CONTRIBUTIONS TO INFERENTIAL PROCEDURES FOR SOME RELIABILITY CHARACTERISTICS

THESIS

SUBMITTED TO

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Under the Supervision of

Dr. Surinder Kumar

Professor and Head

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BABASAHEB BHIMRAO AMBEDKAR UNIVERSITY
(A CENTRAL UNIVERSITY)
Vidya Vihar, Raebareli Road, Lucknow-226025 (U.P.), India

Enrolment No. - 728/18 2021

Dedicated to

My Father

Late Prof. Ajit Chaturvedi

DECLARATION

I, **Aditi Chaturvedi**, Enrolment No. 728/18, hereby declare that the work which is being presented in the thesis entitled **"Contributions to Inferential Procedures for some Reliability Characteristics"** in fulfillment of the requirements for the award of the degree of Doctor of Philosophy and submitted in the Department of Statistics of the Babasaheb Bhimrao Ambedkar University (A Central University), Lucknow is an authentic record of my own work carried out during a period from August, 2018 to August, 2021 under the supervision of Dr. Surinder Kumar, Professor and Head, Department of Statistics, School of Physical & Decision Sciences, Babasaheb Bhimrao Ambedkar University, Lucknow.

The matter presented in this thesis has not been submitted by me for the award of any other degree or diploma to this or any other University.

This is also declared that the thesis is essentially free from all kinds of plagiarism.

Achaturvedi

Date: 20/10/2021

Place: LUCKWOW

(Aditi Chaturvedi)

Research Scholar,

Department of Statistics,

School of Physical & Decision Sciences,

Babasaheb Bhimrao Ambedkar University,

Lucknow-226025, India.

CERTIFICATE

This is to certify that the thesis titled **"Contributions to Inferential Procedures for some Reliability Characteristics"** submitted by **Ms. Aditi Chaturvedi** is an original research work and has not been previously submitted in part or full for the award of any other degree or diploma to this or any other University.

The thesis submitted to Babasaheb Bhimrao Ambedkar University Lucknow satisfies all the requirements as stipulated in the *Master of Philosophy (M.Phil.)/ Doctor of Philosophy (Ph.D.) regulations* amended in *2017* and it is fit for submission and evaluation for the award of the degree of Doctor of Philosophy of the University.

Date: 20/10/2021

Supervisor
(Prof. Surinder Kumar)

Head of the Department

Head
Department of Statistics
Babasaheb Bhimrao Ambedkar University
(A Central University) Lucknow-226025

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AChaturvedi

(Aditi Chaturvedi)

Date: 20/10/2021

Place: LUCKNOW

LIST OF RESEARCH PAPERS

1. LIST OF PUBLISHED PAPERS

- Kumar, S. and Chaturvedi, A., (2020). On a generalization of the positive exponential family of distributions and the estimation of reliability characteristics, *Statistica*, 80(1), 57-77.
- Chaturvedi, A. and Kumar, S., (2021). Estimation and Testing Procedures for the Reliability Characteristics of Chen Distribution Based on Type II Censoring and the Sampling Scheme of Bartholomew, *Statistics, Optimization & Information Computing*, 9(1), 99-122.
- Chaturvedi, A. and Kumar, S., (2020). Estimation Procedures for the Reliability Functions of Kumaraswamy-G Distribution Based on Type II Censoring and the Sampling Scheme of Bartholomew, *Statistics In Transition-New Series.* (Accepted)

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- On the Estimation of the Reliability Characteristics of a Weighted Generalized Positive Exponential Family of Distributions.

3. LIST OF PAPERS PRESENTED IN CONFERENCE

- Attended and presented paper titled "On a Generalization of the Positive Exponential Family of Distributions and the Estimation of Reliability Characterstics" in the "Twenty- Seventh International Conference of Forum for Interdisciplinary Mathematics (FIM) in Conjuction with Third Convention of IARS On Interdisciplinary Mathematics, Statistics and Computational Techniques (IMSCT 2018- FIM XXVII)" from November 02–04, 2018.
- Attended and presented paper titled "On the Estimtion of Stress-Strength Reliability of a Generalization of the Positive Exponential Family of

Distributions" in the "24th International Conference of International Academy of Physical Sciences (CONIAPS XXIV)" from August 09–11, 2019.

- Attended and presented paper titled "Estimation and Testing Procedures for the Reliability Functions of Kumarswamy-G Distributions based on Censorings" in the "2019 International Indian Statistical Association (IISA) Conference" from December 26–30, 2019.
- Attended and presented paper titled "Estimation Procedures for the Reliability Characteristics of Chen Distribution based on Type II Censoring and the Sampling Scheme of Bartholomew" in the "1st International Conference On Advances in Mathematics, Science and Technology (ICAMST-2020)" from September 01–03, 2020.
- Attended and presented paper titled "On the Estimation of the Reliability Characteristics of a Weighted Generalized Positive Exponential Family of Distributions" in the "International Virtual Conference on Prof. C.R. Rao's School of Thought on Statistical Sciences (ICON-CRRAO-STOSS 2020)" from November 21–22 and November 28–29, 2020.

4. BEST PAPER AWARDS

- Received R.S. Verma Best Paper Award (Second Runner-up) for paper entitled "On a Generalization of the Positive Exponential Family of Distributions and the Estimation of Reliability Characteristics" presented in Twenty -Seventh International Conference of Forum for Interdisciplinary Matheatics (FIM) in Conjunction with Third convention of IARS On Interdisciplinary Mathematics, Statistics and Computational Techniques (IMSCT 2018- FIM XXVII) organized by Department of Statistics, University of Jammu, Jammu, India.
- Received Best Theoretical Research Paper award (Second Position) for the paper entitled "On the Estimation of Reliabiity Characteristics of

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Contents

List of Tables

List of Figures

Chapter 1

Introduction and the Related Concepts

1.1 An Overview of Reliability and the Associated Characteristics

The term "reliability" is commonly used in our day-to-day life. We often talk about whether a particular brand of refrigerator, mobile phone, washing machine or television set is reliable or not. When, we say that a television set manufactured by a particular company is reliable, it means that its functioning is hassle-free. However, the possibility of its failure can not be completely ruled out because the performance of that television set will necessarily depend on the quality of its components, some past experiences of its use etc., and we can only conclude that the chances of its failure are very few.

The rapid advancement of technology has put a very strong challenge before manufacturers because the demand for supreme quality products having a long life is the need of the hour. Everyone expects a high degree of reliability of the product to be purchased and this expectation becomes extremely high when the product is useful for some important purposes like defence, security, space research or anything which is directly or indirectly related to the national security of any state. It is worth mentioning here that the manufacturers also understand these demands and they launch their products after assessing the effectiveness, quality, lifetime or other characteristics which affect the reliability of the product.

In order to attract more and more consumers, manufacturers provide warranty or guarantee on their products. Therefore a good knowledge of the failure time distribution of any product becomes a mandatory task for the manufacturers. It leads them to conduct reliability and life testing experiments before launching the products in the market. These experiments are carried out in the supervision of quality control inspectors, scientists, statisticians and other experts. As a result, the reliability and lifetime of the products improve, which is somehow in favour of the consumers as well as manufacturers.

The entire subject of reliability has a wide range of applications in various areas like safety/risk analysis, quality control, environmental protection, engineering technology, transport mechanism, among many others. Due to these applications, there is remarkable progress in the study of reliability in almost all developed and developing countries. The motive is to improve the quality of products and increase their average life through reliability and life testing experiments. The complex mechanism and automation of industrial processes are leading us to increase the reliability of units or systems as well.

One can define the term "reliability" as the probability that a system or device performs its required function properly for a given period of time under the stated operating conditions. To be more specific, we can define the term reliability, denoted by $R(t_\circ)$, as the probability of failure-free operation until time $t_\circ$. Thus, if the lifetime of an item is denoted by X; then the reliability function is defined as

$$R(t_\circ) = P(X > t_\circ), \tag{1.1.1}$$

where P stands for the probability. In the succeeding chapters, $t_\circ$ is replaced by t.

Reliability can also be defined in a stress-strength setup. Suppose that for a component or unit, the r.v. X denotes the amount of strength and the r.v. Y

represents the random stress; then the reliability function is defined as

$$P = P(X > Y). \tag{1.1.2}$$

This is known as the "stress-strength" reliability. It means that a component will fail if the stress exceeds the strength and P serves as the reliability parameter when the amount of strength is more than the stress. The stress-strength model was first introduced by Birnbaum (1956) and developed by Birnbaum and McCarty (1958). One may refer to Kotz *et al.* (2003) for a general overview of the applications of stress-strength reliability.

Now, we define some reliability characteristics which are very important for the study of any system or component.

1.1.1 Hazard Rate Function or Failure Rate Function

The hazard function is defined as the failure of an item at time t given that it has survived till time t. Also referred to as the instantaneous failure rate or instantaneous death rate, the function is represented by $h(t;\theta)$ and is numerically defined as

$$h(t;\theta) = \lim_{\Delta t \to 0} \frac{P(t \le T \le t + \Delta t | T \ge t)}{\Delta t} = \frac{f(t;\theta)}{R(t;\theta)},$$

where, $t > 0$, $f(t;\theta) > 0$, $R(t;\theta) > 0$. The hazard function, when plotted, can take different shapes like constant, increasing, decreasing, bathtub, upside-down bathtub, etc. and are very useful in recognising the lifetime distribution exhibited by the data available with us.

1.1.2 Mean Residual Life (MRL)

Suppose that a unit is of age t, then the remaining life after time t is random. The expected value of the random residual life after time t, is called the Mean Residual Life at time t. Let X denote random life with cdf $F(t)$ with a finite first moment. Let $\bar{F}(t) = 1 - F(t)$. Then MRL is defined as

$$\begin{aligned} m(t) &= E\{X - t|X > t\} \\ &= \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)}dx \\ &= \int_t^\infty \frac{\bar{F}(u)}{\bar{F}(t)}du, \end{aligned}$$

where, $t > 0$, $\bar{F}(t) > 0$. Watson and Wells (1961) discussed the usage of MRL in studying burn-in. Actuaries apply MRL while deciding insurance rates and insurance coverage in life insurance. Bhattacharjee (1982) demonstrated the applications of MLR functions in areas such as optimal disposal of an asset, renewal theory, dynamic programming, and branching processes.

