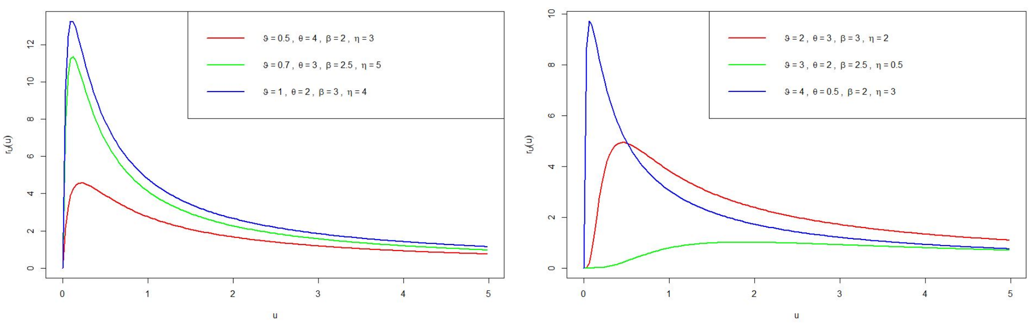


Figure 2.7: Shapes of the hazard rate functions of the TLGLo distribution

Chapter 3

Real data applications for Topp-Leone generated family of distributions

3.1 Introduction

In this chapter, we analyze real data sets to compare the fits of few models of the TL-G family of distributions, named, TLGW distribution (see, Aryal *et al.* (2017)), TL-log logistic distribution, and TLGLo distribution (defined in Subsection 2.3.2). We also provide the applications of the TL-exponential distribution (defined in Subsection 2.3.1) to two real data sets and compare the fits of this distribution with the Lomax distribution (see, Lomax (1954)) and the Burr-XII distribution (see, Burr (1942)). The criteria used for choosing the best fitted distribution are : Akaike information criterion (AIC), Akaike information criterion corrected (AICC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS) statistic, and it's p-value. The model with the smallest values of these statistics and largest p-value is generally better fitted to the data. Throughout the chapter, all the calculations were performed using R-software.

3.1.1 Comparative study on some models of the TL-G family of distributions

In this subsection, we consider few models of the TL-G family of distributions, named, TLGW distribution, TL-log logistic distribution, and TLGLo, respectively, and compare the performances of these models with the help of three real data sets.

The first data set includes 63 observations of the strengths of 1.5 cm of glass fibers collected by the UK National Physical Laboratory and also applied by Smith and Naylor (1987). The data are given as:

$0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.$

The second data set contains an active repair time (in hours) for an airborne communication transceiver recorded by Balakrishnan *et al.* (2009), initially given by Chhikara and Folks (1989). The data are given as:

$0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.$

The third data set contains 63 observations of the gauge lengths of each of the 10 mm recorded by Kundu and Gupta (2009). The data are given as:

$1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522,$

$2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928,$

$2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272,$

$3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871,$

$3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.$

Some descriptive statistics for these data sets are shown in Table 3.1.

Table 3.1: Descriptive statistics for the Data Sets 1, 2, and 3

Descriptive statistics	Data Set 1	Data Set 2	Data Set 3
Minimum	0.550	0.200	1.901
Median	1.590	1.750	2.996
Mean	1.507	3.607	3.059
Maximum	2.240	24.500	5.020
Standard deviation	0.3241257	4.944195	0.6209216

We compare the results of the fits of the TLGW model with p.d.f. given by

$$f_X(x) = 2\vartheta\theta\beta\eta^{\beta}x^{\beta-1}e^{-(\eta x)^{\beta}}(1-e^{-(\eta x)^{\beta}})^{\theta\vartheta-1}[1-(1-e^{-(\eta x)^{\beta}})^{\theta}][2-(1-e^{-(\eta x)^{\beta}})^{\theta}]^{\vartheta-1},$$

$$x>0, \vartheta>0, \theta>0, \beta>0, \eta>0,$$

the TL-log logistic model with p.d.f. given by

$$f_X^{*}(x) = 2\vartheta\theta\beta\eta^{\beta\theta\vartheta}\frac{x^{\beta\theta\vartheta-1}}{[1+(\eta x)^{\beta}]^{1+\theta\vartheta}}\left[1-\left(\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right)^{\theta}\right]\left[2-\left(\frac{(\eta x)^{\beta}}{1+(\eta x)^{\beta}}\right)^{\theta}\right]^{\vartheta-1},$$

$$x>0, \vartheta>0, \theta>0, \beta>0, \eta>0,$$

and the TLGLo model with p.d.f. given by

$$f_X^{**}(x) = 2\vartheta\theta\beta\eta(1+\eta x)^{-(\beta+1)}\left[1-(1+\eta x)^{-\beta}\right]^{\theta\vartheta-1}\left[1-(1-(1+\eta x)^{-\beta})^{\theta}\right]$$

$$\times\left[2-(1-(1+\eta x)^{-\beta})^{\theta}\right]^{\vartheta-1}, \qquad x>0, \vartheta>0, \theta>0, \beta>0, \eta>0.$$

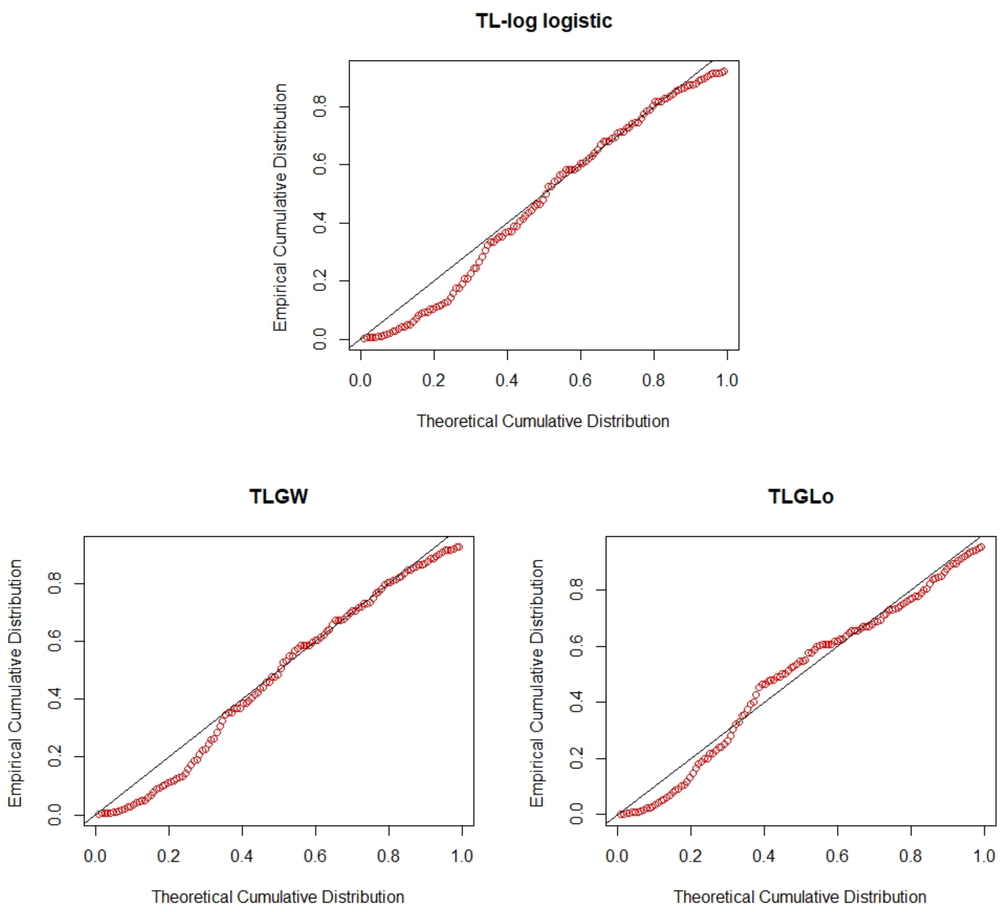


Figure 3.1: The PP-plots of the strength data of glass fibers fitted with the TL-log logistic, TLGW, and TLGLo models

First, we plot the probability-probability plots (PP-plots) for Data Sets 1, 2, and 3 in Figures 3.1, 3.2, and 3.3, respectively, from which we can easily observe that the models TLGW, TL-log logistic, and TLGLo provide a suitable fit to these data sets.

Next, we compute the estimates for the unknown parameters of each model using the method of maximum likelihood (ML). Then, we compare the results through statistics: AIC, AICC, BIC, KS statistic, and its p-value. Also, we determine -log l which indicates the log-likelihood function evaluated at the ML estimates for all data sets (see, Tables 3.2, 3.3, and 3.4).

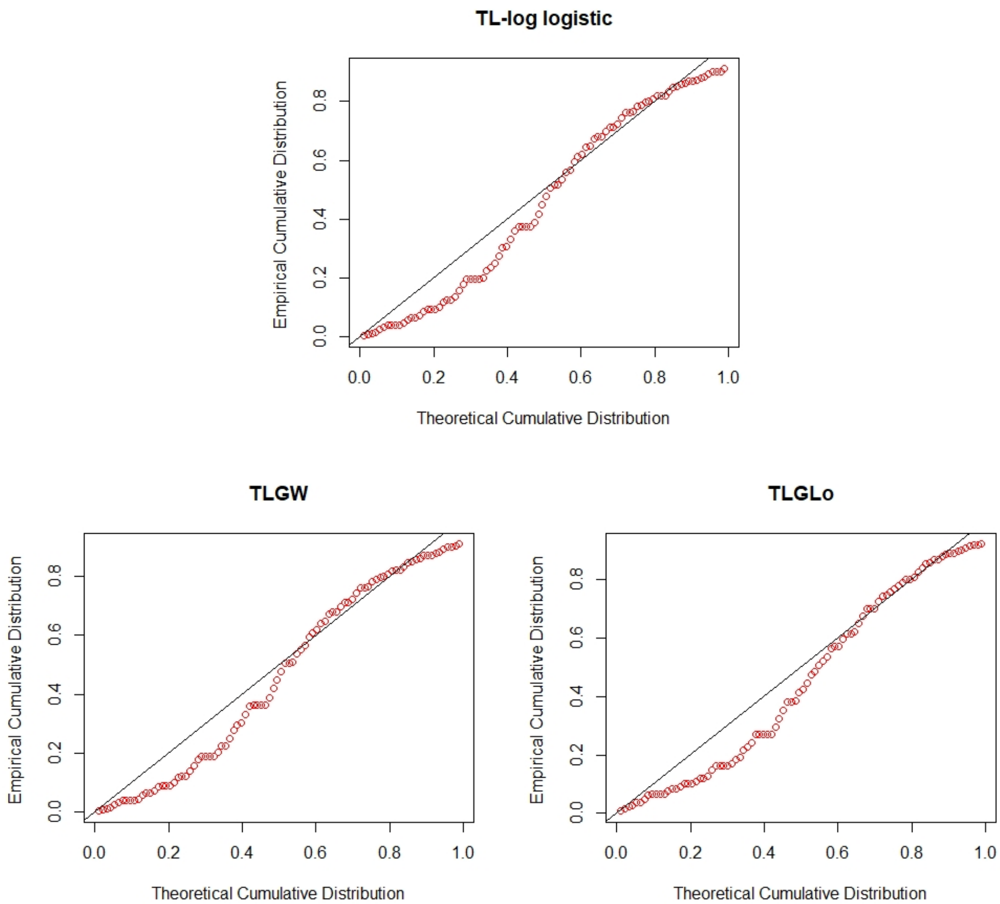


Figure 3.2: The PP-plots for an active repair times data fitted with the TL-log logistic, TLGW, and TLGLo models

