ON A NEW STOCHASTIC ORDER, ITS PROPERTIES, AND APPLICATIONS

THESIS

SUBMITTED TO

BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY

(A CENTRAL UNIVERSITY)

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DECLARATION

I, **Vaishali Gupta**, Enrolment No. 1152/15, hereby declare that the work which is being presented in the thesis entitled **"On a new stochastic order, its properties, and applications"** for the award of the degree of Doctor of Philosophy and submitted in the Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow (U.P.), India, is an authentic record of my own work carried out during the period from July, 2015 to January, 2021 under the supervision of Dr. Amit Kumar Misra, Assistant Professor, Department of Applied Statistics, School for Physical Sciences, Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow (U.P.), India.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma of this or any other Institute. I also declare that the thesis is essentially free from all kinds of plagiarism.

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CERTIFICATE

This is to certify that the thesis titled **"On a new stochastic order, its properties, and applications"** submitted by **Vaishali Gupta**, is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other University.

The thesis submitted to Babasaheb Bhimrao Ambedkar University, Lucknow satisfies all the requirements as stipulated in the *Doctor of Philosophy (Ph.D.) regulations–1999 as amended in 2013* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

Supervisor

Date:

Head of the Department

Dedicated

To

My Beloved Parents

Mrs. Madhuri Gupta and Mr. Ashvani Kumar Gupta

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Chapter 1

Introduction

1.1 Review of the literature

Comparing two or more random variables in some probabilistic sense is one of the significant aims of statistics. These comparisons are made with the help of a few statistical measures associated with these random variables such as mean, dispersion, or variance of these random variables. However, in some applied problems, these measures do not provide fruitful information because once in a while they may have the same values or may not exist. For illustration, (a) let X and Y be two independent random variables following Cauchy distribution then it is well known that mean of this distribution does not exist. Thus, in this case, one can not get any fruitful information by use of means of random quantities; (b) let us consider two independent Weibull random variables X and Y having distribution functions $F_X(x) = 1 - e^{-\frac{x^2}{2}}$ and $F_Y(x) = 1 - e^{-\frac{\sqrt{2}x}{\sqrt{\pi}}}$, $x \geq 0$, respectively. Then, we can easily calculate that the means of these two random variables are equal, i.e., $E(X) = E(Y) = \frac{\sqrt{\pi}}{\sqrt{2}}$. Now, let these two random variables X and Y be the lifetimes of two systems/devices, then we can examine that X and Y have the same expected lifetimes if we are considering means of these two random lifetimes. However, if we consider the failure rates of these two devices at fixed time $x \geq 0$, then the failure rate (or hazard rate) of X and Y are given by $r_X(x) = \frac{f_X(x)}{\bar{F}_X(x)} = x$ and $r_Y(x) = \frac{f_Y(x)}{\bar{F}_Y(x)} = \frac{\sqrt{2}}{\sqrt{\pi}}$, respectively. Now, one can easily examine that $r_X(x) - r_Y(x) \geq 0$, for all $x \in [\frac{\sqrt{2}}{\sqrt{\pi}}, \infty)$ and $r_X(x) - r_Y(x) \leq 0$, for all $x \in [0, \frac{\sqrt{2}}{\sqrt{\pi}}]$. Thus, the failure rate of X is larger than the failure rate of Y, for all $x \in (\frac{\sqrt{2}}{\sqrt{\pi}}, \infty)$ and the failure rate of Y is larger than the failure rate of X, for all $x \in [0, \frac{\sqrt{2}}{\sqrt{\pi}})$.

Hence, the details achieved by the failure rates are more reliable than those given by the means. Because of the necessity described above, Stochastic ordering has been introduced for comparing random variables. A Stochastic order is a partial order that evaluates the concept that one random variable is "larger" than the other using underlying distribution functions in a more complex way. Mainly, they compare random variables with the use of different functions such as likelihood ratio, hazard rate function, residual life function, etc. These functions tell us about the "location" or "magnitude," "variability" or "dispersion" of a random phenomenon. Stochastic orders and their properties have given a wide range of meaningful applications over the last five decades in reliability theory and survival analysis to compare lifetimes of systems/devices/organisms, in economics to compare inequalities in-between income population, in actuarial science to compare portfolios, etc. (see, for example, Bergmann (1991), Singh and Misra (1994), Nanda and Shaked (2001), Arcones, Kvam, and Samaniego (2002), Müller and Stoyan (2002), Boland, Singh, and Cukic (2004), Balakrishnan and Zhao (2011), Li and Li (2013), Bouhadjar, Zeghdoudi, and Remita (2016), Zhang and Balakrishnan (2016), Mahdy (2019), Soltanifar (2019), Hazra and Misra (2020) and references textcited therein).

Several stochastic orders exist in the literature that compares "location" or "magnitude," "variability" or "dispersion" of random variables. Some of them that compare the location of random variables are "the usual stochastic order," "the hazard rate order," "the reversed hazard rate order," "the mean residual life order," and "the likelihood ratio order" and those who compare the variability of random variables are "the increasing convex order," "the increasing concave order," "the dispersive order." We come up with the formal definitions of these stochastic orders in Section 1.2 of the thesis. Also, there are various applications of these orders. More specifically, Belzunce, Martínez-Riquelme, and Mulero (2015) showed comparisons of random variables concerning these stochastic orders with the help of several examples. In which they used different distribution functions to obtain several results. These distributions are normal, Weibull, gamma, lognormal, and Pareto family of distributions. Moreover, they compared distorted distributions and coherent systems, comparing collective and individual risk and shock models.

These stochastic orders are introduced using random variables' marginal distribution and are known as univariate stochastic orders. Consequently, univariate stochastic orders do not care about the mutual relationship between the ordering of random variables. It is easy to avoid dependence structure between the ordering of random variables in various practical situations, but in a few cases, it becomes difficult not to consider dependence between them. To resolve this problem, Shanthikumar and Yao (1991) considered into account the mutual relationship between the ordering of random variables and defined bivariate version of "the usual stochastic order," "the hazard rate order," and "the likelihood ratio order" with the use of the joint distribution of random variables. They also provided bivariate stochastic ordering applications by considering problems in the closed queuing network, stochastic scheduling, and reliability.

Moreover, Boland, Singh, and Cukic (2004) introduced another univariate stochastic order known as the stochastic precedence order, which may depend on the joint distribution of random variables rather than the marginal distribution of random variables, and described its applications in sampling and testing. They showed that the stochastic precedence order provides a basis for exciting criteria under which simple random sampling and stratified sampling may be compared to become easier to choose between these two sampling methods in a given situation. Subsequently, several authors introduced some well-defined bivariate versions of univariate stochastic orders by considering the dependence between the ordering of random variables and named "the joint stochastic orders." They also demonstrated their properties, implications, and applications in various areas.

These areas include analysis of random utility models, portfolio selection, and allocation of redundant components (see, for references, Aly and Kochar (1993), Belzunce, Ortega, Pellerey, and Ruiz (2007), Belzunce, Martínez-Puertas, and Ruiz (2013), Li and You (2014), Li and You (2015), Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Pellerey and Spizzichino (2016), Balakrishnan, Barmalzan, and Kosari (2017), Li and Li (2019), Zhang and Cheung (2020)).

This thesis targets to add some new joint stochastic orders and illustrate their usefulness, properties, implications, and applications in the research field. To make the dissertation more presentable, we provide a few preliminary notation and definitions used throughout the thesis, in Section 1.2. Finally, we provide outline of the dissertation in Section 1.3

1.2 Notation and definitions

All over the thesis, we will adopt the following notation, definitions, and terminology.

1.2.1 Notation and terminology

(i) Let $\mathbb{R}$ indicate the set of all real numbers, i.e., $\mathbb{R} = (-\infty, \infty)$.

(ii) Let $\mathbb{R}_+$ indicate the nonnegative part of set of all real numbers, i.e., $\mathbb{R}_+ = [0, \infty)$.

(iii) Specific (non-bold) letters x, y, etc. will indicate the elements of $\mathbb{R}$ (or $\mathbb{R}_+$).

(iv) Let $\mathbb{R}^2$ indicate the set of all points (x, y) where x and y are real numbers i.e., $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

(v) To indicate maximum and minimum, we use the symbols '$\vee$' and '$\wedge$', respectively. Therefore, $x \vee y$ and $x \wedge y$ indicate, respectively, the maximum and minimum of x and

y, for x, $y \in \mathbb{R}$.

(vi) Considering a differentiable function $\phi(\cdot)$ defined on $\mathbb{R}$, $\phi'(\cdot)$ indicates its derivative.

(vii) The indicator function of a set $A \subseteq \mathbb{R}$ is indicated as

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

(viii) The terms “increasing” and “decreasing” are used in the non-strict sense.

(ix) Considering any random variable A and an event E, we use $[A|E]$ to indicate a random variable whose distribution is same as the conditional distribution of A given E.

(x) The support of all the random variables, if explicitly not mentioned, is assumed to be $\mathbb{R}_+$.

In this sequence, we identify a few essential characteristics of reliability for a nonnegative random variable. For monographs on this topic, see, Barlow and Proschan (1975), Block, Savits, and Singh (1998), Patra and Kundu (2020). Let X be a random variable with common support $\mathbb{R}_+$, we use $F_X(x) = P(X \leq x)$, $x \in \mathbb{R}$ and $\bar{F}_X(x) = 1 - F_X(x)$, $x \in \mathbb{R}$, to indicate, respectively, the distribution function and the survival function of X. For a random variable having Lebésgue probability density function (pdf), we use $f_X(x)$ to indicate the probability density function of X, $x \in \mathbb{R}$. Then,

(i) the **hazard (failure) rate function** of X, indicated by $r_X(\cdot)$, is represented by

$$\begin{aligned} r_X(x) &= \lim_{\varepsilon \to 0^+} \frac{P(X \leq x + \varepsilon | X > x)}{\varepsilon} \\ &= \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\mathrm{d}}{\mathrm{d}x}(-\ln \bar{F}_X(x)), \qquad x \in \mathbb{R}; \end{aligned}$$

(ii) the **reversed hazard (failure) rate function** of X, indicated by $\tilde{r}_X(\cdot)$, is represented by

$$\tilde{r}_X(x) = \lim_{\varepsilon \to 0^+} \frac{P(X > x - \varepsilon | X \leq x)}{\varepsilon}$$

$$= \frac{f_X(x)}{F_X(x)} = \frac{\mathrm{d}}{\mathrm{d}x}(\ln F_X(x)), \qquad x > 0;$$

(iii) the **residual (or remaining) life** of X at a fixed time $t > 0$ is given by the conditional random variable $X_t = [X - t | X > t]$;

(iv) the **mean residual life function** of X, indicated by $m_X(\cdot)$, is represented by

$$m_X(t) = E(X_t) = \frac{1}{\bar{F}_X(t)} \int_t^{\infty} \bar{F}_X(a)\,\mathrm{d}a, \qquad t > 0;$$

(v) the **reversed residual life (or inactivity time)** of X at a fixed time $t > 0$ is given by the conditional random variable $X_{(t)} = [t - X | X \leq t]$;

(vi) the **mean reversed residual life (inactivity time function)** of X, indicated by $\tilde{m}_X(\cdot)$, is represented by

$$\tilde{m}_X(t) = E(X_{(t)}) = \frac{1}{F_X(t)} \int_0^t F_X(a)\,\mathrm{d}a, \qquad t > 0.$$

For a discussion of the formal definitions of several univariate stochastic orders, let us consider a random vector (X,Y) with the marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, the survival functions $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$, the probability density functions $f_X(\cdot)$ and $f_Y(\cdot)$, the hazard rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, the reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$, the mean residual life functions $m_X(\cdot)$ and $m_Y(\cdot)$, and the mean inactivity life functions $\tilde{m}_X(\cdot)$ and $\tilde{m}_Y(\cdot)$, respectively.

Now, we consider some formal definitions of several univariate stochastic orders which are used further in the thesis. To understand the equivalent definitions, properties, interpretations, and applications of these orders, we refer the reader to Chapter 1 of Müller and Stoyan (2002), Chapter 1, 2, 4 of Shaked and Shanthikumar (2007), and Chapter 2 of Belzunce, Martínez-Riquelme, and Mulero (2015).

1.2.2 Univariate stochastic orders

X is said to be smaller than Y in the

(i) **likelihood ratio order** (indicated as $X \leq_{\text{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}_+$;

(ii) **usual stochastic order** (indicated as $X \leq_{\text{st}} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x)$, $\forall x \in \mathbb{R}$, or equivalently, if $F_Y(x) \leq F_X(x)$, $\forall x \in \mathbb{R}$;

(iii) **hazard rate order** (indicated as $X \leq_{\text{hr}} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $r_X(x) \geq r_Y(x), \forall x \in \mathbb{R}_+$;

(iv) **reversed hazard rate order** (indicated as $X \leq_{\text{rh}} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x), \forall x \in (0,\infty)$;

(v) **mean residual life order** (indicated as $X \leq_{\text{mrl}} Y$) if $\int_x^\infty \bar{F}_Y(t)\,\mathrm{d}t / \int_x^\infty \bar{F}_X(t)\,\mathrm{d}t$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $m_X(x) \leq m_Y(x), \forall x \in \mathbb{R}_+$;

(vi) **mean inactivity time order** (indicated as $X \leq_{\text{mit}} Y$) if $\int_0^x F_Y(t)\,\mathrm{d}t / \int_0^x F_X(t)\,\mathrm{d}t$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{m}_X(x) \geq \tilde{m}_Y(x), \forall x \in (0,\infty)$;

(vii) **stochastic precedence order** (indicated as $X \leq_{\text{sp}} Y$) if $P(Y < X) \leq P(X < Y)$;

(viii) **residual stochastic precedence order** (indicated as $X \leq_{\text{rsp}} Y$) if

$$\int_t^\infty (\bar{F}_Y(x) f_X(x) - \bar{F}_X(x) f_Y(x))\,\mathrm{d}x \geq 0, \text{ for all } t \geq 0,$$

or equivalently, if

$$\int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)(r_X(x) - r_Y(x))\,\mathrm{d}x \geq 0, \text{ for all } t \geq 0;$$

(ix) **inactivity stochastic precedence order** (indicated as $X \leq_{\text{isp}} Y$) if

$$\int_0^t (F_Y(x) f_X(x) - F_X(x) f_Y(x))\,\mathrm{d}x \leq 0, \text{ for all } t > 0,$$

or equivalently, if

$$\int_0^t F_X(x) F_Y(x)(\tilde{r}_X(x) - \tilde{r}_Y(x))\,\mathrm{d}x \leq 0, \text{ for all } t > 0.$$

It is important to mention here that the residual stochastic precedence order and the inactivity stochastic precedence order are those univariate stochastic orders which were described by Zardasht and Asadi (2010) and Abouelmagd, Hamed, Ebraheim, and Afify (2018), respectively. They used the terms residual probability (rpr) order in place of residual stochastic precedence order and inactivity probability (ipr) order in place of inactivity stochastic precedence order. Also, Abouelmagd, Hamed, Ebraheim, and Afify (2018) give reverse conditions for the inactivity stochastic precedence order, i.e., they are just opposite to each other because of the intuitive counter definition of the inactive probability order. For a detailed description, see the paragraph just after Example 3.1.1.

The succeeding implications among these orders are well defined (see, Shaked and Shanthikumar (2007), Misra, Gupta, and Chanchal (2020), Misra, Gupta, and Misra (2020)).

$$
\begin{array}{ccccccc}
 & & & & X \leq_{\text{rsp}} Y & & \\
 & & & & \Uparrow & & \\
 & & X \leq_{\text{lr}} Y & \Rightarrow & X \leq_{\text{hr}} Y & \Rightarrow & X \leq_{\text{mrl}} Y \\
 & & \Downarrow & & \Downarrow & & \\
X \leq_{\text{isp}} Y & \Leftarrow & X \leq_{\text{rh}} Y & \Rightarrow & X \leq_{\text{st}} Y & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & X \leq_{\text{mit}} Y & & X \leq_{\text{sp}} Y & &
\end{array}
$$

For a detailed description of the stochastic precedence order, we refer the readers to Singh and Misra (1994) and Boland, Singh, and Cukic (2004). The term “probabilistic relation” (indicated as $\leq_{\text{pr}}$) for the stochastic precedence order was used by Mi (1999) and Valdés, Arango, Zequeira, and Brito (2010). In general, there is no relationship between $X \leq_{\text{st}} Y$ and $X \leq_{\text{sp}} Y$ (see, Blyth (1972)). However, if X and Y are independent, then $X \leq_{\text{st}} Y$ implies $X \leq_{\text{sp}} Y$ (see, Boland, Singh, and Cukic (2004)). Also, under certain conditions, $X \leq_{\text{sp}} Y$ may imply $X \leq_{\text{rsp}} Y$ and $X \leq_{\text{isp}} Y$. For details, see, Theorem 2.2.2 and Theorem 3.3.2.

In the following subsection, we discuss the bivariate stochastic orders (or joint stochastic orders). These orders consider the dependence structure of random variables. For a detailed description of these orders, we refer the researcher to Shanthikumar and Yao (1991), Aly and Kochar (1993), Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Pellerey and Spizzichino (2016), Balakrishnan, Barmalzan, and Kosari (2017), Misra, Gupta, and Misra (2020), and Misra, Gupta, and Chanchal (2020). For a random vector (X,Y) having a Lebésgue pdf, we use $f_{X,Y}(\cdot,\cdot)$ to denote the joint density of $(X,Y) \in \mathbb{R}^2$.

1.2.3 Bivariate stochastic orders

Given a random vector (X,Y), X is said to be smaller than Y in the

(i) **joint likelihood ratio order** (indicated as $X \leq_{\text{lr:j}} Y$) if $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ (see, Shanthikumar and Yao (1991));

(ii) **joint hazard rate order** (indicated as $X \leq_{\text{hr:j}} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x)$ is decreasing in x, whenever $0 \leq x \leq y$ (see, Shanthikumar and Yao (1991));

(iii) **weak joint hazard rate order** (indicated as $X \leq_{\text{hr:wj}} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ (see, Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016));

(iv) **joint weak reversed hazard rate order** (indicated as $X \leq_{\text{rh:wj}} Y$) if $F_{X,Y}(x,y) - F_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$ (see, Balakrishnan, Barmalzan, and Kosari (2017));

(v) **joint hazard rate order** (indicated as $X \leq_{\text{jhr}} Y$) if $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y \geq 0$, $\forall x \geq 0$ (see, Misra, Gupta, and Misra (2020));

(vi) **joint reversed hazard rate order** (indicated as $X \leq_{\text{jrh}} Y$) if $\int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y \leq 0$, $\forall x \geq 0$ (see, Misra, Gupta, and Chanchal (2020));

(vii) **residual stochastic precedence order** (indicated as $X \leq_{\text{rsp}} Y$) if $P(X_t < Y_t) \geq P(Y_t < X_t)$ for all $t \geq 0$, or equivalently, if $\int_t^{\infty}\int_x^{\infty}(f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y\,\mathrm{d}x \geq 0,$ for all $t \geq 0$ (see, Misra, Gupta, and Misra (2020));

(viii) **inactivity stochastic precedence order** (indicated as $X \leq_{\text{isp}} Y$) if $P(X_{(t)} < Y_{(t)}) \leq P(Y_{(t)} < X_{(t)})$ for all $t > 0$, or equivalently, if $\int_0^t\int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right)\mathrm{d}y\,\mathrm{d}x \leq 0,$ for all $t > 0$ (see, Misra, Gupta, and Chanchal (2020));

(ix) **joint stochastic order** (indicated as $X \leq_{\text{st:j}} Y$) if $(X, -Y) \leq_{\text{st}} (Y, -X)$, i.e., $Eg(X,Y) \leq Eg(Y,X),\ \forall g \in \{g(x,y) : g(x,y) - g(y,x)$ is increasing in $x, \forall x\}$ (see, Shanthikumar and Yao (1991)).

Joint likelihood ratio order ($\leq_{\text{lr:j}}$), joint hazard rate order ($\leq_{\text{hr:j}}$), and joint stochastic order ($\leq_{\text{st:j}}$), mentioned above, are the first bivariate stochastic orders given by Shanthikumar and Yao (1991). It is appropriate to mention here if X and Y are independent, then the joint likelihood ratio order, the joint hazard rate (weak joint hazard rate) order, the joint weak reversed hazard rate (joint reversed hazard rate) order, and the joint stochastic order are equivalent to the likelihood ratio order, the hazard rate order, the reversed hazard rate order, and the usual stochastic order, respectively. Also, if X and Y are independent, then the residual stochastic precedence order and the inactivity stochastic precedence order are equivalent to their univariate versions.

The following implications among bivariate stochastic orders are well defined (see, Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Balakrishnan, Barmalzan, and Kosari (2017), Misra, Gupta, and Misra (2020), and Misra, Gupta, and Chanchal (2020)).

$$\begin{array}{ccccccc}
X \leq_{\text{lr:j}} Y & \Rightarrow & X \leq_{\text{hr:j}} Y & \Rightarrow & X \leq_{\text{hr:wj}} Y & \Rightarrow & X \leq_{\text{jhr}} Y \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
X \leq_{\text{rh:wj}} Y & & X \leq_{\text{st:j}} Y & \Rightarrow & X \leq_{\text{st}} Y & & X \leq_{\text{rsp}} Y \\
\Downarrow & & & & & & \\
X \leq_{\text{jrh}} Y & & & & & & \\
\Downarrow & & & & & & \\
X \leq_{\text{isp}} Y & & & & & &
\end{array}$$

Also,

$$\begin{array}{ccccc}
 & & X \leq_{\text{hr:wj}} Y & & \\
 & & \Downarrow & & \\
X \leq_{\text{rsp}} Y & \Rightarrow & X \leq_{\text{sp}} Y & \Leftarrow & X \leq_{\text{isp}} Y
\end{array}$$

For dependent random variables X and Y, Shanthikumar and Yao (1991) have proved that there is no implication between $X \leq_{\text{lr:j}} [\leq_{\text{hr:j}}]\, Y$ and $X \leq_{\text{lr}} [\leq_{\text{hr}}]\, Y$. Recently, Pellerey and Zalzadeh (2015) given conditions under which $X \leq_{\text{lr:j}} [\leq_{\text{hr:j}}]\, Y$ implies $X \leq_{\text{lr}} [\leq_{\text{hr}}]\, Y$. Authors also given conditions under which $X \leq_{\text{st}} [\leq_{\text{hr}}, \leq_{\text{lr}}]\, Y$ implies $X \leq_{\text{st:j}} [\leq_{\text{hr:j}}, \leq_{\text{lr:j}}]\, Y$. Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016) given few conditions using the joint density of X and Y, under which $X \leq_{\text{hr:j}} Y$ and $X \leq_{\text{hr:wj}} Y$ are equivalent. They also provided conditions under which $X \leq_{\text{hr}} [\leq_{\text{hr:wj}}]\, Y$ implies $X \leq_{\text{hr:wj}} [\leq_{\text{hr}}]\, Y$. Balakrishnan, Barmalzan, and Kosari (2017) given conditions under which $X \leq_{\text{rh}} [\leq_{\text{rh:wj}}]\, Y$ implies $X \leq_{\text{rh:wj}} [\leq_{\text{rh}}]\, Y$.