1.1.3 Mean Time to Failure (MTTF)

Mean time to failure or Mean Time to System failure (MTSF) is defined as the expected time to failure for a non-repairable system. The MTTF of a unit is defined as

$$MTTF = E(T) = \int_0^\infty tf(t)dt,$$

where $t > 0$, $f(t) > 0$.

1.2 Lifetime Distributions

The distributions which represent the life data of an item or group of individuals are called lifetime distributions. Some of the commonly used distributions are discussed below:

1.2.1 Exponential Distribution

The exponential distribution, due to its simplicity, is one of the most widely used distributions. The pdf of exponential distribution is given by

$$f(t) = \lambda e^{-\lambda t}; \ \lambda > 0, \ t > 0.$$

The reliability function, $R(t)$ and the hazard rate -function, $h(t)$ are respectively given by

$$R(t) = exp(-\lambda t)$$

and

$$h(t) = \lambda \text{ (constant)}.$$

The Mean and Variance of this distribution are given by $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ respectively. It has constant hazard rate and is often suitable to model the lifetimes of components having a long chance-failure region. An important property of this distribution is that the failure process represented by it has memoryless property, which does not hold for any other continuous distribution.

1.2.2 Weibull Distribution

Another widely used distribution in reliability analysis is Weibull distribution. The distribution is flexible in the sense that it can be used to describe all the three regions of the bathtub curve. The pdf of this distribution is given by:

$$f(t) = \frac{\beta t^{\beta-1}}{\alpha^{\beta}} exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]; \;\; \alpha, \beta > 0, \;\; t > 0,$$

where α and β are shape and scale parameters, respectively.
The Mean and Variance of this distribution are given by $\alpha\Gamma\left(1 + \frac{1}{\beta}\right)$ and $\alpha^2 \times \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \left(\Gamma\left(1 + \frac{1}{\beta}\right)\right)^2\right]$ respectively. The reliability function, $R(t)$ and the hazard rate function, $h(t)$ are respectively given by

$$R(t) = exp\left[\left(-\frac{t}{\alpha}\right)^{\beta}\right]$$

and

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}.$$

For $0 < \beta < 1$, the Weibull distribution exhibits Decreasing Failure Rate (DFR) while for $\beta = 1$, it reduces to exponential distribution and for $\beta > 1$, it exhibits Increasing Failure Rate (IFR). Weibull distribution is widely used in corrosion resistance studies, studies related to time to failure of different types of hardware, e.g., capacitors, electron tubes, ball bearings, motors, etc. and components and parts of a system.

1.2.3 Lognormal Distribution

Another widely used distribution in reliability engineering is Lognormal distribution. The pdf of distribution is given by

$$f(t) = \frac{1}{\sigma_t t\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma_t^2}(lnt - \mu_t)^2\right], \; 0 < t < \infty, \; -\infty < \mu_t < \infty, \; \sigma_t > 0,$$

where $\mu_t = E(lnt)$ and $\sigma_t^2 = Var(lnt)$. The reliability function, $R(t)$ and the hazard function, $h(t)$ are respectively given by

$$R(t) = 1 - \Phi\left(\frac{log\, t - \mu}{\sigma}\right)$$

and

$$h(t) = \frac{\frac{1}{\sigma_t t\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma_t^2}(lnt - \mu_t)^2\right]}{1 - \Phi\left(\frac{log\, t - \mu}{\sigma}\right)},$$

where $\Phi(\cdot)$ is the cumulative standard normal function. The failure rate of this distribution initially increases over time and then decreases. The values of μ_t and σ_t^2 determine the rate of increase and decrease of the failure rate. This model is appropriate in situations where small multiplicative errors result in failure of processes. It can also be used to model the data of failure of components due to mechanical fatigue or to model the lifetime of electronic components. It is popularly used as the prior distribution in Bayesian Analysis.

1.2.4 Inverse Gaussian Distribution

When the product experiences high occurences of early repairs or failures, such as, accelerated life testing and repair time situation and it is expected that failure rate will be nonmonotonic, i.e., first increasing and then decreasing, the Inverse Gaussian may prove to be a good choice for lifetime model. The pdf of Inverse Gaussian distribution is given by

$$f(t;\mu,\lambda) = \left\{\frac{\lambda}{2\pi t^3}\right\}^{1/2} \exp\left\{\frac{-\lambda\,(t-\mu)^2}{2t\mu^2}\right\}, x > 0,$$

where, $\mu > 0$ and $\lambda > 0$ are the location and scale parameters respectively. The mean and the variance of this distribution are given by μ and $\frac{\mu^3}{\lambda}$ respectively. The reliability function, $R(t)$ and the hazard function, $h(t)$ are respectively given by

$$R(t) = \Phi\left[\sqrt{\frac{\lambda}{t}}\left(1 - \frac{t}{\mu}\right)\right] - exp\left(\frac{2\lambda}{\mu}\right) \cdot \Phi\left[-\sqrt{\frac{\lambda}{t}}\left(1 + \frac{t}{\mu}\right)\right]$$

and

$$h(t) = \left(\frac{\lambda}{2\pi t^3}\right)^{1/2} \cdot \frac{\exp\left\{\frac{-\lambda(t-\mu)^2}{2t\mu^2}\right\}}{\Phi\left[\sqrt{\frac{\lambda}{t}}\left(1 - \frac{t}{\mu}\right)\right] - exp\left(\frac{2\lambda}{\mu}\right) \cdot \Phi\left[-\sqrt{\frac{\lambda}{t}}\left(1 + \frac{t}{\mu}\right)\right]}.$$

The benefit of this distribution is that unlike lognormal distribution the physical aspect of Brownian motion or any Gaussian process which inspired the Gaussian distribution as the first passage time distribution shows its natural applicability in studying life testing.

A large amount of work surrounding these distributions is available in the literature of reliability theory. Kelley *et al.* (1976), Sathe and Shah (1981) and Tong (1974, 1975) considered the stress-strength reliability estimation under exponential distribution. Constantine *et al.* (1986) worked on the same problem under gamma distribution. Kundu and Gupta (2006) dealt with the estimation of stress-strength reliability parameter under Weibull model. Krishnamoorthy *et al.* (2007) considered the reliability estimation in two-parameter exponential stress-strength model. Some more references include Chao (1982), Guo and Krishnamoorthy (2004), Baklizi and El-Masri (2004) and others. One may refer to Rostamian and Nematollahi (2019) for a quick review of the inverse Gaussian stress-strength model.

1.3 Censoring Schemes

In real life, due to cost and time factors or due to the destructive property of some life experiments, it may not be possible or desirable to observe each and every unit available for testing experiment. Such experiments, which are terminated before the failure of all the items put on test, either intentionally or unintentionally, are known as censored experiments and the samples obtained after censoring are known as censored samples. Different kinds of censoring schemes are available according to the requirements of the experimenter(s). Some of the censoring schemes used in the thesis are discussed below:

1.3.1 Type I Censoring

Suppose n items, say, $X_1, X_2, \ldots, X_n$, are put on a test and the test is terminated as soon as it reaches a pre-assigned time point, say $t_\circ$. Let the lifetimes of n items be i.i.d. with pdf $f(x)$ and cdf $F(x)$. The number of failed items, say m, during this time period are not replaced and constitute the sample. The data consists of the failure times $x_{(1)} < x_{(2)} < \ldots < x_{(m)}$ of m items that failed before $t_\circ$. The likelihood function is given by

$$L(\underset{\sim}{x}) = \frac{n!}{(n-m)!}\prod_{i=1}^{n} f(x_{(i)})[1-F(t_\circ)]^{n-m};\;\; 0 < x_{(1)} < x_{(2)} < \ldots < x_{(m)} < t_\circ < \infty.$$

This type of censoring is known as Type I censoring or Time Censoring. Here, the number of the failures observed is random. One may refer to Cohen (1965), Sirvanci and Yang (1984) and Balakrishnan and Aggarwala (2000) for a quick review on the type I censoring.

1.3.2 Type II Censoring

Let us consider the situation where the failed items are not replaced. Suppose n items, say, $X_1, X_2, \ldots, X_n$ are put on a test and the test is terminated as soon as a fixed number of items, say r fail. The data consists of the lifetimes of the first r items, say, $x_{(1)} < x_{(2)} < \ldots < x_{(r)}$, that failed. The remaining $(n-r)$ items are removed from the experiment. The joint distribution of $x_{(1)} < x_{(2)} < \ldots < x_{(r)}$ is given by

$$L(\underset{\sim}{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i)[1-F(x_{(r)})]^{(n-r)};\quad x_{(1)} < x_{(2)} < \ldots < x_{(r)}.$$

This type of censoring is known as Type II censoring or Failure Censoring. Here, the number of failures is pre-assigned while termination time to obtain the failures is random. A good amount of work on type II censoring comprises of Sinha (1986), Wingo (1993), Balakrishnan and Aggarwala (2000), Shah and Patel (2011) and the references therein.