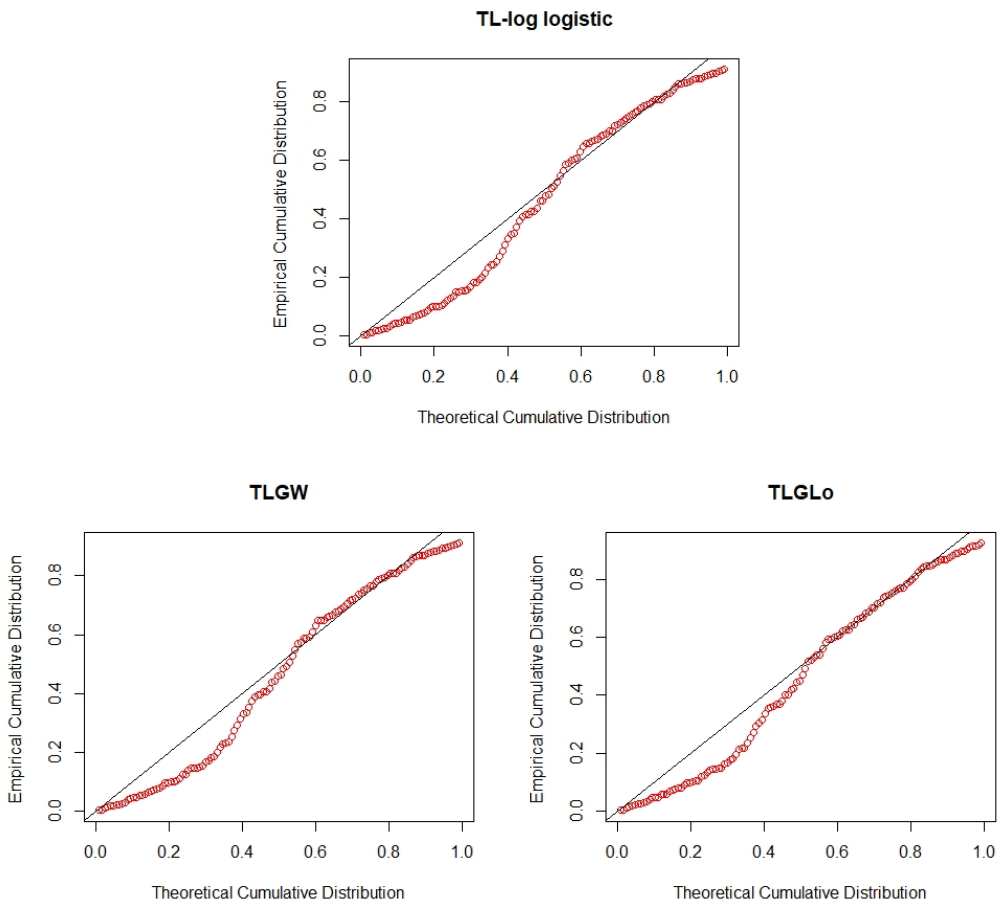


Figure 3.3: The PP-plots for the data of the gauge lengths for the TL-log logistic, TLGW, and TLGLo distributions

Table 3.2: ML-estimates and log-likelihood functions for the TL-log logistic, TLGW, and TLGLo models, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for Data Set 1

Models	ML-estimates	-log l	AIC	AICC	BIC	KS statistic	p-value
TL-log logistic	$\hat{\vartheta} = 0.1964573$ $\hat{\theta} = 1.6199747$ $\hat{\beta} = 14.5917323$ $\hat{\eta} = 0.5459271$	-12.6589	**33.3178**	**34.00746**	**41.89034**	**0.12698**	**0.69**
TLGW	$\hat{\vartheta} = 0.4127334$ $\hat{\theta} = 1.8399951$ $\hat{\beta} = 6.4448543$ $\hat{\eta} = 0.5491863$	-14.56609	37.13218	37.82184	45.70472	0.14286	0.5412
TLGLo	$\hat{\vartheta} = 5.481529e-03$ $\hat{\theta} = 6.743062e+03$ $\hat{\beta} = 7.930990e+00$ $\hat{\eta} = 4.409398e-01$	-17.86159	43.72318	44.41284	52.29572	0.22222	0.08909

Table 3.3: ML-estimates and log-likelihood functions for the TL-log logistic, TLGW, and TLGLo models, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for Data Set 2

Models	ML-estimates	-log l	AIC	AICC	BIC	KS statistic	p-value
TL-log logistic	$\hat{\vartheta} = 0.1048540$ $\hat{\theta} = 92.3674740$ $\hat{\beta} = 0.9768692$ $\hat{\eta} = 7.4358103$	-99.55465	**207.1093**	**208.0849**	**214.4239**	**0.086957**	**0.995**
TLGW	$\hat{\vartheta} = 3.013868$ $\hat{\theta} = 20.536544$ $\hat{\beta} = 0.181533$ $\hat{\eta} = 571.376529$	-99.81147	207.6229	208.5985	214.9375	0.1087	0.9487
TLGLo	$\hat{\vartheta} = 3.539238e-04$ $\hat{\theta} = 2.670313e+03$ $\hat{\beta} = 1.079512e+03$ $\hat{\eta} = 2.455555e-04$	-104.972	217.944	218.9196	225.2586	0.15217	0.6612

Table 3.4: ML-estimates and log-likelihood functions for the TL-log logistic, TLGW, and TLGLo models, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for Data Set 3

Models	ML-estimates	-log l	AIC	AICC	BIC	KS statistic	p-value
TL-log logistic	$\hat{\vartheta} = 0.1324324$ $\hat{\theta} = 30.9651194$ $\hat{\beta} = 5.6684962$ $\hat{\eta} = 0.4442241$	-56.44526	**120.8905**	**121.5802**	**129.4631**	**0.079365**	**0.9888**
TLGW	$\hat{\vartheta} = 13.49923566$ $\hat{\theta} = 0.02088991$ $\hat{\beta} = 28.27730026$ $\hat{\eta} = 0.20264199$	-66.73314	141.4663	142.1559	150.0388	0.19048	0.2032
TLGLo	$\hat{\vartheta} = 7.271080e-02$ $\hat{\theta} = 1.395350e+03$ $\hat{\beta} = 1.942276e+02$ $\hat{\eta} = 8.457492e-03$	-59.38903	126.7781	127.4677	135.3506	0.12698	0.69

Since the results from Tables 3.2, 3.3, and 3.4 show that the TL-log logistic model has the smallest values of AIC, AICC, BIC, KS statistic, and the largest p-value, so, this could be considered as the best model as compared to the TLGW and TLGLo models for Data Sets 1, 2, and 3.

3.1.2 Comparative study on the TL-exponential with the Lomax distribution and the Burr-XII distribution

In this subsection, we present the applications of the TL-exponential distribution to two real data sets. We also compare the results of the fits of the TL-exponential distribution having p.d.f.

$$f(x) = 2\vartheta\mu e^{-2\mu x}(1-e^{-2\mu x})^{\vartheta-1}, \qquad x>0,\, \mu>0,\, \vartheta>0,$$

with the Lomax distribution having p.d.f.

$$g(x) = \vartheta\mu(1+\mu x)^{-(\vartheta+1)}, \qquad x>0, \vartheta>0, \mu>0,$$

and with the Burr-XII distribution having p.d.f.

$$h(x) = \vartheta\mu x^{\mu-1}(1+x^{\mu})^{-(\vartheta+1)}, \qquad x>0, \vartheta>0, \mu>0.$$

We consider the fourth data set consists of 20 observations which describes the relaxation times (in minutes) of 20 patients who were receiving an analgesic as recorded by Gross and Clark (1975, p. 105). The data are given as:

$1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.$

The fifth data set consists of 31 observations, which is the strength data of glass of the aircraft window recorded by Fuller *et al.* (1994). The data are given as:

$18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.$

Some descriptive statistics for the Data Sets 4 and 5 are given below in Table 3.5.

Table 3.5: Descriptive statistics for Data Sets 4 and 5

Descriptive statistics	Data Set 4	Data Set 5
Minimum	1.100	18.83
Median	1.700	29.90
Mean	1.900	30.81
Maximum	4.100	45.38
Standard deviation	0.7041232	7.253381

The fitting of these distributions to both the data sets are shown in Figures 3.4 and 3.5 by using PP-plots, which indicate that the TL-exponential distribution can also be a good option to fit the data as well as other distributions.

We compute the ML-estimates for the unknown parameters of each distribution and compare the results based on AIC, AICC, BIC, KS statistics, and its p-values for both the data sets (see Table 3.6). The values of -log l for each distribution are also calculated for both the data sets.

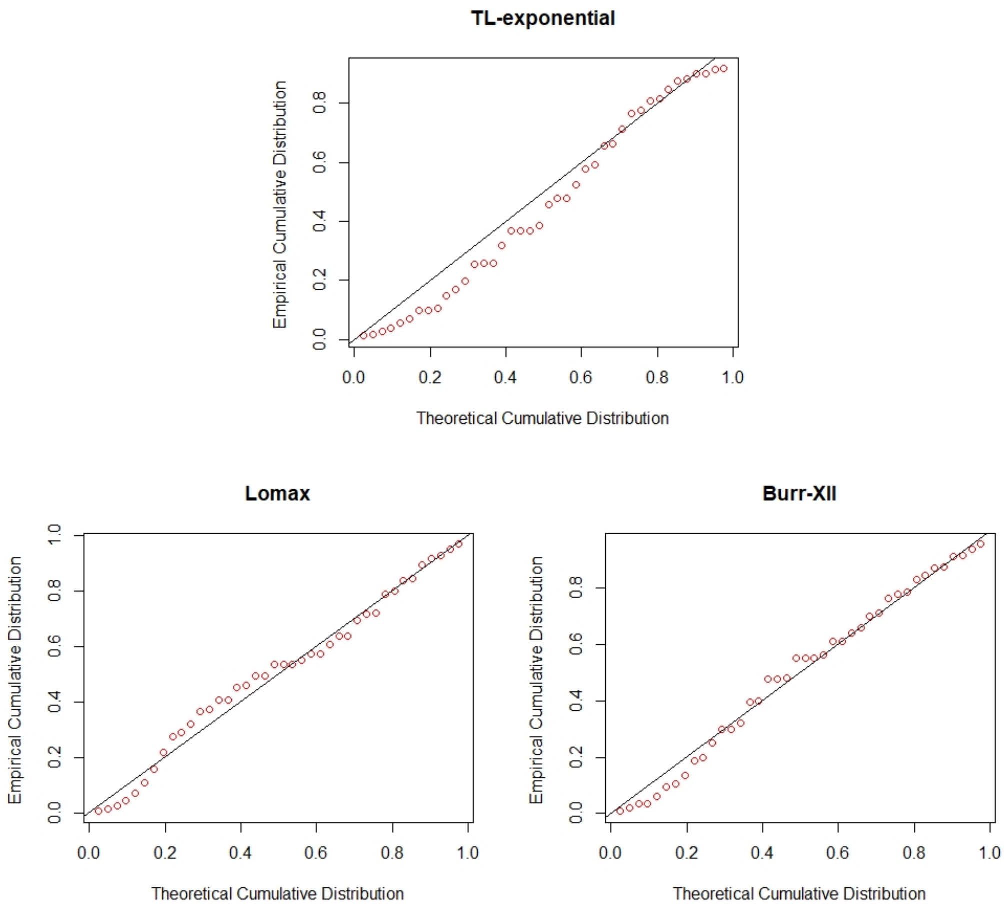


Figure 3.4: The PP-plots of relaxation times of patients data fitted with the TL-exponential, Lomax, and Burr-XII distributions