It is important to mention here that joint hazard rate order ($\leq_{\text{jhr}}$), joint reversed hazard rate order ($\leq_{\text{jrh}}$), residual stochastic precedence order ($\leq_{\text{rsp}}$), and inactivity stochastic precedence order ($\leq_{\text{isp}}$), mentioned above, are new stochastic orders defined and studied in the thesis.

1.3 Thesis outline

In this section, we summarize all the chapters of the thesis to understand our work better.

In Chapter 2, we introduce a new joint stochastic order based on the residual lifetimes of two nonnegative dependent random variables and the stochastic precedence order. We name this order as "residual stochastic precedence order." This order describes the bivariate version of the residual probability order, introduced by Zardasht and Asadi (2010). Section 2.2 is based on the relationships between the residual stochastic precedence order and the other existing well-known stochastic orders. We also develop some characterization results of this new stochastic order. In Section 2.3, we discuss some preservation properties of this order and provide some results. In addition, we study one of its possible applications in reliability theory, which is given in Section 2.4. For this, we compare the lifetimes of the two series systems, each consisting of two components, and having the lifetimes $\wedge\{X_1, X_2\}$ and $\wedge\{X_1, X_3\}$, respectively, with respect to the residual stochastic precedence order.

Chapter 3 is devoted to the extension of inactivity probability order, defined by Abouelmagd, Hamed, Ebraheim, and Afify (2018), to the case of nonindependent random variables. It is done by defining a new joint stochastic order based on the inactivity times of two nonnegative dependent random variables and the stochastic precedence order. We name it "inactivity stochastic precedence order." In Section 3.2, we recall some useful definitions and implications of univariate stochastic orders as well as bivariate stochastic orders used in this chapter. Section 3.3 aims to consider the relationships between this new stochastic order and the other existing well known stochastic orders. This section also discuss some characterization results of this order. In Section 3.4, we develop preservation properties of the new stochastic order and give the results. Section 3.5 is devoted to discussion this order using real data set as well as simulated data by use of few examples.

Motivated by the importance of joint stochastic orders, we consider two new joint stochastic orders named as "joint hazard rate order" and "joint reversed hazard rate order," which are defined by Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020), respectively, in Chapter 4. In Section 4.2, we examine the relationship of two new stochastic orders with the other existing well-defined stochastic orders. We also discuss a significant result with the help of two newly defined stochastic orders: the residual stochastic precedence order and the inactivity stochastic precedence order. Some implications results are also discussed with the help of examples.

In Chapter 5, we try to discuss applications of the residual stochastic precedence order and the inactivity stochastic precedence order, which are defined by Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020), respectively, on Covid-19 data. We consider the Covid-19 data over different countries globally and try to find out the COVID-19 effect on males and females by using these two stochastic orders. Section 5.2 is based upon the description of data which is obtained from the website `https://globalhealth5050.org/covid19/sex-disaggregated-data-tracker/` and this data is updated as on October 20, 2020. Also, it is used to perform the analysis and statistical methods. In Section 5.3, we calculate the descriptive statistics of the Covid-19 data and obtain some results by using R-software, and compare the results from same data which is till May, 2020. We also discuss the interpretations of these results.

Chapter 2

Residual Stochastic Precedence Order

2.1 Introduction

Let (X,Y) be a random vector with the Lebésgue probability density function $f_{X,Y}(x,y)$, distribution function $F_{X,Y}(x,y)$, and survival function $\bar{F}_{X,Y}(x,y) = P(X > x, Y > y)$, $x \in \mathbb{R}$, $y \in \mathbb{R}$. For convenience of notation, we assume that the distributional support of the random vector (X,Y) is $[0,\infty) \times [0,\infty)$, although most of the results derived in this chapter can be extended to situations where the support is different from $[0,\infty) \times [0,\infty)$.

In many situations, comparisons of the magnitudes of the random variables X and Y may be of interest and the goal may be to show that one random variable is larger than the other in some stochastic sense. A common approach to achieve this goal is the use of theory of stochastic orders that has been extensively studied in the literature. Stochastic orders are used to compare random variables in a more complex way when single measures, such as, mean, variance are not very informative. There are several stochastic orders defined in the literature. Some of them are the usual stochastic order, the hazard rate or-

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der, the reversed hazard rate order, the likelihood ratio order, the stochastic precedence order; see Müller and Stoyan (2002), Boland, Singh, and Cukic (2004), Shaked and Shanthikumar (2007), and Belzunce, Martínez-Riquelme, and Mulero (2015) for the details of various stochastic orders. These orders have applications in many disciplines such as reliability theory, survival analysis, economics, actuarial science, operations research, epidemiology, and management science (see for example, Singh and Misra (1994), Nanda and Shaked (2001), Arcones, Kvam, and Samaniego (2002), Müller and Stoyan (2002), Boland, Singh, and Cukic (2004), Misra and Misra (2011), Misra, Misra, and Dhariyal (2011), Balakrishnan and Zhao (2011), Li and Li (2013), and references cited therein). To make the chapter comprehensive, let us recall the definitions of the stochastic orders used in the chapter.

Definition 2.1.1. *Let (X,Y) be a random vector with marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, survival functions $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$, probability density functions $f_X(\cdot)$ and $f_Y(\cdot)$, hazard rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$, and mean residual life functions $m_X(\cdot)$ and $m_Y(\cdot)$. Then X is said to be smaller than Y in the*

(i) *likelihood ratio order (written as $X \leq_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}_+ \equiv [0,\infty)$;*

(ii) *usual stochastic order (written as $X \leq_{st} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x), \forall x \in \mathbb{R}_+$;*

(iii) *hazard rate order (written as $X \leq_{hr} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $r_X(x) \geq r_Y(x), \forall x \in \mathbb{R}_+$;*

(iv) *reversed hazard rate order (written as $X \leq_{rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x), \forall x \in (0,\infty)$;*

(v) *mean residual life order (written as $X \leq_{mrl} Y$) if $\int_x^\infty \bar{F}_Y(t)\,dt / \int_x^\infty \bar{F}_X(t)\,dt$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $m_X(x) \leq m_Y(x), \forall x \in \mathbb{R}_+$.*

Let us also recall the following relationships that are well known (see, Shaked and Shanthikumar (2007)).

$$
\begin{array}{ccccc}
X \leq_{\rm lr} Y & \Rightarrow & X \leq_{\rm hr} Y & \Rightarrow & X \leq_{\rm mrl} Y \\
\Downarrow & & \Downarrow & & \\
X \leq_{\rm rh} Y & \Rightarrow & X \leq_{\rm st} Y & &
\end{array}
$$

Shanthikumar and Yao (1991) pointed out that these univariate stochastic orders are basically defined with the help of marginal distributions of the random variables, and therefore, any dependence between the random variables is no way reflected in the ordering of random variables. For this purpose, many authors have introduced stochastic orders which take care of the dependence structure and are commonly known as *joint stochastic orders*. They also demonstrated applications of these orders in many areas, more specifically, analysis of random utility models, allocation of redundant components, and portfolio selection (see, for example, Shanthikumar and Yao (1991), Aly and Kochar (1993), Belzunce, Ortega, Pellerey, and Ruiz (2007), Belzunce, Martínez-Puertas, and Ruiz (2013), Li and You (2014), Li and You (2015), Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Pellerey and Spizzichino (2016), Balakrishnan, Barmalzan, and Kosari (2017)). We recall the following definitions of some of the joint stochastic orders.

Definition 2.1.2. *Given a random vector (X,Y), X is said to be smaller than Y in the*

(i) *joint likelihood ratio order (written as $X \leq_{lr:j} Y$) if $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$; (Shanthikumar and Yao (1991))*

(ii) *joint hazard rate order (written as $X \leq_{hr:j} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x)$ is decreasing in x, whenever $0 \leq x \leq y$; (Shanthikumar and Yao (1991))*

(iii) *weak joint hazard rate order (written as $X \leq_{hr:wj} Y$) if $\bar{F}_{X,Y}(x,y) - \bar{F}_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$; (Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016))*

(iv) *joint weak reversed hazard rate order (written as $X \leq_{rh:wj} Y$) if $F_{X,Y}(x,y) - F_{X,Y}(y,x) \geq 0$, whenever $0 \leq x \leq y$; (Balakrishnan, Barmalzan, and Kosari (2017))*

(v) *joint stochastic order (written as $X \leq_{st:j} Y$) if $(X,-Y) \leq_{st} (Y,-X)$, i.e., $Eg(X,Y) \leq$*

$Eg(Y,X)$, $\forall g \in \{g(x,y) : g(x,y) - g(y,x)$ is increasing in $x, \forall x\}$; (Shanthikumar and Yao (1991)).

Let us also recall the following relationships that are well known (see, Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016)).

$$\begin{array}{ccccc} X \leq_{\text{lr:j}} Y & \Rightarrow & X \leq_{\text{hr:j}} Y & \Rightarrow & X \leq_{\text{st:j}} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq_{\text{hr:wj}} Y & \Rightarrow & X \leq_{\text{st}} Y \end{array}$$

One can also show that $X \leq_{\text{lr:j}} Y \Rightarrow X \leq_{\text{rh:wj}} Y$. It is worth mentioning that, if X and Y are independent, then the joint likelihood ratio order, the joint hazard rate (weak joint hazard rate) order, the joint weak reversed hazard rate order, and the joint stochastic order are equivalent to the likelihood ratio order, the hazard rate order, the reversed hazard rate order, and the usual stochastic order, respectively. However, for dependent X and Y, we recall that, Shanthikumar and Yao (1991) have shown that there is no implication between $X \leq_{\text{lr:j}}$ $[\leq_{\text{hr:j}}]$ Y and $X \leq_{\text{lr}} [\leq_{\text{hr}}]$ Y. Recently, Pellerey and Zalzadeh (2015) provided conditions under which $X \leq_{\text{lr:j}} [\leq_{\text{hr:j}}]$ Y implies $X \leq_{\text{lr}} [\leq_{\text{hr}}]$ Y. Authors also provided conditions under which $X \leq_{\text{st}} [\leq_{\text{hr}}, \leq_{\text{lr}}]$ Y implies $X \leq_{\text{st:j}} [\leq_{\text{hr:j}}, \leq_{\text{lr:j}}]$ Y. Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016) provided some conditions, on the joint density of X and Y, under which $X \leq_{\text{hr:j}} Y$ and $X \leq_{\text{hr:wj}} Y$ are equivalent. They also provided conditions under which $X \leq_{\text{hr}} [\leq_{\text{hr:wj}}]$ Y implies $X \leq_{\text{hr:wj}} [\leq_{\text{hr}}]$ Y. Balakrishnan, Barmalzan, and Kosari (2017) provided conditions under which $X \leq_{\text{rh}} [\leq_{\text{rh:wj}}]$ Y implies $X \leq_{\text{rh:wj}} [\leq_{\text{rh}}]$ Y.

Note that in the above definitions of the joint stochastic orders, we have $P(X = Y) = 0$. For the case when $P(X = Y)$ may take a positive value, Boland, Singh, and Cukic (2004) gave the definition of the stochastic precedence order. According to their definition, X is said to be smaller than Y in the stochastic precedence order (written as $X \leq_{\text{sp}} Y$) if $P(X < Y) \geq P(Y < X)$. In the case of (X,Y) having a Lebésgue pdf, this definition reduces to that of Arcones, Kvam, and Samaniego (2002), i.e., X is said to be smaller than Y in the stochastic precedence order if $P(X \leq Y) \geq \frac{1}{2}$.

It is well known in the literature that $X \leq_{\text{hr:wj}} [\leq_{\text{st:j}}] Y$ implies $X \leq_{\text{sp}} Y$ (see Pellerey and Spizzichino (2016)). We recall that, in general, there is no relationship between $X \leq_{\text{st}} Y$ and $X \leq_{\text{sp}} Y$. However, if X and Y are independent, then $X \leq_{\text{st}} Y$ implies $X \leq_{\text{sp}} Y$ (see, Boland, Singh, and Cukic (2004)). It is worth mentioning that Mi (1999) and Valdés, Arango, Zequeira, and Brito (2010) use the term "probabilistic relation" (written as $\leq_{\text{pr}}$) instead of stochastic precedence order.

Now consider that X and Y denote the random lifetimes of two components/ systems. Then, instead of comparing X and Y, many times it may be of interest to compare the respective residual lifetimes, say $X_t = [X - t | X > t, Y > t]$ and $Y_t = [Y - t | X > t, Y > t]$, $t \geq 0$, due to the interest in comparing the lifetimes of the components/systems beyond time t (see, Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016)). Note that when X and Y are independent, $X_t = [X - t | X > t]$ and $Y_t = [Y - t | Y > t]$, $t \geq 0$, and their comparisons are of interest (see for example, Gupta (2013), Gupta, Misra, and Kumar (2015), and references cited therein). For more details about residual lifetime of a random variable see, Guess and Proschan (1988), Shaked and Shanthikumar (2007) (Chapters 1 and 2), Nanda, Bhattacharjee, and Balakrishnan (2010), and Cai and Zheng (2012).

Certain comparisons of X_t and Y_t are related with the well-defined stochastic orders. For example, when X and Y are independent, $X_t \leq_{\text{st}} Y_t$ for all $t \geq 0$ is equivalent to say that $X \leq_{\text{hr}} Y$. In fact, some joint stochastic orders between X and Y are also related with the certain comparisons between X_t and Y_t. Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016) have shown that:

i. $X \leq_{\text{hr:wj}} Y \Leftrightarrow X_t \leq_{\text{st}} Y_t$, $t \geq 0$;

ii. $X \leq_{\text{hr:wj}} Y \Leftrightarrow X_t \leq_{\text{hr:wj}} Y_t$, $t \geq 0$;

iii. $X \leq_{\text{hr:j}} Y \Leftrightarrow X_t \leq_{\text{hr:j}} Y_t$, $t \geq 0$;

iv. $X \leq_{\text{hr:j}} Y \Leftrightarrow X_t \leq_{\text{st:j}} Y_t$, $t \geq 0$.

Motivated from this, in this chapter, we define (and study) the "residual stochastic precedence order" by comparing X_t and Y_t with respect to the stochastic precedence order. The definition is as follows.

Definition 2.1.3. *X is said to be smaller than Y in the residual stochastic precedence order (written as $X \leq_{rsp} Y$) if $P(X_t < Y_t) \geq P(Y_t < X_t)$ for all $t \geq 0$.*

It is worth mentioning that the above definition does not require (X,Y) to have a Lebésgue pdf. When (X,Y) has a Lebésgue pdf, this reduces to the definition of the residual probability order (denoted by $\leq_{\mathrm{rpr}}$) defined by Zardasht and Asadi (2010). They studied this order for the case when X and Y are independent random variables and derived some interesting results. Kayid, Izadkhah, and Alshami (2014) further studied the residual probability order and derived some results on characterizations and implications of the order. They, too, derived the results under the assumption that random variables X and Y are independent.

It is clear from Definition 2.1.3 that $X \leq_{\mathrm{rsp}} Y$ is equivalent to say that $X_t \leq_{\mathrm{sp}} Y_t$ for all $t \geq 0$. Since at time $t = 0$, the residual lifetimes of X and Y are the same as X and Y, respectively, it follows, on taking $t = 0$, that $X \leq_{\mathrm{rsp}} Y$ implies $X \leq_{\mathrm{sp}} Y$. Clearly, the converse of this implication may not always be true (see, Example 2.2.2 in the sequel). However, we will show, in Theorem 2.2.2, that $X \leq_{\mathrm{sp}} Y$ may imply $X \leq_{\mathrm{rsp}} Y$ under certain conditions. In this chapter we discuss various properties of the residual stochastic precedence order.

The rest of the chapter is organized as follows. Section 2.2 is devoted to some results on characterizations and implications of the residual stochastic precedence order. In Section 2.3, preservation properties of this order under some reliability operations such as monotone transformation, convolution and mixture have been derived. Finally, one possible application of the residual stochastic precedence order in reliability theory is given in Section 2.4.

Throughout this chapter we use the following notation which we recall here. The terms "increasing" and "decreasing" are used in the non-strict sense. For any random variable W and an event E, we use $[W|E]$ to represent a random variable whose distribution is same as the conditional distribution of W given E. For any random variable W, we use $F_W(\cdot)$ and $\bar{F}_W(\cdot)$ to denote, respectively, the distribution function and the survival function of W. For a random variable having a Lebésgue pdf, we use $f_W(\cdot)$, $r_W(\cdot)$, $\tilde{r}_W(\cdot)$ and $m_W(\cdot)$ to denote the probability density function, the hazard rate function, the reversed hazard rate function and the mean residual life function of W, respectively. Similarly, for a random vector (W_1,W_2) having a Lebésgue pdf, we use $f_{W_1,W_2}(\cdot,\cdot)$ to denote the joint density of (W_1,W_2). The support of all the random variables, if explicitly not mentioned, is assumed to be $\mathbb{R}_+ \equiv [0,\infty)$. While referring to the previous works, we use $\leq_{\text{rsp}}$ for $\leq_{\text{rpr}}$.

2.2 Characterizations and implications

The aim of this section is to focus on the relationships between the residual stochastic precedence order and the other existing well-known stochastic orders. Some characterization results are also obtained. Throughout this section, we assume that (X,Y) has the Lebésgue pdf, $f_{X,Y}(\cdot,\cdot)$ in $\mathbb{R}^2$.

The following theorem describes the equivalent conditions for the residual stochastic precedence order. The proof is trivial and hence omitted.

Theorem 2.2.1. *The following statements are equivalent:*

(i) $X \leq_{rsp} Y$.

(ii) $P(t<X<Y)-P(t<Y<X)\geq 0$, *for all* $t\geq 0$.

(iii) $\int_t^\infty \int_x^\infty (f_{X,Y}(x,y)-f_{X,Y}(y,x))\,dy\,dx \geq 0$, *for all* $t\geq 0$.

The following corollary directly follows from the above theorem.

Corollary 2.2.1. *If* $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \geq 0$, $\forall x \geq 0$, *then* $X \leq_{rsp} Y$.

Corollary 2.2.1 motivates us to introduce a new joint hazard rate order based on the joint distribution of random variables. The definition is as follows.

Definition 2.2.1. *X is said to be smaller than Y in the joint hazard rate order (written as* $X \leq_{jhr} Y$*) if* $\int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \geq 0$, $\forall x \geq 0$.

Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016) have proved that if $X \leq_{\text{hr:wj}} Y$, then $\int_x^\infty f_{X,Y}(x,z)\,\mathrm{d}z \geq \int_x^\infty f_{X,Y}(z,x)\,\mathrm{d}z$ for all $x \geq 0$. Using this, it is direct to see that $X \leq_{\text{hr:wj}} Y$ implies $X \leq_{\text{jhr}} Y$. Also, using Definition 2.2.1 and Corollary 2.2.1, we get that $X \leq_{\text{jhr}} Y$ implies $X \leq_{\text{rsp}} Y$. Thus,

$$X \leq_{\text{hr:wj}} Y \Rightarrow X \leq_{\text{jhr}} Y \Rightarrow X \leq_{\text{rsp}} Y,$$

which in turn implies that $X_t \leq_{\text{st}} Y_t \Rightarrow X_t \leq_{\text{sp}} Y_t$, $\forall t \geq 0$. Note that the joint hazard rate order is weaker than the weak joint hazard rate order and is stronger than the residual stochastic precedence order. It is also important to note that $X \leq_{\text{jhr}} Y$ and $X \leq_{\text{hr}} Y$ are equivalent when X and Y are independent. We are working on the properties of the joint hazard rate order.

As we have seen that $X \leq_{\text{jhr}} Y$ implies $X \leq_{\text{rsp}} Y$, one may ask whether the converse also holds. The following counterexample gives the answer in negation.