1.3.3 Random Censoring

In random censoring, some of the items under observation are either lost or removed randomly from the study before they experience failure. For example, in the study of the life of electric bulbs, some of the bulbs under observation may break before they experience failure. In such a case, the exact time of survival of the items is unknown, and hence, they are known as randomly censored observations. Let the failure times, $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.s. with pdf and cdf denoted by $f_X(x)$ and $F_X(x)$, respectively, where $x > 0$. Associated with these failure times $T_1, T_2, \ldots, T_n$ are i.i.d. censoring times with pdf and cdf $f_T(t)$ and $F_T(t)$, respectively, where $t > 0$. In random censoring, the censoring time of items is statistically independent of the failure times. We observe failure or censored time $Y_i = min(X_i, T_i)$; $i = 1, 2, \ldots, n$ and the corresponding censor indicators $D_i = 1(0)$ if failure (censoring) occurs. Since X_i and T_i are assumed to be independent, so will be Y_i and D_i, $i = 1, 2, \ldots, n$. As a result, the joint pdf of Y and D is given by

$$L(y, d) = \{f_X(y)\,(1 - F_T(y))\}^d \, \{f_T(y)\,(1 - F_X(y))\}^{1-d} \tag{1.3.1}$$

One may refer to Ghitany and Al-Awadhi (2002), Friesl and Hurt (2007), Wu *et al.* (2007), Einmahl *et al.* (2008), Wang and Veraverbeke (2009), Danish and Aslam (2013), Garg *et al.* (2016, 2020) and Krishna and Goel (2018) for a detailed overview of the random censoring scheme.

1.4 Classical Theory of Estimation

1.4.1 Some Important Concepts and Results

Definition 1.4.1. ***Point Estimator:*** *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n) \sim P_{\boldsymbol{\theta}}$*, where* $\boldsymbol{X}$ *is a vector of random variables and* $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ *is a vector of unknown* *parameters, $\theta \in \Theta$. A statistic $\delta(\boldsymbol{X})$ is called the point estimator of ϕ, where ϕ is a real-valued function defined on Θ, if $\delta : \mathfrak{X} \to \Theta$ where $\mathfrak{X}$ denotes the space of values of $\boldsymbol{X}$.*

Definition 1.4.2. ***Unbiasedness:*** *A function $T(\boldsymbol{X})$ is known as the unbiased estimator of a parameter θ if*

$$bias T_\theta(\boldsymbol{X}) = 0 \quad \forall\ \theta \in \Theta,$$

or

$$E_\theta T(\boldsymbol{X}) = \theta \quad \forall\ \theta \in \Theta.$$

Definition 1.4.3. ***Consistency:*** *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common cdf $F_{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \Theta$. A sequence of point estimators $T_n = T_n(X_1, X_2, \ldots, X_n)$ is said to be a consistent estimator for $\phi(\boldsymbol{\theta})$ if*

$$T_n \xrightarrow{P} \phi(\boldsymbol{\theta}) \qquad \text{as } n \to \infty,$$

for each fixed $\boldsymbol{\theta} \in \Theta$.

Theorem 1.4.1. *Let T_n be a sequence of estimators such that $ET_n \to \phi(\boldsymbol{\theta})$ and $var(T_n) \to 0$ as $n \to \infty$, then T_n is said to be consistent for $\phi(\boldsymbol{\theta})$.*

Definition 1.4.4. ***Sufficiency:*** *The concept of sufficiency was introduced in order to reduce the volume and maintenance cost of the data collected to estimte the parameter θ. A function $T(\boldsymbol{X})$ is known as the sufficient statistic for the family of distributions $\mathcal{F} = F_\theta$, $\theta \in \Theta$ if the conditional distribution of $\boldsymbol{X}$ given $T = t$ is independent of θ $\forall\, t$.*

Another simple criterion for determining sufficient statistic is given in the Theorem below:

Theorem 1.4.2. ***Factorization theorem:*** *Let $X_1, X_2, \ldots, X_n$ be a sequence of r.v.s with pmf $p_\theta(x_1, x_2, \ldots, x_n)$ where $\theta \in \Theta$. The statistic $T(X_1, X_2, \ldots, X_n)$ is said to be sufficient statistic for the parameter θ if and only if*

$$p_\theta(x_1, x_2, \ldots, x_n) = k(x_1, x_2, \ldots, x_n) h_\theta(T(x_1, x_2, \ldots, x_n)),$$

where k is a non-negative function of $x_1, x_2, \ldots, x_n$ only and is independent of θ while h_θ is a non-consistent and non-negative function of θ and $T(x_1, x_2, \ldots, x_n)$ only.

Definition 1.4.5. ***Completeness:*** *The family of pmfs or pdfs $\{f_\theta(x),\ \theta \in \Theta\}$ is said to be complete if for some function $h(x)$*

$$E_\theta h(x) = 0 \ \forall\, \theta \in \Theta$$

implies that

$$P_\theta \left\{h(x) = 0\right\} = 1 \ \forall\, \theta \in \Theta.$$

A statistic $T(X)$ which belongs to family of pmfs or pdfs $\{f_\theta(x),\ \theta \in \Theta\}$ is said to be complete if its family of distributions is complete.

Definition 1.4.6. ***Efficiency:*** *Let T_1 and T_2 be two unbiased estimators for a parameter θ. Let $E_\theta T_1^2 < \infty$ and $E_\theta T_2^2 < \infty$. The efficiency of T_1 relative to T_2 is defined as*

$$eff_\theta(T_1|T_2) = \frac{Var_\theta(T_2)}{Var_\theta(T_1)}$$

and T_1 is said to be more efficient than T_2 if

$$eff_\theta(T_1|T_2) > 1.$$

In this thesis, we have used Mean Square Error (MSE) to compare the performance of the estimators. MSE of an estimator T of a parameter θ is defined as

$$MSE(T) = E_\theta \left[(T-\theta)^2\right].$$

1.4.2 Uniformly Minimum Variance Unbiased (UMVU) Estimator

Definition: Let U' be the class of all unbiased estimators T of $\theta \in \Theta$ where $E_\theta T^2 < \infty \ \forall\ \theta \in \Theta$. An estimator $T' \in U'$ is called a UMVU estimator of θ if

$$E_\theta(T' - \theta)^2 \leq E_\theta(T-\theta)^2$$

$\forall\ \theta \in \Theta$ and every $T \in U'$.

Theorem 1.4.3. *Let us consider a family of pdfs* $\{F_\theta : \theta \in \Theta\}$ *and let* k *be any statistic belonging to a non-empty class* U *of all unbiased estimators of* θ *with* $E_\theta k^2 < \infty$*. Let* T *be a sufficient statistic for* $\{F_\theta,\, \theta \notin \Theta\}$*. Then the conditional expectation* $E_\theta \{k|T\}$ *will be independent of* θ *and is also an unbiased estimator of* θ*. Further,*

$$E_\theta(E\{k|T\} - \theta)^2 \leq E_\theta(k - \theta)^2 \;\; \forall\, \theta \in \Theta.$$

Theorem 1.4.4. ***Lehmann-Scheffé Theorem*** *If* T *is a complete sufficient statistic and* k *is an unbiased estimator of* θ*, then there exists a unique UMVU estimator of* θ *and it is given by* $E\{k|T\}$*.*

1.4.3 Maximum Likelihood (ML) Estimator

The principle of maximum likelihood initially considers the sample as the representative of the population and chooses that estimator as the representative of the parameter which maximizes the pdf or pmf $f_\theta(x)$.

Definition 1.4.7. ***Likelihood Function:*** *Let* $(X_1, X_2, \ldots, X_n)$ *be a random sample drawn from the population with pdf (pmf)* $f(x; \boldsymbol{\theta})$*, where* $\boldsymbol{\theta}$ *is the vector of unknown parameters* $\theta \in \Theta$*. The joint distribution of sample observations is given by*

$$L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i}^{n} f(x_i, \boldsymbol{\theta}) \tag{1.4.1}$$

Equation (1.4.1) represents the joint density of $\boldsymbol{\theta}$ *for given* $\boldsymbol{x}$ *and is known as the Likelihood Function.*

Definition 1.4.8. *The value* $\boldsymbol{\theta}_\circ \in \Theta$ *is said to be ML estimator of* $\boldsymbol{\theta} \in \Theta$*, given the sample of observations* $\boldsymbol{x}$*, if*

$$L(\boldsymbol{\theta}_\circ; \boldsymbol{x}) = \max_{\theta \in \Theta} L(\boldsymbol{\theta}; \boldsymbol{x}) \tag{1.4.2}$$

Notice that the range of the parameters $\boldsymbol{\theta}$ *and their ML estimator* $\boldsymbol{\theta}_\circ$ *must be the same and should lie in* Θ*.*

For simplicity, one can write (1.4.2) as

$$log\ L(\boldsymbol{\theta}_\circ; \boldsymbol{x}) = \underset{\theta \in \Theta}{max}\ log\ L(\boldsymbol{\theta}; \boldsymbol{x}) \tag{1.4.3}$$

since log is a monotonic function. This function is known as log-likelihood function. Hence, ML estimator of $\boldsymbol{\theta}$ can be obtained from the solution of the equations

$$\frac{\partial}{\partial \theta_i} log\ L(\boldsymbol{\theta}; \boldsymbol{x}) = 0; \;\; i = 1, 2, \ldots, k. \tag{1.4.4}$$

1.4.4 Method of Moment (MM) Estimator

MM estimator is one of the oldest and simplest method of estimation. Let us consider a random sample $X_1, X_2, \ldots, X_n$ of size n drawn from the population with pdf or pmf $f(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of k unknown parameters, i.e., $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, $\boldsymbol{\theta} \in \Theta$. MM estimator of the k parameters are obtained by solving the following k equations simultaneously,

$$\begin{aligned} m_1^{'} &= \mu_1^{'}(\boldsymbol{\theta}) \\ m_2^{'} &= \mu_2^{'}(\boldsymbol{\theta}) \\ &\vdots \\ m_k^{'} &= \mu_k^{'}(\boldsymbol{\theta}), \end{aligned}$$

where $\mu_j^{'}$ and $m_j^{'}$; $j = 1, 2, \ldots, k$ are population and sample moments respectively, and are defined as

$$m_j^{'} = \frac{1}{n} \sum_{i=1}^{n} X_i^j \quad \text{and} \quad \mu_j^{'}(\boldsymbol{\theta}) = E_\theta(X^j).$$

1.4.5 Interval Estimation and Hypothesis Testing

In certain situations, the experimenter may be interested in constructing a family of sets that contain the true value of unknown parameter with a specified (high) probability. For example, if X represents the length of life of a unit, the experimenter will be interested in the lower bound $\underline{\theta}$ for the θ, where θ is the mean of X. Since $\underline{\theta} = \underline{\theta}(X)$ will be a function of the observations, one cannot always make sure that $\underline{\theta}(X) \leq \theta$ with probability 1. Instead, one can choose a number $1-\alpha$ such that $P_\theta \left\{\underline{\theta}(X) \leq \theta\right\} \geq 1-\alpha \; \forall \; \theta$.