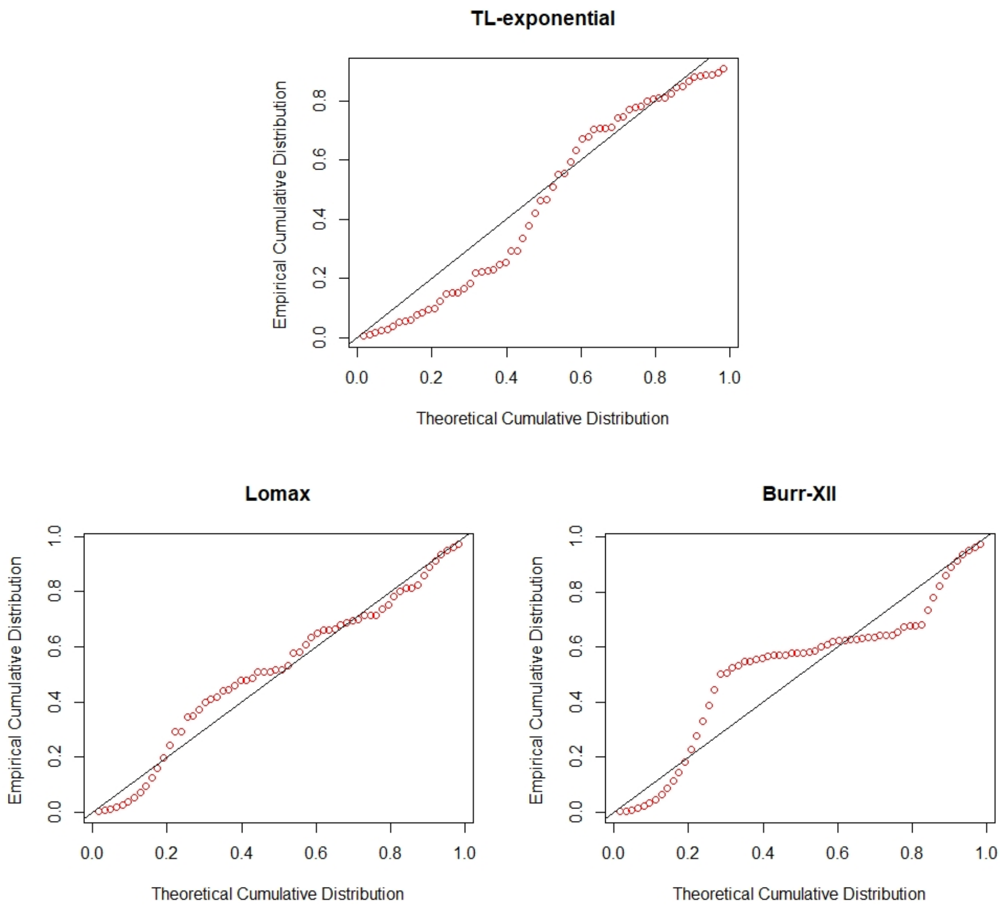


Figure 3.5: The PP-plots of the strength data of glass of the aircraft window fitted with the TL-exponential, Lomax, and Burr-XII distributions

Table 3.6: ML-estimates and log-likelihood functions for the TL-exponential, Lomax, and Burr-XII distributions, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for Data Sets 4 and 5

	Models	ML-estimates	-log l	AIC	AICC	BIC	KS statistic	p-value
Data Set 4	TL-exponential	$\hat{\vartheta} = 36.682456$ $\hat{\mu} = 1.117614$	-16.26061	**36.52122**	**37.2271**	**38.51268**	**0.15**	**0.978**
	Lomax	$\hat{\vartheta} = 1.374595e+03$ $\hat{\mu} = 3.821833e-04$	-32.84344	69.68688	70.39276	71.67834	0.4	0.08152
	Burr-XII	$\hat{\vartheta} = 0.01277514$ $\hat{\mu} = 132.81326644$	-21.20715	46.4143	47.12018	48.40576	0.25	0.5596
Data Set 5	TL-exponential	$\hat{\vartheta} = 93.75757060$ $\hat{\mu} = 0.08300174$	-104.1343	**212.2686**	**212.6972**	**215.1366**	**0.12903**	**0.9634**
	Lomax	$\hat{\vartheta} = 4.634453e+02$ $\hat{\mu} = 6.918740e-05$	-137.2984	278.5968	279.0254	281.4648	0.45161	0.003178
	Burr-XII	$\hat{\vartheta} = 0.05783577$ $\hat{\mu} = 5.08354219$	-174.3869	352.7738	353.2024	355.6418	0.54839	0.000125

From Table 3.6, it can be easily seen that the TL-exponential distribution has the smallest AIC, AICC, BIC, KS statistic, and the largest p-value as compared to the Lomax and Burr-XII distributions, and therefore, the TL-exponential distribution can be considered as the best distribution among the three.

Chapter 4

Stochastic comparisons of series and parallel systems with Topp-Leone generated family of distributions

4.1 Introduction

In statistics, applied probability, actuarial science, reliability theory, and many other related areas, order statistics play a prominent role (one may see, David and Nagaraja (2003) and Balakrishnan and Rao (1998a,b)). Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$ order statistics of the random variables $X_1, X_2, \ldots, X_n$. In reliability theory, the k^{th} order statistic is related to the lifetime of $(n-k+1)$-out-of-n system. Specifically, $X_{n:n}$ (when $k = n$) and $X_{1:n}$ (when $k = 1$) denote the lifetimes of parallel and series systems, respectively. The role of order statistics has been widely discussed in the literature and they found to be fruitful in comparisons of lifetimes of series and parallel systems consisting of heterogeneous components (see, for example, Pledger and Proschan (1971), Proschan and

Sethuraman (1976), Kochar and Korwar (1996), Dykstra *et al.* (1997), Khaledi and Kochar (2000), Khaledi and Kochar (2006), Balakrishnan (2007), Di Crescenzo and Pellerey (2011), and Nadarajah *et al.* (2017)).

Many researchers have worked upon the stochastic comparisons between the lifetimes of different systems where the random lifetimes of components follow various lifetime distributions, for example, Dykstra *et al.* (1997) and Khaledi and Kochar (2000) compared two parallel systems with heterogeneous exponential distributed components. Khaledi and Kochar (2006), Fang and Tang (2014), and Torrado and Kochar (2015) considered the case when components of the system follow heterogeneous Weibull distributions. Moreover, several results have been derived for the heterogeneous generalized exponential distributions, gamma, Fréchet, and Pareto type distributions (see, for examples, Balakrishnan and Zhao (2013), Balakrishnan *et al.* (2014), Gupta *et al.* (2015), and Patra *et al.* (2018)). For more results related to the stochastic comparisons of this type, we refer to Barmalzan *et al.* (2016), Fang *et al.* (2016), Fang and Wang (2017), and Nadarajah *et al.* (2017).

Apart from these types of comparisons, the families of distributions have also been considered. Some well known families of lifetime distributions are exponentiated Weibull (Mudholkar and Srivastava (1993)) and generalized exponential (Gupta and Kundu (1998)), etc. Recently, Kayal (2018) studied the stochastic relations among series and parallel systems with Kumaraswamy generalized family of distributions.

A vast literature is available on stochastic comparisons between order statistics from two heterogeneous distributions. Readers may refer to Chapter 1 of the thesis to review the literature on this topic.

It is important to mention that the notion of majorization is one of the useful tools to compare lifetimes of series and parallel systems (see, for example, Patra *et al.* (2018), Kayal (2018)), and references cited therein). The concept of majorization deals with the diversity of components of vectors in $\mathbb{R}^n$ (see Subsection 1.2.4 of Chapter 1).

This chapter aims to consider the stochastic comparisons of series and parallel systems with respect to the likelihood ratio order, the hazard rate order, and the usual stochastic order using vector majorization technique, where the components of the systems follow Topp-Leone generated (TL-G) family of distributions. Let X be a random variable following the TL-G family of distribution with parameters θ and ϑ. Recall from Section 2.1 of Chapter 2, the p.d.f. and the c.d.f. of X are given by

$$f(x;\vartheta,\theta,\zeta)=2\vartheta\theta g(x;\zeta)G(x;\zeta)^{\theta\vartheta-1}(1-G(x;\zeta)^{\theta})(2-G(x;\zeta)^{\theta})^{\vartheta-1},\quad x\geq 0,\ \theta,\vartheta>0, \tag{4.1.1}$$

and

$$F(x;\vartheta,\theta,\zeta)=(G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta},\quad x\geq 0,\ \theta,\vartheta>0, \tag{4.1.2}$$

respectively, where $G(x;\zeta)$ and $g(x;\zeta)$ represent the c.d.f. and the p.d.f. of the base-line distribution, respectively, and ζ contains the parameters which specify the base-line distribution. For ease of notation, we write $X\sim$TL-$G(\vartheta,\theta,\zeta)$ for a random variable X having p.d.f. written as (4.1.1).

In the following section, we provide some ordering results on taking into account the usual stochastic order, and the hazard rate order (the likelihood ratio order) for the comparisons of series (parallel) systems with TL-G distributed components.

4.2 Results based on stochastic comparisons

The main focus of this section is the comparisons of the lifetimes of series and parallel systems having independent and heterogeneous TL-G distributed components under two setups: (i) when there is heterogeneity in one parameter while another is fixed, and the base-line distributions are same; and (ii) when there is heterogeneity in one parameter while another is fixed, and the base-line distributions are different.

4.2.1 For the same base-line distribution

The following theorem demonstrates the hazard rate ordering of series systems when the parameter $\underline{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ varies.