Example 2.2.1. Let

$$f_{X,Y}(x,y) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha) x e^{-x^2} e^{-\frac{y}{2}}, & \text{if } x \geq 0, y \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ is an arbitrary fixed number. Then,

$$f_{X,Y}(y,x) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha) y e^{-y^2} e^{-\frac{x}{2}}, & \text{if } x \geq 0, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for $x \geq 0$, let

$$\begin{aligned}\Delta_1(x) &= \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y \\ &= (1-\alpha)\left(xe^{-x^2}\int_x^\infty e^{-\frac{y}{2}}\,\mathrm{d}y - e^{-\frac{x}{2}}\int_x^\infty ye^{-y^2}\,\mathrm{d}y\right) \\ &= \frac{1}{2}(1-\alpha)(4x-1)e^{-(x^2+\frac{x}{2})}.\end{aligned}$$

One can easily see that $\Delta_1(x) > 0$ for $x > \frac{1}{4}$ and $\Delta_1(x) < 0$ for $x < \frac{1}{4}$. Thus, from Definition 2.2.1, we conclude that neither $X \leq_{\mathrm{jhr}} Y$ nor $Y \leq_{\mathrm{jhr}} X$. Now, for $t \geq 0$, let

$$\begin{aligned}\Delta_2(t) &= \int_t^\infty \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_t^\infty \frac{1}{2}(1-\alpha)(4x-1)e^{-(x^2+\frac{x}{2})}\,\mathrm{d}x. \qquad (2.2.1)\end{aligned}$$

Our aim is to show that $\Delta_2(t) \geq 0$, $\forall t \geq 0$. It is easy to see that $\Delta_2'(t) = -\frac{1}{2}(1-\alpha)(4t-1)e^{-\frac{1}{2}(2t^2+t)}$, $t \geq 0$. Clearly, $\Delta_2'(t) > 0$ for $t < \frac{1}{4}$ and $\Delta_2'(t) < 0$ for $t > \frac{1}{4}$, which implies that $\Delta_2(t)$ is increasing for $t < \frac{1}{4}$ and is decreasing for $t > \frac{1}{4}$. Also, $\lim_{t\to\infty} \Delta_2(t) = 0$. Then, it is sufficient to show that $\Delta_2(0) \geq 0$. We have

$$\begin{aligned}\Delta_2(0) &= (1-\alpha)\int_0^\infty \frac{1}{2}(4x-1)e^{-\frac{1}{2}(2x^2+x)}\,\mathrm{d}x \\ &= (1-\alpha)\int_0^\infty \frac{1}{2}\left[(4x+1)e^{-\frac{1}{2}(2x^2+x)} - 2e^{-\frac{1}{2}(2x^2+x)}\right]\mathrm{d}x \\ &= (1-\alpha)\left[\int_0^\infty \frac{1}{2}(4x+1)e^{-\frac{1}{2}(2x^2+x)}\,\mathrm{d}x - \int_0^\infty e^{\frac{1}{16}-\left(x+\frac{1}{4}\right)^2}\,\mathrm{d}x\right] \\ &= (1-\alpha)\left[1 - e^{\frac{1}{16}}\int_{\sqrt{2}/4}^\infty \frac{1}{\sqrt{2}}e^{-\frac{1}{2}v^2}\,\mathrm{d}v\right] \\ &= (1-\alpha)\left[1-\sqrt{\pi}e^{\frac{1}{16}}\left(1-\Phi(\sqrt{2}/4)\right)\right] \quad (\text{where } \Phi(\cdot) \text{ is the cdf of } N(0,1)) \\ &= 0.3147(1-\alpha) \\ &> 0.\end{aligned}$$

Therefore, $\Delta_2(t) \geq 0$, $\forall t \geq 0$, and on using Theorem 2.2.1, we have that $X \leq_{\mathrm{rsp}} Y$. Thus, we conclude that $X \leq_{\mathrm{rsp}} Y$ does not imply $X \leq_{\mathrm{jhr}} Y$. □

As an immediate consequence of Theorem 2.2.1, we have the following corollary which is given in Zardasht and Asadi (2010) as a remark.

Corollary 2.2.2. *Let X and Y be independent random variables. Then the following statements are equivalent:*

(i) $X \leq_{rsp} Y$.

(ii) $\int_t^\infty [\bar{F}_Y(x) f_X(x) - \bar{F}_X(x) f_Y(x)]\, dx \geq 0$, *for all* $t \geq 0$.

(iii) $\int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)\,[r_X(x) - r_Y(x)]\, dx \geq 0$, *for all* $t \geq 0$.

Zardasht and Asadi (2010) have proved (in Theorem 6) that if X and Y are independent random variables with $X \leq_{\text{hr}} Y$, then $X \leq_{\text{rsp}} Y$; which directly follows from part (iii) of the above corollary. Zardasht and Asadi (2010) have also shown, with the help of counterexample, that the converse of this result does not hold. We provide one more example to show the same.

Example 2.2.2. Let X and Y be two independent Weibull random variables with $\bar{F}_X(x) = e^{-x^2}$ and $\bar{F}_Y(x) = e^{-\frac{x}{2}}$, $x \geq 0$, respectively. Then, $r_X(x) = 2x$ and $r_Y(x) = \frac{1}{2}$, $x > 0$. It is easy to see that $r_X(x) - r_Y(x)$ takes positive as well as negative values for $x \in (0, \infty)$, i.e., neither $X \leq_{\text{hr}} Y$ nor $Y \leq_{\text{hr}} X$. Now, for $t \geq 0$, let

$$
\begin{aligned}
\Delta_3(t) &= \int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)\,[r_X(x) - r_Y(x)]\,\mathrm{d}x \\
&= \int_t^\infty e^{-x^2} e^{-\frac{x}{2}} \left(2x - \frac{1}{2}\right) \mathrm{d}x \\
&= \int_t^\infty \frac{1}{2}(4x-1)e^{-\frac{1}{2}(2x^2+x)}\,\mathrm{d}x \\
&= \frac{\Delta_2(t)}{1-\alpha} \qquad \text{(using Eq. (2.2.1))} \\
&\geq 0, \forall t \geq 0 \qquad \text{(as shown in Example 2.2.1)}
\end{aligned}
$$

Now, from Corollary 2.2.2, we have $X \leq_{\text{rsp}} Y$. Thus, we conclude that $X \leq_{\text{rsp}} Y$ does not imply that $X \leq_{\text{hr}} Y$. □

As we know that $X \leq_{\text{hr}} [\leq_{\text{rh}}]\, Y$ implies $X \leq_{\text{st}} Y$, one may be interested in the relationship between the usual stochastic order and the residual stochastic precedence order. Moreover, one may also be interested in the relationship between the reversed hazard rate order and the residual stochastic precedence order. The following example shows, for independent random variables, that the reversed hazard rate order does not imply the residual stochastic precedence order, and hence, the usual stochastic order also does not imply the residual stochastic precedence order. Since, for independent random variables, the usual stochastic order implies the stochastic precedence order, this example also shows that the stochastic precedence order does not imply the residual stochastic precedence order.

Example 2.2.3. Let X and Y be two independent random variables with $F_X(x) = 1 - \bar{F}_X(x) = 1 - e^{-x}$ and $F_Y(x) = 1 - \bar{F}_Y(x) = 1 - (1 + xe^{-x})e^{-x}$, $x \geq 0$, respectively. Then, for $x > 0$,

$$\frac{F_Y(x)}{F_X(x)} = 1 - \frac{xe^{-2x}}{1 - e^{-x}} = 1 - \psi_1(x),$$

where

$$\psi_1(x) = \frac{xe^{-2x}}{1 - e^{-x}} = \frac{1}{e^x} \cdot \frac{x}{e^x - 1},\ x > 0.$$

We will show that $\psi_1(x)$ is decreasing in $x \in (0, \infty)$. Since it is a product of two positive functions and one of them $\left(\text{i.e., } \frac{1}{e^x}\right)$ is decreasing in $x \in (0, \infty)$, it is sufficient to show that $\frac{x}{e^x - 1}$ is decreasing in $x \in (0, \infty)$. Let

$$\psi_2(x) = \frac{x}{e^x - 1},\ x > 0.$$

On taking derivative with respect to x, we get

$$(e^x - 1)^2 \psi_2'(x) = e^x - 1 - xe^x = \psi_3(x), \text{ say}.$$

Then,

$$\psi_3'(x) = e^x - xe^x - e^x = -xe^x < 0, \quad \forall x > 0,$$

which implies that $\psi_3(x)$ is decreasing in $x \in (0, \infty)$. Therefore $\psi_3(x) < \lim_{x \to 0^+} \psi_3(x) = 0$, $\forall x \in (0, \infty)$, which further implies that $\psi_2'(x) < 0$, $\forall x \in (0, \infty)$. Therefore, $\psi_2(x)$ is decreasing in $(0, \infty)$ and hence $\psi_1(x)$ is decreasing in $x \in (0, \infty)$. Thus, $\frac{F_Y(x)}{F_X(x)} = 1 - \psi_1(x)$ is increasing in $x \in (0, \infty)$, which implies that $X \leq_{\text{rh}} Y$.

It is easy to verify that $r_X(x) = 1$ and $r_Y(x) = \frac{1+2xe^{-x}-e^{-x}}{1+xe^{-x}}$, $x > 0$, respectively. Now, for $t \geq 0$, let

$$\begin{aligned}
\Delta_4(t) &= \int_t^{\infty} \bar{F}_X(x)\bar{F}_Y(x)\left[r_X(x) - r_Y(x)\right] \mathrm{d}x \\
&= \int_t^{\infty} e^{-x} \cdot (1+xe^{-x})e^{-x} \cdot \frac{-xe^{-x}+e^{-x}}{1+xe^{-x}} \mathrm{d}x \\
&= \int_t^{\infty} e^{-3x}(1-x)\,\mathrm{d}x \\
&= \frac{1}{3}e^{-3t}\left(\frac{2}{3} - t\right).
\end{aligned}$$

Clearly, $\Delta_4(t) \geq 0$, if $0 \leq t \leq \frac{2}{3}$ and $\Delta_4(t) < 0$, if $t > \frac{2}{3}$. Therefore, on using Corollary 2.2.2, we get that $X \nleq_{\mathrm{rsp}} Y$. Thus $X \leq_{\mathrm{rh}} Y$ does not imply that $X \leq_{\mathrm{rsp}} Y$, and hence, $X \leq_{\mathrm{st}} Y$ also does not imply that $X \leq_{\mathrm{rsp}} Y$. Since, for independent random variables, the usual stochastic order implies the stochastic precedence order, we conclude that $X \leq_{\mathrm{sp}} Y$ also does not imply that $X \leq_{\mathrm{rsp}} Y$. □

Now, with the help of the following counterexample, we show that, if X and Y are dependent random variables, then the usual stochastic order and the residual stochastic precedence order may exist simultaneously, but the mean residual life order (and hence the hazard rate order) may not exist.

Example 2.2.4. Let

$$f_{X,Y}(x,y) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha)\lambda_1\lambda_2 e^{-\lambda_1 x}e^{-\lambda_2 y}, & \text{if } x \geq 0, y \geq 0, \lambda_1 > \lambda_2 > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ is an arbitrary fixed number. Then,

$$f_{X,Y}(y,x) = \begin{cases} \alpha e^{-(x+y)} + (1-\alpha)\lambda_1\lambda_2 e^{-\lambda_1 y}e^{-\lambda_2 x}, & \text{if } x \geq 0, y \geq 0, \lambda_1 > \lambda_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, For $x \geq 0$, let

$$\begin{aligned}\Delta_5(x) &= \int_x^\infty \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) \mathrm{d}y \\ &= (1-\alpha)\lambda_1\lambda_2 \left(e^{-\lambda_1 x}\int_x^\infty e^{-\lambda_2 y}\,\mathrm{d}y - e^{-\lambda_2 x}\int_x^\infty e^{-\lambda_1 y}\,\mathrm{d}y\right) \\ &= (1-\alpha)\,e^{-(\lambda_1+\lambda_2)x}(\lambda_1-\lambda_2) \\ &> 0, \forall x \geq 0.\end{aligned}$$

This implies that $X \leq_{\text{jhr}} Y$, which further implies that $X \leq_{\text{rsp}} Y$.

One can easily verify that the marginal probability density functions and the survival functions of X and Y are given by

$$f_X(x) = \alpha e^{-x} + (1-\alpha)\lambda_1 e^{-\lambda_1 x},\ x \geq 0,\ \lambda_1 > 0,$$

$$f_Y(x) = \alpha e^{-x} + (1-\alpha)\lambda_2 e^{-\lambda_2 x},\ x \geq 0,\ \lambda_2 > 0,$$

and

$$\bar{F}_X(x) = \alpha e^{-x} + (1-\alpha)e^{-\lambda_1 x},\ x \geq 0,$$

$$\bar{F}_Y(x) = \alpha e^{-x} + (1-\alpha)e^{-\lambda_2 x},\ x \geq 0.$$

Clearly, X and Y are not independent random variables. Since $\lambda_1 > \lambda_2$, it follows that $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \geq 0$, i.e., $X \leq_{\text{st}} Y$.

Now we show that the mean residual life order (and hence the hazard rate order) between X and Y does not exist. It is easily verifiable that

$$m_X(x) = \frac{\int_x^\infty \bar{F}_X(t)\,\mathrm{d}t}{\bar{F}_X(x)} = \frac{\alpha e^{-x} + \dfrac{(1-\alpha)}{\lambda_1}e^{-\lambda_1 x}}{\alpha e^{-x} + (1-\alpha)e^{-\lambda_1 x}},\ x > 0,$$

and

$$m_Y(x) = \frac{\int_x^\infty \bar{F}_Y(t)\,\mathrm{d}t}{\bar{F}_Y(x)} = \frac{\alpha e^{-x} + \dfrac{(1-\alpha)}{\lambda_2}e^{-\lambda_2 x}}{\alpha e^{-x} + (1-\alpha)e^{-\lambda_2 x}},\ x > 0,$$

respectively.

Now, for $\lambda_1 = 2, \lambda_2 = 1.5,$ and $\alpha = 0.5$, we have

$$m_X(x) = \frac{0.5e^{-x} + 0.25e^{-2x}}{0.5\,(e^{-x} + e^{-2x})},$$

and

$$m_Y(x) = \frac{0.5e^{-x} + 0.33e^{-1.5x}}{0.5\,(e^{-x} + e^{-1.5x})},$$

respectively. Then, it is easy to calculate that

$$m_X(1.5) = 0.9088,\ m_Y(1.5) = 0.8909,\ m_X(0.5) = 0.811 \text{ and } m_Y(0.5) = 0.851,$$

which implies that $m_X(1.5) > m_Y(1.5)$ and $m_X(0.5) < m_Y(0.5)$. Thus, neither $m_X(x) \geq m_Y(x)$ nor $m_X(x) \leq m_Y(x)$ for all $x > 0$, which implies that neither $X \leq_{\text{mrl}} Y$ nor $Y \leq_{\text{mrl}} X$. Hence, $X \nleq_{\text{hr}} Y$. □

Remark 2.2.1. Zardasht and Asadi (2010), in Remark 4, have mentioned that if the random variables X and Y are independent and if $F_Y(\cdot)$ is concave, then $X \leq_{\text{rsp}} Y$ implies $X \leq_{\text{mrl}} Y$. It can be easily verified that in Example 2.2.4, $F_Y(x) = 1 - \alpha e^{-x} - (1-\alpha)e^{-\lambda_2 x}$, $x \geq 0$, is concave and $X \leq_{\text{rsp}} Y$ but $X \nleq_{\text{mrl}} Y$, which contradicts the remark of Zardasht and Asadi (2010). This is due to the dependence between X and Y. As a consequence, Theorem 7 of Zardasht and Asadi (2010), which says that $X \leq_{\text{rsp}} Y \Leftrightarrow F_Y(X) \leq_{\text{mrl}} F_Y(Y)$, does not hold when X and Y are not independent.

Recall that we have mentioned in the introduction that $X \leq_{\text{rsp}} Y$ implies $X \leq_{\text{sp}} Y$ but $X \leq_{\text{sp}} Y$ does not imply $X \leq_{\text{rsp}} Y$ in general. One may be curious to know the conditions under which the stochastic precedence order implies the residual stochastic precedence. The following theorem provides such a condition.

Theorem 2.2.2. *Let $X \leq_{sp} Y$. If $\frac{\int_x^\infty f_{X,Y}(x,y)\,dy}{\int_x^\infty f_{X,Y}(y,x)\,dy}$ is increasing in $x > 0$, then $X \leq_{rsp} Y$.*

Proof. Let

$$\Delta_5(t) = \int_t^\infty \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y\,\mathrm{d}x,\ t \geq 0.$$

To prove that $X \leq_{\rm rsp} Y$, using Theorem 2.2.1, we have to show that $\Delta_5(t) \geq 0$, $\forall t \geq 0$, which is equivalent to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x} \geq 1,\ \forall t \geq 0.$$

Since $X \leq_{\rm sp} Y$ implies that $\frac{P(Y>X)}{P(X>Y)} \geq 1$, it is sufficient to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x} \geq \frac{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t \geq 0,$$

which is equivalent to show that

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x} \geq \frac{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t \geq 0.$$

Let Z_1 and Z_2 be two random variables having probability density functions

$$f_{Z_1}(x) = \frac{\int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ x > 0$$

and

$$f_{Z_2}(x) = \frac{\int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x},\ x > 0,$$

respectively. Then,

$$\frac{f_{Z_2}(x)}{f_{Z_1}(x)} = \frac{\int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y}{\int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y} \times \frac{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}.$$

Since $\frac{\int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y}{\int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y}$ is increasing in $x > 0$, it follows that $\frac{f_{Z_2}(x)}{f_{Z_1}(x)}$ is increasing in $x > 0$, and therefore, $Z_1 \leq_{\rm lr} Z_2$, and hence, $Z_1 \leq_{\rm st} Z_2$. Therefore, $\bar{F}_{Z_2}(t) \geq \bar{F}_{Z_1}(t)$, $\forall t \geq 0$, i.e.,

$$\frac{\int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x} \geq \frac{\int_t^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_x^\infty f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t \geq 0.$$

Hence, the theorem follows. □

The following corollary is an immediate consequence of Theorem 2.2.2.

Corollary 2.2.3. *Let X and Y be independent random variables with $X \leq_{sp} Y$. If $\frac{r_X(x)}{r_Y(x)}$ is increasing in $x > 0$, then $X \leq_{rsp} Y$.*

Remark 2.2.2. It is worth mentioning that the assumption that $\frac{r_X(x)}{r_Y(x)}$ is increasing in $x > 0$, or equivalently, that $\frac{r_Y(x)}{r_X(x)}$ is decreasing in $x > 0$, is related to the notion of the relative aging introduced by Kalashnikov and Rachev (1986) (see, for more details, Hazra and Nanda (2016), Misra and Francis (2018), Misra, Francis, and Naqvi (2017), Remark 2.3 of Misra, Misra, and Dhariyal (2011), and Sengupta and Deshpande (1994)).

2.3 Preservation properties

Under well-defined reliability operations, the preservation properties of stochastic orders has acquired noticeable attention in the literature of reliability theory (see, for reference, Alzaid, Kim, and Proschan (1991), Bartoszewicz and Skolimowska (2006), and Misra, Gupta, and Dhariyal (2008)). In this section, we derive some preservation properties of the residual stochastic precedence order.

The following theorem, which strengthens Theorem 9 of Kayid, Izadkhah, and Alshami (2014), proves that the residual stochastic precedence order is preserved under strictly increasing transformations.

Theorem 2.3.1. *Let $X \leq_{rsp} Y$. If $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function, then $\phi(X) \leq_{rsp} \phi(Y)$.*

Proof. For $t \geq 0$, let

$$\begin{aligned}\Delta_6(t) &= P(t < \phi(X) < \phi(Y)) - P(t < \phi(Y) < \phi(X)) \\ &= P(t^* < X < Y) - P(t^* < Y < X) \qquad (\text{where } t^* = \phi^{-1}(t) \geq 0) \\ &\geq 0,\end{aligned}$$

where the last inequality follows from the assumption that $X \leq_{\text{rsp}} Y$ and Theorem 2.2.1. Thus, using Theorem 2.2.1, we conclude that $\phi(X) \leq_{\text{rsp}} \phi(Y)$. □

It is worth mentioning that the above preservation property also holds for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, Theorems 1.B.2, 1.B.43, 1.C.8 and 2.A.19 in Shaked and Shanthikumar (2007)).

Next result shows that the residual stochastic precedence order is preserved under convolution.

Theorem 2.3.2. *Let X, Y and Z be jointly distributed nonnegative random variables and let, for each $z \geq 0$, (X_z, Y_z) be a random vector having the same joint distribution as the conditional distribution of $(X, Y|Z = z)$. If $X_z \leq_{rsp} Y_z$, $\forall z \geq 0$, then $X + Z \leq_{rsp} Y + Z$.*

Proof. For $t \geq 0$, let

$$\begin{aligned}
\Delta_7(t) &= P(t < X + Z < Y + Z) - P(t < Y + Z < X + Z) \\
&= \int_0^\infty [P(t < X + Z < Y + Z|Z = z) - P(t < Y + Z < X + Z|Z = z)] \, \mathrm{d}F_Z(z) \\
&= \int_0^\infty [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)] \, \mathrm{d}F_Z(z) \\
&= \int_0^t [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)] \, \mathrm{d}F_Z(z) \\
&\quad + \int_t^\infty [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)] \, \mathrm{d}F_Z(z) \\
&= \int_0^t [P(t - z < X_z < Y_z) - P(t - z < Y_z < X_z)] \, \mathrm{d}F_Z(z) \\
&\quad + \int_t^\infty [P(X_z < Y_z) - P(Y_z < X_z)] \, \mathrm{d}F_Z(z) \\
&\geq 0, \ \forall t \geq 0,
\end{aligned}$$

where the nonnegativity of the first integral follows from the assumption that $X_z \leq_{\mathrm{rsp}} Y_z$ and Theorem 2.2.1, and that of the second integral follows from the fact that $X_z \leq_{\mathrm{rsp}} Y_z$ implies $X_z \leq_{\mathrm{sp}} Y_z$. Since $\Delta_7(t) \geq 0$ for all $t \geq 0$, the result follows from Theorem 2.2.1. □

The following corollary directly follows from the above theorem.

Corollary 2.3.1. *Let X, Y and Z be jointly distributed nonnegative random variables such that (X, Y) is independent of Z. If $X \leq_{rsp} Y$, then $X + Z \leq_{rsp} Y + Z$.*

Note that the results similar to the above one also holds for the hazard rate order, the reversed hazard rate order, and the mean residual life order (see Lemmas 1.B.3, 1.B.44 and 2.A.8 in Shaked and Shanthikumar (2007)).

In the following theorems, we discuss the preservation of the residual stochastic precedence order under mixtures.