Definition 1.4.9. ***Confidence Interval.*** *Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from a population with pdf $f(x;\theta)$, $\theta \in \Theta$. Let $T_1 = t_1(X_1, X_2, \ldots, X_n)$ and $T_2 = t_2(X_1, X_2, \ldots, X_n)$ be the two statistics satisfying $T_1 \leq T_2$ such that*

$$P_\theta \left[T_1 \leq \phi(\theta) \leq T_2\right] = 1-\alpha \; \forall \; \theta \in \Theta,$$

where $1-\alpha$ is independent of θ. The random interval (T_1, T_2) is called $100(1-\alpha)\%$ confidence interval for $\phi(\theta)$. The statistic T_1 is called the lower confidence limit, while the statistic T_2 is called the upper confidence limit for $\phi(\theta)$.

Definition 1.4.10. ***Pivotal Quantity.*** *Let us suppose that there exists a statistic T and a function $\phi(T,\theta)$ of T and θ, which is measurable for each $\theta \in \Theta$ and the distribution of which is independent of T and θ. This statistic $\phi(T,\theta)$ is known as pivotal quantity.*

Definition 1.4.11. ***Asymptotic Confidence Interval.*** *Sometimes we can find a sequence of estimators $\{t_n\}$ that are asymptotically normally distributed about the true value of parameter with an asymptotic variance $\sigma_n^2(\theta)$, which depends on n and θ. Further, under certain regularity conditions, ML estimators are asymptotically normally distribution with variance*

$$\sigma_n^2(\theta) = \frac{1}{nE_\theta \left[\frac{\partial log f(x;\theta)}{\partial\theta}\right]^2} = -\frac{1}{nE_\theta \left[\frac{\partial^2 log f(x;\theta)}{\partial\theta^2}\right]}.$$

In this case. $\frac{t_n-\theta}{\sigma_n^2(\theta)}$ can be taken as a pivotal quantity and a $100(1-\alpha)\%$ confidence interval of θ is given by

$$\left\{T_n - \tau_{\alpha/2}\sigma_n(\theta), T_n + \tau_{\alpha/2}\sigma_n(\theta)\right\}.$$

Hypothesis Testing

Sometimes we are interested in testing a simple hypothesis $H_\circ : \theta = \theta_\circ$ against a composite alternative hypothesis $H_1 : \theta \neq \theta_\circ$ known as both-sided alternative or

against one-sided composite alternative $H_1 : \theta < \theta_\circ$ ($H_1 : \theta > \theta_\circ$). The problem of testing $H_\circ$ against H_1 can be described as follows:
Let $\underset{\sim}{x} = (x_1, x_2 \ldots, x_n)$ be the sample observations based on which we decide the rejection or acceptance of $H_\circ$. Let X_n, the sample space of random vector $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$, be divided into two disjoint subsets ω and $\omega^c = X_n - \omega$. We reject $H_\circ$ if $\underset{\sim}{x} \in \omega$ and ω is known as the critical region. We accept $H_\circ$ if $\underset{\sim}{x} \in \omega^c$ and hence ω^c is known as the acceptance region. Such a test is known as the non-randomised test of $H_\circ$ against H_1. Let $\delta(\underset{\sim}{x})$ be a function denoting the probability of rejecting $H_\circ$ when $\underset{\sim}{x}$ is the sample observation. Then, for a non-randomised test

$$\delta(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \underset{\sim}{x} \in \omega \\ 0 & \text{if } \underset{\sim}{x} \in \omega^c. \end{cases}$$

There has been a remarkable progress in the literature of classical statistical inference regarding the parameters of reliability models. Authors have introduced the estimation and testing procedures under various scenarios. Cohen (1951) estimated the parameters of logarithmic-normal distributions by the maximum likelihood approach. Epstein and Sobel (1953, 1954) and Epstein (1960) have considered the parameter estimation under one and two-parameter exponential populations using the maximum likelihood approach. Cohen (1960) considered the parameter estimation problems under Poisson distribution. Harter and Moore (1966) have used the local maximum likelihood method to estimate the parameters of three-parameter lognormal populations from complete and censored samples. Calitz (1973) did the maximum likelihood estimation of the parameters of a three-parameter lognormal distribution and showed that this method outperforms any other alternative estimates such as the method of moments. Similar type of problems under various models have been considered by other authors as well. In this direction, some important citations include Lemon (1975), Cohen and Whitten (1980, 1986), Grubbs (1971), Antle (1971), Lawless (1978), Johnson and Kotz (1970), Engelhardt and Bain (1974) and others.

The use of UMVU estimator in estimating the parameters of reliability models is also abundant. Tyagi and Bhattacharya (1989a) considered the estimation of parametric functions in Maxwell failure distribution using UMVU estimator. Kumar and Bhattacharya (1989) considered the UMVU estimation of the reliability function in negative binomial distribution. Chaturvedi and Rani (1997) developed inferential procedures under a family of lifetime distributions. Chaturvedi and Rani (1998) have proposed the generalized Maxwell failure distribution and obtained the UMVUE of the associated reliability function. Chaturvedi and Kumar (1999) obtained the UMVU estimator of reliability function of one-parameter exponential distribution under Type I and Type II censorings.

Moreover, the use of classical inferential procedures in reliability theory has been rapidly growing since last two decades. Authors have introduced several probability distributions, families of distributions, censoring schemes, improved estimators, estimation techniques and studied them thoroughly. Plenty of papers are available in this direction. One may refer to Chaturvedi and Tomer (2002, 2003), Chaturvedi *et al.* (2007, 2016), Chaturvedi and Sharma (2010), Chaturvedi and Alam (2010), Chaturvedi and Kumari (2016, 2018), Chaturvedi and Vyas (2017), Chaturvedi and Malhotra (2018, 2019, 2020), Krishna *et al.* (2015, 2017), Krishna and Goel (2020), Garg *et al.* (2020) and references therein.

1.5 Bayesian Method of Estimation

Bayesian Theory plays a vital role in the reliability theory. In this method, it is preassumed that the unknown parameter θ, where $\theta \in \Omega$ and can be vector-valued, is itself a random variable with pdf $h(\theta)$. Let us suppose that $X_1, X_2, \ldots, X_n$ is a random sample drawn from pdf $f(x|\theta)$. Then the joint distribution of X_i, $i = 1, 2, \ldots, n$ conditional on θ is

$$f(x_1, x_2, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta).$$

Hence, the joint pdf of $X_1, X_2, \ldots, X_n$ and θ is

$$f(x_1, x_2, \ldots, x_n|\theta)h(\theta) = \left\{ \prod_{i=1}^{n} f(x_i|\theta) \right\} h(\theta).$$

The marginal distribution of $X_1, X_2, \ldots, X_n$ is given by

$$\int_{\Omega} f(x_1, x_2, \ldots, x_n|\theta)h(\theta)d\theta,$$

and hence, the conditional distribution of θ given $x_1, x_2, \ldots, x_n$ is given by

$$\pi(\theta|x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n|\theta)h(\theta)}{\int_{\Omega} f(x_1, x_2, \ldots, x_n|\theta)h(\theta)d\theta}.$$

Here, $h(\theta)$ is known as the prior distribution of θ and $\pi(\theta|x_1, x_2, \ldots, x_n)$ is known as the posterior distribution of θ. Once we have obtained the posterior distribution of θ, the Bayes estimator of $\phi(\theta)$, a function of θ, with Squared Error Loss Function (SELF) is given by

$$\begin{aligned} \phi^*(\theta) &= E\left\{\phi(\theta)|(x_1, x_2, \ldots, x_n)\right\} \\ &= \int_{\Omega} \phi(\theta)\pi(\theta|x_1, x_2, \ldots, x_n)d\theta. \end{aligned}$$

Similarly, the Bayes estimator of θ, denoted by θ^*, is given by

$$\theta^* = \int_{\Omega} \theta\, \pi(\theta|x_1, x_2, \ldots, x_n)d\theta.$$

1.5.1 Choice of Prior distribution

Numerous rules have been suggested to decide the prior distribution $h(\theta)$ or a class of priors $g(\theta)$. However, no clear solution exist to this problem.

Raiffa and Schlaifer (1961) proposed "conjugate priors" which contain a range of prior distributions and are relatively easy to handle. A more general rule was introduced by Jeffreys which prescribed $g(\theta) \propto \sqrt{I(\theta)}$, where $I(\theta)$ is the Fisher's Information matrix and is defined as

$$I(\theta) = E\left[\left(\frac{\partial\, log\, f(x|\theta)}{\partial\, \theta}\right)^2\right] = E\left(\frac{\partial^2\, log\, f(x|\theta)}{\partial\, \theta^2}\right),$$

where θ is the only unknown parameter. Another important concept is that of improper priors or quasi priors. These priors arise when $g(\theta)$ is not a probability distribution, i.e., $g(\theta) > 0$ but $\int_{\Omega} g(\theta)d\theta \neq 1$.

The research on Bayesian reliability started with the pioneering work of Bhattacharya (1967). He considered the estimation of the parameter and reliability function of the exponential distribution using the uniform and beta priors. Bhattacharya and Kumar (1986) and Bhattacharya and Tyagi (1988) have further extended this work for other priors. Harris and Singpurwalla (1968) estimated the reliability functions of exponential and Weibull distributions using uniform and gamma priors. One may refer to Soland (1969), Canfield (1970), Canavos and Tsokos (1971), Lwin and Singh (1974), Padgett (1981), Martz and Waller (1982), Tyagi and Bhattacharya (1989b), Chaturvedi and Rani (1998), Bernardo and Smith (2000) and many other authors for a comprehensive overview on the problems of Bayesian reliability under various scenarios. These authors have estimated the parameters and reliability functions of several distributions under different loss structures and priors.