Theorem 4.2.1. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the two pairs of independent random variables with $X_h \sim$ TL-$G(\vartheta_h, \theta, \zeta)$ and $Y_h \sim$ TL-$G(\vartheta_h^*, \theta, \zeta)$ for $h = 1, 2, \ldots, n$, respectively. Then, for a fixed $\theta > 0$ and for any fixed ζ, we have*

$$\underline{\vartheta}^* = (\vartheta_1^*, \vartheta_2^*, \ldots, \vartheta_n^*) \overset{m}{\preceq} (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \underline{\vartheta} \quad \Longrightarrow \quad X_{1:n} \leq_{hr} Y_{1:n}.$$

Proof. It is well defined for a series system that the sum of the hazard rate functions of each components is equal to the hazard rate function of the system. Therefore, for $x > 0$, the hazard rate function of $X_{1:n}$ is given by

$$\begin{aligned}
r_{X_{1:n}}(x) &= \sum_{h=1}^{n} \frac{f(x;\vartheta_h,\theta,\zeta)}{\bar{F}(x;\vartheta_h,\theta,\zeta)} \\
&= \sum_{h=1}^{n} \frac{2\vartheta_h\theta g(x;\zeta)G(x;\zeta)^{\theta\vartheta_h-1}(1-G(x;\zeta)^{\theta})(2-G(x;\zeta)^{\theta})^{\vartheta_h-1}}{1-G(x;\zeta)^{\theta\vartheta_h}(2-G(x;\zeta)^{\theta})^{\vartheta_h}} \\
&= 2\theta g(x;\zeta)G(x;\zeta)^{\theta-1}(1-G(x;\zeta)^{\theta}) \sum_{h=1}^{n} \frac{\vartheta_h G(x;\zeta)^{\theta(\vartheta_h-1)}(2-G(x;\zeta)^{\theta})^{\vartheta_h-1}}{1-G(x;\zeta)^{\theta\vartheta_h}(2-G(x;\zeta)^{\theta})^{\vartheta_h}} \\
&= 2\theta g(x;\zeta)G(x;\zeta)^{\theta-1}(1-G(x;\zeta)^{\theta}) \sum_{h=1}^{n} \frac{\vartheta_h (G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta_h-1}}{1-(G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta_h}} \\
&= 2\theta g(x;\zeta)G(x;\zeta)^{\theta-1}(1-G(x;\zeta)^{\theta}) \sum_{h=1}^{n} z(\vartheta_h),
\end{aligned}$$

where, for fixed $x > 0$, $\theta > 0$, and for any fixed ζ,

$$z(\vartheta) = \frac{\vartheta(G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta-1}}{1-(G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta}))^{\vartheta}}, \quad \vartheta > 0.$$

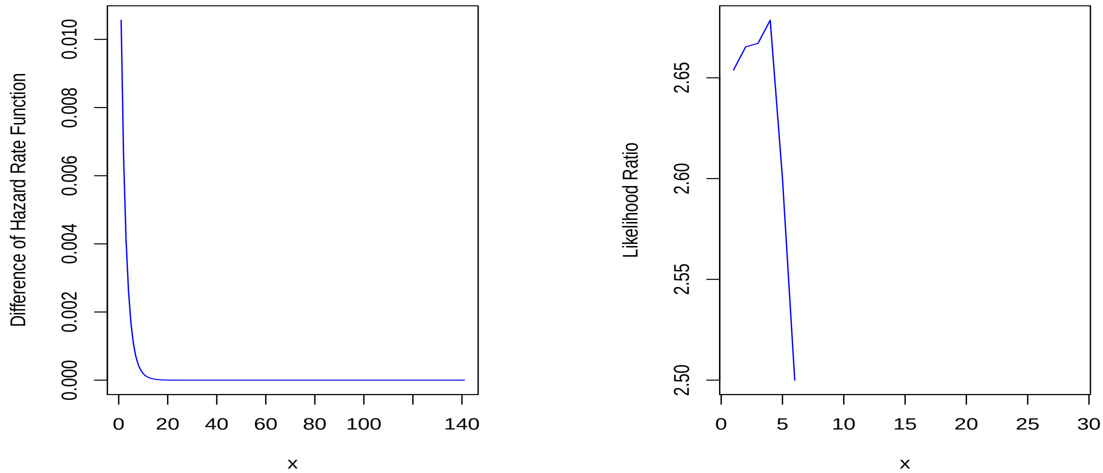
On taking $t = G(x;\zeta)^{\theta}(2-G(x;\zeta)^{\theta})$ and using Lemma 1.3.3, it follows that $z(\vartheta)$ is convex in ϑ. Now, on using Lemma 1.3.4, we conclude that $\sum_{h=1}^{n} z(\vartheta_h)$ is Schur-convex on $(0,\infty)^n$, which implies that if $\underline{\vartheta}^* \overset{\text{m}}{\preceq} \underline{\vartheta}$, then $r_{X_{1:n}}(x) \geq r_{Y_{1:n}}(x)$. Hence the theorem follows. □

To discuss the above theorem, we present the following example.

Example 4.2.1. Assume that $G(x;\zeta) = 1 - e^{-x}$, $x \geq 0$. Let X_1, X_2 and Y_1, Y_2 be the two pairs of independent random variables with $X_h \sim$ TL-$G(\vartheta_h, \theta, \zeta)$ and $Y_h \sim$ TL-$G(\vartheta_h^*, \theta, \zeta)$ for $h = 1, 2$, respectively. Assume $\vartheta_1 = 1, \vartheta_2 = 9$, $\vartheta_1^* = 4, \vartheta_2^* = 6$, and $\theta = 0.5$. Clearly, $(\vartheta_1^*, \vartheta_2^*) \overset{\text{m}}{\preceq} (\vartheta_1, \vartheta_2)$, and therefore, using Theorem 4.2.1, we have $X_{1:2} \leq_{\text{hr}} Y_{1:2}$. This can also be concluded from the Figure 4.1 (a) where we plot $r_{X_{1:2}}(x) - r_{Y_{1:2}}(x)$ which is nonnegative for $x \geq 0$. □

One may be interested to know whether Theorem 4.2.1 can be true for the likelihood ratio order. The following counterexample gives the answer in negation.

Counterexample 4.2.1. Continuing with the Example 4.2.1, if we plot $\frac{f_{Y_{1:2}}(x)}{f_{X_{1:2}}(x)}$, we get the Figure 4.1 (b), which shows that as x increases, the ratio first increases and then decreases. Hence the result in Theorem 4.2.1 cannot be strengthened to the likelihood ratio order. □



(a) Plot of $r_{X_{1:2}}(x) - r_{Y_{1:2}}(x)$

(b) Plot of $\frac{f_{Y_{1:2}}(x)}{f_{X_{1:2}}(x)}$

Figure 4.1: Plots of $r_{X_{1:2}}(x) - r_{Y_{1:2}}(x)$ and $\frac{f_{Y_{1:2}}(x)}{f_{X_{1:2}}(x)}$

In the next theorems, we provide the stochastic comparisons of parallel systems with respect to the usual stochastic order and the likelihood ratio order when the parameters $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\underline{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ vary, respectively.

Theorem 4.2.2. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the two pairs of independent random variables with $X_h \sim TL\text{-}G(\vartheta, \theta_h, \zeta)$ and $Y_h \sim TL\text{-}G(\vartheta, \theta_h^*, \zeta)$ for $h = 1, 2, \ldots, n$, re-*

spectively. Then, for a fixed $\vartheta > 0$ *and for any fixed* ζ*, we have*

$$\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \preceq_w (\theta_1^*, \theta_2^*, \ldots, \theta_n^*) = \underline{\theta}^* \quad \Longrightarrow \quad X_{n:n} \leq_{st} Y_{n:n}.$$

Proof. The distribution function of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} G(x;\zeta)^{\theta_i \vartheta}(2 - G(x;\zeta)^{\theta_i})^{\vartheta} = \varphi(\underline{\theta}), \text{ say.} \tag{4.2.1}$$

Clearly, $\varphi : (0,\infty)^n \to \mathbb{R}$ is a symmetric function on $(0,\infty)^n$. Partially differentiating $\varphi(\underline{\theta})$ with respect to θ_h, we get

$$\begin{aligned}
\frac{\partial \varphi(\underline{\theta})}{\partial \theta_h} &= \left[\prod_{\substack{i=1\\ i\neq h}}^{n} G(x;\zeta)^{\theta_i\vartheta}(2 - G(x;\zeta)^{\theta_i})^{\vartheta}\right]\Big(-\vartheta G(x;\zeta)^{\theta_h(\vartheta+1)}(2 - G(x;\zeta)^{\theta_h})^{\vartheta-1}\ln G(x;\zeta) \\
&\quad + (2 - G(x;\zeta)^{\theta_h})^{\vartheta} G(x;\zeta)^{\theta_h\vartheta}\ln G(x;\zeta)^{\vartheta}\Big) \\
&= \left[\prod_{\substack{i=1\\ i\neq h}}^{n} G(x;\zeta)^{\theta_i\vartheta}(2 - G(x;\zeta)^{\theta_i})^{\vartheta}\right]\vartheta G(x;\zeta)^{\theta_h\vartheta}(2 - G(x;\zeta)^{\theta_h})^{\vartheta}\ln G(x;\zeta) \\
&\quad \times \left(1 - \frac{G(x;\zeta)^{\theta_h}}{2 - G(x;\zeta)^{\theta_h}}\right) \\
&= 2\vartheta F_{X_{n:n}}(x)\ln G(x;\zeta)\left(\frac{1 - G(x;\zeta)^{\theta_h}}{2 - G(x;\zeta)^{\theta_h}}\right).
\end{aligned}$$

It is easy to see that $\frac{\partial \varphi(\underline{\theta})}{\partial \theta_h} < 0$. Therefore, $\varphi(\underline{\theta})$ is decreasing in θ_h. For $\theta_h \neq \theta_l$, we have

$$\begin{aligned}
&(\theta_h - \theta_l)\left(\frac{\partial \varphi(\underline{\theta})}{\partial \theta_h} - \frac{\partial \varphi(\underline{\theta})}{\partial \theta_l}\right) \\
&\quad = 2\vartheta(\theta_h - \theta_l)F_{X_{n:n}}(x)\ln G(x;\zeta)\left(\frac{1 - G(x;\zeta)^{\theta_h}}{2 - G(x;\zeta)^{\theta_h}} - \frac{1 - G(x;\zeta)^{\theta_l}}{2 - G(x;\zeta)^{\theta_l}}\right) \\
&\quad = 2\vartheta(\theta_h - \theta_l)F_{X_{n:n}}(x)\ln G(x;\zeta)\left(\frac{G(x;\zeta)^{\theta_l} - G(x;\zeta)^{\theta_h}}{\left(2 - G(x;\zeta)^{\theta_h}\right)\left(2 - G(x;\zeta)^{\theta_l}\right)}\right) \\
&\quad \leq 0.
\end{aligned}$$

On using Lemma 1.3.1, $\varphi(\underline{\theta})$ is Schur-concave in $\underline{\theta}$. Thus, $-\varphi(\underline{\theta})$ is increasing in θ_h and Schur-convex in $\underline{\theta}$. Now, using Lemma 1.3.2, it follows that $\underline{\theta} \preceq_w \underline{\theta}^*$ implies $-\varphi(\underline{\theta}) \leq -\varphi(\underline{\theta}^*)$, or equivalently, $\varphi(\underline{\theta}^*) \leq \varphi(\underline{\theta})$. Therefore, $F_{Y_{n:n}}(x) \leq F_{X_{n:n}}(x)$ and hence $X_{n:n} \leq_{\text{st}} Y_{n:n}$. □

The following corollary immediately follows from Theorem 4.2.2.