Theorem 2.3.3. *Consider a family of absolutely continuous survival functions $\{\bar{G}_\theta, \theta \in \mathfrak{X}\}$ of nonnegative random variables $\{X(\theta), \theta \in \mathfrak{X}\}$, where $\mathfrak{X}$ is a subset of the real line $\mathbb{R}$. Let Θ_1 and Θ_2 be two random variables with support $\mathfrak{X}$, and with distribution functions H_1 and H_2, respectively. Let Y_1 and Y_2 be two independent random variables and let the survival function of Y_i be given by*

$$\bar{F}_i(x) = \int_{\mathfrak{X}} \bar{G}_\theta(x)\, dH_i(\theta), \qquad x \in \mathbb{R}; \qquad i = 1, 2.$$

If

$$X(\theta_1) \leq_{rsp} X(\theta_2), \quad \textit{whenever} \quad \theta_1 \leq \theta_2,$$

and $\Theta_1 \leq_{lr} \Theta_2$, then

$$Y_1 \leq_{rsp} Y_2.$$

Proof. Let $\{g_\theta, \theta \in \mathfrak{X}\}$ be the family of probability density functions of $\{X(\theta), \theta \in \mathfrak{X}\}$. Then, the pdf of Y_i is given by

$$f_i(x) = \int_{\mathfrak{X}} g_\theta(x)\, \mathrm{d}H_i(\theta), \qquad x \in \mathbb{R}; \qquad i = 1, 2.$$

Let, for a fixed $t \geq 0$,

$$\begin{aligned}
\Delta_8(t) &= P(t < Y_1 < Y_2) - P(t < Y_2 < Y_1) \\
&= \iint\limits_{t<y_1<y_2} f_1(y_1) f_2(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 - \iint\limits_{t<y_2<y_1} f_1(y_1) f_2(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \\
&= \iint\limits_{t<y_1<y_2} \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, \mathrm{d}H_1(\theta_1)\, \mathrm{d}H_2(\theta_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \\
&\quad - \iint\limits_{t<y_2<y_1} \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, \mathrm{d}H_1(\theta_1)\, \mathrm{d}H_2(\theta_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \\
&= \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} \Big[\iint\limits_{t<y_1<y_2} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 - \iint\limits_{t<y_2<y_1} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \Big]\, \mathrm{d}H_1(\theta_1)\, \mathrm{d}H_2(\theta_2) \\
&= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))]\, \mathrm{d}H_1(\theta_1)\, \mathrm{d}H_2(\theta_2) \\
&= E[\phi_2(\Theta_1, \Theta_2)] - E[\phi_1(\Theta_1, \Theta_2)],
\end{aligned}$$

where, for $\theta_1, \theta_2 \in \mathfrak{X}$,

$$\phi_1(\theta_1, \theta_2) = P(t < X(\theta_2) < X(\theta_1)) \quad \text{and} \quad \phi_2(\theta_1, \theta_2) = P(t < X(\theta_1) < X(\theta_2)).$$

Let $\Delta\phi_{21}(\theta_1, \theta_2) = \phi_2(\theta_1, \theta_2) - \phi_1(\theta_1, \theta_2)$. Since $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$, for all $\theta_1 \leq \theta_2$, it follows from Theorem 2.2.1 that $\Delta\phi_{21}(\theta_1, \theta_2) \geq 0$ whenever $\theta_1 \leq \theta_2$. It is also easy to verify that $\Delta\phi_{21}(\theta_1, \theta_2) = -\Delta\phi_{21}(\theta_2, \theta_1)$ for all θ_1, θ_2. Now, on using Theorem 1.C.22 of Shaked and Shanthikumar (2007), we have $\Delta_8(t) \geq 0$. Since $t \geq 0$ is arbitrary, we have $\Delta_8(t) \geq 0, \forall t \geq 0$. Now, on using Theorem 2.2.1, we conclude that $Y_1 \leq_{\text{rsp}} Y_2$. □

It is worth mentioning that the above theorem generalizes Theorem 5 of Kayid, Izadkhah, and Alshami (2014). Further, note that Theorem 2.3.3 also holds for the usual stochastic order, the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, Theorems 1.A.6, 1.B.14, 1.B.52, 1.C.17 and 2.A.15 in Shaked and Shanthikumar (2007)).

One may ask whether the assumption $\Theta_1 \leq_{\text{lr}} \Theta_2$, in the previous theorem, can be relaxed to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}]\, \Theta_2$. The following counterexample shows that this may not always be true.

Example 2.3.1. Let the survival function of $X(\theta)$ be $\bar{G}_\theta(x) = e^{-\frac{x}{\theta}}$, $\theta \geq 0$. For $\theta_1 \neq \theta_2$, assume that $X(\theta_1)$ and $X(\theta_2)$ are independent random variables. It is easy to see that $X(\theta_1) \leq_{\text{hr}} X(\theta_2)$ for all $\theta_1 \leq \theta_2$, which implies that $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$ whenever $\theta_1 \leq \theta_2$. Now, let Θ_1 and Θ_2 be two independent gamma random variables with respective probability density functions given by

$$h_1(\theta) = \frac{1}{4} e^{-\frac{\theta}{4}}, \theta > 0,$$

and

$$h_2(\theta) = \frac{1}{6} \theta^3 e^{-\theta}, \theta > 0.$$

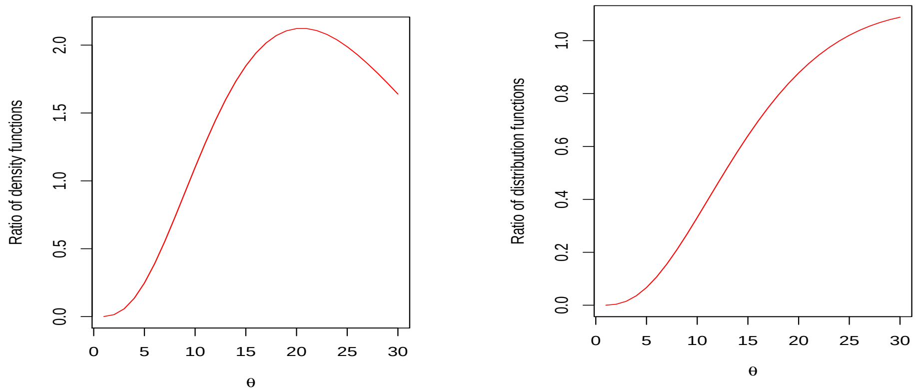
The distribution functions of Θ_1 and Θ_2 are given by

$$H_1(\theta) = 1 - e^{-\frac{\theta}{4}}, \theta \geq 0,$$

and

$$H_2(\theta) = \frac{6-(\theta^3+3\theta^2+6\theta+6)e^{-\theta}}{6}, \theta \geq 0,$$

respectively. Then, it can be verified that $\frac{h_2(\theta)}{h_1(\theta)}$ is not a monotone function of $\theta \in [0,\infty)$ (see Figure 2.1 (a)), which implies that $\Theta_1 \nleq_{\mathrm{lr}} \Theta_2$. It can also be verified that $\frac{H_2(\theta)}{H_1(\theta)}$ is increasing in $\theta \in [0,\infty)$ (see Figure 2.1 (b)), which implies that $\Theta_1 \leq_{\mathrm{rh}} \Theta_2$.



(a) Plot of $\frac{h_2(\theta)}{h_1(\theta)}$ **(b)** Plot of $\frac{H_2(\theta)}{H_1(\theta)}$

Figure 2.1: Plots of ratio of density functions and distribution functions

Now, let Y_1 and Y_2 be two independent random variables as defined in Theorem 2.3.3, and let, for $t \geq 0$,

$$\begin{aligned}\Delta_9(t) &= P(t<Y_1<Y_2) - P(t<Y_2<Y_1)\\ &= \int_0^\infty\int_0^\infty [P(t<X(\theta_1)<X(\theta_2)) - P(t<X(\theta_2)<X(\theta_1))]\, h_1(\theta_1)h_2(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.\end{aligned}$$

One can easily verify that

$$P(t<X(\theta_1)<X(\theta_2)) - P(t<X(\theta_2)<X(\theta_1)) = \frac{\theta_2-\theta_1}{\theta_2+\theta_1}e^{-\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)t}.$$

Therefore,

$$\Delta_9(t) = \frac{1}{24}\int_0^\infty\int_0^\infty \frac{\theta_2-\theta_1}{\theta_2+\theta_1}\,\theta_2^3\, e^{-\left(\frac{t}{\theta_1}+\frac{\theta_1}{4}\right)} e^{-\left(\frac{t}{\theta_2}+\theta_2\right)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

Now, using R-software, we obtain that $\Delta_9(0.5) = 0.01409676$ and $\Delta_9(1) = -0.009706716$, which implies that $\Delta_9(t) \not\geq 0$, $\forall t \geq 0$. Now, on using Theorem 2.2.1, we conclude that $Y_1 \not\leq_{\text{rsp}} Y_2$.

The following theorem provides conditions under which the assumption $\Theta_1 \leq_{\text{lr}} \Theta_2$ can be relaxed to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \, \Theta_2$.

Theorem 2.3.4. *Consider the setup of Theorem 2.3.3. If, for each* $\theta_1 \, [\theta_2]$, $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))$ *is increasing [decreasing] in* $\theta_2 \in [\theta_1, \infty) \cap \mathfrak{X} \, [\theta_1 \in (-\infty, \theta_2) \cap \mathfrak{X}]$, $\forall t \geq 0$, *and* $\Theta_1 \leq_{hr} [\leq_{rh}] \, \Theta_2$, *then* $Y_1 \leq_{rsp} Y_2$.

The proof of the above theorem can be written down using lines similar to those used in the proof of Theorem 2.3.3, and using Theorem 1.B.10 [1.B.48] of Shaked and Shanthikumar (2007).

Remark 2.3.1. Observe that the assumption $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1))$ is increasing [decreasing] in $\theta_2 \in [\theta_1, \infty) \cap \mathfrak{X} \, [\theta_1 \in (-\infty, \theta_2) \cap \mathfrak{X}]$, $\forall t \geq 0$ implies that $P(t < X(\theta_1) < X(\theta_2)) - P(t < X(\theta_2) < X(\theta_1)) \geq 0$, $\forall t \geq 0$, i.e., $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$, for all $\theta_1 \leq \theta_2$. Thus, to relax $\Theta_1 \leq_{\text{lr}} \Theta_2$ to $\Theta_1 \leq_{\text{hr}} [\leq_{\text{rh}}] \, \Theta_2$, an assumption stronger than $X(\theta_1) \leq_{\text{rsp}} X(\theta_2)$, whenever $\theta_1 \leq \theta_2$, is being used.

Theorem 2.3.5. *Let* X *and* Y *be independent random variables having absolutely continuous survival functions* $\bar{F}_X$ *and* $\bar{F}_Y$*, respectively, and let* W *be a random variable, independent of* X *and* Y*, with the survival function* $p\bar{F}_X + (1-p)\bar{F}_Y$ *for some* $p \in (0,1)$. *If* $X \leq_{rsp} Y$*, then* $X \leq_{rsp} W \leq_{rsp} Y$.

Proof. Clearly, the pdf of W is given by $pf_X + (1-p)f_Y$ for some $p \in (0,1)$. Now, first we show that $X \leq_{\text{rsp}} W$. For $t \geq 0$, we have

$$\begin{aligned}\Delta_{10}(t) &= \int_t^\infty [(p\bar{F}_X(x) + (1-p)\bar{F}_Y(x)) f_X(x) - \bar{F}_X(x)(pf_X(x) + (1-p)f_Y(x))] \, \mathrm{d}x \\ &= (1-p) \int_t^\infty [\bar{F}_Y(x) f_X(x) - \bar{F}_X(x) f_Y(x)] \, \mathrm{d}x\end{aligned}$$

$$\geq 0, \ \forall t \geq 0,$$

where the last inequality follows from the assumption that $X \leq_{\text{rsp}} Y$ and Corollary 2.2.2. Since $\Delta_{10}(t) \geq 0$ for all $t \geq 0$, again using Corollary 2.2.2, it follows that $X \leq_{\text{rsp}} W$. Similarly we can show that $W \leq_{\text{rsp}} Y$, and on combining the two results, we get that $X \leq_{\text{rsp}} W \leq_{\text{rsp}} Y$. □

It is worth mentioning that the theorems similar to above one also holds for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, Theorems 1.B.22, 1.C.30 and 2.A.18 in Shaked and Shanthikumar (2007)).

2.4 Application

In this section, we discuss an application of the residual stochastic precedence order in reliability theory.

Let X_1, X_2, and X_3 be nonnegative random variables. Consider the two series systems, each consisting of two components. Let the random lifetimes of two components of the first system be X_1 and X_2, and let that of the two components of the second system be X_1 and X_3. Then the lifetimes of the two systems are given by $\wedge\{X_1, X_2\}$ and $\wedge\{X_1, X_3\}$, respectively, where the symbol '$\wedge$' represents *minimum*. We are interested in the comparison of the lifetimes of the two systems with respect to the residual stochastic precedence order. The following theorem provides sufficient condition for the comparison.

Theorem 2.4.1. *Let X_1, X_2, and X_3 be jointly distributed nonnegative random variables having a joint Lebésgue density, and let, for each $x \geq 0$, $(X_2^*(x), X_3^*(x))$ be a random vector having the same joint distribution as the conditional distribution of $(X_2, X_3 | X_1 = x)$. If $X_2^*(x) \leq_{jhr} X_3^*(x)$, $\forall x \geq 0$, then $\wedge\{X_1, X_2\} \leq_{rsp} \wedge\{X_1, X_3\}$.*

Proof. For $t \geq 0$, let

$$\begin{aligned}
\Delta_{11}(t) &= P\left(t < \wedge\{X_1,X_2\} < \wedge\{X_1,X_3\}\right) - P\left(t < \wedge\{X_1,X_3\} < \wedge\{X_1,X_2\}\right) \\
&= P\left(t < X_2 < X_1, t < X_2 < X_3\right) - P\left(t < X_3 < X_1, t < X_3 < X_2\right) \\
&= \int_t^\infty \big[P\left(t < X_2 < X_1, t < X_2 < X_3 | X_1 = x\right) \\
&\qquad\qquad - P\left(t < X_3 < X_1, t < X_3 < X_2 | X_1 = x\right)\big] f_{X_1}(x)\,\mathrm{d}x \\
&= \int_t^\infty \big[P\left(t < X_2^*(x) < x, t < X_2^*(x) < X_3^*(x)\right) \\
&\qquad\qquad - P\left(t < X_3^*(x) < x, t < X_3^*(x) < X_2^*(x)\right)\big] f_{X_1}(x)\,\mathrm{d}x \\
&= \int_t^\infty \int_t^{x_1} \left[\int_{x_2}^\infty \left(f_{X_2^*(x),X_3^*(x)}(x_2,x_3) - f_{X_2^*(x),X_3^*(x)}(x_3,x_2)\right)\mathrm{d}x_3\right] f_{X_1}(x)\,\mathrm{d}x_2\,\mathrm{d}x \\
&\geq 0, \forall t \geq 0,
\end{aligned}$$

where the last inequality follows from the assumption that $X_2^*(x) \leq_{\text{jhr}} X_3^*(x)$, $\forall x \geq 0$. Now, on using Theorem 2.2.1, we conclude that $\wedge\{X_1,X_2\} \leq_{\text{rsp}} \wedge\{X_1,X_3\}$. □

The following corollaries directly follow from the above theorem.

Corollary 2.4.1. *Let X_1, X_2, and X_3 be jointly distributed nonnegative random variables, having a joint Lebésgue density, such that (X_2,X_3) is independent of X_1. If $X_2 \leq_{jhr} X_3$, then $\wedge\{X_1,X_2\} \leq_{rsp} \wedge\{X_1,X_3\}$.*

Corollary 2.4.2. *Let X_1, X_2, and X_3 be nonnegative independent random variables having Lebésgue densities. If $X_2 \leq_{hr} X_3$, then $\wedge\{X_1,X_2\} \leq_{rsp} \wedge\{X_1,X_3\}$.*

It is worth mentioning that the similar kind of comparisons, with respect to different stochastic orders, are available in the literature. See, for example, Ding, Da, and Li (2014), Misra and Misra (2012), and references cited therein.

Chapter 3

Inactivity Stochastic Precedence Order

3.1 Introduction

Let X be a nonnegative continuous random variable representing the lifetime of a unit. For a fixed $t > 0$, the inactivity time of the unit at time t is defined as $X_{(t)} = [t - X | X \leq t]$, which represents the time elapsed since failure of the unit, given that the unit has failed by the time t. Nanda, Singh, Misra, and Paul (2003) used the term 'reversed residual life' to interpret the random variable $X_{(t)}$. The idea of the inactivity time has been used extensively during the last few decades in reliability theory, life testing applications, survival analysis, risk theory, etc. (see, for example, Andersen, Borgan, Gill, and Keiding (1993), Li and Lu (2003), Eryilmaz (2010), Mahdy (2012), Gupta, Misra, and Kumar (2015), Bayramoglu and Ozkut (2016), and references cited therein).

To compare two or more random variables, the role of stochastic orders is very well known in the fields of survival analysis, reliability theory, and economics. For an extensive monograph on stochastic orders, see, Shaked and Shanthikumar (2007). Let X and Y

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denote the random lifetimes of two units, i.e., X and Y are nonnegative continuous random variables. Many times, instead of making direct comparison between X and Y, researchers have compared their corresponding inactivity times $X_{(t)} = [t - X | X \leq t]$ and $Y_{(t)} = [t - Y | Y \leq t]$, which resulted in the development of various stochastic orders (see, for instance, Nanda, Singh, Misra, and Paul (2003), Kayid and Ahmad (2004), Ahmad, Kayid, and Pellerey (2005), Ahmad and Kayid (2005), Li and Xu (2006), Yasaei Sekeh, Mohtashami Borzadaran, and Rezaei Roknabadi (2013), Kayid and Izadkhah (2014), Zhang and Balakrishnan (2016) Arriaza, Sordo, and Suárez-Llorens (2017), Abouelmagd, Hamed, Ebraheim, and Afify (2018), Kayid, Izadkhah, and Alfifi (2018), and references cited therein). In this chapter, our interest is in the *inactivity probability order* which is recently defined and studied by Abouelmagd, Hamed, Ebraheim, and Afify (2018). According to them, X is said to be smaller than Y in the inactivity probability order (denoted by $X \leq_{\text{ipr}} Y$) if, for all $t > 0$, $P(X_{(t)} > Y_{(t)}) \leq 0.5$. They studied this order for the case when X and Y are independent, and derived some useful results. It is worth mentioning here that the definition of the inactivity probability order, given by Abouelmagd, Hamed, Ebraheim, and Afify (2018), is equivalent to say that $X_{(t)}$ is smaller than $Y_{(t)}$, for all $t > 0$, in the stochastic precedence order given by Arcones, Kvam, and Samaniego (2002). There exists another definition of the stochastic precedence order which says that: X is said to be smaller than Y in the stochastic precedence order (denoted by $X \leq_{\text{sp}} Y$) if $P(X < Y) \geq P(Y < X)$. This definition was given by Boland, Singh, and Cukic (2004), and they have shown that it takes care of the dependence structure between X and Y.

The main purpose of this chapter is to define (and study) the *inactivity stochastic precedence order* by comparing $X_{(t)}$ and $Y_{(t)}$, for all $t > 0$, with respect to the stochastic precedence order given by Boland, Singh, and Cukic (2004). The definition is as follows.

Definition 3.1.1. *X is said to be smaller than Y in the inactivity stochastic precedence order (written as $X \leq_{isp} Y$) if $P(X_{(t)} < Y_{(t)}) \leq P(Y_{(t)} < X_{(t)})$ for all $t > 0$, or equivalently, if $Y_{(t)} \leq_{sp} X_{(t)}$ for all $t > 0$.*

The inspiration to define the inactivity stochastic precedence order as above comes from the fact that the above definition is better than that given by Abouelmagd, Hamed, Ebraheim, and Afify (2018), for it considers the dependence structure between the random variables X and Y. The following example, which is similar to the Example 2 of Boland, Singh, and Cukic (2004), justifies our claim.

Example 3.1.1. Let U and V be two independent exponential random variables with means $\frac{1}{2}$ and 1, respectively. Define $X = \min(U, (U+V)/2)$ and $Y = U$. Then X and Y are both continuous but highly dependent. In fact, for $t > 0$, we have

$$P(X = Y \le t) = P(U < V, U \le t) = \int_0^t e^{-u} \cdot 2e^{-2u}\,\mathrm{d}u = \tfrac{2}{3}(1 - e^{-3t}),$$

$$P(X < Y \le t) = P(V < U \le t) = \int_0^t (1 - e^{-u}) \cdot 2e^{-2u}\,\mathrm{d}u = (1 - e^{-2t}) - \tfrac{2}{3}(1 - e^{-3t}),$$

$$P(Y < X \le t) = 0,$$

and therefore,

$$P(X \le t, Y \le t) = P(X = Y \le t) + P(X < Y \le t) + P(Y < X \le t) = 1 - e^{-2t}.$$

Now, it is easy to see that

$$P(X_{(t)} < Y_{(t)}) - P(Y_{(t)} < X_{(t)}) = \frac{P(Y < X \le t) - P(X < Y \le t)}{P(X \le t, Y \le t)} < 0,\ \forall t > 0,$$

i.e., $X \le_{\text{isp}} Y$. But

$$P(X_{(t)} < Y_{(t)}) = \frac{P(Y < X \le t)}{P(X \le t, Y \le t)} = 0 < \frac{1}{2},\ \forall t > 0,$$

i.e., $Y \le_{\text{ipr}} X$, and

$$P(Y_{(t)} < X_{(t)}) = \frac{P(X < Y \le t)}{P(X \le t, Y \le t)} = \frac{(1 - e^{-2t}) - \frac{2}{3}(1 - e^{-3t})}{1 - e^{-2t}} = 1 - \frac{2(1 - e^{-3t})}{3(1 - e^{-2t})},\ \forall t > 0.$$

Our aim is to show that $P(Y_{(t)} < X_{(t)}) < \frac{1}{2}, \forall t > 0$, which is equivalent to show that $3e^{-2t} - 4e^{-3t} + 1 > 0$, $\forall t > 0$. Let $\psi(t) = 3e^{-2t} - 4e^{-3t} + 1$, $t > 0$. Then, $\psi'(t) = 6e^{-2t}(2e^{-t} - 1)$, $t > 0$. One can easily see that $\psi'(t) > 0$ if $t < \ln 2$ and $\psi'(t) < 0$ if $t > \ln 2$, which implies that $\psi(t)$ is increasing in $t \in (0, \ln 2)$ and is decreasing in $t \in (\ln 2, \infty)$. Also, $\lim_{t \to 0^+} \psi(t) = 0$ and $\lim_{t \to \infty} \psi(t) = 1$. Therefore, $\psi(t) > 0$, $\forall t > 0$, which implies that $X \le_{\text{ipr}} Y$. Thus, X and Y are equivalent according to the notion of inactivity probability order defined by Abouelmagd, Hamed, Ebraheim, and Afify (2018). □

For the independent random variables X and Y, the inactivity probability order and the inactivity stochastic precedence order must be same. However, one can verify that if X and Y are independent, then $X \leq_{\text{isp}} Y$ is equivalent to say that $Y \leq_{\text{ipr}} X$, i.e., they are just opposite to each other. This is due to the counter intuitive definition of the inactive probability order. Recall that $X \leq_{\text{ipr}} Y$ if, for all $t > 0$, $P(X_{(t)} > Y_{(t)}) \leq 0.5$, or equivalently, if, for all $t > 0$, $P(Y_{(t)} > X_{(t)}) \geq 0.5$, which means that there is a greater chance that $Y_{(t)}$ exceeds $X_{(t)}$ than $X_{(t)}$ exceeds $Y_{(t)}$. Also, observe that the large inactivity time implies small lifetime. Therefore, it would have been better if Abouelmagd, Hamed, Ebraheim, and Afify (2018) define that $Y \leq_{\text{ipr}} X$ if, for all $t > 0$, $P(X_{(t)} > Y_{(t)}) \leq 0.5$.