Moreover, some recent advancements on the use of Bayesian inferential procedures in reliability theory comprises of Chaturvedi and Singh (2006), Chaturvedi *et al.* (2007, 2019), Chaturvedi and Kumari (2017, 2018), Krishna and Goel (2018) and Tomer and Panwar (2020).

1.6 R Software Packages

R packages are a collection of R functions, compiled code and sample data. They are stored under a directory called the *library* in the R environment. Listed below are some of the important R packages that we have used to perform simulations.

1. **rootSolve:** This package has been used to find roots of non-linear equations by the Newton-Raphson method. The function *uniroot* of this package is used to find all the roots of one non-linear equation, while the function *multiroot* is used to find n roots based on n non-linear equations.

2. **extraDistr:** This package has been used to implement density function, distribution function, quantile function and random generation for a number of univariate and multivariate distributions.

3. **maxLik:** This package consists of a set of functions and tools to find Maximum Likelihood (ML) estimators. The package focuses on the non-linear

optimization from the ML viewpoint. Further, it provides several useful wrappers and tools, like BHHH algorithm, variance-covariance matrix and standard errors.

4. **ggplot2:** This package has been used to create graphics, based on *The Grammar of Graphics*. While using the package, we provide the data, give instructions on how to map variables to aesthetics, what graphical primitives to use and it takes care of the details.

1.7 Contents of the Thesis

In Chapter 2, we consider Chen distribution and derive UMVU and ML estimators of the parameter λ, hazard rate $h(t)$ and the two measures of reliability, namely $R(t)$ and P, under Type II censoring scheme and the sampling scheme of Bartholomew. We also develop interval estimates of the reliability measures. Procedures for testing the hypotheses related to different parametric functions have also been developed. Through simulation experiments, a comparison of different methods of point estimate and average confidence length has been carried out. The analysis of a real data set is presented for illustration purpose.

In Chpter 3, we consider Kumaraswamy-G distributions and derive UMVU and ML estimators of the two measures of reliability, namely $R(t)$ and P under Type II censoring scheme and the sampling scheme of Bartholomew (1963). We also develop interval estimates of the reliability measures. A comparative study of different methods of point estimation has been done through simulation studies. The analysis of a real data set is presented for illustration purpose.

In Chapter 4, the classical and Bayesian estimation procedures for Kumaraswamy distribution under random censoring scheme are considered. Maximum likelihood estimates and Bayesian estimates using Importance Sampling and Gibbs sampling technique of the parameters, reliability function, $R(t)$, failure rate function, $h(t)$ and MTSF are derived. Asymptotic confidence intervals for the parameters based on the observed Fisher's information matrix are obtained. Highest posterior density credible intervals for the parameters are constructed using MCMC method. Estimated Time on Test of items is also discussed. To compare the three estimates developed, a Monte Carlo simulation study is also carried out. Finally, the analysis of randomly censored real data set is discussed for the illustration purposes.

In Chapter 5, a generalization of positive exponential family of distributions developed by Liang (2008) is taken into consideration. Its properties are studied. Two measures of reliability are discussed, namely $R(t)$ and P. UMVU estimators, ML estimators and MM estimators are developed for the reliability functions. The performances of three types of estimators are compared through Monte Carlo Simulation. Real life data sets are also analysed.

In Chapter 6, a weighted generalization of positive exponential family of distributions is taken into consideration and its properties are studied. Considering two measures of reliability, namely $R(t)$ and P, their UMVU estimators, ML estimators and MM estimators are developed and the performance of the estimators are investigated using Monte Carlo Simulation. We investigate two empirical data sets to illustrate the proposed approach.

Chapter 2

Inferential Procedures for the Reliability Characteristics of Chen Distribution Based on Censored Observations

2.1 Introduction

In the reliability literature, we have many such distributions (e.g. generalized exponential, gamma, Weibull and lognormal) whose hazard rate functions are constant, increasing or decreasing in nature. These are the most commonly used models, and we analyze various real-life phenomena using them. However, these models are not suitable if the data sets exhibit a bathtub-shaped hazard rate. Authors have introduced some probability models to analyze real data with bathtub-shaped failure, for instance, modified Weibull (see Lai *et al.* (2003)) and extended Weibull (see Marshall and Olkin (1997)), but still they are not suitable to produce a good bathtub shape of the failure rates.

Chen (2000) proposed a two-parameter lifetime distribution with an increasing or bathtub shape failure rate function. The hazard rate of this distribution first decreases, then remains constant and then increases. Chen distribution is an appropriate model for the analysis of electronic and mechanical products and the lifetime of humans. Further, it can be used for modelling positively skewed data, apart from the well-known models such as lognormal and gamma. This distribution is flexible in nature in the sense that it has two parameters and the confidence intervals for the shape parameter as well as the joint confidence regions have the closed form.

A random variable (r.v.) X is said to follow the Chen distribution, if its probability density function (pdf) is of the form:

$$f(x;\lambda,\beta) = \lambda\beta e^{x^\beta} x^{\beta-1} e^{\lambda(1-e^{x^\beta})}, \tag{2.1.1}$$

and cumulative distribution function (cdf) is of the form:

$$F(x) = 1 - \exp[\lambda(1-e^{x^\beta})]; x > 0. \tag{2.1.2}$$

Moreover, the hazard rate $h(t)$ of the distribution (2.1.1) corresponding to time 't' is given by:

$$h(t) = \lambda\beta t^{\beta-1} e^{t^\beta}; t > 0. \tag{2.1.3}$$

Wu (2008) obtained ML estimators of the unknown parameters β and λ of the distribution (2.1.1) based on progressive censoring and discussed the interval estimation. Rastogi *et al.* (2012) considered the Bayesian estimation for different symmetric and asymmetric loss functions. Ahmed (2014) proposed Bayes estimates of unknown parameters β and λ under balanced squared-error loss function. Bayesian estimation for the discrete Chen distribution was discussed by Kinachi (2016). Khan and Sharma (2016) derived recurrence relations for single and product moments of generalized order statistics from Chen distribution. Kayal *et al.* (2017) obtained one-sample and two-sample Bayes predictive estimates and constructed prediction intervals of censored observations under progressive censoring. Moreover, Kayal *et al.* (2019a) have considered the estimation of unknown parameters β and λ using both classical and Bayesian approaches under Type I progressive hybrid censoring scheme. They have considered the problem of optimal censoring as well. Kayal *et al.* (2019b) have considered the problem of estimating the reliability in a multicomponent stress-strength model based on Chen distribution. Algarni *et al.* (2020) obtained Empirical Bayes estimators of the scale parameter, reliability and hazard rate functions of Chen distribution under the condition when a sample is obtained from a Type I censoring scheme.

The literature on estimation procedures for Chen distribution discussed above mainly focuses on maximum likelihood or Bayesian/empirical Bayesian procedures. However, developing and investigating properties of UMVU estimators for the parameters of Chen distribution under various sampling schemes is an area that still remains unexplored. The present chapter is an attempt to fill this gap.

In this chapter, we construct point estimation techniques for the reliability functions using Type II censoring and the sampling scheme of Bartholomew (1963) with the help of a concept proposed by Chaturvedi and Tomer (2003), which is simpler and not time-consuming. In this concept, we first obtain the estimator of the powers of parameter λ and then with the help of this estimator we obtain estimator of the pdf. The estimator of the pdf is further used to obtain estimator of $R(t)$ and P. The chapter is organized as follows: In Section 2.2, we provide ML estimators and UMVU estimators of parameter λ^q, hazard rate $h(t)$ and the reliability functions $R(t)$ and P based on Type II censoring scheme assuming β to be known. We also provide exact confidence intervals for λ, $R(t)$ and P. Further, we develop testing procedures for λ, when β is known. In Section 2.3, we obtain ML estimators and UMVU estimators of λ^q, hazard rate $h(t)$ and the reliability functions $R(t)$ and P based on the censoring scheme of Bartholomew assuming β to be known. We also provide testing procedures for λ based on this censoring scheme for known β case. In Section 2.4, we provide extensive sets of simulation studies followed by a real data example in Section 2.5. We end with a brief set of conclusions in Section 2.6.

2.2 Estimation and Testing Procedures Based on Type II Censoring Scheme

Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let us denote by $0 < X_{(1)} \leq X_{(2)} \leq ... \leq X_{(r)}$, $0 < r < n$, the lifetimes of first r failures. Obviously, $(n-r)$ items survived until $X_{(r)}$.

2.2.1 UMVU and ML Estimators of λ^q, $R(t)$, P and $h(t)$

In this section, we obtain the UMVU estimators and ML estimators of λ^q, $R(t)$, P and $h(t)$ under the assumption that β is known. We first provide an important lemma, which will be helpful in proving the main results of this section.

Lemma 2.2.1. *Let*

$$S_{(r)} = \sum_{i=1}^{r}(e^{x_{(i)}^{\beta}} - 1) + (n-r)(e^{x_{(r)}^{\beta}} - 1).$$

Then, $S_{(r)}$ is complete and sufficient for the distribution given at (2.1.1). Moreover, the pdf of S_r is given by

$$g_{S_{(r)}}(s;\lambda) = \frac{s^{r-1}\lambda^{r}e^{-\lambda s}}{\Gamma(r)}, s > 0, \alpha > 0, r > 0, \tag{2.2.1}$$

where, $\Gamma(\cdot)$ denotes the Gamma function.