Corollary 4.2.1. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the two pairs of independent random variables with $X_h \sim$ TL-G$(\vartheta, \theta_h, \zeta)$ and $Y_h \sim$ TL-G$(\vartheta, \theta_h^*, \zeta)$, $h = 1, 2, \ldots, n$, respectively. Then, for a fixed $\vartheta > 0$ and for any fixed ζ, we have*

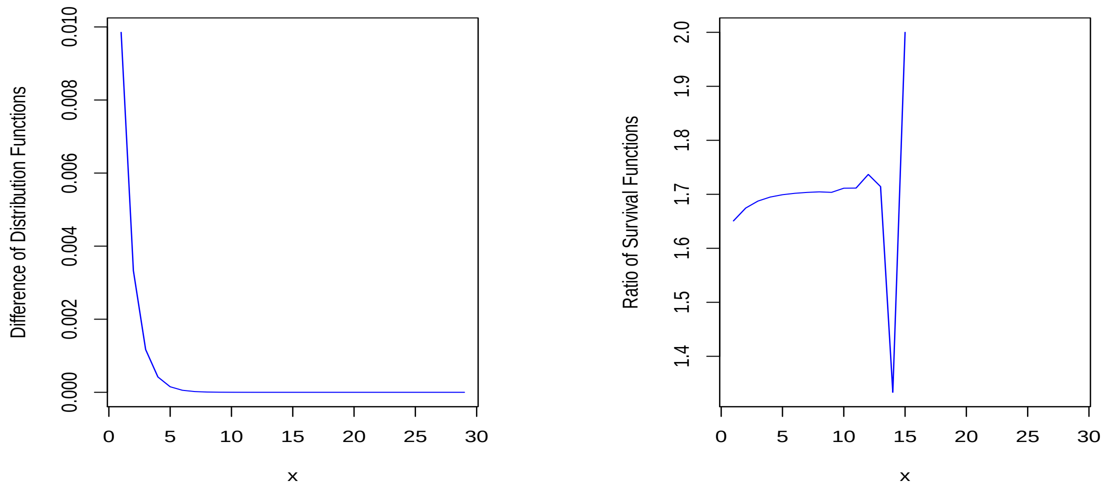
$$\underline{\theta} \overset{m}{\preceq} \underline{\theta}^* \implies X_{n:n} \leq_{st} Y_{n:n}.$$

The following example illustrates the result established in Theorem 4.2.2.

Example 4.2.2. Assume that $G(x;\zeta) = 1 - e^{-x}$, $x \geq 0$. Let X_1, X_2 and Y_1, Y_2 be the two pairs of independent random variables with $X_h \sim$ TL-$G(\vartheta, \theta_h, \zeta)$ and $Y_h \sim$ TL-$G(\vartheta, \theta_h^*, \zeta)$ for $h = 1, 2$, respectively. Assume $\theta_1 = 0.1$, $\theta_2 = 0.4$, $\theta_1^* = 0.2$, $\theta_2^* = 0.5$, and $\vartheta = 0.5$. Clearly, $(\theta_1, \theta_2) \preceq_{\text{w}} (\theta_1^*, \theta_2^*)$, and therefore, using Theorem 4.2.2, we have $X_{2:2} \leq_{\text{st}} Y_{2:2}$. This can also be seen from the Figure 4.2 (a) where we plot $F_{X_{2:2}}(x) - F_{Y_{2:2}}(x)$ which is nonnegative for $x \geq 0$. □

The following counterexample shows that the result in Theorem 4.2.2 may not hold for the hazard rate order.

Counterexample 4.2.2. Continuing with the Example 4.2.2, if we plot $\frac{\bar{F}_{Y_{2:2}}(x)}{\bar{F}_{X_{2:2}}(x)}$, we get the Figure 4.2 (b), which shows that the ratio is not monotone in x. Thus, the result in Theorem 4.2.2 cannot be strengthened to the hazard rate order. □



(a) Plot of $F_{X_{2:2}}(x) - F_{Y_{2:2}}(x)$

(b) Plot of $\frac{\bar{F}_{Y_{2:2}}(x)}{\bar{F}_{X_{2:2}}(x)}$

Figure 4.2: Plots of $F_{X_{2:2}}(x) - F_{Y_{2:2}}(x)$ and $\frac{\bar{F}_{Y_{2:2}}(x)}{\bar{F}_{X_{2:2}}(x)}$

Next result is a generalization of Theorem 4.2.2 to a wide range of scale parameters.

Theorem 4.2.3. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the two pairs of independent random variables with $X_h \sim TL\text{-}G(\vartheta, \theta_h, \zeta)$ and $Y_h \sim TL\text{-}G(\vartheta, \theta_h^*, \zeta)$, $h = 1, 2, \ldots, n$, respectively. For a fixed $\vartheta > 0$ and for any fixed ζ, if $(\theta_1, \theta_2, \ldots, \theta_n) \leq (\theta_1^*, \theta_2^*, \ldots, \theta_n^*)$, that is, $\theta_h \leq \theta_h^*$, $h = 1, 2, \ldots, n$, then $X_{n:n} \leq_{st} Y_{n:n}$.*

Proof. In the proof of Theorem 4.2.2, we have shown that $\varphi(\underline{\theta})$, given by (4.2.1), is decreasing in each θ_h, $h \in \{1, 2, \ldots, n\}$. Therefore, $\varphi(\underline{\theta}^*) \leq \varphi(\underline{\theta})$, or equivalently, $F_{Y_{n:n}}(x) \leq F_{X_{n:n}}(x)$. Hence the required result holds. □

The following theorem deals with the comparison when the parameter $\underline{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ varies.

Theorem 4.2.4. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the two pairs of independent random variables with $X_h \sim TL\text{-}G(\vartheta_h, \theta, \zeta)$ and $Y_h \sim TL\text{-}G(\vartheta_h^*, \theta, \zeta)$, $h = 1, 2, \ldots, n$, respectively. Then, for a fixed $\theta > 0$ and for any fixed ζ, $X_{n:n} \leq_{lr} Y_{n:n}$ if, and only if, $\sum_{h=1}^n \vartheta_h \leq \sum_{h=1}^n \vartheta_h^*$.*

Proof. It is easy to verify that the probability density functions of $X_{n:n}$ and $Y_{n:n}$ are given by

$$f_{X_{n:n}}(x) = 2\theta\left(\sum_{h=1}^n \vartheta_h\right) g(x;\zeta) G(x;\zeta)^{\theta \sum_{h=1}^n \vartheta_h - 1} (2 - G(x;\zeta)^\theta)^{\sum_{h=1}^n \vartheta_h} \left(\frac{1 - G(x;\zeta)^\theta}{2 - G(x;\zeta)^\theta}\right), \; x > 0,$$

and

$$f_{Y_{n:n}}(x) = 2\theta\left(\sum_{h=1}^n \vartheta_h^*\right) g(x;\zeta) G(x;\zeta)^{\theta \sum_{h=1}^n \vartheta_h^* - 1} (2 - G(x;\zeta)^\theta)^{\sum_{h=1}^n \vartheta_h^*} \left(\frac{1 - G(x;\zeta)^\theta}{2 - G(x;\zeta)^\theta}\right), \; x > 0,$$

respectively. Then, for $x > 0$,

$$\frac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)} = \left(\frac{\sum_{h=1}^n \vartheta_h^*}{\sum_{h=1}^n \vartheta_h}\right) (G(x;\zeta)^\theta (2 - G(x;\zeta)^\theta))^{\sum_{h=1}^n \vartheta_h^* - \sum_{h=1}^n \vartheta_h}.$$

Using the fact that $G(x;\zeta)^\theta$ is an increasing function of x and the observation that $y(2-y)$ is an increasing function of $y \in (0,1)$, we conclude that the ratio $\frac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)}$ is increasing in x if, and only if, $\sum_{h=1}^n \vartheta_h \leq \sum_{h=1}^n \vartheta_h^*$, which proves the desired result. □

4.2.2 For different base-line distributions

Till now we have derived the results when TL-G family of distributions have the same base-line distributions. Now, we present the results for the case when TL-G family of distributions have different base-line distributions. Let X_1^* and X_2^* be the two random variables having the cumulative distribution functions $G_1(\cdot)$ and $G_2(\cdot)$, respectively. Also, assume that $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ be the two pairs of independent random variables following the TL-G family of distributions with base-line distributions $G_1(\cdot)$ and $G_2(\cdot)$, respectively, and we denote $U_h \sim$ TL-$G(\vartheta_h, \theta, G_1)$ and $V_h \sim$ TL-$G(\vartheta_h^*, \theta, G_2)$ for $h = 1, 2, \ldots, n$. The reliability functions of $U_{1:n}$ and $V_{1:n}$ are respectively given by

$$\bar{F}_{U_{1:n}}(x) = \prod_{h=1}^{n} \left[1 - \left((G_1(x))^{\theta} \left(2 - (G_1(x))^{\theta} \right) \right)^{\vartheta_h} \right], \; x \geq 0$$

and

$$\bar{F}_{V_{1:n}}(x) = \prod_{h=1}^{n} \left[1 - \left((G_2(x))^{\theta} \left(2 - (G_2(x))^{\theta} \right) \right)^{\vartheta_h^*} \right], \; x \geq 0.$$

The following theorem provides the conditions under which $U_{1:n} \leq_{\text{st}} V_{1:n}$.

Theorem 4.2.5. *Let $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ be the two pairs of independent random variables with $U_h \sim$ TL-$G(\vartheta_h, \theta, G_1)$ and $V_h \sim$ TL-$G(\vartheta_h^*, \theta, G_2)$ for $h = 1, 2, \ldots, n$, respectively, and let $\underline{\vartheta}^* \overset{m}{\preceq} \underline{\vartheta}$. Then, for a fixed $\theta > 0$,*

$$X_1^* \leq_{st} X_2^* \implies U_{1:n} \leq_{st} V_{1:n}.$$

Proof. Let $Z_1, Z_2, \ldots, Z_n$ be the set of independent random variable with $Z_h \sim$ TL-$G(\vartheta_h^*, \theta, G_1)$ for $h = 1, 2, \ldots, n$. On using Theorem 4.2.1, we have $U_{1:n} \leq_{\text{hr}} Z_{1:n}$, which implies that

$U_{1:n} \leq_{\text{st}} Z_{1:n}$. Also, the reliability function of $Z_{1:n}$ is given by

$$\bar{F}_{Z_{1:n}}(x) = \prod_{h=1}^{n} \left[1 - \left((G_1(x))^{\theta} \left(2 - (G_1(x))^{\theta} \right) \right)^{\vartheta_h^*} \right], \; x \geq 0.$$

Since $X_1^* \leq_{\text{st}} X_2^*$ implies that $G_2(x) \leq G_1(x)$ for all $x \geq 0$, which further implies that $(G_2(x))^{\theta} \leq (G_1(x))^{\theta}$ for all $x \geq 0$. Now, using the observation that $y(2-y)$ is an increasing function of $y \in (0,1)$, we have

$$\prod_{h=1}^{n} \left[1 - \left((G_1(x))^{\theta} \left(2 - (G_1(x))^{\theta} \right) \right)^{\vartheta_h^*} \right] \leq \prod_{h=1}^{n} \left[1 - \left((G_2(x))^{\theta} \left(2 - (G_2(x))^{\theta} \right) \right)^{\vartheta_h^*} \right], \quad \forall x \geq 0, \quad (4.2.2)$$

i.e., $\bar{F}_{Z_{1:n}}(x) \leq \bar{F}_{V_{1:n}}(x)$ for all $x \geq 0$. Therefore, $Z_{1:n} \leq_{\text{st}} V_{1:n}$. Thus, we have $U_{1:n} \leq_{\text{st}} Z_{1:n} \leq_{\text{st}} V_{1:n}$. Hence the result follows. □

The following theorem provides the sufficient conditions for the comparison of parallel systems.