Moreover, it can be verified that $X \leq_{\text{isp}} Y$ implies $X \leq_{\text{sp}} Y$. Clearly, the converse of this implication may not always be true. However, we will show, in Theorem 3.3.2, that $X \leq_{\text{sp}} Y$ may imply $X \leq_{\text{isp}} Y$ under certain conditions.

The rest of the chapter is planned as follows. Section 3.2 contains some definitions which will be used throughout this chapter. Section 3.3 provides some results on characterizations and implications of the inactivity stochastic precedence order. Section 3.4 is devoted to the preservation properties of this order under some reliability operations such as monotone transformations, convolution and mixture. Finally, in Section 3.5, we discuss some applications of the inactivity stochastic precedence order.

Let us recall the following notation which are used throughout this chapter. The terms 'increasing' and 'decreasing' stands for nondecreasing and nonincreasing, respectively. For any random variable U and an event E, we use $[U|E]$ to represent a random variable whose distribution is the conditional distribution of U given E. For a random variable U, we use $f_U(\cdot)$, $F_U(\cdot)$, $\bar{F}_U(\cdot)$, $r_U(\cdot)$, $\tilde{r}_U(\cdot)$ and $\tilde{m}_U(\cdot)$ to denote the probability density function, the distribution function, the survival function, the hazard rate function, the reversed hazard rate function and the mean inactivity time function of U, respectively. We use $f_{U_1,U_2}(\cdot,\cdot)$ to denote the joint probability density function of the random variables U_1 and U_2. The support of all the random variables, unless explicitly mentioned, is assumed to be $\mathbb{R}_+ \equiv [0,\infty)$. While referring to the previous works, we use $\leq_{\text{isp}}$ for $\leq_{\text{ipr}}$ whenever X and Y are independent.

3.2 Preliminaries

In this section, we recall some definitions to be used throughout the chapter. Firstly, we recollect the definitions of the univariate stochastic orders.

Definition 3.2.1. *X is said to be smaller than Y in the*

(i) *likelihood ratio order (written as $X \leq_{lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}_+$;*

(ii) *usual stochastic order (written as $X \leq_{st} Y$) if $F_Y(x) \leq F_X(x), \forall x \in \mathbb{R}_+$;*

(iii) *hazard rate order (written as $X \leq_{hr} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $r_X(x) \geq r_Y(x), \forall x \in \mathbb{R}_+$;*

(iv) *reversed hazard rate order (written as $X \leq_{rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x), \forall x \in (0,\infty)$;*

(v) *mean inactivity time order (written as $X \leq_{mit} Y$) if $\int_0^x F_Y(t)\,dt / \int_0^x F_X(t)\,dt$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{m}_X(x) \geq \tilde{m}_Y(x), \forall x \in (0,\infty)$;*

(vi) *stochastic precedence order (written as $X \leq_{sp} Y$) if $P(Y < X) \leq P(X < Y)$.*

Let us also recollect the following implications that are well known.

$$\begin{array}{ccccc} X \leq_{\text{lr}} Y & \Rightarrow & X \leq_{\text{rh}} Y & \Rightarrow & X \leq_{\text{mit}} Y \\ \Downarrow & & \Downarrow & & \\ X \leq_{\text{hr}} Y & \Rightarrow & X \leq_{\text{st}} Y & & \end{array}$$

Moreover, $X \leq_{\mathrm{st}} Y$ implies $X \leq_{\mathrm{sp}} Y$ when X and Y are independent. For more details on various stochastic orders, one may refer to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Belzunce, Martínez-Riquelme, and Mulero (2015). For the notion of stochastic precedence order, readers may refer to Boland, Singh, and Cukic (2004).

Note that these univariate stochastic orders, except the stochastic precedence order, are based on the comparison between the marginal distributions of random variables, without taking into account the mutual dependence between them. For taking into account the dependence structure, various authors introduced the joint stochastic orders and studied their properties (see, for example, Shanthikumar and Yao (1991), Aly and Kochar (1993), Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Pellerey and Spizzichino (2016)). In this chapter, we are mainly concerned with the following definition recently given by Balakrishnan, Barmalzan, and Kosari (2017).

Definition 3.2.2. *X is said to be smaller than Y in the joint weak reversed hazard rate order (written as $X \leq_{rh:wj} Y$) if $F(x,y) \geq F(y,x)$ for all $x \leq y$.*

Remark 3.2.1. Balakrishnan, Barmalzan, and Kosari (2017) have proved that if $X \leq_{\mathrm{rh:wj}} Y$, then $\int_0^y f(x,\nu)\,\mathrm{d}\nu \leq \int_0^y f(\nu,x)\,\mathrm{d}\nu$ for all $y \geq 0$. While going through the proof of this result, we found that there is a typo and the result should be as follows: If $X \leq_{\mathrm{rh:wj}} Y$, then $\int_0^y f(y,\nu)\,\mathrm{d}\nu \leq \int_0^y f(\nu,y)\,\mathrm{d}\nu$ for all $y \geq 0$.

3.3 Characterizations and implications

In this section, our main aim is to discuss the relationships between the inactivity stochastic precedence order and the other existing well-known stochastic orders.

The succeeding theorem narrates the equivalent conditions for the inactivity stochastic precedence order. The proof is trivial and hence deleted.

Theorem 3.3.1. *The following declarations are equivalent:*

(i) $X \leq_{isp} Y$.

(ii) $P(Y < X \leq t) - P(X < Y \leq t) \leq 0$, *for all* $t > 0$.

(iii) $\int_0^t \int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy\, dx \leq 0$, *for all* $t > 0$.

The following corollary directly follows from the above theorem.

Corollary 3.3.1. *If* $\int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy \leq 0$, $\forall x \geq 0$, *then* $X \leq_{isp} Y$.

The Corollary 3.3.1 inspires us to define a new joint reversed hazard rate order based on the joint distribution of random variables. The definition is as follows.

Definition 3.3.1. *X is said to be smaller than Y in the joint reversed hazard rate order (written as $X \leq_{jrh} Y$) if* $\int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy \leq 0$, $\forall x \geq 0$.

In view of the Remark 3.2.1, it is easy to see that $X \leq_{\text{rh:wj}} Y$ implies $X \leq_{\text{jrh}} Y$. Also, utilizing Definition 3.3.1 and Corollary 3.3.1, we get that $X \leq_{\text{jrh}} Y$ implies $X \leq_{\text{isp}} Y$. Thus, the joint reversed hazard rate order is weaker than the joint weak reversed hazard rate order and is stronger than the inactivity stochastic precedence order. It is also important to observe that $X \leq_{\text{jrh}} Y$ and $X \leq_{\text{rh}} Y$ are equivalent when X and Y are independent. We are investigating the properties of the joint reversed hazard rate order, and the findings will be reported in future.

As we have seen that $X \leq_{\text{jrh}} Y$ implies $X \leq_{\text{isp}} Y$, one may be curious to know whether the converse also holds. The following counterexamples give the answer in negation.

Example 3.3.1. Let

$$f_{X,Y}(x,y) = \begin{cases} e^{-2(y-x)} e^{-x}, & \text{if } y > x > 0, \\ \frac{1}{4} e^{-\frac{1}{2}(x-y)} e^{-y}, & \text{if } x > y > 0. \end{cases}$$

Then,

$$f_{X,Y}(y,x) = \begin{cases} e^{-2(x-y)}e^{-y}, & \text{if } x > y > 0, \\ \frac{1}{4}e^{-\frac{1}{2}(y-x)}e^{-x}, & \text{if } y > x > 0. \end{cases}$$

Now, for $x \geq 0$, let

$$\begin{aligned} \Delta_1(x) &= \int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) \mathrm{d}y \\ &= \int_0^x \left(\frac{1}{4}e^{-\frac{1}{2}(x-y)}e^{-y} - e^{-2(x-y)}e^{-y}\right) \mathrm{d}y \\ &= e^{-2x} - \frac{3}{2}e^{-x} + \frac{1}{2}e^{-\frac{x}{2}}. \end{aligned}$$

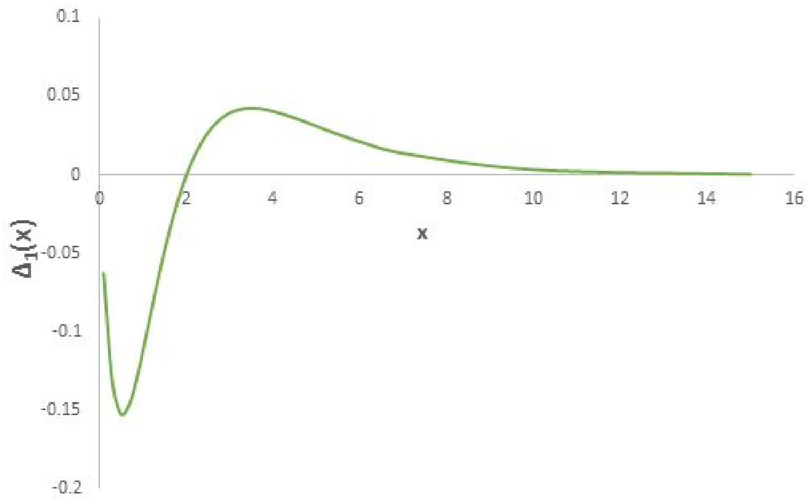


Figure 3.1: Plot of $\Delta_1(x)$

It can be verified that $\Delta_1(x)$ takes positive as well as negative values for $x \geq 0$, (see Figure 3.1). Thus, from the Definition 3.3.1, we conclude that neither $X \leq_{\text{jrh}} Y$ nor $Y \leq_{\text{jrh}} X$.

Now, for $t > 0$, let

$$\begin{aligned} \Delta_2(t) &= \int_0^t \int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) \mathrm{d}y\,\mathrm{d}x \\ &= \int_0^t \left(e^{-2x} - \frac{3}{2}e^{-x} + \frac{1}{2}e^{-\frac{x}{2}}\right) \mathrm{d}x \end{aligned}$$

$$= \frac{3}{2}e^{-t} - \frac{1}{2}e^{-2t} - e^{-\frac{t}{2}}$$
$$\leq 0, \forall t > 0, \qquad \text{(see Figure 3.2)}.$$

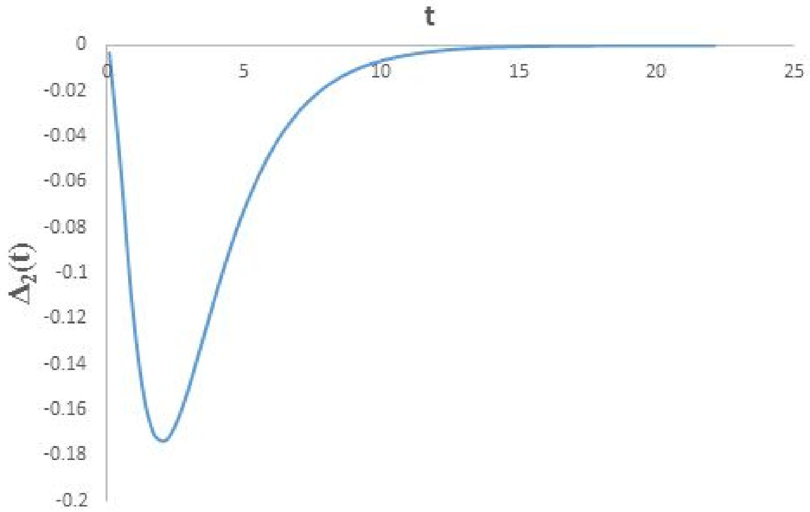


Figure 3.2: Plot of $\Delta_2(t)$

Now, on using Theorem 3.3.1, we have that $X \leq_{\text{isp}} Y$. Thus, we conclude that $X \leq_{\text{isp}} Y$ does not imply $X \leq_{\text{jrh}} Y$. □

Example 3.3.2. Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x^2y^2}\left(\alpha e^{-\left(\frac{1}{x}+\frac{1}{y}\right)} + (1-\alpha)\frac{1}{x^2y^3}e^{-\frac{1}{2x}}e^{-\frac{1}{y^2}}\right), & \text{if } x>0, y>0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ is an arbitrary fixed number. Then,

$$f_{X,Y}(y,x) = \begin{cases} \frac{1}{x^2y^2}\left(\alpha e^{-\left(\frac{1}{x}+\frac{1}{y}\right)} + (1-\alpha)\frac{1}{y^2x^3}e^{-\frac{1}{2y}}e^{-\frac{1}{x^2}}\right), & \text{if } x>0, y>0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for $x > 0$, let

$$\Delta_3(x) = \int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, dy$$
$$= (1-\alpha)\left(\frac{1}{x^2}e^{-\frac{1}{2x}}\int_0^x \frac{1}{y^3}e^{-\frac{1}{y^2}}\, dy - \frac{1}{x^3}e^{-\frac{1}{x^2}}\int_0^x \frac{1}{y^2}e^{-\frac{1}{2y}}\, dy\right)$$

$$= (1-\alpha)\left(\frac{1}{2x^2}e^{-\frac{1}{2x}}\int_{\frac{1}{x^2}}^{\infty} e^{-u}\,\mathrm{d}u - \frac{2}{x^3}e^{-\frac{1}{x^2}}\int_{\frac{1}{2x}}^{\infty} e^{-u}\,\mathrm{d}u\right)$$
$$= (1-\alpha)\left(\frac{1}{2x^2}-\frac{2}{x^3}\right)e^{-\frac{1}{2x^2}-\frac{1}{x^2}}$$
$$= \frac{2}{x^3}(1-\alpha)(x-4)\,e^{-\frac{1}{2x^2}-\frac{1}{x^2}}$$

It can be verified that $\Delta_3(x)$ takes positive values for $x \geq 4$ and negative values for $0 < x < 4$. Thus, from the Definition 3.3.1, we conclude that neither $X \leq_{\text{jrh}} Y$ nor $Y \leq_{\text{jrh}} X$. Now, for $t > 0$, let

$$\begin{aligned}
\Delta_4(t) &= \int_0^t \int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^t \left((1-\alpha)\left(\frac{1}{2x^2}-\frac{2}{x^3}\right)e^{-\frac{1}{2x^2}-\frac{1}{x^2}}\right)\mathrm{d}x \\
&= \int_0^t (1-\alpha)e^{-\frac{1}{2x}}\cdot e^{-\frac{1}{x^2}}\cdot\frac{1}{2x^2}\left(1-\frac{4}{x}\right)\mathrm{d}x \qquad (3.3.1)\\
&= \int_{\frac{1}{t}}^{\infty} \frac{1}{2}(1-\alpha)(1-4y)e^{-\frac{1}{2}(y+2y^2)}\,\mathrm{d}y.
\end{aligned}$$

Our aim is to show that $\Delta_4(t) \leq 0$, $\forall t > 0$. It is easy to see that

$$\Delta_4'(t) = \frac{1}{2t^2}\left(1-\frac{4}{t}\right)(1-\alpha)e^{-\frac{1}{2}\left(\frac{1}{t}+\frac{2}{t^2}\right)}, \quad t > 0.$$

Clearly, $\Delta_4'(t) < 0$ for $t < 4$ and $\Delta_4'(t) > 0$ for $t > 4$, which implies that $\Delta_4(t)$ is decreasing for $t < 4$ and is increasing for $t > 4$. Also, $\lim_{t\to 0^+}\Delta_4(t) = 0$. Then, it is sufficient to show that $\lim_{t\to\infty}\Delta_4(t) \leq 0$. We have

$$\begin{aligned}
\lim_{t\to\infty}\Delta_4(t) &= \int_0^{\infty} \frac{1}{2}(1-\alpha)(1-4y)e^{-\frac{1}{2}(y+2y^2)}\,\mathrm{d}y \\
&= -(1-\alpha)\int_0^{\infty}\frac{1}{2}\left[(4y+1)e^{-\frac{1}{2}(y+2y^2)} - 2e^{-\frac{1}{2}(y+2y^2)}\right]\mathrm{d}y \\
&= -(1-\alpha)\left(\int_0^{\infty}\frac{1}{2}(4y+1)e^{-\frac{1}{2}(y+2y^2)}\,\mathrm{d}y + \int_0^{\infty} e^{\frac{1}{16}-\left(y+\frac{1}{4}\right)^2}\mathrm{d}y\right) \\
&= (1-\alpha)\left(-1+e^{\frac{1}{16}}\int_{1/4}^{\infty} e^{-v^2}\,\mathrm{d}v\right)
\end{aligned}$$

$$\leq (1-\alpha)\left(-1+e^{\frac{1}{16}}\int_{0}^{\infty} e^{-v^2}\,\mathrm{d}v\right)$$

$$= (1-\alpha)\left(-1+e^{\frac{1}{16}}\frac{\sqrt{\pi}}{2}\right)$$

$$= -0.0566(1-\alpha)$$

$$< 0.$$

Therefore, $\Delta_4(t) \leq 0$ for all $t > 0$. Now, from Corollary 3.3.2, we have $X \leq_{\text{isp}} Y$. □

As a direct consequence of Theorem 3.3.1, we have the following corollary.

Corollary 3.3.2. *Let X and Y be independent random variables. The following declarations are equivalent:*

(i) $X \leq_{isp} Y$.

(ii) $\int_0^t [F_Y(x)f_X(x) - F_X(x)f_Y(x)]\, dx \leq 0$, *for all* $t > 0$.

(iii) $\int_0^t F_X(x)F_Y(x)\,[\tilde{r}_X(x) - \tilde{r}_Y(x)]\, dx \leq 0$, *for all* $t > 0$.

It is worth noting here that Abouelmagd, Hamed, Ebraheim, and Afify (2018) have also given the above equivalent conditions but with reversed inequalities in parts (ii) and (iii). Their results are just opposite to that of ours due to the reason explained in the paragraph just after the Example 3.1.1. Abouelmagd, Hamed, Ebraheim, and Afify (2018) have also shown (in Theorem 6) that if X and Y are independent random variables with $X \leq_{\text{rh}} Y$, then $X \leq_{\text{isp}} Y$. It can be easily verified that there is a typo in this result and it should be: If $X \leq_{\text{rh}} Y$, then $Y \leq_{\text{isp}} X$. Similar but just opposite result follows from the part (iii) of the above corollary, which says that if $X \leq_{\text{rh}} Y$, then $X \leq_{\text{isp}} Y$. Now, with the help of following example, we show that the converse of this result does not hold.

Example 3.3.3. Let X and Y be two independent random variables with $F_X(x) = e^{-\frac{1}{2x}}$ and $F_Y(x) = e^{-\frac{1}{x^2}}$, $x > 0$, respectively. Then, $\tilde{r}_X(x) = \frac{1}{2x^2}$ and $\tilde{r}_Y(x) = \frac{2}{x^3}$, $x > 0$. It is easy to verify that $\tilde{r}_X(x) - \tilde{r}_Y(x)$ takes negative values for $x \in (0,4)$ and positive values for $x \in (4,\infty)$, i.e., neither $X \leq_{\text{rh}} Y$ nor $Y \leq_{\text{rh}} X$. Now, for $t > 0$, let

$$\begin{aligned}
\Delta_5(t) &= \int_0^t F_X(x)F_Y(x)\,[\tilde{r}_X(x) - \tilde{r}_Y(x)]\,\mathrm{d}x \\
&= \int_0^t e^{-\frac{1}{2x}} \cdot e^{-\frac{1}{x^2}} \cdot \frac{1}{2x^2}\left(1 - \frac{4}{x}\right)\mathrm{d}x \\
&= \frac{\Delta_4(t)}{(1-\alpha)} \qquad \text{(using Eq. (3.3.1))} \\
&= -0.0566 \\
&< 0 \qquad \text{(as shown in Example 3.3.2)}
\end{aligned}$$

Now, from Corollary 3.3.2, we have $X \leq_{\text{isp}} Y$. Thus, we conclude that $X \leq_{\text{isp}} Y$ does not imply that $X \leq_{\text{rh}} Y$. □

Since $X \leq_{\text{rh(hr)}} Y$ implies $X \leq_{\text{st}} Y$, one may be concerned with the relationship between the usual stochastic order and the inactivity stochastic precedence order. Moreover, one may also be concerned with the relationship between the hazard rate order and the inactivity stochastic precedence order. The following example shows, for independent random variables, that the hazard rate order does not imply the inactivity stochastic precedence order, and hence, the usual stochastic order also does not imply the inactivity stochastic precedence order.