Proof. Using (2.1.1) the joint pdf of $0 < X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)} < \infty$ is given by

$$f(x_{(1)}, x_{(2)}, ..., x_{(n)}; \alpha, \beta) = n!\lambda^{n}\beta^{n}e^{\sum_{i=1}^{n}x_{(i)}^{\beta}}\prod_{i=1}^{n}x_{(i)}^{\beta-1}exp(-\lambda\sum_{i=1}^{n}(e^{x_{(i)}^{\beta}} - 1)). \tag{2.2.2}$$

Integrating out $x_{(r+1)}, x_{(r+2)}, ..., x_{(n)}$ from (2.2.2) over the region $x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n)} < \infty$ the joint pdf of $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(r)}$ comes out to be

$$h(x_{(1)}, x_{(2)}, ..., x_{(r)}; \lambda, \beta) = \frac{n!}{(n-r)!}\lambda^{r}\beta^{r}e^{\sum_{i=1}^{r}x_{(i)}^{\beta-1}}\prod_{i=1}^{r}x_{(i)}^{\beta-1}e^{-\lambda s_{r}}. \tag{2.2.3}$$

It follows from (2.2.3) and Fisher-Neyman factorization theorem (see Rohatgi and Saleh (2012), pp. 347) that S_r is sufficient for the distribution given in (2.1.1). Moreover, if we consider the transformation $Z_i = (n-i+1)\left\{U_{(i)} - U_{(i-1)}\right\}$, $i = 1, 2, ...r$; $U_o = 0$, where $U_{(i)} = e^{x_{(i)}^{\beta}} - 1$, then $Z_i's$ are independent and identically distributed (i.i.d.) r.v.'s, each having exponential distribution with mean life $1/\alpha$. It is easy to see that $\sum_{i=1}^{r} Z_i = S_r$. Lemma 2.2.1 now follows from the additive property of gamma distribution (see Coetzee (1996), pp. 170). For known β, we can observe that the distribution of S_r belongs to one-parameter exponential family and hence, it is also complete (see Rohatgi and Saleh (2012), pp. 347). The pdf of $S_{(r)}$ can now be used to obtain the UMVU estimator of λ^q.

Theorem 2.2.2. *For $q \in (-\infty, \infty)$, the UMVU estimator of λ^q is given by*

$$\widetilde{\lambda}_{II}^{q} = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-q)} S_{(r)}^{-q} & ; r - q > 0 \\ 0 & ; otherwise. \end{cases}$$

Proof. We have from (2.2.1),

$$\begin{aligned} E\left(S_{(r)}^{-q}\right) &= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty e^{-\lambda s} s^{r-q-1} ds \\ &= \frac{\Gamma(r-q)}{\Gamma(r)} \lambda^q, r > q. \end{aligned}$$

The Lehmann-Scheffe theorem can now be used to prove the statement of the above theorem [see Rohatgi & Saleh (2012)].

Corollary 2.2.2.1. *At a specified point x, the UMVU estimator of $f(x)$ is given by:*

$$\widetilde{f}_{II}(x; \lambda, \beta) = \begin{cases} (r-1)\frac{e^{x^\beta}\beta x^{\beta-1}}{S_{(r)}} \left(1 - \frac{e^{x^\beta}-1}{S_{(r)}}\right)^{r-2} & ; x < \left\{ln(1 + S_{(r)})\right\}^{1/\beta} \\ 0 & ; otherwise. \end{cases}$$

Proof. We can write the pdf given in (2.1.1) as

$$f(x; \lambda, \beta) = \lambda \beta e^{x^\beta} x^{\beta-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left\{\lambda(e^{x^\beta} - 1)\right\}^i.$$

Making use of the UMVU estimator of λ^q given in Theorem 2.2.2, the UMVU estimator of the sampled pdf at a specified point 'x' is given by

$$\begin{aligned}\widetilde{f}_{II}(x;\lambda,\beta) &= \beta e^{x^\beta} x^{\beta-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (e^{x^\beta}-1)^i \widetilde{\lambda^{i+1}} \\ &= \beta e^{x^\beta} x^{\beta-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (e^{x^\beta}-1)^i \frac{\Gamma(r)}{\Gamma(r-i-1)} S_{(r)}^{-(i-1)},\end{aligned}$$

and hence the result follows.

Theorem 2.2.3. *The UMVU estimator of $R(t)$ at a specified point t is*

$$\widetilde{R}(t)_{II} = \begin{cases} \left(1 - \frac{e^{t^\beta}-1}{S_{(r)}}\right)^{r-1} & ; e^{t^\beta} - 1 < S_{(r)} \\ 0 & ; \textit{otherwise.} \end{cases}$$

Proof. On using Corollary 2.2.2.1 the UMVU estimator of $R(t)$ is given by

$$\begin{aligned}\widetilde{R}(t)_{II} &= \int_t^\infty \widetilde{f}(x;\lambda,\beta)dx \\ &= (r-1)\int_t^{(ln(1+S_{(r)}))^{1/\beta}} \frac{e^{x^\beta}\beta x^{\beta-1}}{S_{(r)}} \left(1 - \frac{e^{x^\beta}-1}{S_{(r)}}\right)^{r-2} dx.\end{aligned}$$

The result now follows by substituting $(e^{x^\beta}-1)/S_{(r)} = y$ in the above expression.

Let X and Y be two independent random variables following the classes of distributions $f_1(x;\lambda_1,\beta_1)$ and $f_2(y;\lambda_2,\beta_2)$, respectively, where

$$f_1(x;\lambda_1,\beta_1) = \lambda_1\beta_1 e^{x^{\beta_1}} x^{\beta_1-1} e^{\lambda_1(1-e^{x^{\beta_1}})}; x>0, \lambda_1, \beta_1 > 0 \tag{2.2.4}$$

and

$$f_2(y;\lambda_2,\beta_2) = \lambda_2\beta_2 e^{y^{\beta_2}} y^{\beta_2-1} e^{\lambda_2(1-e^{y^{\beta_2}})}; y>0, \lambda_2, \beta_2 > 0. \tag{2.2.5}$$

Let n items on X and m items on Y are put on a life test and the termination numbers for X and Y are r and r', respectively. Let us define

$$S_{(r)} = \sum_{i=1}^{r} (e^{x_{(i)}^{\beta_1}} - 1) + (n-r)(e^{x_{(r)}^{\beta_1}} - 1),$$

$$T_{(r')} = \sum_{j=1}^{r'} (e^{y_{(j)}^{\beta_2}} - 1) + (m-r')(e^{y_{(r')}^{\beta_2}} - 1).$$

It follows from Corollary 2.2.2.1 that the UMVU estimators of $f_1(x;\lambda_1,\beta_1)$ and $f_2(y;\lambda_2,\beta_2)$ based on Type II censoring at specified points x and y, respectively, are given by

$$\widetilde{f}_{1II}(x;\lambda_1,\beta_1)=(r-1)\frac{e^{x^{\beta_1}}\beta_1 x^{\beta_1-1}}{S_{(r)}}\left(1-\frac{e^{x^{\beta_1}}-1}{S_{(r)}}\right)^{r-2};x<\left\{ln(1+S_{(r)})\right\}^{1/\beta_1}, \tag{2.2.6}$$

$$\widetilde{f}_{2II}(y;\lambda_2,\beta_2)=(r'-1)\frac{e^{y^{\beta_2}}\beta_2 y^{\beta_2-1}}{T_{(r')}}\left(1-\frac{e^{y^{\beta_2}}-1}{T_{(r')}}\right)^{r'-2};y<\left\{ln(1+T_{(r')})\right\}^{1/\beta_2}. \tag{2.2.7}$$

In the following theorem, we obtain the UMVU estimator of P.

Theorem 2.2.4. *The UMVU estimator of P, when X and Y belong to different families of distributions, is given by*

$$\widetilde{P}_{II}=\begin{cases}\displaystyle\int_{z=0}^{c}\frac{1}{B(1,r'-1)}\left[1-\frac{exp(ln(zT_{(r')}+1))^{\beta_1/\beta_2}}{S_r}\right]^{r-1}(1-z)^{r'-2}dz\\ If\ \ (ln(1+S_{(r)}))^{1/\beta_1}<(ln(1+T_{(r')}))^{1/\beta_2}\\ \displaystyle\int_{z=0}^{1}\frac{1}{B(1,r'-1)}\left[1-\frac{exp(ln(zT_{(r')}+1))^{\beta_1/\beta_2}}{S_r}\right]^{r-1}(1-z)^{r'-2}dz\\ If\ \ (ln(1+S_{(r)}))^{1/\beta_1}\geq(ln(1+T_{(r')}))^{1/\beta_2},\end{cases}$$

where $c=[exp(ln(1+S_{(r)}))^{\beta_2/\beta_1}-1]/T_{(r')}$.

Proof. The UMVU estimator of P can be written in terms of $\widetilde{R}(y;\lambda_1,\beta_1)_{II}$ as follows:

$$\begin{aligned}\widetilde{P}_{II}&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}_{II}(x;\lambda_1,\beta_1)\widetilde{f}_{II}(y;\lambda_2,\beta_2)dxdy\\&=\int_{y=0}^{\infty}\widetilde{R}_{II}(y;\lambda_1,\beta_1)\widetilde{f}_{II}(y;\lambda_2,\beta_2)dy,\end{aligned}$$

which, on using Theorem 2.2.3 and (2.2.7), leads to

$$\widetilde{P}_{II} = \int_{y=0}^{\infty} \left(1 - \frac{e^{y^{\beta_1}} - 1}{S_{(r)}}\right)^{r-1} (r'-1)\frac{e^{y^{\beta_2}}\beta_2 y^{\beta_2 - 1}}{T_{(r')}}\left(1 - \frac{e^{y^{\beta_2}} - 1}{T_{(r')}}\right)^{r'-2} dy,$$

$$where \;\; y < \left\{ln(1+S_{(r)})\right\}^{1/\beta_1}, y < \left\{ln(1+T_{(r')})\right\}^{1/\beta_2},$$

$$= \int_{y=0}^{c'} \left(1 - \frac{e^{y^{\beta_1}} - 1}{S_{(r)}}\right)^{r-1} (r'-1)\frac{e^{y^{\beta_2}}\beta_2 y^{\beta_2 - 1}}{T_{(r')}}\left(1 - \frac{e^{y^{\beta_2}} - 1}{T_{(r')}}\right)^{r'-2} dy, \tag{2.2.8}$$

$$where \;\; c' = min\left[\left\{ln(1+S_{(r)})\right\}^{1/\beta_1}, \left\{ln(1+T_{(r')})\right\}^{1/\beta_2}\right].$$