Theorem 4.2.6. *Let $W_1, W_2, \ldots, W_n$ and $W_1^*, W_2^*, \ldots, W_n^*$ be the two sets of independent random variables with $W_h \sim$ TL-$G(\vartheta, \theta_h, G_1)$ and $W_h^* \sim$ TL-$G(\vartheta, \theta_h^*, G_2)$ for $h = 1, 2, \ldots, n$, respectively. For a fixed $\vartheta > 0$, if*

(i) $\underline{\theta} \preceq_w \underline{\theta}^*$, then $X_1^* \leq_{st} X_2^*$ implies $W_{n:n} \leq_{st} W_{n:n}^*$;

(ii) $(\theta_1, \theta_2, \ldots, \theta_n) \leq (\theta_1^*, \theta_2^*, \ldots, \theta_n^*)$, i.e., if $\theta_h \leq \theta_h^*$, $h = 1, 2, \ldots, n$, then $X_1^* \leq_{st} X_2^*$ implies $W_{n:n} \leq_{st} W_{n:n}^*$.

Proof. Let $Z_1^*, Z_2^*, \ldots, Z_n^*$ be the set of independent random variable with $Z_h^* \sim$ TL-$G(\vartheta, \theta_h^*, G_1)$

for $h = 1, 2, \ldots, n$. On using the arguments similar to that used in the proof of Theorem 4.2.5, the part (i) and part (ii) follows from the Theorem 4.2.2 and Theorem 4.2.3, respectively. □

Chapter 5

Stochastic comparison results for the allocation of one active/standby redundancy in series systems

5.1 Introduction

Suppose that we have a series system with n components $C_1, C_2, \ldots, C_n$. Further suppose that we have a spare R. Let X_1, $X_2, \ldots, X_n$ and X be the lifetimes of components $C_1, C_2, \ldots, C_n$, and spare R, respectively. Assume that the spare R is assigned either to component C_1 or to component C_2. Let $W = \min\{X_3, \ldots, X_n\}$. Then, for the case of active redundancy, the lifetimes of two resulting systems are given by

$$S_1 = \wedge\{\vee(X_1, X), X_2, W\} \quad \text{and} \quad S_2 = \wedge\{X_1, \vee(X_2, X), W\}, \tag{5.1.1}$$

respectively, and the symbols $\wedge$ and $\vee$ denote the *minimum* and *maximum*, respectively. In the case of standby redundancy, the lifetimes of two resulting systems are given by

$$T_1 = \wedge\{X_1 + X, X_2, W\} \quad \text{and} \quad T_2 = \wedge\{X_1, X_2 + X, W\}, \tag{5.1.2}$$

respectively.

Several researchers have considered the above setups in the past few decades to establish several comparison results based on different stochastic orders. To do so they have assumed that the lifetimes $X_1, X_2, \ldots, X_n$ and X are statistically independent. For a bit of information, in the active redundancy case, Boland *et al.* (1992) and Singh and Misra (1994) established that if $X_1 \leq_{\text{st}} X_2$, then $S_2 \leq_{\text{st}} S_1$ and $S_2 \leq_{\text{sp}} S_1$, respectively. For two-component series systems ($n = 2$), Li and Hu (2008) proved that if $X_1 \leq_{\text{icv}} X_2$, then $S_2 \leq_{\text{icv}} S_1$. Also, they showed that if $X_1 \leq_{\text{icv}} X_2$ and if X and X_1 (or X_2) have convex survival functions, then $S_2 \leq_{\text{sp}} S_1$. Further, Zhao *et al.* (2012) proved that $S_2 \leq_{\text{lr}} S_1$ under the exponential framework. Moreover, You and Li (2014) extended the result of Zhao *et al.* (2012) from exponential distributions to PHR models. Recently, for the n-component series system, Zhao *et al.* (2016) extended the results of Zhao *et al.* (2012).

For the standby redundancy case, Boland *et al.* (1992) proved that if $X_1 \leq_{\text{hr}} X_2$, then $T_2 \leq_{\text{st}} T_1$. Singh and Misra (1994) showed that if $X_1 \leq_{\text{st}} X_2$, then $T_2 \leq_{\text{sp}} T_1$. Li and Hu (2008) showed that if $X_1 \leq_{\text{icv}} X_2$ and if $X_1, X_3, \ldots, X_n$ have convex survival functions, then $T_2 \leq_{\text{sp}} T_1$. Further, for two-component series systems, Li *et al.* (2011) strengthened the results of Singh and Misra (1994) by showing that $X_1 \leq_{\text{sp}} X_2$ if, and only if, $T_2 \leq_{\text{sp}}$

T_1. Zhao *et al.* (2012) improved the result of Boland *et al.* (1992) under the exponential framework. Recently, Zhao *et al.* (2016) strengthened the result of Zhao *et al.* (2012) for the n-component series system.

From the literature, one can see that the comparisons between S_1 and S_2, and between T_1 and T_2 have been made with respect to the different stochastic orders. These comparisons have been made under the assumption that the random lifetimes $X_1, X_2, \ldots, X_n$ and X are independent. One may be interested in the comparison results when these random variables are not independent. With this motivation, we deal with the problem of allocation of one active redundancy as well as one standby redundancy under the setups mentioned in (5.1.1) and (5.1.2). In this chapter, comparisons have been made between S_1 and S_2, and between T_1 and T_2 with respect to the residual stochastic precedence and the inactivity stochastic precedence orders. These orders have been recently introduced by Misra *et al.* (2020a,b). In comparison to the univariate stochastic orders which depend only on the marginal distributions, these newly defined orders have a special concern while comparing random variables as they take care of the dependence structure between the random variables.

Section 5.2 of the chapter presents the comparison results related to the allocation of one active spare in the n-component series systems. In Section 5.3, we provide the comparison results on the allocation of one standby spare in the n-component series systems. In Section 5.4, the comparison results between two parallel systems with respect to the inactivity stochastic precedence order have been derived.

Let the p.d.f., the c.d.f., and the survival functions of a random variable X be denoted by $f_X(\cdot)$, $F_X(\cdot)$, and $\bar{F}_X(\cdot)$, respectively. For a random variable X and an event E, we use $(X|E)$ to denote a random variable with the distribution same as the conditional distribution of X given E. The joint p.d.f. of a random vector (X,Y) is given by $f_{X,Y}(\cdot,\cdot)$.

5.2 Allocation of one active redundancy in the n-component series systems

This section is devoted to the results of the stochastic comparisons between S_1 and S_2 (described in Equation (5.1.1)).

The following theorem provides sufficient conditions under which $S_2 \leq_{\text{rsp}} S_1$.

Theorem 5.2.1. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. For each x, $w \geq 0$, let (X_1^*, X_2^*) be a random vector having the same joint distribution as the conditional distribution of $(X_1, X_2|X = x, W = w)$. If $X_1^* \leq_{jhr} X_2^*$, $\forall x, w \geq 0$, then $S_2 \leq_{rsp} S_1$.*

Proof. For $t \geq 0$, let

$$\begin{aligned}
\Delta_1(t) &= P(t < S_2 < S_1) - P(t < S_1 < S_2) \\
&= P(t < \wedge\{X_1, \vee(X_2, X), W\} < \wedge\{\vee(X_1, X), X_2, W\}) \\
&\quad - P(t < \wedge\{\vee(X_1, X), X_2, W\} < \wedge\{X_1, \vee(X_2, X), W\}) \\
&= P(t < X_1 < W < X_2 < X) - P(t < X_2 < W < X_1 < X)
\end{aligned}$$

$$
\begin{aligned}
&+P(t<X_1<X_2<X<W)-P(t<X_2<X_1<X<W)\\
&+P(t<X_1<W<X<X_2)-P(t<X_2<W<X<X_1)\\
&+P(t<X_1<X<X_2<W)-P(t<X_2<X<X_1<W)\\
&+P(t<X_1<X<W<X_2)-P(t<X_2<X<W<X_1)\\
&+P(t<X_1<X_2<W<X)-P(t<X_2<X_1<W<X) \qquad (5.2.1)
\end{aligned}
$$

$$
\begin{aligned}
=&\int_t^\infty\int_t^x [P(t<X_1<W<X_2<X|X=x,W=w)\\
&\qquad -P(t<X_2<W<X_1<X|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty [P(t<X_1<X_2<X<W|X=x,W=w)\\
&\qquad -P(t<X_2<X_1<X<W|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x [P(t<X_1<W<X<X_2|X=x,W=w)\\
&\qquad -P(t<X_2<W<X<X_1|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty [P(t<X_1<X<X_2<W|X=x,W=w)\\
&\qquad -P(t<X_2<X<X_1<W|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty [P(t<X_1<X<W<X_2|X=x,W=w)\\
&\qquad -P(t<X_2<X<W<X_1|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x [P(t<X_1<X_2<W<X|X=x,W=w)\\
&\qquad -P(t<X_2<X_1<W<X|X=x,W=w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
=&\int_t^\infty\int_t^x [P(t<X_1^*<w<X_2^*<x)-P(t<X_2^*<w<X_1^*<x)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty [P(t<X_1^*<X_2^*<x)-P(t<X_2^*<X_1^*<x)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x [P(t<X_1^*<w<x<X_2^*)-P(t<X_2^*<w<x<X_1^*)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^\infty\int_x^\infty [P(t<X_1^*<x<X_2^*<w)-P(t<X_2^*<x<X_1^*<w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty [P(t<X_1^*<x<w<X_2^*)-P(t<X_2^*<x<w<X_1^*)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x [P(t<X_1^*<X_2^*<w)-P(t<X_2^*<X_1^*<w)]f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x\\
=&\int_t^\infty\int_t^x\int_t^w\int_w^x (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty\int_t^x\int_{x_1}^x (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x\int_t^w\int_x^\infty (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty\int_t^x\int_x^w (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty\int_t^x\int_w^\infty (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_t^x\int_t^w\int_{x_1}^w (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
=&\int_t^\infty\int_t^x\int_t^w\left[\int_{x_1}^\infty (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))\,\mathrm{d}x_2\right]f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_t^\infty\int_x^\infty\int_t^x\left[\int_{x_1}^\infty (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))\,\mathrm{d}x_2\right]f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}w\,\mathrm{d}x\\
&\qquad\qquad\big(\text{on combining } 1^{\text{st}}, 3^{\text{rd}} \text{ and } 6^{\text{th}} \text{ terms, and } 2^{\text{nd}}, 4^{\text{th}} \text{ and } 5^{\text{th}} \text{ terms}\big).
\end{aligned}
$$

Since $X_1^* \le_{\text{jhr}} X_2^*$, we have $\int_{x_1}^\infty (f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))\,\mathrm{d}x_2 \ge 0$, $\forall x_1 \ge 0$, which implies that $\Delta_1(t) \ge 0$, $\forall t \ge 0$. Now, on using Definition 1.2.2 (iv), we conclude that $S_2 \le_{\text{rsp}} S_1$. □

Some simple consequences of the above theorem are given as following corollaries.

Corollary 5.2.1. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. Assume that (X_1, X_2) is independent of (X, W). If $X_1 \le_{jhr} X_2$, then $S_2 \le_{rsp} S_1$.*

Proof. Consider (X_1^*, X_2^*) as defined in Theorem 5.2.1. If (X_1, X_2) is independent of (X, W), then it is direct to see that $X_1^* \leq_{\text{jhr}} X_2^*$, $\forall x, w \geq 0$, is equivalent to $X_1 \leq_{\text{jhr}} X_2$. Hence the result follows from Theorem 5.2.1. □

Corollary 5.2.2. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. Assume that X_1, X_2, and (X, W) are independent. If $X_1 \leq_{hr} X_2$, then $S_2 \leq_{rsp} S_1$.*

Proof. If X_1 and X_2 are independent, then $X_1 \leq_{\text{jhr}} X_2$ is equivalent to $X_1 \leq_{\text{hr}} X_2$. Hence the result follows from Corollary 5.2.1. □

Corollary 5.2.3. *Let $X_1, X_2, \ldots, X_n$ and X be non-negative independent random variables. If $X_1 \leq_{hr} X_2$, then $S_2 \leq_{rsp} S_1$.*

Proof. The proof directly follows from Corollary 5.2.2. □

In the following theorem, we provide appropriate conditions under which S_2 is smaller than S_1 with respect to the inactivity stochastic precedence order.