Example 3.3.4. Let X and Y be two independent random variables with $F_X(x) = 1 - e^{-\left(\frac{5}{2}x^2+10x\right)}$ and $F_Y(x) = 1 - e^{-\left(\frac{3}{2}x^2+10x\right)}$, $x \geq 0$, respectively. Then, $r_X(x) = 5x+10$ and $r_Y(x) = 3x+10$, $x \geq 0$. Clearly, $r_X(x) - r_Y(x) = 2x \geq 0$, $\forall x \geq 0$, i.e., $X \leq_{\text{hr}} Y$. Now, for $t > 0$, let

$$\begin{aligned}
\Delta_6(t) &= \int_0^t [F_Y(x)f_X(x) - F_X(x)f_Y(x)]\,\mathrm{d}x \\
&= \int_0^t \Big[\left(1 - e^{-\left(\frac{3}{2}x^2+10x\right)}\right) \cdot (5x+10)e^{-\left(\frac{5}{2}x^2+10x\right)}
\end{aligned}$$

$$- \left(1 - e^{-\left(\frac{5}{2}x^2+10x\right)}\right) \cdot (3x+10)e^{-\left(\frac{3}{2}x^2+10x\right)}\Big] \mathrm{d}x$$

$$= \int_0^t \left((5x+10)e^{-\left(\frac{5}{2}x^2+10x\right)} - (3x+10)e^{-\left(\frac{3}{2}x^2+10x\right)} - 2xe^{-(4x^2+20x)}\right) \mathrm{d}x$$

$$= \int_0^{\frac{5}{2}t^2+10t} e^{-u}\,\mathrm{d}u - \int_0^{\frac{3}{2}t^2+10t} e^{-v}\,\mathrm{d}v - \int_0^t (2x+5)e^{-(4x^2+20x)}\,\mathrm{d}x + 5\int_0^t e^{-(4x^2+20x)}\,\mathrm{d}x$$

$$= e^{-\left(\frac{3}{2}t^2+10t\right)} - e^{-\left(\frac{5}{2}t^2+10t\right)} - \frac{1}{4}\left(1 - e^{-(4t^2+20t)}\right) + \frac{5}{2}e^{25}\int_5^{2t+5} e^{-\nu^2}\,\mathrm{d}\nu$$

$$= e^{-\left(\frac{3}{2}t^2+10t\right)} - e^{-\left(\frac{5}{2}t^2+10t\right)} - \frac{1}{4}\left(1 - e^{-(4t^2+20t)}\right) + \frac{5\sqrt{\pi}e^{25}}{4}\,\mathrm{Erf}(5, 2t+5),$$

where $\mathrm{Erf}(5,2t+5) = \frac{2}{\sqrt{\pi}}\int_5^{2t+5} e^{-\nu^2}\,\mathrm{d}\nu$ is known as the error function.

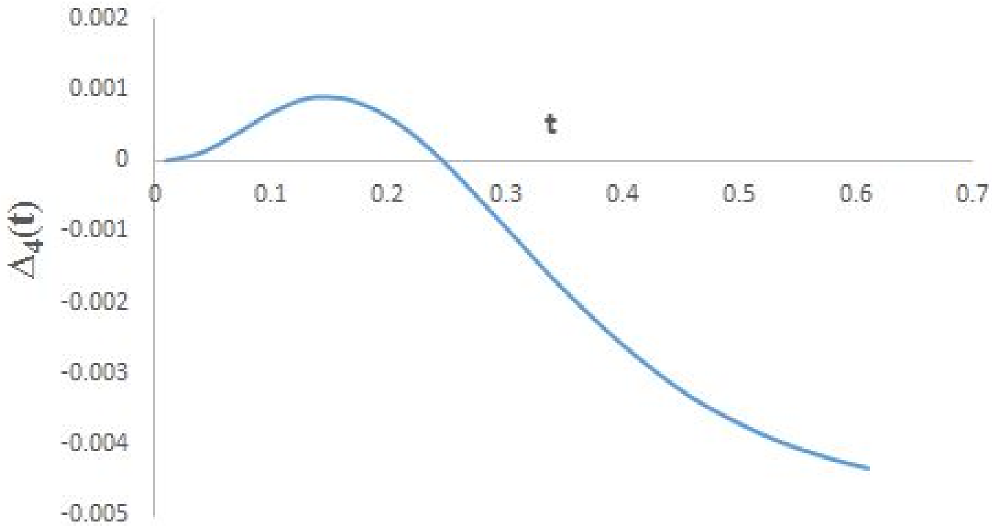


Figure 3.3: Plot of $\Delta_6(t)$

Now, with the help of Microsoft Excel, we plot $\Delta_6(t)$ as given in Figure 3.3. Clearly, $\Delta_6(t)$ takes positive as well as negative values for $t > 0$. Therefore, on using Corollary 3.3.2, we get that $X \not\leq_{\mathrm{isp}} Y$. Thus $X \leq_{\mathrm{hr}} Y$ does not imply that $X \leq_{\mathrm{isp}} Y$, and hence, $X \leq_{\mathrm{st}} Y$ also does not imply that $X \leq_{\mathrm{isp}} Y$. □

Now, the following example shows that, if X and Y are dependent random variables, then the usual stochastic order and the inactivity stochastic precedence order may exist simultaneously, but the mean inactivity time order (and hence the reversed hazard rate order) may not exist.

Example 3.3.5. Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x^2y^2}\left(\alpha e^{-\left(\frac{1}{x}+\frac{1}{y}\right)} + 8(1-\alpha)e^{-\frac{2}{x}}e^{-\frac{4}{y}}\right), & \text{if } x>0, y>0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ is an arbitrary fixed number. Then,

$$f_{X,Y}(y,x) = \begin{cases} \frac{1}{x^2y^2}\left(\alpha e^{-\left(\frac{1}{x}+\frac{1}{y}\right)} + 8(1-\alpha)e^{-\frac{2}{y}}e^{-\frac{4}{x}}\right), & \text{if } x>0, y>0, \\ 0, & \text{otherwise.} \end{cases}$$

Now, for $x>0$, let

$$\begin{aligned} \Delta_7(x) &= \int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) \mathrm{d}y \\ &= \frac{8(1-\alpha)}{x^2}\left(e^{-\frac{2}{x}}\int_0^x \frac{1}{y^2}e^{-\frac{4}{y}}\,\mathrm{d}y - e^{-\frac{4}{x}}\int_0^x \frac{1}{y^2}e^{-\frac{2}{y}}\,\mathrm{d}y\right) \\ &= \frac{8(1-\alpha)}{x^2}\left(e^{-\frac{2}{x}}\int_{\frac{1}{x}}^{\infty} e^{-4u}\,\mathrm{d}u - e^{-\frac{4}{x}}\int_{\frac{1}{x}}^{\infty} e^{-2u}\,\mathrm{d}u\right) \\ &= -\frac{2(1-\alpha)}{x^2}e^{-\frac{6}{x}} \\ &< 0, \forall x>0. \end{aligned}$$

Also, $\Delta_7(0)=0$. Thus, $\Delta_7(x) \le 0$, $\forall x \ge 0$, which implies that $X \le_{\text{jrh}} Y$, which further implies that $X \le_{\text{isp}} Y$.

One can easily verify that the marginal distribution functions of X and Y are given by

$$F_X(x) = \alpha e^{-\frac{1}{x}} + (1-\alpha)e^{-\frac{2}{x}},\ x>0,$$

and

$$F_Y(x) = \alpha e^{-\frac{1}{x}} + (1-\alpha)e^{-\frac{4}{x}},\ x>0.$$

Clearly, X and Y are not independent random variables. Also, one can easily see that $F_X(x) \ge F_Y(x)$ for all $x \ge 0$, i.e., $X \le_{\text{st}} Y$.

Now we show that the mean inactivity time order (and hence the reversed hazard rate order) between X and Y does not exist. For $x > 0$, let

$$\begin{aligned}\Delta_8(x) &= \frac{\int_0^x F_Y(t)\,\mathrm{d}t}{\int_0^x F_X(t)\,\mathrm{d}t} \\ &= \frac{\int\limits_0^x \left(\alpha e^{-\frac{1}{t}} + (1-\alpha)e^{-\frac{4}{t}}\right)\mathrm{d}t}{\int\limits_0^x \left(\alpha e^{-\frac{1}{t}} + (1-\alpha)e^{-\frac{2}{t}}\right)\mathrm{d}t} \\ &= \frac{\alpha \int\limits_{\frac{1}{x}}^{\infty} \frac{1}{u^2} e^{-u}\,\mathrm{d}u + 4(1-\alpha)\int\limits_{\frac{4}{x}}^{\infty} \frac{1}{u^2} e^{-u}\,\mathrm{d}u}{\alpha \int\limits_{\frac{1}{x}}^{\infty} \frac{1}{u^2} e^{-u}\,\mathrm{d}u + 2(1-\alpha)\int\limits_{\frac{2}{x}}^{\infty} \frac{1}{u^2} e^{-u}\,\mathrm{d}u} \\ &= \frac{\alpha\Gamma\left(-1,\frac{1}{x}\right) + 4(1-\alpha)\Gamma\left(-1,\frac{4}{x}\right)}{\alpha\Gamma\left(-1,\frac{1}{x}\right) + 2(1-\alpha)\Gamma\left(-1,\frac{2}{x}\right)},\end{aligned}$$

where $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}\,\mathrm{d}u$ is known as the upper incomplete gamma function. For $\alpha = 0.3$, with the help of R-software, we plot $\Delta_8(x)$ as given in Figure 3.4.

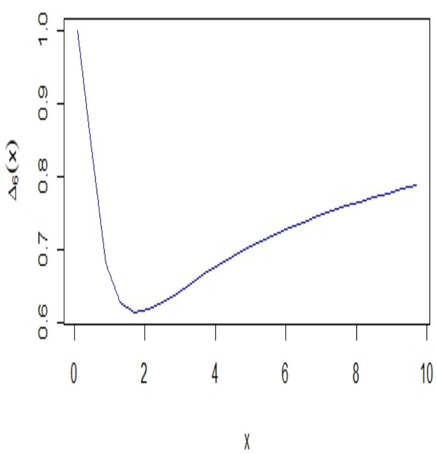


Figure 3.4: Plot of $\Delta_8(x)$

Clearly, $\Delta_8(x)$ is not a monotone function of x, which implies that neither $X \leq_{\text{mit}} Y$ nor $Y \leq_{\text{mit}} X$. □

Recall that we have pointed out in the introduction that $X \leq_{\text{isp}} Y$ implies $X \leq_{\text{sp}} Y$. One may be interested to know whether the converse holds. The following theorem shows that under certain conditions, the stochastic precedence order implies the inactivity stochastic precedence order.

Theorem 3.3.2. *Let $X \leq_{sp} Y$. If $\frac{\int_0^x f_{X,Y}(x,y)\,dy}{\int_0^x f_{X,Y}(y,x)\,dy}$ is increasing in $x > 0$, then $X \leq_{isp} Y$.*

Proof. Let

$$\Delta_9(t) = \int_0^t \int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y\,\mathrm{d}x,\ t > 0.$$

To prove that $X \leq_{\text{isp}} Y$, using Theorem 3.3.1, we have to show that $\Delta_9(t) \leq 0$, $\forall t > 0$, which is equivalent to show that

$$\frac{\int_0^t \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^t \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x} \leq 1,\ \forall t > 0.$$

Since $X \leq_{\text{sp}} Y$ implies that $\frac{P(Y<X)}{P(X<Y)} \leq 1$, it is sufficient to show that

$$\frac{\int_0^t \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^t \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x} \leq \frac{\int_0^\infty \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t > 0,$$

which is equivalent to show that

$$\frac{\int_0^t \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x} \leq \frac{\int_0^t \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t > 0.$$

Let Z_1 and Z_2 be two random variables having probability density functions

$$f_{Z_1}(x) = \frac{\int_0^x f_{X,Y}(y,x)\,\mathrm{d}y}{\int_0^\infty \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ x > 0$$

and

$$f_{Z_2}(x) = \frac{\int_0^x f_{X,Y}(x,y)\,\mathrm{d}y}{\int_0^\infty \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x},\ x > 0,$$

respectively. Then,

$$\frac{f_{Z_2}(x)}{f_{Z_1}(x)} = \frac{\int_0^x f_{X,Y}(x,y)\,\mathrm{d}y}{\int_0^x f_{X,Y}(y,x)\,\mathrm{d}y} \times \frac{\int_0^\infty \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}.$$

Since $\frac{\int_0^x f_{X,Y}(x,y)\,\mathrm{d}y}{\int_0^x f_{X,Y}(y,x)\,\mathrm{d}y}$ is increasing in $x > 0$, it follows that $\frac{f_{Z_2}(x)}{f_{Z_1}(x)}$ is increasing in $x > 0$, and therefore, $Z_1 \leq_{\text{lr}} Z_2$, and hence, $Z_1 \leq_{\text{st}} Z_2$. Therefore, $F_{Z_2}(t) \leq F_{Z_1}(t)$, $\forall t > 0$, i.e.,

$$\frac{\int_0^t \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x} \leq \frac{\int_0^t \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^\infty \int_0^x f_{X,Y}(y,x)\,\mathrm{d}y\,\mathrm{d}x},\ \forall t > 0.$$

Hence, the theorem follows. □

The following corollary is an immediate consequence of Theorem 3.3.2.

Corollary 3.3.3. *Let X and Y be independent random variables with $X \leq_{sp} Y$. If $\frac{\tilde{r}_X(x)}{\tilde{r}_Y(x)}$ is increasing in $x > 0$, then $X \leq_{isp} Y$.*

Remark 3.3.1. It is worth mentioning that the assumption that $\frac{\tilde{r}_X(x)}{\tilde{r}_Y(x)}$ is increasing in $x > 0$ is related to the notion of the relative reversed hazard rate order introduced by Rezaei, Gholizadeh, and Izadkhah (2015).

3.4 Preservation properties

Preservation properties of stochastic orders under some reliability operations, such as increasing transformations, convolutions and mixtures, have been an important topic of interest for last 3 decades (see, for reference, Alzaid, Kim, and Proschan (1991), Bartoszewicz and Skolimowska (2006), and Misra, Gupta, and Dhariyal (2008)). In this section, we discuss some preservation properties of the inactivity stochastic precedence order.

The following theorem proves that the inactivity stochastic precedence order is preserved under strictly increasing transformations.

Theorem 3.4.1. *Let $X \leq_{isp} Y$. If $\phi : (0,\infty) \to (0,\infty)$ is a strictly increasing function, then $\phi(X) \leq_{isp} \phi(Y)$.*

Proof. For $t > 0$, let

$$\begin{aligned}
\Delta_{10}(t) &= P(\phi(Y) < \phi(X) \leq t) - P(\phi(X) < \phi(Y) \leq t) \\
&= P(Y < X \leq t^*) - P(X < Y \leq t^*) \qquad (\text{where } t^* = \phi^{-1}(t) > 0) \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the assumption that $X \leq_{\text{isp}} Y$ and Theorem 3.3.1. Thus, using Theorem 3.3.1, we conclude that $\phi(X) \leq_{\text{isp}} \phi(Y)$. □

It is worth mentioning that the above preservation property also holds for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean inactivity time order (see, Theorem 3.1 in Li and Xu (2006), and Theorems 1.B.2, 1.B.43 and 1.C.8 in Shaked and Shanthikumar (2007)).

Next result shows that the inactivity stochastic precedence order is preserved under convolution.

Theorem 3.4.2. *Let X, Y and Z be jointly distributed nonnegative random variables and let, for each $z \geq 0$, (X_z, Y_z) be a random vector having the same joint distribution as the conditional distribution of $(X, Y|Z = z)$. If $X_z \leq_{isp} Y_z$, $\forall z \geq 0$, then $X + Z \leq_{isp} Y + Z$.*

Proof. For $t > 0$, let

$$\begin{aligned}
\Delta_{11}(t) &= P(Y+Z < X+Z \leq t) - P(X+Z < Y+Z \leq t) \\
&= \int_0^\infty [P(Y+Z < X+Z \leq t|Z=z) - P(X+Z < Y+Z \leq t|Z=z)]\, f_Z(z)\,\mathrm{d}z \\
&= \int_0^\infty [P(Y_z < X_z \leq t-z) - P(X_z < Y_z \leq t-z)]\, f_Z(z)\,\mathrm{d}z \\
&= \int_0^t [P(Y_z < X_z \leq t-z) - P(X_z < Y_z \leq t-z)]\, f_Z(z)\,\mathrm{d}z \\
&\quad + \int_t^\infty [P(Y_z < X_z \leq t-z) - P(X_z < Y_z \leq t-z)]\, f_Z(z)\,\mathrm{d}z \\
&= \int_0^t [P(Y_z < X_z \leq t-z) - P(X_z < Y_z \leq t-z)]\, f_Z(z)\,\mathrm{d}z \\
&\leq 0, \ \forall t > 0,
\end{aligned}$$

where the last inequality follows from the assumption that $X_z \leq_{\text{isp}} Y_z$ and Theorem 3.3.1. Since $\Delta_{11}(t) \leq 0$ for all $t > 0$, the result follows from Theorem 3.3.1. □

The following corollary directly follows from the above theorem.

Corollary 3.4.1. *Let X, Y and Z be jointly distributed nonnegative random variables such that (X, Y) is independent of Z. If $X \leq_{isp} Y$, then $X + Z \leq_{isp} Y + Z$.*

Note that the results similar to the above one also hold for the hazard rate order, the reversed hazard rate order, and the mean inactivity time order (see, Theorem 3.1 in Kayid and Ahmad (2004), and Lemmas 1.B.3 and 1.B.44 in Shaked and Shanthikumar (2007)).

In the following theorems, we discuss the preservation of the inactivity stochastic precedence order under mixtures.

Theorem 3.4.3. *Consider a family of distribution functions* $\{G_\theta, \theta \in \mathfrak{X}\}$ *of nonnegative random variables* $\{X(\theta), \theta \in \mathfrak{X}\}$*, where* $\mathfrak{X}$ *is a subset of the real line* $\mathbb{R}$*. Let* Θ_1 *and* Θ_2 *be two random variables with support* $\mathfrak{X}$*, and with distribution functions* H_1 *and* H_2*, respectively. Let* Y_1 *and* Y_2 *be two independent random variables and let the distribution function of* Y_i *be given by*

$$F_i(x) = \int_{\mathfrak{X}} G_\theta(x)\, dH_i(\theta), \qquad x \in \mathbb{R}; \qquad i = 1,2.$$

If

$$X(\theta_1) \leq_{isp} X(\theta_2), \quad \theta_1 \leq \theta_2,$$

and $\Theta_1 \leq_{lr} \Theta_2$*, then*

$$Y_1 \leq_{isp} Y_2.$$

Proof. Let $h_i(\theta)$ be the probability density function of Θ_i, $i = 1,2$, and let $\{g_\theta, \theta \in \mathfrak{X}\}$ be the family of probability density functions of $\{X(\theta), \theta \in \mathfrak{X}\}$. Then, the density function of Y_i be given by

$$f_i(x) = \int_{\mathfrak{X}} g_\theta(x) h_i(\theta)\, \mathrm{d}\theta, \qquad x \in \mathbb{R}; \qquad i = 1,2.$$

Let, for $t > 0$,

$$\begin{aligned}
\Delta_{12}(t) &= P(Y_2 < Y_1 \leq t) - P(Y_1 < Y_2 \leq t) \\
&= \iint\limits_{0 \leq y_2 < y_1 \leq t} f_1(y_1) f_2(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 - \iint\limits_{0 \leq y_1 < y_2 \leq t} f_1(y_1) f_2(y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \\
&= \iint\limits_{0 \leq y_2 < y_1 \leq t} \int\limits_{\mathfrak{X}} \int\limits_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2) h_1(\theta_1) h_2(\theta_2)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}y_1\, \mathrm{d}y_2
\end{aligned}$$

$$- \iint\limits_{0\le y_1<y_2\le t} \int_{\mathfrak{X}} \int_{\mathfrak{X}} g_{\theta_1}(y_1) g_{\theta_2}(y_2) h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2\, dy_1\, dy_2$$

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} \Big[\iint\limits_{0\le y_2<y_1\le t} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dy_1\, dy_2 - \iint\limits_{0\le y_1<y_2\le t} g_{\theta_1}(y_1) g_{\theta_2}(y_2)\, dy_1\, dy_2 \Big]$$

$$\times h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times I(\theta_2 \le \theta_1) h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

$$+ \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times I(\theta_1 \le \theta_2) h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

(where $I(a \le b) = 1,$ if $a \le b$; $0,$ otherwise)

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times I(\theta_2 \le \theta_1) h_1(\theta_1) h_2(\theta_2)\, d\theta_2\, d\theta_1$$

$$+ \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times I(\theta_1 \le \theta_2) h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

(changing the order of integration in the first part)

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_1) < X(\theta_2) \le t) - P(X(\theta_2) < X(\theta_1) \le t)]$$

$$\times I(\theta_1 \le \theta_2) h_1(\theta_2) h_2(\theta_1)\, d\theta_1\, d\theta_2$$

$$+ \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times I(\theta_1 \le \theta_2) h_1(\theta_1) h_2(\theta_2)\, d\theta_1\, d\theta_2$$

(interchanging θ_1 and θ_2 in the first part)

$$= \int_{\mathfrak{X}} \int_{\mathfrak{X}} [P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t)]$$

$$\times [h_1(\theta_1)h_2(\theta_2) - h_1(\theta_2)h_2(\theta_1)]\, I(\theta_1 \le \theta_2)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2.$$

Since $X(\theta_1) \le_{\mathrm{isp}} X(\theta_2)$, $\theta_1 \le \theta_2$, it follows from Theorem 3.3.1 that

$$P(X(\theta_2) < X(\theta_1) \le t) - P(X(\theta_1) < X(\theta_2) \le t) \le 0,\ \forall t > 0,$$

and $\Theta_1 \le_{\mathrm{lr}} \Theta_2$ implies that $h_1(\theta_1)h_2(\theta_2) - h_1(\theta_2)h_2(\theta_1) \ge 0$ for $\theta_1 \le \theta_2$. Therefore, $\Delta_{12}(t) \le 0, \forall t > 0$. Now, on using Theorem 3.3.1, we conclude that $Y_1 \le_{\mathrm{isp}} Y_2$. □

It is worth mentioning that the above theorem generalizes the Theorem 8 of Abouelmagd, Hamed, Ebraheim, and Afify (2018). Further, note that Theorem 3.4.3 also holds for the usual stochastic order, the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean inactivity time order (see, Theorem 3.3 in Kayid and Ahmad (2004), and Theorems 1.A.6, 1.B.14, 1.B.52 and 1.C.17 in Shaked and Shanthikumar (2007)).