From (2.2.8), for $\left\{ln(1+S_{(r)})\right\}^{1/\beta_1} < \left\{ln(1+T_{(r')})\right\}^{1/\beta_2}$, we have

$$\widetilde{P}_{II} = \int_{y=0}^{\left\{ln(1+S_{(r)})\right\}^{1/\beta_1}} \left(1 - \frac{e^{y^{\beta_1}} - 1}{S_{(r)}}\right)^{r-1} (r'-1)\frac{e^{y^{\beta_2}}\beta_2 y^{\beta_2 - 1}}{T_{(r')}}\left(1 - \frac{e^{y^{\beta_2}} - 1}{T_{(r')}}\right)^{r'-2} dy, \tag{2.2.9}$$

and for $\left\{ln(1+S_{(r)})\right\}^{1/\beta_1} \geq \left\{ln(1+T_{(r')})\right\}^{1/\beta_2}$, we have

$$\widetilde{P_{II}} = \int_{y=0}^{\left\{ln(1+T_{(r')})\right\}^{1/\beta_2}} \left(1 - \frac{e^{y^{\beta_1}} - 1}{S_{(r)}}\right)^{r-1} (r'-1)\frac{e^{y^{\beta_2}}\beta_2 y^{\beta_2 - 1}}{T_{(r')}}\left(1 - \frac{e^{y^{\beta_2}} - 1}{T_{(r')}}\right)^{r'-2} dy. \tag{2.2.10}$$

The result follows by substituting $(e^{y^{\beta_2}} - 1)/T_{(r')} = z$ in (2.2.9) and (2.2.10), respectively, for both the cases.

Corollary 2.2.4.1. *When X and Y are from the same family of distributions, i.e., when $\beta_1 = \beta_2$, the UMVU estimator of P is given by*

$$\widetilde{P}_{II} = \begin{cases} \frac{1}{B(1,r'-1)} \sum_{i=0}^{r'-2} (-1)^i \binom{r'-2}{i} \left(\frac{S_{(r)}}{T_{(r')}}\right)^{i+1} B(i+1,r); & S_{(r)} < T_{(r')} \\ \\ \frac{1}{B(1,r'-1)} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \left(\frac{T_{(r')}}{S_{(r)}}\right)^{j} B(j+1,r'-1); & S_{(r)} \geq T_{(r')}. \end{cases}$$

From Theorem 2.2.4, when $S_r < T_s$,

$$\widetilde{P}_{II} = \int_{z=0}^{S_r/T_s} \frac{1}{B(1,s-1)}\left[1 - \frac{zT_s}{S_r}\right]^{r-1} (1-z)^{s-2} dz,$$

and the first assertion follows by substituting $\frac{zT_s}{S_r} = w$ and the binomial expansion of $\left(1 - \frac{S_r w}{T_s}\right)^{s-2}$.

When $S_r > T_s$,

$$\widetilde{P}_{II} = \int_{z=0}^{1} \frac{1}{B(1, s-1)} \left[1 - \frac{zT_s}{S_r}\right]^{r-1} (1-z)^{s-2} dz,$$

and the second assertion follows by binomial expansion of $\left(1 - \frac{zT_s}{S_r}\right)^{r-1}$.

Theorem 2.2.5. *The UMVU estimator of $h(t)$ at a specified point t is given by*

$$\widetilde{h}(t)_{II} = \frac{r-1}{S_{(r)}} e^{t^\beta} \beta t^{\beta-1}.$$

Proof. Using (2.1.3), we can write the UMVU estimator of $h(t)$ at a specified point t as

$$\widetilde{h}(t)_{II} = \widetilde{\lambda} e^{t^\beta} \beta t^{\beta-1}. \quad (2.2.11)$$

Substituting the UMVU estimator of λ in (2.2.11), we obtain the required result.

Theorem 2.2.6. *For $q \in (-\infty, \infty)$, the ML estimator of λ^q is given by:*

$$\widehat{\lambda}_{II}^q = \left(\frac{r}{S_{(r)}}\right)^q.$$

Proof. We take natural log of both sides of (2.2.3) and differentiate it with respect to λ. On equating the differential with respect to zero, we get

$$\widehat{\lambda}_{II} = \frac{r}{S_{(r)}}, \quad (2.2.12)$$

and hence the theorem follows.

Corollary 2.2.6.1. *The ML estimator of $f(x; \alpha, \beta)$ at a specified point $'x'$ is given by*

$$\widehat{f}(x)_{II} = \frac{r}{S_{(r)}} \beta e^{x^\beta} x^{\beta-1} e^{\frac{r}{S_{(r)}}(1-e^{x^\beta})}.$$

Proof. The ML estimator of $f(x)$ is given by

$$\widehat{f}(x)_{II} = \widehat{\lambda} \beta e^{x^\beta} x^{\beta-1} e^{\widehat{\lambda}(1-e^{x^\beta})}.$$

From (2.2.12) and the invariance property of ML estimators, the result follows.

Theorem 2.2.7. *The ML estimator of* $R(t)$ *is given by*

$$\widehat{R}(t)_{II} = exp\left\{\frac{-r}{S_{(r)}}(e^{t^{\beta}} - 1)\right\}.$$

Proof. From (2.1.2), the reliability function $R(t)$ for Chen distribution is given by

$$R(t) = \exp[\lambda(1 - e^{x^{\beta}})]. \tag{2.2.13}$$

From (2.2.12), (2.2.13) and invariance property of ML estimator, the result follows.

Theorem 2.2.8. *When* X *and* Y *belong to different families of distributions, the ML estimator of* P *is given by*

$$\widehat{P}_{II} = \int_{z=0}^{\infty} exp\left[\frac{r}{S_{(r)}}\left\{1 - exp\left(ln\left(\frac{zT_{(r')}}{r'} + 1\right)\right)^{\beta_1/\beta_2}\right\}\right] e^{-z}dz.$$

Proof. We know that

$$\begin{aligned}\widehat{P}_{II} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widehat{f}_{II}(x;\lambda_1,\beta_1)\widehat{f}_{II}(y;\lambda_2,\beta_2)dxdy \\ &= \int_{y=0}^{\infty} \widehat{R}(y;\lambda_1,\beta_1)_{II}\widehat{f}_{II}(y;\lambda_2,\beta_2)dy,\end{aligned}$$

which on using Corollary 2.2.6.1 and Theorem 2.2.7, gives

$$\widehat{P}_{II} = \int_{y=0}^{\infty} exp\left\{\frac{-r}{S_{(r)}}(e^{y^{\beta_1}} - 1)\right\}\frac{r'}{T_{(r')}}e^{y^{\beta_2}}\beta_2 y^{\beta_2 - 1} exp\left\{\frac{-r'}{T_{(r')}}(e^{y^{\beta_2}} - 1)\right\}dy,$$

and the result follows by substituting $r'(e^{y^{\beta_2}} - 1)/T_{(r')} = z$ in the above expression.

Corollary 2.2.8.1. *When* X *and* Y *belong to the same family of distributions, i.e. when* $\beta_1 = \beta_2$*, the ML estimator of* P *is given by*

$$\widehat{P}_{II} = \frac{r'S_{(r)}}{r'S_{(r)} + rT_{(r')}}. \tag{2.2.14}$$

Proof. The proof is on similar lines as that of Theorem 2.2.8.

2.2.2 Exact Confidence Intervals for λ, $R(t)$ and P

Now, we consider the problem of constructing a two-sided confidence interval for λ (β known). The confidence interval is obtained by using pivotal quantity $2\lambda S_{(r)}$. If we define $\chi^2(\nu)$ as the value of χ^2 such that

$$P(\chi^2 > \chi^2(\nu)) = \int_{\chi^2(\nu)}^{\infty} P(\chi^2)d\chi^2 = \nu, \tag{2.2.15}$$

where, $P(\chi^2)$ is the pdf of χ^2 distribution with $2r$ degrees of freedom, then by using the fact that $2\lambda S_{(r)} \sim \chi^2_{2r}$, the confidence interval is given by

$$P\left(\frac{\chi^2\left(1-\frac{\nu}{2}\right)}{2S_{(r)}} \leq \lambda \leq \frac{\chi^2\left(\frac{\nu}{2}\right)}{2S_{(r)}}\right) = 1-\nu, \tag{2.2.16}$$

where $\chi^2\left(\frac{\nu}{2}\right)$ and $\chi^2\left(1-\frac{\nu}{2}\right)$ are obtained by using (2.2.15). Thus for known β, $100(1-\nu)\%$ confidence interval for λ is given by

$$\left(\frac{\chi^2\left(1-\frac{\nu}{2}\right)}{2S_{(r)}}, \frac{\chi^2\left(\frac{\nu}{2}\right)}{2S_{(r)}}\right).$$

The problem of obtaining the confidence interval for the reliability function $R(t) = exp\left\{-\lambda\left(e^{t^\beta}-1\right)\right\}$ can be solved by noting that $R(t_\circ;\lambda)$ is a decreasing function of λ. Thus $\Psi_1(x_1, x_2, ..., x_n) \leq exp\left\{-\lambda\left(e^{t^\beta}-1\right)\right\}$ is equivalent to $\lambda \leq ln(\Psi_1(x_1, x_2, ..., x_n))/1-e^{t^\beta}$ and $\Psi_2(x_1, x_2, ..., x_n) \geq exp\left\{-\lambda\left(e^{t^\beta}-1\right)\right\}$ is equivalent to $\lambda \geq ln(\Psi_2(x_1, x_2, ..., x_n))/1-e^{t^\beta}$.