Theorem 5.2.2. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. For each x, $w \geq 0$, let (X_1^*, X_2^*) be a random vector having the same joint distribution as the conditional distribution of $(X_1, X_2 | X = x, W = w)$. If $X_1^* \leq_{lr:j} X_2^*$, $\forall x$, $w \geq 0$, then $S_2 \leq_{isp} S_1$.*

Proof. For $t > 0$, let

$$\Delta_2(t) = P(S_2 < S_1 \leq t) - P(S_1 < S_2 \leq t)$$

$$
\begin{aligned}
&= P(\wedge\{X_1, \vee(X_2, X), W\} < \wedge\{\vee(X_1, X), X_2, W\} \leq t) \\
&\quad - P(\wedge\{\vee(X_1, X), X_2, W\} < \wedge\{X_1, \vee(X_2, X), W\} \leq t) \\
&= P(X_1 < W < X < X_2, W \leq t) - P(X_2 < W < X < X_1, W \leq t) \\
&\quad + P(X_1 < X < W < X_2, X \leq t) - P(X_2 < X < W < X_1, X \leq t) \\
&\quad + P(X_1 < X < X_2 < W, X \leq t) - P(X_2 < X < X_1 < W, X \leq t) \\
&\quad + P(X_1 < W < X_2 < X, W \leq t) - P(X_2 < W < X_1 < X, W \leq t) \\
&\quad + P(X_1 < X_2 < W < X, X_2 \leq t) - P(X_2 < X_1 < W < X, X_1 \leq t) \\
&\quad + P(X_1 < X_2 < X < W, X_2 \leq t) - P(X_2 < X_1 < X < W, X_1 \leq t) \qquad (5.2.2) \\
&= \int_0^\infty \int_0^{\wedge(x,t)} [P(X_1 < W < X < X_2 | X = x, W = w) \\
&\qquad\qquad - P(X_2 < W < X < X_1 | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1 < X < W < X_2 | X = x, W = w) \\
&\qquad\qquad - P(X_2 < X < W < X_1 | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1 < X < X_2 < W | X = x, W = w) \\
&\qquad\qquad - P(X_2 < X < X_1 < W | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x \\
&\quad + \int_0^\infty \int_0^{\wedge(x,t)} [P(X_1 < W < X_2 < X | X = x, W = w) \\
&\qquad\qquad - P(X_2 < W < X_1 < X | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x \\
&\quad + \int_0^\infty \int_0^x [P(X_1 < X_2 < W < X, X_2 \leq t | X = x, W = w) \\
&\qquad\qquad - P(X_2 < X_1 < W < X, X_1 \leq t | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x \\
&\quad + \int_0^\infty \int_x^\infty [P(X_1 < X_2 < X < W, X_2 \leq t | X = x, W = w) \\
&\qquad\qquad - P(X_2 < X_1 < X < W, X_1 \leq t | X = x, W = w)] f_{X,W}(x,w)\, \mathrm{d}w\, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^{\wedge(x,t)} [P(X_1^* < w < x < X_2^*) - P(X_2^* < w < x < X_1^*)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1^* < x < w < X_2^*) - P(X_2^* < x < w < X_1^*)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1^* < x < X_2^* < w) - P(X_2^* < x < X_1^* < w)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^\infty \int_0^{\wedge(x,t)} [P(X_1^* < w < X_2^* < x) - P(X_2^* < w < X_1^* < x)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^\infty \int_0^x [P(X_1^* < X_2^* < w, X_2^* \le t) - P(X_2^* < X_1^* < w, X_1^* \le t)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^\infty \int_x^\infty [P(X_1^* < X_2^* < x, X_2^* \le t) - P(X_2^* < X_1^* < x, X_1^* \le t)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&= \int_0^t \int_0^x [P(X_1^* < w < x < X_2^*) - P(X_2^* < w < x < X_1^*)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_t^\infty \int_0^t [P(X_1^* < w < x < X_2^*) - P(X_2^* < w < x < X_1^*)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1^* < x < w < X_2^*) - P(X_2^* < x < w < X_1^*)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty [P(X_1^* < x < X_2^* < w) - P(X_2^* < x < X_1^* < w)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_0^x [P(X_1^* < w < X_2^* < x) - P(X_2^* < w < X_1^* < x)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_t^\infty \int_0^t [P(X_1^* < w < X_2^* < x) - P(X_2^* < w < X_1^* < x)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^\infty \int_0^x [P(X_1^* < X_2^* < w, X_2^* \le t) - P(X_2^* < X_1^* < w, X_1^* \le t)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^\infty \int_x^\infty [P(X_1^* < X_2^* < x, X_2^* \le t) - P(X_2^* < X_1^* < x, X_1^* \le t)] f_{X,W}(x,w)\,\mathrm{d}w\,\mathrm{d}x \\
&= \int_0^t \int_0^x \int_x^\infty \int_0^w (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_t^\infty \int_0^t \int_x^\infty \int_0^w (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty \int_w^\infty \int_0^x (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_x^\infty \int_x^w \int_0^x (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_0^t \int_0^x \int_w^x \int_0^w (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x \\
&\quad + \int_t^\infty \int_0^t \int_w^x \int_0^w (f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1)) f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\infty\int_0^x\int_0^{\wedge(w,t)}\int_0^{x_2}(f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x\\
&+\int_0^\infty\int_x^\infty\int_0^{\wedge(x,t)}\int_0^{x_2}(f_{X_1^*,X_2^*}(x_1,x_2)-f_{X_1^*,X_2^*}(x_2,x_1))f_{X,W}(x,w)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}w\,\mathrm{d}x\\
&\geq 0,\ \forall t>0,
\end{aligned}
$$

where the last inequality derives from the assumption that $X_1^* \leq_{\text{lr:j}} X_2^*$ $\big($i.e., $f_{X_1^*,X_2^*}(x_1,x_2) - f_{X_1^*,X_2^*}(x_2,x_1) \geq 0$, for $x_1 \leq x_2\big)$. Now, on using the Definition 1.2.2 (v), we conclude that $S_2 \leq_{\text{isp}} S_1$. □

In following corollaries, we provide some simple consequences of the above theorem.

Corollary 5.2.4. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. Assume that (X_1, X_2) is independent of (X, W). If $X_1 \leq_{lr:j} X_2$, then $S_2 \leq_{isp} S_1$.*

Proof. Consider (X_1^*, X_2^*) as defined in Theorem 5.2.2. If (X_1, X_2) is independent of (X, W), then it is direct to see that $X_1^* \leq_{\text{lr:j}} X_2^*$, $\forall x, w \geq 0$, is equivalent to $X_1 \leq_{\text{lr:j}} X_2$. Hence the result follows from Theorem 5.2.2. □

Corollary 5.2.5. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. Assume that X_1, X_2, and (X, W) are independent. If $X_1 \leq_{lr} X_2$, then $S_2 \leq_{isp} S_1$.*

Proof. If X_1 and X_2 are independent, then $X_1 \leq_{\text{lr:j}} X_2$ is equivalent to $X_1 \leq_{\text{lr}} X_2$. Hence the result follows from Corollary 5.2.4. □

Corollary 5.2.6. *Let $X_1, X_2, \ldots, X_n$ and X be non-negative independent random variables. If $X_1 \leq_{lr} X_2$, then $S_2 \leq_{isp} S_1$.*

Proof. The proof directly follows from Corollary 5.2.5. □

5.3 Allocation of one standby redundancy in the n-component series systems

In this section, we discuss the stochastic relations between T_1 and T_2 (described in Equation (5.1.2)).

In the following theorem, we provide sufficient conditions under which $T_2 \leq_{\mathrm{rsp}} T_1$.

Theorem 5.3.1. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3, \ldots, X_n\}$. Assume that W is independent of (X_1, X_2). If $X_1 \leq_{jhr} X_2$, then $T_2 \leq_{rsp} T_1$.*

Proof. For $t \geq 0$, let

$$\begin{aligned}
\Delta_3(t) &= P(t < T_2 < T_1) - P(t < T_1 < T_2) \\
&= P(t < \wedge\{X_1, X_2 + X, \ldots, X_n\} < \wedge\{X_1 + X, X_2, \ldots, X_n\}) \\
&\quad - P(t < \wedge\{X_1 + X, X_2, \ldots, X_n\} < \wedge\{X_1, X_2 + X, \ldots, X_n\}) \\
&= P(t < X_1 < X_2, X_1 < W) - P(t < X_2 < X_1, X_2 < W) \\
&= \int_t^\infty \int_{x_1}^\infty P(t < X_1 < X_2, X_1 < W | X_1 = x_1, X_2 = x_2) f_{X_1,X_2}(x_1, x_2)\, \mathrm{d}x_2\, \mathrm{d}x_1
\end{aligned}$$

$$
\begin{aligned}
&-\int_t^\infty\int_{x_2}^\infty P(t<X_2<X_1,X_2<W|X_1=x_1,X_2=x_2)f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&=\int_t^\infty\int_{x_1}^\infty \bar{F}_W(x_1)f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1-\int_t^\infty\int_{x_2}^\infty \bar{F}_W(x_2)f_{X_1,X_2}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&=\int_t^\infty\left[\int_{x_1}^\infty (f_{X_1,X_2}(x_1,x_2)-f_{X_1,X_2}(x_2,x_1))\,\mathrm{d}x_2\right]\bar{F}_W(x_1)\,\mathrm{d}x_1\\
&\geq 0,\ \forall t\geq 0,
\end{aligned}
$$

where the last inequality derives from the assumption that $X_1 \leq_{\mathrm{jhr}} X_2$ $\left(\text{i.e., } \int_{x_1}^\infty f_{X_1,X_2}(x_1,x_2)-f_{X_1,X_2}(x_2,x_1)\,\mathrm{d}x_2 \geq 0,\ \forall x_1\geq 0\right)$. Now, on using Definition 1.2.2 (iv), we conclude that $T_2 \leq_{\mathrm{rsp}} T_1$. □

The following corollary directly follows from the fact that $X_1 \leq_{\mathrm{jhr}} X_2$ is equivalent to $X_1 \leq_{\mathrm{hr}} X_2$ when X_1 and X_2 are independent, and using Theorem 5.3.1.

Corollary 5.3.1. *Let $X_1, X_2, \ldots, X_n$ and X be jointly distributed non-negative random variables and let $W = \wedge\{X_3,\ldots,X_n\}$. Assume that X_1, X_2, and W are independent. If $X_1 \leq_{hr} X_2$, then $T_2 \leq_{rsp} T_1$.*

5.4 Stochastic comparisons of two parallel systems

Now, we compare two parallel systems made up of two components with respect to the inactivity stochastic precedence order. Let X_1, X_2, and X_3 be nonnegative random variables. Consider the two parallel systems, each consisting of two components, and having the lifetimes $\vee\{X_1,X_2\}$ and $\vee\{X_1,X_3\}$, respectively. We are interested in finding the conditions under which $\vee\{X_1,X_2\} \leq_{\mathrm{isp}} \vee\{X_1,X_3\}$. The following theorem provides such conditions.