Theorem 3.4.4. *Let X and Y be independent random variables and let W be a random variable, independent of X and Y, with the distribution function $pF_X + (1-p)F_Y$ for some $p \in (0,1)$. If $X \le_{isp} Y$, then $X \le_{isp} W \le_{isp} Y$.*

Proof. Clearly, the probability density function of W is given by $pf_X + (1-p)f_Y$ for some $p \in (0,1)$. Now, first we show that $X \le_{\mathrm{isp}} W$. For $t > 0$, let

$$\begin{aligned}\Delta_{13}(t) &= \int_0^t [(pF_X(x) + (1-p)F_Y(x))\, f_X(x) - F_X(x)\, (pf_X(x) + (1-p)f_Y(x))]\, \mathrm{d}x \\ &= (1-p)\int_0^t [F_Y(x)f_X(x) - F_X(x)f_Y(x)]\, \mathrm{d}x \\ &\le 0,\ \forall t > 0,\end{aligned}$$

where the last inequality follows from the assumption that $X \le_{\mathrm{isp}} Y$ and Corollary 3.3.2. Since $\Delta_{13}(t) \le 0$ for all $t > 0$, again using Corollary 3.3.2, it follows that $X \le_{\mathrm{isp}} W$. Similarly we can show that $W \le_{\mathrm{isp}} Y$, and on combining the two results, we get that $X \le_{\mathrm{isp}} W \le_{\mathrm{isp}} Y$. □

It is worth mentioning that the theorems similar to above one also holds for the hazard rate order and the likelihood ratio order (see, Theorems 1.B.22 and 1.C.30 in Shaked and Shanthikumar (2007)).

3.5 Applications

In this section, we provide examples using real dataset as well as simulated data to discuss this new stochastic order.

With the help of following example, we try to discuss existence of the inactivity stochastic precedence order. For this purpose, we consider a dataset in medical research coming from a cross-over study which is an important class of statistical methods. Cross-Over study is prevalent for experiments in various scientific areas such as psychology, pharmaceutical science, and medicine (see, for example, Greene, Kerr, McIntosh, and Prescott (1981), White, Lewith, Hopwood, and Prescott (2003), Senn, Rolfe, and Julious (2010), and references cited therein). In this type of study each experimental unit (patient) receives different treatments during different period of time, i.e., during the period of experiment the patients cross-over from one treatment to another.

Example 3.5.1. Consider the dataset used by Senn, Lillienthal, Patalano, and Till (1997) and is available at `http://www.senns.demon.co.uk/Data/Selipati.xls`. In this data, a fraction of patients suffering from asthma have been bring upon many doses of two formulations of formoterol (a strong cure acting on asthma) and a placebo according to irregular order. An indicator illustrating the log area under the curve for forced expiratory volume in one second (denoted by log-AUC) has been reported after each treatment. In our analysis, we compare the log-AUC after the placebo treatment and after the MTA24 treatment, i.e., two treatments of each patient have been compared.

It is well known that $X \leq_{\mathrm{rh}} Y$ holds if, and only if, the *P-P* plot is star-shaped with respect to $(0,0)$ (see Müller and Stoyan (2002, p.11)), and recall that $X \leq_{\mathrm{rh}} Y$ implies

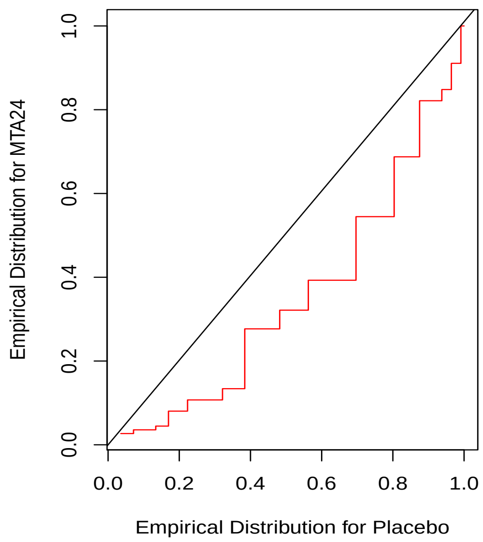


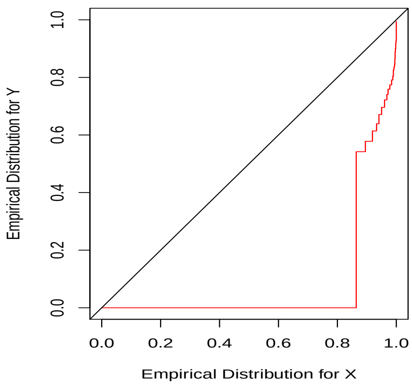
Figure 3.5: *P-P* plot of the empirical distribution functions of the log-AUC calculates under placebo and MTA24 treatments.

$X \leq_{\text{isp}} Y$. Figure 3.5 represents the *P-P* plot of the empirical distribution functions of placebo and MTA24 treatments. Clearly, the *P-P* plot is star-shaped with respect to $(0,0)$. Thus, the reversed hazard rate order holds between placebo and MTA24 treatments, and hence the inactivity stochastic precedence order also holds between them. □

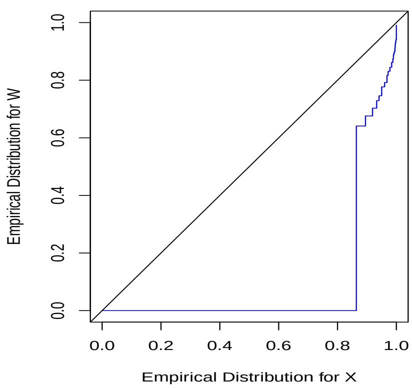
The following example discusses Theorem 3.4.4 for a simulated data.

Example 3.5.2. In this example, we generate data from R-software for random variables X, Y, and W, where X and Y follow exponential distributions with means 0.5 and 1.25, respectively, and W follows Weibull distribution with shape and scale parameters 0.95 and 1, respectively. It can be easily verified that the distribution function of W is a mixture of distribution functions of X and Y.

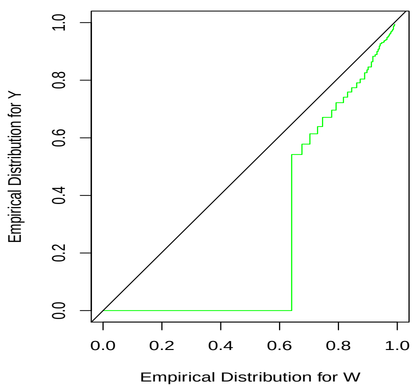
From Figure 3.6 (a), we can conclude that $X \leq_{\text{rh}} Y$, and hence $X \leq_{\text{isp}} Y$. Now, from Figure 3.6 (b) and 3.6 (c) as well as from Theorem 3.4.4, we can conclude that $X \leq_{\text{isp}} W \leq_{\text{isp}} Y$. □



(a) P-P plot of the empirical distribution functions of X and Y.



(b) P-P plot of the empirical distribution functions of X and W.



(c) P-P plot of the empirical distribution functions of W and Y.

Figure 3.6: P-P plots of the empirical distribution functions

Chapter 4

Properties of Joint Hazard Rate and Joint Reversed Hazard Rate Orders

4.1 Introduction

Comparisons among independent random variables, their comparative properties, and applications using univariate stochastic orders have been extensively used for more than a decade. To learn these orders with their detailed illustrations, see, Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Belzunce, Martínez-Riquelme, and Mulero (2015). It is to highlight that numerous stochastic orders based on independent random variables are introduced in the literature using marginal distributions among random variables, with the avoidance of their mutual non-independence. It is the case of several applied situations, but in some cases, it becomes compulsory to take care of their mutual non-independence. Because of this issue, Shanthikumar and Yao (1991) think about this situation and established replaceable versions of few stochastic orders such as *the likelihood ratio order, the hazard rate order,* and *the usual stochastic order,* and also give their applications in areas like stochastic scheduling problem, allocation problem by considering mutual non-independence between random variables. Moreover, different authors also have introduced a well-established new version of stochastic orders named as *joint stochastic orders,* which provide a new account for stochastic comparisons of dependent random variables. They are also defined their properties and applications in different areas, more specifically, analysis of random utility models, portfolio selection, and allocation of redundant components (see, for references, Aly and Kochar (1993), Belzunce, Ortega, Pellerey, and Ruiz (2007), Belzunce, Martínez-Puertas, and Ruiz (2013), Li and You (2014), Li and You (2015), Belzunce, Martínez-Riquelme, Pellerey, and Zalzadeh (2016), Pellerey and Spizzichino (2016), Balakrishnan, Barmalzan, and Kosari (2017)).

Recently, Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020) introduced two new weak versions of the joint weak hazard (reversed hazard) rate orders named as *the joint hazard (reversed hazard) rate order.* Motivated by the importance of joint stochastic orders, we investigate some contributing properties, implications, and relationships of these joint stochastic orders with other comparable joint stochastic orders in this chapter.

We start with the notation that will be taken in this chapter. For a random vector (T_1, T_2) with Lebésgue pdf, we take $f_{T_1,T_2}(\cdot,\cdot)$ to indicate the joint density of (T_1, T_2). $\mathbb{R}_+ \equiv [0, \infty)$ is assumed to be the support of all the random variables.

Now, we recollect the definitions of the above mention two new joint stochastic orders given by Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020), respectively, which are beneficial for the comparisons of dependent random variables.

Definition 4.1.1. *For a given random vector (X,Y), X is said to be smaller than Y in the*

(i) *"joint hazard rate order" (denoted as $X \leq_{jhr} Y$) if $\int_x^{\infty}(f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \geq 0$, $\forall x \geq 0$;*

(ii) *"joint reversed hazard rate order" (denoted as $X \leq_{jrh} Y$) if $\int_0^{x}(f_{X,Y}(x,y) - f_{X,Y}(y,x))\,dy \leq 0$, $\forall x \geq 0$.*

Also, we recollect that if X and Y are not dependent random variables, then there is no difference between bivariate versions and univariate versions of these orders.

4.2 Relations among dependent random variables

In this section, we aim to examine the relationship of two new stochastic orders with the another existing well-introduced stochastic orders, and also we discuss an important result with the help of two newly defined bivariate stochastic orders which are the residual stochastic precedence order and the inactivity stochastic precedence order.

For this purpose, we first reconsider the formal definitions of all the stochastic orders through which implications are shown.

Definition 4.2.1. *For a given random vector (X,Y), X is taken to be smaller than Y in the*

(a) *"joint likelihood ratio order" (denoted as $X \leq_{lr:j} Y$) if $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$ for $x \leq y$, or equivalently, if $f_{X,Y}(x,y) - f_{X,Y}(y,x) \leq 0$ for $y \leq x$ (see, Shanthikumar and Yao (1991);*

(b) *"residual stochastic precedence order" (denoted as $X \leq_{rsp} Y$) if $P(X_t < Y_t) \geq P(Y_t < X_t)$ for all $t \geq 0$, or equivalently, if $\int_t^\infty \int_x^\infty \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) dy\, dx \geq 0$, for all $t \geq 0$ (see, Misra, Gupta, and Misra (2020);*

(c) *"inactivity stochastic precedence order" (denoted as $X \leq_{isp} Y$) if $P(X_{(t)} < Y_{(t)}) \leq P(Y_{(t)} < X_{(t)})$ for all $t > 0$, or equivalently, if $\int_0^t \int_0^x \left(f_{X,Y}(x,y) - f_{X,Y}(y,x)\right) dy\, dx \leq 0$, for all $t > 0$ (see, Misra, Gupta, and Chanchal (2020).*

We want to examine whether there is a relationship between the inactivity stochastic precedence order and the residual stochastic precedence order. The theorem given below helps us to understand the relationship.

Theorem 4.2.1. *If $X \leq_{rsp} Y$, then $\frac{1}{X} \geq_{isp} \frac{1}{Y}$.*

Proof. Since $X \leq_{\text{rsp}} Y$, we have $P(X_t < Y_t) - P(Y_t < X_t) \geq 0$, for all $t \geq 0$, or equivalently,

$$\frac{P(t \leq X < Y) - P(t \leq Y < X)}{P(X \geq t, Y \geq t)} \geq 0, \quad \text{for all } t \geq 0.$$

Let

$$X = \frac{1}{X^*} \quad \text{and} \quad Y = \frac{1}{Y^*}.$$

Then,

$$\frac{P\left(t \leq \frac{1}{X^*} < \frac{1}{Y^*}\right) - P\left(t \leq \frac{1}{Y^*} < \frac{1}{X^*}\right)}{P\left(\frac{1}{X^*} \geq t, \frac{1}{Y^*} \geq t\right)} \geq 0, \quad \text{for all } t \geq 0,$$

which implies

$$\frac{P\left(Y^* < X^* < \frac{1}{t}\right) - P\left(X^* < Y^* < \frac{1}{t}\right)}{P\left(X^* \leq \frac{1}{t}, Y^* \leq \frac{1}{t}\right)} \geq 0, \quad \text{for all } t > 0.$$

This implies $X^* \geq_{\text{isp}} Y^*$, hence $\frac{1}{X} \geq_{\text{isp}} \frac{1}{Y}$. □

Let us also recall the following implications that are known (see, Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020)).

$$\begin{array}{ccccc} X \leq_{\text{lr:j}} Y & \Rightarrow & X \leq_{\text{jhr}} Y & \Rightarrow & X \leq_{\text{rsp}} Y \\ \Downarrow & & & & \\ X \leq_{\text{jrh}} Y & \Rightarrow & X \leq_{\text{isp}} Y & & \end{array}$$

Because of the Definitions 4.1.1 and 4.2.1, it is very clear that $X \leq_{\text{lr:j}} Y$ implies $X \leq_{\text{jhr(jrh)}} Y$. Thus, the joint hazard rate (joint reversed hazard rate) order is not stronger than the joint likelihood ratio order but this is stronger than the residual stochastic precedence order (inactivity stochastic precedence order). Someone may be interested whether the joint hazard (reversed hazard) rate order implies the joint likelihood ratio order. We show this implications with the help of succeeding counterexample(s).

The following counterexample shows that the joint hazard rate order (and hence the residual stochastic precedence order) may exist when the joint likelihood ratio order does not exist.

Example 4.2.1. Let the joint pdf of two random variables X and Y be:

$$f_{X,Y}(x,y) = \begin{cases} xy^2, & \text{if } 0 \le x \le 1, 0 \le y \le 1, \\ \dfrac{y}{x^2}, & \text{if } 0 \le y < 1 < x < \infty, \\ \frac{1}{3}e^{-(x-1)}e^{-(y-1)}, & \text{if } 1 \le x < \infty, 1 \le y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$f_{X,Y}(y,x) = \begin{cases} x^2y, & \text{if } 0 \le x \le 1, 0 \le y \le 1, \\ \dfrac{x}{y^2}, & \text{if } 0 \le x < 1 < y < \infty, \\ \frac{1}{3}e^{-(x-1)}e^{-(y-1)}, & \text{if } 1 \le x < \infty, 1 \le y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and therefore,

$$f_{X,Y}(x,y) - f_{X,Y}(y,x) = \begin{cases} xy(y-x), & \text{if } 0 \le x \le 1, 0 \le y \le 1, \\ -\dfrac{x}{y^2}, & \text{if } 0 \le x < 1 < y < \infty, \\ \dfrac{y}{x^2}, & \text{if } 0 \le y < 1 < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f_{X,Y}(x,y) - f_{X,Y}(y,x)$ takes both positive and negative values for $x \le y$. Thus, from Definition 4.2.1, we interpret that neither $Y \le_{\text{lr:j}} X$ nor $X \le_{\text{lr:j}} Y$. Let

$$\Delta_1(x) = \int_x^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y, \quad x \ge 0.$$

Since $f_{X,Y}(x,y) - f_{X,Y}(y,x) = 0$ if $1 \le x \le y < \infty$, it follows that $\Delta_1(x) = 0$, $\forall x \ge 1$. Now, for $0 \le x < 1$, we have

$$\begin{aligned} \Delta_1(x) &= \int_x^1 (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y + \int_1^\infty (f_{X,Y}(x,y) - f_{X,Y}(y,x))\,\mathrm{d}y \\ &= \int_x^1 (xy(y-x))\,\mathrm{d}y + \int_1^\infty \left(-\frac{x}{y^2}\right)\mathrm{d}y \end{aligned}$$

$$(\text{since } f_{X,Y}(x,y) - f_{X,Y}(y,x) = xy(y-x) \text{ if } 0 < x < 1, 0 < y < 1 \text{ and}$$
$$f_{X,Y}(x,y) - f_{X,Y}(y,x) = -\frac{x}{y^2} \text{ if } 0 < x < 1 < y < \infty)$$
$$= x.\frac{1}{3}(1-x^3) - x^2.\frac{1}{2}(1-x^2) - x$$
$$= \frac{x^4}{6} - \frac{x^2}{2} - \frac{2x}{3}.$$

Our aim is to show that $\Delta_1(x) \leq 0$, for $0 \leq x < 1$. It is direct to see that $\Delta_1'(x) = \frac{2}{3}x^3 - x - \frac{2}{3}$, $0 \leq x < 1$ and $\Delta_1''(x) = 2x^2 - 1 = 2\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right)$, $0 \leq x < 1$. Clearly, $\Delta_1''(x) < 0$ for $x \in \left[0, \frac{1}{\sqrt{2}}\right)$ and $\Delta_1''(x) > 0$ for $x \in \left[\frac{1}{\sqrt{2}}, 1\right)$, which implies that $\Delta_1'(x)$ is decreasing in $x \in \left[0, \frac{1}{\sqrt{2}}\right)$ and increasing in $x \in \left[\frac{1}{\sqrt{2}}, 1\right)$, and therefore

$$\Delta_1'(x) \leq \max\{\Delta_1'(0), \Delta_1'(1)\} = \max\{-\tfrac{2}{3}, -1\} = -\tfrac{2}{3} < 0, \quad x \in [0,1),$$

which further implies that $\Delta_1(x)$ is decreasing in $x \in [0,1)$. Therefore, $\Delta_1(x) \leq \Delta_1(0) = 0$, $\forall x \in [0,1)$. Thus, from Definition 4.1.1, we get that $Y \leq_{\text{jhr}} X$ and hence $Y \leq_{\text{rsp}} X$. □

The following counterexample shows that the joint reversed hazard rate order (and hence the inactivity stochastic precedence order) may exist when the joint likelihood ratio order does not exist.

Example 4.2.2. Let the joint pdf of two random variables X and Y be:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3x^2y^2} e^{-\left(\frac{1}{x}-1\right)} e^{-\left(\frac{1}{y}-1\right)}, & \text{if } 0 < x \leq 1, 0 < y \leq 1, \\ \frac{1}{y^3}, & \text{if } 0 < x < 1 < y < \infty, \\ \frac{1}{x^3y^4}, & \text{if } 1 \leq x < \infty, 1 \leq y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$f_{X,Y}(y,x) = \begin{cases} \frac{1}{3x^2y^2} e^{-\left(\frac{1}{x}-1\right)} e^{-\left(\frac{1}{y}-1\right)}, & \text{if } 0 < x \leq 1, 0 < y \leq 1, \\ \frac{1}{x^3}, & \text{if } 0 < y < 1 < x < \infty, \\ \frac{1}{x^4y^3}, & \text{if } 1 \leq x < \infty, 1 \leq y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and therefore,

$$f_{X,Y}(x,y) - f_{X,Y}(y,x) = \begin{cases} \frac{1}{y^3}, & \text{if } 0 < x < 1 < y < \infty, \\ -\frac{1}{x^3}, & \text{if } 0 < y < 1 < x < \infty, \\ \frac{1}{x^3y^3}\left(\frac{1}{y} - \frac{1}{x}\right), & \text{if } 1 \leq x < \infty, 1 \leq y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f_{X,Y}(x,y) - f_{X,Y}(y,x)$ takes both positive and negative values for $y \leq x$. Thus, from Definition 4.2.1, we interpret that neither $X \leq_{\text{lr:j}} Y$ nor $Y \leq_{\text{lr:j}} X$. Let

$$\Delta_2(x) = \int_0^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, \mathrm{d}y, \quad x \geq 0.$$

Since $f_{X,Y}(x,y) - f_{X,Y}(y,x) = 0$ if $0 \leq y \leq x < 1$, it follows that $\Delta_2(x) = 0$, for all $0 \leq x < 1$. Now, for $x \geq 1$, we have

$$\begin{aligned} \Delta_2(x) &= \int_0^1 (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, \mathrm{d}y + \int_1^x (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, \mathrm{d}y \\ &= \int_0^1 \left(-\frac{1}{x^3}\right) \mathrm{d}y + \int_1^x \left(\frac{1}{x^3y^3}\left(\frac{1}{y} - \frac{1}{x}\right)\right) \mathrm{d}y \\ &= -\frac{1}{x^3} - \frac{1}{3x^3}\left(\frac{1}{x^3} - 1\right) + \frac{1}{2x^4}\left(\frac{1}{x^2} - 1\right) \\ &= \frac{1}{6x^6} - \frac{1}{2x^4} - \frac{2}{3x^3} \\ &= \frac{1}{6x^6}(1 - 3x^2 - 4x^3) \\ &\leq \frac{1}{6x^6}(1 - 3 - 4) \qquad \text{(since } x \geq 1\text{)} \\ &= -\frac{1}{x^6} \end{aligned}$$

$$< 0.$$

Therefore, $\Delta_2(x) \leq 0$, $\forall x \geq 1$. Thus, from Definition 4.1.1, we get that $X \leq_{\text{jrh}} Y$, and hence $X \leq_{\text{isp}} Y$. □

Although, from theory, we know that the joint likelihood ratio order implies the residual stochastic precedence order, we provide such an example.

Example 4.2.3. Let the joint pdf of two random variables X and Y be:

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(3x+2y)}[1+\alpha(1-2e^{-3x})(1-2e^{-2y})], & \text{if } x \geq 0,\, y \geq 0,\, \alpha \in (-1,1), \\ 0; & \text{otherwise.} \end{cases}$$

Then,

$$f_{X,Y}(y,x) = \begin{cases} 6e^{-(2x+3y)}[1+\alpha(1-2e^{-2x})(1-2e^{-3y})], & \text{if } x \geq 0,\, y \geq 0,\, \alpha \in (-1,1), \\ 0; & \text{otherwise,} \end{cases}$$

and therefore,

$$\begin{aligned} f_{X,Y}(x,y) - f_{X,Y}(y,x) = {} & 6e^{-(3x+2y)}[1+\alpha(1-2e^{-3x})(1-2e^{-2y})] \\ & - 6e^{-(2x+3y)}[1+\alpha(1-2e^{-2x})(1-2e^{-3y})] \end{aligned}$$

With the help of Figure 4.1, we can clearly see that $f_{X,Y}(x,y) - f_{X,Y}(y,x) \geq 0$ for all $x \leq y$.

Thus, from Definition 4.2.1, we conclude that $X \leq_{\text{lr:j}} Y$. Now, for $t \geq 0$ and $\alpha = 0.5$, we have

$$\begin{aligned} \Delta_3(t) &= \int_t^{\infty}\int_x^{\infty} (f_{X,Y}(x,y) - f_{X,Y}(y,x))\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_t^{\infty}\int_x^{\infty} \Big(6e^{-(3x+2y)}[1+0.5(1-2e^{-3x})(1-2e^{-2y})] \\ &\quad - 6e^{-(2x+3y)}[1+0.5(1-2e^{-2x})(1-2e^{-3y})]\Big)\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_t^{\infty}\int_x^{\infty} (6e^{-(3x+2y)})\, \mathrm{d}y\, \mathrm{d}x + \int_t^{\infty}\int_x^{\infty} (3e^{-(3x+2y)})\, \mathrm{d}y\, \mathrm{d}x \end{aligned}$$

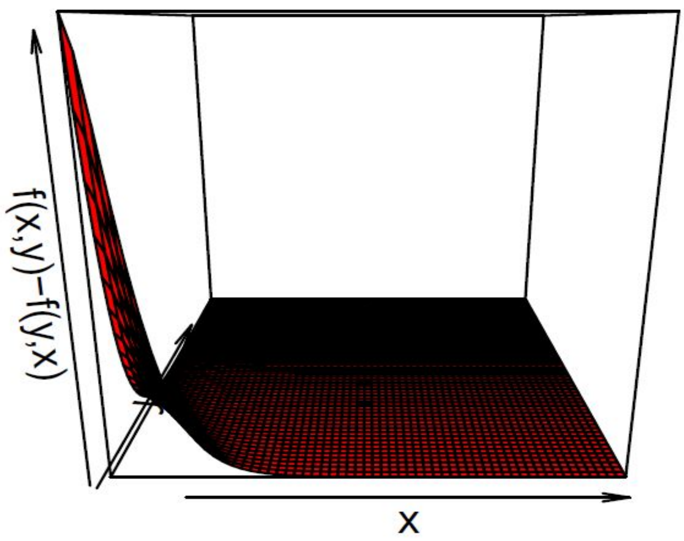


Figure 4.1: Plot of $f_{X,Y}(x,y) - f_{X,Y}(y,x)$ for all $x \leq y$

$$
\begin{aligned}
&- \int_t^\infty \int_x^\infty (6e^{-(3x+4y)})\,\mathrm{d}y\,\mathrm{d}x - \int_t^\infty \int_x^\infty (6e^{-(6x+2y)})\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_t^\infty \int_x^\infty (12e^{-(6x+4y)})\,\mathrm{d}y\,\mathrm{d}x - \int_t^\infty \int_x^\infty (6e^{-(2x+3y)})\,\mathrm{d}y\,\mathrm{d}x \\
&- \int_t^\infty \int_x^\infty (3e^{-(2x+3y)})\,\mathrm{d}y\,\mathrm{d}x + \int_t^\infty \int_x^\infty (6e^{-(4x+3y)})\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_t^\infty \int_x^\infty (6e^{-(2x+6y)})\,\mathrm{d}y\,\mathrm{d}x - \int_t^\infty \int_x^\infty (12e^{-(4x+6y)})\,\mathrm{d}y\,\mathrm{d}x \\
= {}& 9\int_t^\infty e^{-3x}\left(\int_x^\infty e^{-2y}\,\mathrm{d}y\right)\mathrm{d}x - 6\int_t^\infty e^{-3x}\left(\int_x^\infty e^{-4y}\,\mathrm{d}y\right)\mathrm{d}x \\
&- 6\int_t^\infty e^{-6x}\left(\int_x^\infty e^{-2y}\,\mathrm{d}y\right)\mathrm{d}x + 12\int_t^\infty e^{-6x}\left(\int_x^\infty e^{-4y}\,\mathrm{d}y\right)\mathrm{d}x \\
&- 9\int_t^\infty e^{-2x}\left(\int_x^\infty e^{-3y}\,\mathrm{d}y\right)\mathrm{d}x + 6\int_t^\infty e^{-4x}\left(\int_x^\infty e^{-3y}\,\mathrm{d}y\right)\mathrm{d}x \\
&+ 6\int_t^\infty e^{-2x}\left(\int_x^\infty e^{-6y}\,\mathrm{d}y\right)\mathrm{d}x - 12\int_t^\infty e^{-4x}\left(\int_x^\infty e^{-6y}\,\mathrm{d}y\right)\mathrm{d}x \\
= {}& 9\int_t^\infty \frac{e^{-5x}}{2}\,\mathrm{d}x - 6\int_t^\infty \frac{e^{-7x}}{4}\,\mathrm{d}x - 6\int_t^\infty \frac{e^{-8x}}{2}\,\mathrm{d}x + 12\int_t^\infty \frac{e^{-10x}}{4}\,\mathrm{d}x \\
&- 9\int_t^\infty \frac{e^{-5x}}{3}\,\mathrm{d}x + 6\int_t^\infty \frac{e^{-7x}}{3}\,\mathrm{d}x + 6\int_t^\infty \frac{e^{-8x}}{6}\,\mathrm{d}x - 12\int_t^\infty \frac{e^{-10x}}{6}\,\mathrm{d}x \\
= {}& \frac{9}{10}e^{-5t} - \frac{3}{14}e^{-7t} - \frac{3}{8}e^{-8t} + \frac{3}{10}e^{-10t} - \frac{3}{5}e^{-5t} + \frac{2}{7}e^{-7t} + \frac{1}{8}e^{-8t} - \frac{1}{5}e^{-10t}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-5t}\left(\frac{3}{10}+\frac{1}{14}e^{-2t}-\frac{2}{8}e^{-3t}+\frac{1}{10}e^{-5t}\right)\\
&\geq e^{-5t}\left(\frac{2}{8}(1-e^{-3t})+\frac{1}{14}e^{-2t}+\frac{1}{10}e^{-5t}\right)\left(\text{since } \frac{3}{10}\geq\frac{2}{8}\right)\\
&\geq 0.
\end{aligned}
$$

Thus, from Definition 4.2.1, we conclude that $X \leq_{\mathrm{rsp}} Y$.

Chapter 5

Applications of Residual Stochastic Precedence and Inactivity Stochastic Precedence Orders on Covid-19 Data

5.1 Introduction

In the nature, human beings and viruses thrives together from the inception. Some viruses affect the human being while others do not. Moreover, effect of viruses on human life can also be categorized as mild, severe, and mortal. There are also families of viruses that can cause effect on both animals and humans. Coronaviruses are a part of such massive family of viruses. Coronaviruses can cause infections basically effecting the respiratory system of humans. Sometimes, it can be as common cold and in some situations it becomes severe. In the past, humanity has faced such diseases originating from coronaviruses like "Middle East Respiratory Syndrome (MERS)" and "Severe Acute Respiratory Syndrome (SARS)," see for references `https://www.cdc.gov/coronavirus/mers/index.html`, https://www.who.int/ith/diseases/sars/en/, and https://www.cdc.gov/sars/index.html.

In December 2019, a new type of coronavirus spread in Wuhan, China. Afterward, it affected all over the world, and later it was declared as a pandemic by WHO. It was named as "2019-nCoV" and the disease caused by it is known as "COVID-19" (see, for reference, COVID and Team (2020), Zheng, Ma, Zhang, and Xie (2020)). Fatigue, fever, runny nose, and cough are the prevalent symptoms of this disease. Sometimes, patients may also suffer from nasal congestion, sore throat, or diarrhea. The severity of this virus is usually mild in the beginning and worsen gradually. However, some of the infected people do not develop any symptoms. About 80% of the infected people recover from the disease without having need of any particular treatment. In one of every six infected people, the "COVID-19" progresses to severe stage and affects the respiratory system like difficulty in breathing. Also, people of older age and people with comorbidities such as diabetes, high blood pressure, or heart problems are more prone to develop a severe stage of this disease. For more piece of information on "COVID-19," see, "https://www.cdc.gov/," "https://www.mygov.in/covid-19," and "https://www.who.int/emergencies/diseases/novel-coronavirus-2019."

Different organizations, governments, drug companies are trying to develop vaccine and treatment for "COVID-19" on a fast track mode. Several clinical trials are ongoing to evaluate the efficacy of potential vaccines/drugs such as Russia's Sputnik V, AZD1222 of University of Oxford, AstraZeneca, and mRNA-1273 of Moderna, NIAID, BARDA, etc. However, the final results of any of them are yet to come and approve by WHO.

"COVID-19" shows its effects on different strata of people in different ways. Like, its effects on another gender, other age groups are not the same in terms of mortality or recovery rate. We consider the data over different countries globally and try to determine the "COVID-19" effect on males and females. It is challenging to say globally that who is most infected between males and females due to the coronavirus. However, we try to find a clear trend between males and females in different countries. This trend may be varied by country and show changes over time. For this purpose, we use stochastic orders that are used to compare two or more random quantities in a more complex way with the help of underlying distribution functions. These orders are prevalent over last some decades in actuarial science, biology, economics, reliability theory, queuing theory, survival analysis, operations research, and management science (see, for example, Singh and Misra (1994), Nanda and Shaked (2001), Müller and Stoyan (2002), Boland, Singh, and Cukic (2004), Shaked and Shanthikumar (2007), Misra and Misra (2011), Misra and Misra (2012), Balakrishnan and Zhao (2011), Arriaza, Sordo, and Suárez-Llorens (2017), Misra and Francis (2018), Hazra and Misra (2019), and references cited therein). The goal is to show that one random variable (i.e., proportion of males) is larger/smaller than the other (i.e., proportion of females) in some stochastic sense (i.e., mortality rate/survival rate).

For better understanding of ongoing work, let us recall the definitions of some stochastic orders which are used here.

Definition 5.1.1. *Let (X,Y) be a random vector having marginal distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, probability density functions $f_X(\cdot)$ and $f_Y(\cdot)$, survival functions $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$, hazard rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, and reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$. Then X is said to be smaller than Y in the*

(i) *hazard rate order (written as $X \leq_{hr} Y$) if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, if $r_X(x) \geq r_Y(x), \forall x \in \mathbb{R}_+$;*

(ii) *reversed hazard rate order (written as $X \leq_{rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in (0,\infty)$, or equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x), \forall x \in (0,\infty)$;*

(iii) *residual stochastic precedence order (written as $X \leq_{rsp} Y$) if*

$$\int_t^\infty \bar{F}_X(x)\bar{F}_Y(x)\left[r_X(x) - r_Y(x)\right] dx \geq 0, \text{ for all } t \geq 0;$$

(iv) *inactivity stochastic precedence order (written as* $X \leq_{isp} Y$*) if*

$$\int_0^t F_X(x)F_Y(x)\left[\tilde{r}_X(x)-\tilde{r}_Y(x)\right]dx \leq 0, \text{ for all } t>0.$$

Let us also recall the following relationships that are well known (see, Misra, Gupta, and Misra (2020) and Misra, Gupta, and Chanchal (2020)).

$$X \leq_{\mathrm{hr}} Y \quad \Rightarrow \quad X \leq_{\mathrm{rsp}} Y$$

$$X \leq_{\mathrm{rh}} Y \quad \Rightarrow \quad X \leq_{\mathrm{isp}} Y$$

5.2 Materials and methods

In this section, we discuss about the data which is used for the analysis and statistical methods which are used to perform the analysis.

5.2.1 Data source and description

"Covid-19" data used here is obtained from the website `https://globalhealth5050.org/covid19/sex-disaggregated-data-tracker/` updated as on October 20, 2020. This data is disaggregated by both age and sex on confirmed cases and deaths from those nations where this data is reported. Moreover, it is reported per $100,000$ men and women in the population. The data shows total cases, total death cases, total portion of cases of males and females (indicated as CMP and CFP), total percentage of death of males and females (indicated as DMP and DFP), and deaths among confirmed cases of males and females (indicated as DACCM and DACCF). Table 5.1 shows the preview of the data.

5.2.2 Underlying theory

In this section, we discuss *P*-*P* plot and also discuss the stochastic orders that are obtained with the help of *P*-*P* plot.

5.2.2.1 *P*-*P* plot

P-*P* plot is used to visualize the stochastic comparison of the "location" or "magnitudes" of two or more random variables. Let X and Y be two continuous random variables having cumulative distribution functions F_X and F_Y, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, and density functions f_X and f_Y, respectively. We use the notation F^{-1} to denote the right continuous inverse of F. Graph of F_Y versus F_X is visualized with the help of *P*-*P* plot and expressed in the function form as $F_Y F_X^{-1}$. We uses the hazard rate order and the reversed hazard rate order, identified with the help of the *P*-*P* plot, and see how it can help in understanding the findings of data (see, for reference, Wilk and Gnanadesikan (1968), Dewan and Kochar (2013), Chambers (2018)).

5.2.2.2 Stochastic orders detected with the help of *P*-*P* plot

It is already proved that $X \leq_{\text{hr[rh]}} Y$ holds if, and only if, the *P*-*P* plot is star-shaped with respect to $(1,1)[(0,0)]$ (see Müller and Stoyan (2002, p.11)), and recall that $X \leq_{\text{hr[rh]}} Y$ implies $X \leq_{\text{rsp[isp]}} Y$.

5.3 Results and discussions

In this section, we first calculate the descriptive statistics of the Covid-19 data which is shown with the help of Table 5.2. Then, we obtain results by using R-software and also give its interpretation.

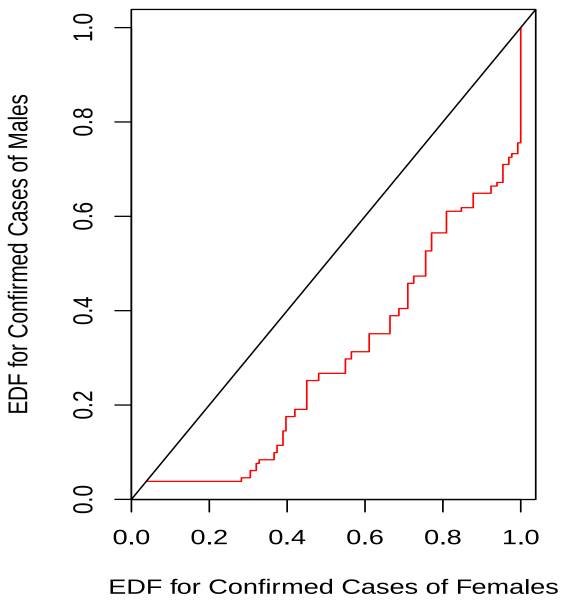


Figure 5.1: *P-P* plot of empirical distribution functions of percentage of confirmed cases of males and females

Figure 5.1 represents the *P-P* plot of the empirical distribution functions of the percentage of confirmed cases of males and females. In this Figure, we can see that *P-P* plot lies entirely below the 45 degree line which shows that F_Y is stochastically greater than F_X and also if we join any chord from $(1,1)$ any point on the P-P plot is entirely above the *P-P* plot which shows hazard rate order holds between them. However, if we join chords from $(0,0)$ to any point on the *P-P* plot then some of them lies entirely below the plot which confirms that the reversed hazard rate order does not holds between them. Thus, *P-P* plot is star-shaped for $(1,1)$ and also the *P-P* plot is not star-shaped for $(0,0)$. Thus, the hazard rate order holds between the percentage of confirmed cases of males and females, and hence the residual stochastic precedence order also holds between them. Therefore, the percentage of confirmed "Covid-19" cases of females is higher than the percentage of confirmed “Covid-19” cases of males, or we can say that the cases of females is higher than that of males due to the coronavirus updated as on October 20, 2020, which may show changes over time. However, this result is just the opposite when data is considered till May 2020 (see, Figure 5.2).

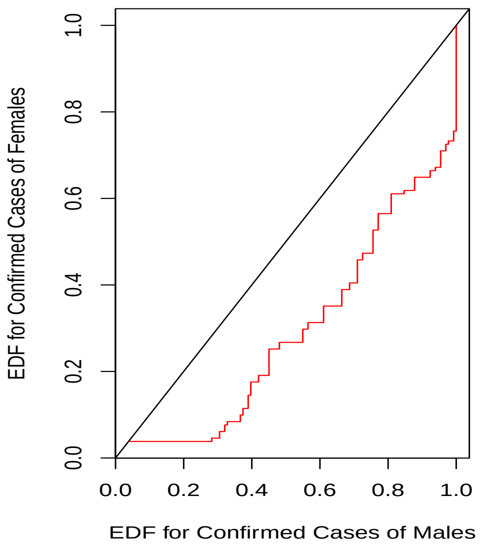


Figure 5.2: *P-P* plot of empirical distribution functions of percentage of confirmed cases of males and females till May 2020

Figure 5.3 represents the *P-P* plot of the empirical distribution functions of the death percentage of males and females. Again, in this Figure, we can see that *P-P* plot lies entirely below the 45 degree line which shows that F_Y is stochastically greater than F_X and also if we join any chord from $(1,1)$ any point on the P-P plot is entirely above the *P-P* plot which shows hazard rate order holds between them. However, if we join chords from $(0,0)$ to any point on the *P-P* plot then all chords does not lies entirely below the plot which confirms that the reversed hazard rate order does not holds between them. Thus, *P-P* plot is star-shaped for $(1,1)$ and also the *P-P* plot is not star-shaped for $(0,0)$. Thus, again the hazard rate order holds between the percentage of death of males and females, and hence the residual stochastic precedence order also holds between them. Therefore, the percentage of death of females is higher than the percentage of death of males due to

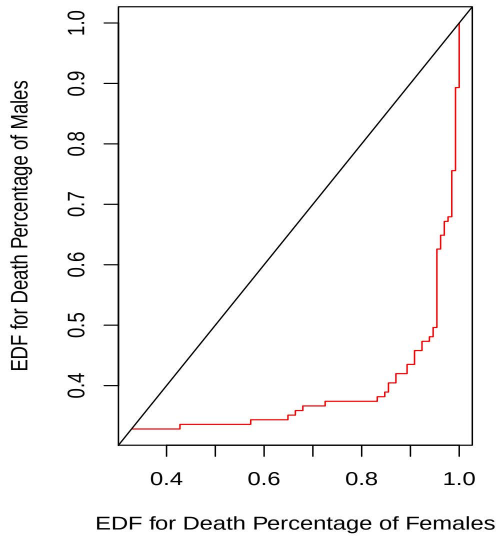


Figure 5.3: *P*-*P* plot of empirical distribution functions of percentage of death of males and females

"Covid-19," or we can say that the mortality rate of females is higher than that of males due to the coronavirus updated as on October 20, 2020, which may show changes over time. However, again this result is just the opposite when data is considered till May 2020 (see, Figure 5.4).

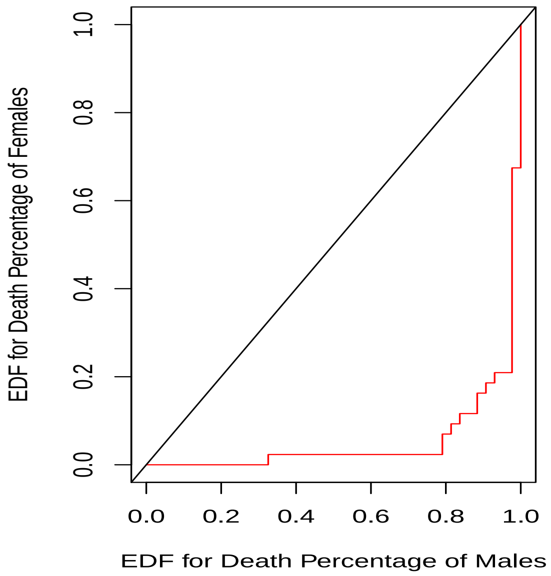


Figure 5.4: *P*-*P* plot of empirical distribution functions of percentage of death of males and females till May 2020

Table 5.1: Preview of Covid-19 data updated as on October 20, 2020

Country	Cases	CMP	CFP	Deaths	DMP	DFP	DACCM	DACCF
Afghanistan	39093	69.75	30.25	1378	74.82	25.18	3.78	2.93
Albania	15752	48	52	429	67	33	3.8	1.73
Algeria	53225	53.93	46.07	0	0	0	0	0
Argentina	928391	50.64	49.36	8597	57	43	2.21	1.71
Armenia	0	0	0	1026	54.29	45.71	0	0
Australia	27301	48.61	51.39	896	48.1	51.9	3.25	3.31
Austria	57762	52	48	879	57	43	1.67	1.36
Azerbaijan	40119	46	54	0	0	0	0	0
Bahrain	6081	88	12	0	0	0	0	0
Bangladesh	379738	71	29	5555	77	23	1.59	1.16
Belgium	165113	44.5	55.5	10186	47.39	52.61	6.57	5.85
Belize	2585	53.89	46.11	39	61.54	38.46	1.72	1.26
Bhutan	309	69.58	30.42	0	0	0	0	0
Bosnia	12710	52	48	354	66	34	3.54	1.98
Botswana	48	83	17	15	40	60	0	0
Brazil	114617	46	54	142764	57.96	42.04	0	0
Bulgaria	2443	49	51	0	0	0	0	0
Burkina	2305	66.33	33.67	55	74.51	25.49	4.63	3

Table 5.2: Descriptive Statistics of the Covid-19 data

Statistics	CMP	CFP	DMP	DFP	DACCM	DACCF
1st Quartile	47.03	35.95	0	0	0	0
Median	52.00	42.84	54.06	35.80	1.510	1.030
Mean	53.35	58.89	39.68	27.49	2.183	1.543
3rd Quartile	58.89	52.00	62.23	43.83	3.200	2.235
Maximum	91.00	60.00	77.00	69.00	13.150	9.670
Standard deviation	15.04697	13.63813	28.91892	20.80978	2.902319	1.998846

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