Theorem 5.4.1. *Let X_1, X_2, and X_3 be jointly distributed nonnegative random variables.*

For each $x_1 \geq 0$*, let* (X_2^*, X_3^*) *be a random vector having the same joint distribution as the conditional distribution of* $(X_2, X_3 | X_1 = x_1)$*. If* $X_2^* \leq_{jrh} X_3^*$*,* $\forall x_1 \geq 0$*, then*

$$\vee\{X_1, X_2\} \leq_{isp} \vee\{X_1, X_3\}.$$

Proof. For $t > 0$, let

$$\begin{aligned}
\Delta_4(t) &= P(\vee\{X_1,X_3\} < \vee\{X_1,X_2\} \leq t) - P(\vee\{X_1,X_2\} < \vee\{X_1,X_3\} \leq t) \\
&= P(X_3 < X_2 \leq t, X_1 < X_2 \leq t) - P(X_2 < X_3 \leq t, X_1 < X_3 \leq t) \\
&= \int_0^t \big[P(X_3 < X_2 \leq t, x_1 < X_2 \leq t | X_1 = x_1) \\
&\qquad - P(X_2 < X_3 \leq t, x_1 < X_3 \leq t | X_1 = x_1)\big] f_{X_1}(x_1)\,\mathrm{d}x_1 \\
&= \int_0^t \big[P(X_3^* < X_2^* \leq t, x_1 < X_2^* \leq t) - P(X_2^* < X_3^* \leq t, x_1 < X_3^* \leq t)\big] f_{X_1}(x_1)\,\mathrm{d}x_1 \\
&= \int_0^t \int_{x_1}^t \left[\int_0^{x_2} \left(f_{X_2^*,X_3^*}(x_2,x_3) - f_{X_2^*,X_3^*}(x_3,x_2)\right) \mathrm{d}x_3\right] f_{X_1}(x_1)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&\leq 0, \; \forall t > 0,
\end{aligned}$$

where the last inequality follows from the assumption that $X_2^* \leq_{\mathrm{jrh}} X_3^*$ $\left(\text{i.e., } \int_0^{x_2} f_{X_2^*,X_3^*}(x_2,x_3) - f_{X_2^*,X_3^*}(x_3,x_2)\,\mathrm{d}x_3 \leq 0,\ \forall x_2 \geq 0\right)$. Now, on using Definition 1.2.2 (v), we conclude that $\vee\{X_1, X_2\} \leq_{\mathrm{isp}} \vee\{X_1, X_3\}$. □

The following corollaries are simple consequences of the above theorem.

Corollary 5.4.1. *Let* X_1*,* X_2*, and* X_3 *be jointly distributed nonnegative random variables such that* (X_2, X_3) *is independent of* X_1*. If* $X_2 \leq_{jrh} X_3$*, then* $\vee\{X_1, X_2\} \leq_{isp} \vee\{X_1, X_3\}$.

Proof. Consider (X_2^*, X_3^*) as defined in Theorem 5.4.1. If (X_2, X_3) is independent of X_1, then it is easy to see that $X_2^* \leq_{\text{jrh}} X_3^*$, for all $x_1 \geq 0$, is equivalent to $X_2 \leq_{\text{jrh}} X_3$. Hence the result follows from Theorem 5.4.1. □

Corollary 5.4.2. *Let X_1, X_2, and X_3 be nonnegative independent random variables. If $X_2 \leq_{rh} X_3$, then $\vee\{X_1, X_2\} \leq_{isp} \vee\{X_1, X_3\}$.*

Proof. If X_2 and X_3 are independent, then $X_2 \leq_{\text{jrh}} X_3$ is equivalent to $X_2 \leq_{\text{rh}} X_3$. Hence the result follows from Corollary 5.4.1. □

Chapter 6

Discussions and conclusions

In Chapter 1, we have provided an introduction about stochastic orders and its applications in different fields of probability and statistics. Also, we discussed some univariate and bivariate stochastic orders related to our study. We have presented an introduction about redundancy allocations and a lifetime distribution, namely, Topp-Leone generated family of distributions, and reviewed the literature related to these problems. This chapter contains some basic notation, definitions, and useful lemmas relevant to the thesis.

In past several years, various authors have worked upon many lifetime distributions and its generated families in reliability theory (for a literature review, see Chapter 1 of the thesis). With this motivation, we considered Topp-Leone generated family of distributions and provided some stochastic properties of this family of distributions in Chapter 2. In this chapter, we have presented that the reversed hazard rate function of the TL-G family of distributions is decreasing if the reversed hazard rate function of the base-line distribution is decreasing. It is popularly known that the decreasing reversed hazard rate implies increasing expected inactivity time. Hence, the expected inactivity time of the TL-G family of distributions is increasing. It is well known that the likelihood ratio order implies the hazard rate order as well as the usual stochastic order but the likelihood ratio order neither imply nor implied by the dispersive order as well as by the star-shaped order (see, Subsection 1.2.2). The comparison of two random variables from the TL-G family of distributions with respect to the likelihood ratio order has been made for $\theta_1 = \theta_2$ (> 0). So, one may be interested in comparing these random variables in terms of the likelihood ratio order when $\theta_1 \neq \theta_2$ (> 0). With the help of a counterexample, we have shown that the likelihood ratio order does not exist for this situation. Hence, one may be interested in the some weaker stochastic orders. So, we compared two random variables from this family of distributions with respect to the dispersive and star-shaped orders. Moreover, we provided the examples to support the existence of these orders.

Since the TL-G family of distributions generate other new distributions by choosing different $G(\cdot;\zeta)$, we have defined two members of the TL-G the family of distributions, named, the TL-log logistic, and the TLGLo distributions, using the log-logistic distribution and the Lomax distribution as the base-line distributions, respectively. Also, we have graphically provided different shapes of the density functions and the hazard rate functions for both the distributions with different values of parameters.

Furthermore, a particular case of this family of distributions, called TL-exponential distribution, has been considered by choosing exponential distribution as a base-line distribution and $\theta = 1$ in Equation 2.1.1. We studied some reliability characteristics such as the hazard rate function, the reversed hazard rate function, the mean residual life function, and the expected inactivity time (defined in Subsection 1.2.1)of this distribution. Also, we derived the expression to show the different shapes of the hazard rate function of the TL-exponential distribution. We showed that the hazard rate function of the TL-exponential distribution is strictly increasing (strictly decreasing, constant) in $x \in (0, \infty)$ for $\vartheta > 1$ ($\vartheta < 1, \vartheta = 1$) for any $\mu > 0$. Further, we derived the specific expressions of the mean residual life function and the expected inactivity time for TL-exponential distribution. The expression for Renyi entropy measure for this distribution has been obtained.

Then, in Chapter 3, we have provided the real data applications of TLGW, TL-log logistic, and TLGLo distributions using three real data sets to compare the fits of these distributions. To choose the best fitted distribution, we have used the statistics AIC, AICC, BIC, and KS statistic with its p-value. In general, smallest the values of AIC, AICC, BIC, KS statistic and largest p-value of the distribution present the best fitting of that distribution. Also, we have computed the ML-estimates for the unknown parameters and determined -log l for each distribution. First, we have plotted the PP-plots for each distribution from which we can say that all the models provide a suitable fit to these data sets. Then, we have used the criteria AIC, AICC, BIC, and KS statistic with its p-value to choose the best fitted distribution among the three distributions for these data sets. With the help of all three real data sets, we have observed that the model TL-log logistic has the smallest values of AIC, AICC, BIC, KS statistic, and the largest p-value (see, Tables 3.2, 3.3, and 3.4) , which implies that the TL-log logistic model is the best model as compared to the TLGW and TLGLo models for these data sets. Also, we have provided the applications of the TL-exponential distribution (defined in Chapter 2) with two real data sets and compared the fits of this distribution with two another distributions, named, the Lomax and Burr-XII distributions. We have shown that the TL-exponential distribution has the smallest values of AIC, AICC, BIC, KS statistic, and the largest p-value as compared to the Lomax and Burr-XII distributions (see, Table 3.6), and hence, the TL-exponential distribution can be considered as the best model among the Lomax and Burr-XII distributions.

The study presents the importance of the TL-G family of distributions, and we may further research on different aspects of the TL-log logistic and the TLGLo models which have not been considered in detail.

In Chapter 4, we have worked on the applications of the TL-G family of distributions in reliability theory. For this purpose, we have compared the lifetimes of two series and parallel systems with components having lifetimes from TL-G family of distributions, with respect to some stochastic orders. Using the vector majorization technique (see, Subsection 1.2.4), we have presented the comparison results with heterogeneity in one parameter while another is fixed. The results are presented in two different setups: (i) when the base-line distributions are fixed; and (ii) when the base-line distributions are different. First, we have considered two pairs of independent random variables having fixed base-line distributions and established the hazard rate order between the lifetimes of two series systems. We have also established the usual stochastic order and the likelihood ratio order between the lifetimes of two parallel systems.

As it is well-known that the likelihood ratio order is stronger than the hazard rate order and the hazard rate order is stronger than the usual stochastic order. So, one may be interested to know whether the hazard rate order between the comparisons of the lifetimes of two series systems (i.e., $X_{1:n} \leq_{\text{hr}} Y_{1:n}$) can be strengthened to the likelihood ratio order, and the usual stochastic order between the comparisons of the lifetimes of two parallel systems (i.e., $X_{n:n} \leq_{\text{st}} Y_{n:n}$) can be extended to the hazard rate order. We have given counterexamples to demonstrate that the answers to these questions are negative.

Next, we have considered two pairs of independent random variables having different base-line distributions. In this case, we have provided the sufficient conditions under which the usual stochastic order holds for the comparisons of the lifetimes of the series and parallel systems. Now, the question arises whether these results may be extended to the hazard rate order to the likelihood ratio order. We will try to extend these results. We have shown that $X_{1:2} \leq_{\text{hr}} Y_{1:2}$ and $X_{2:2} \leq_{\text{st}} Y_{2:2}$ through examples and figures.

There are numerous authors who have worked upon the stochastic comparisons of series and parallel systems with respect to different stochastic orders (for literature review, see, Chapter 1 of the thesis). In Chapter 5, we have considered recently defined orders, namely, the residual stochastic precedence order and the inactivity stochastic precedence order (see, Definition 1.2.2 (iv) and (v)). These orders have a special concern while comparing random variables as they take care of the dependence structure between the random variables. In this chapter, we have discussed about the applications of these orders in reliability theory. The active and standby redundancy allocations for n-component series systems have been discussed in terms of the residual stochastic precedence and the inactivity stochastic precedence orders. We have found the sufficient conditions under which S_2 is smaller than S_1 in terms of the residual stochastic precedence and the inactivity stochastic precedence orders for n-component series systems. Also, we derived the sufficient conditions under which T_2 is smaller than T_1 in terms of the residual stochastic precedence order for n-component series systems. We will try to obtain the conditions under which $T_2 \leq_{\text{isp}} T_1$ for n-component series systems. Also, we will try to find out conditions under which $T_2 \leq_{\text{rsp}} T_1$ and $T_2 \leq_{\text{isp}} T_1$ for n-component parallel systems. In this chapter, we have also compared two parallel systems having two components with respect to the inactivity stochastic precedence order.

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