Therefore, the expression

$$P\left(\Psi_1(x_1, x_2, ..., x_n) \leq exp\left\{-\lambda\left(e^{t_\circ^\beta}-1\right)\right\} \leq \Psi_2(x_1, x_2, ..., x_n)\right) = 1-\nu$$

is equivalent to

$$P\left(\frac{ln\Psi_2(x_1, x_2, ..., x_n)}{1-e^{t_\circ^\beta}} \leq \lambda \leq \frac{ln\Psi_1(x_1, x_2, ..., x_n)}{1-e^{t_\circ^\beta}}\right) = 1-\nu. \tag{2.2.17}$$

Comparing (2.2.16) and (2.2.17), it immediately follows that

$$\chi^2\left(1-\frac{\nu}{2}\right)/2S_{(r)} = ln\Psi_2(x_1, x_2, ..., x_n)/1-e^{t_\circ^\beta}$$

and

$$\chi^2\left(\frac{\nu}{2}\right)/2S_{(r)} = ln\Psi_1(x_1, x_2, ..., x_n)/1-e^{t_\circ^\beta}.$$

Therefore

$$\Psi_1 = exp\left[\frac{(1-e^{t_\circ^\beta})\chi^2\left(\frac{\nu}{2}\right)}{2S_{(r)}}\right] \text{ and } \Psi_2 = exp\left[\frac{(1-e^{t_\circ^\beta})\chi^2\left(1-\frac{\nu}{2}\right)}{2S_{(r)}}\right].$$

Thus, $(1-\nu)100\%$ confidence interval for $R(t_\circ, \lambda)$ is given by

$$\left(exp\left[\frac{(1-e^{t_\circ^\beta})\chi^2\left(\frac{\nu}{2}\right)}{2S_{(r)}}\right], exp\left[\frac{(1-e^{t_\circ^\beta})\chi^2\left(1-\frac{\nu}{2}\right)}{2S_{(r)}}\right]\right). \tag{2.2.18}$$

In order to obtain the confidence interval for P, we utilize the fact that

$$\frac{2\lambda_1 S_{(r)}/2r}{2\lambda_2 T_{(r')}/2r'} \sim F_{2r,2r'}.$$

Thus, the confidence interval for P is given by

$$P\left(F\left(1-\frac{\nu}{2}\right) \leq F \leq F\left(\frac{\nu}{2}\right)\right) = 1-\nu,$$

which can further be written as

$$P\left[\left(\frac{rT_{(r')}F(1-\frac{\nu}{2})}{r'S_{(r)}}+1\right)^{-1} \leq \frac{\lambda_2}{\lambda_1+\lambda_2} \leq \left(\frac{rT_{(r')}F(\frac{\nu}{2})}{r'S_{(r)}}+1\right)^{-1}\right] = 1-\nu.$$

Therefore, $(1-\nu)100\%$ confidence interval for P is given by

$$\left[\left(\frac{rT_{(r')}F(1-\frac{\nu}{2})}{r'S_{(r)}}+1\right)^{-1}, \left(\frac{rT_{(r')}F(\frac{\nu}{2})}{r'S_{(r)}}+1\right)^{-1}\right]. \tag{2.2.19}$$

2.2.3 Hypothesis Testing

Under this section, hypothesis testing of the following three cases are considered:

1. Testing of $H_\circ : \lambda = \lambda_\circ$ against $H_1 : \lambda \neq \lambda_\circ$.

2. Testing of $H_\circ : \lambda \leq \lambda_\circ$ against $H_1 : \lambda > \lambda_\circ$.

3. Testing of $H_\circ : P = P_\circ$ against $H_1 : P \neq P_\circ$.

In life-testing studies, one of the most essential hypothesis is $H_\circ : \lambda = \lambda_\circ$ against $H_1 : \lambda \neq \lambda_\circ$. We presuppose that β is known. Using (2.2.3), it follows that the likelihood function for observing λ is given by

$$L(\lambda; \underset{\sim}{x}, \beta) = \frac{n!}{(n-r)!}\lambda^r \beta^r e^{\sum_{i=1}^{r} x_{(i)}^{\beta-1}} \prod_{i=1}^{r} x_{(i)}^{\beta-1} e^{-\lambda S_{(r)}}. \tag{2.2.20}$$

Now,

$$\sup_{\Theta_\circ} L(\lambda; \underset{\sim}{x}, \beta) = \frac{n!}{(n-r)!}\lambda_\circ^r \beta^r e^{\sum_{i=1}^{r} x_{(i)}^{\beta-1}} \prod_{i=1}^{r} x_{(i)}^{\beta-1} e^{-\lambda_\circ S_{(r)}}; \;\; \Theta_\circ = \{\lambda : \lambda = \lambda_\circ\} \tag{2.2.21}$$

and

$$\sup_{\Theta} L(\lambda; \underset{\sim}{x}, \beta) = \frac{n!}{(n-r)!}\left(\frac{r}{S_{(r)}}\right)^r exp\left\{-r\right\}. \tag{2.2.22}$$

Therefore, the likelihood ratio is given by

$$\begin{aligned}\Phi(\underset{\sim}{x}) &= \sup_{\Theta_\circ} L(\lambda; \underset{\sim}{x}, \beta) / \sup_{\Theta} L(\lambda; \underset{\sim}{x}, \beta) \\ &= \left(\frac{\lambda_\circ S_{(r)}}{r}\right)^r e^{-(r+\lambda_\circ S_{(r)})}. \end{aligned} \tag{2.2.23}$$

On the right-hand side of (2.2.23), the first term is an increasing function of $S_{(r)}$ while, the second term is a monotonically decreasing function of $S_{(r)}$. Denoting by χ^2_{2r}, the Chi-square statistics with $2r$ degrees of freedom and using the fact that $2\lambda_\circ S_{(r)} \sim \chi^2_{2r}$, the critical region is given by

$$\left\{0 < S_{(r)} < k_\circ\right\} \cup \left\{k_\circ^{'} < S_{(r)} < \infty\right\},$$

where $k_\circ$ and $k_\circ^{'}$ are obtained such that $P\left[\chi^2_{2r} < 2\lambda_\circ k_\circ \ \text{ or } \ 2\lambda_\circ k_\circ^{'} < \chi^2_{2r}\right] = \nu$.

Thus,

$$k_\circ = \frac{1}{2\lambda_\circ}\chi^2_{2r}\left(1 - \frac{\nu}{2}\right) \quad \text{and} \quad k_\circ^{'} = \frac{1}{2\lambda_\circ}\chi^2_{2r}\left(\frac{\nu}{2}\right).$$

Another important hypothesis in life testing experiments is $H_\circ : \lambda \leq \lambda_\circ$ against $H_1 : \lambda > \lambda_\circ$. Again, we presuppose that β is known. It follows from (2.2.20) that, for $\lambda_1 < \lambda_2$,

$$\frac{h(x_{(1)}, x_{(2)}, ..., x_{(r)}, \beta, \lambda_2)}{h(x_{(1)}, x_{(2)}, ..., x_{(r)}, \beta, \lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^r exp(-(\lambda_2 - \lambda_1)S_{(r)}). \tag{2.2.24}$$

It follows from (2.2.24) that $h(x_{(1)}, x_{(2)}, ..., x_{(r)}, \lambda, \beta)$ has Monotone Likelihood Ratio (MLR) in S_r. As a result, for testing $H_\circ : \lambda \leq \lambda_\circ$ against $H_1 : \lambda > \lambda_\circ$ (see Lehmann (1959), pp. 88), the uniformly most powerful critical region (UMPCR) is given by

$$\phi(x_{(1)}, x_{(2)}, ..., x_{(r)}) = \begin{cases} 1, S_r \leq k_\circ^{''} \\ 0, \text{otherwise}, \end{cases}$$

where $k_\circ^{''}$ is obtained such that $P[\chi^2_{2r} < 2\lambda_\circ k_\circ^{''}] = \nu$.

Therefore,

$$k_\circ^{''} = \frac{1}{2\lambda_\circ}\chi^2_{2r}(1 - \nu).$$

Suppose, we want to test $H_\circ : P = P_\circ$ against $H_1 : P \neq P_\circ$ based on Type II censoring. We assume that $\beta_1 = \beta_2 = \beta$ and is known. It follows that $H_\circ$ is equivalent to $\lambda_1 = k\lambda_2$. It can be shown that, under $H_\circ$

$$\widehat{\lambda_1} = \frac{k(r+r')}{kS_{(r)} + T_{(r')}},$$

and

$$\widehat{\lambda_2} = \frac{r+r'}{kS_{(r)} + T_{(r')}}.$$

For a generic constant K, the likelihood of observing λ_1 and λ_2, based on $x_{(1)}, x_{(2)}, ..., x_{(r)}$ and $y_{(1)}, y_{(2)}, ..., y_{(r')}$ is given by

$$L(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, ..., x_{(r)}, y_{(1)}, y_{(2)}, ..., y_{(r')}) = K\lambda_1^r \lambda_2^{r'} exp[-(\lambda_1 S_{(r)} + \lambda_2 T_{(r')})]. \tag{2.2.25}$$

Thus,

$$\sup_{\Theta_\circ} L(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, ..., x_{(r)}, y_{(1)}, y_{(2)}, ..., y_{(r')}) = \frac{Kk^r e^{-(r+r')}}{(kS_{(r)} + T_{(r')})^{(r+r')}}, \tag{2.2.26}$$

$$\sup_{\Theta} L(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, ..., x_{(r)}, y_{(1)}, y_{(2)}, ..., y_{(s)}) = \frac{Ke^{-(r+r')}}{S_{(r)}^r T_{(r')}^{r'}}. \tag{2.2.27}$$

From (2.2.26) and (2.2.27), the likelihood ratio criterion is

$$\lambda^*(\lambda_1, \lambda_2 | x_{(1)}, x_{(2)}, ..., x_{(r)}, y_{(1)}, y_{(2)}, ..., y_{(r')}) = K \frac{\left(\frac{kS_{(r)}}{T_{(r')}}\right)^r}{\left[1 + \frac{kS_{(r)}}{T_{(r')}}\right]^{r+r'}}. \tag{2.2.28}$$

Let F-statistic with $(a;b)$ degrees of freedom be denoted by $F_{a,b}(\cdot)$. Making use of the fact that $\frac{S_r}{T_{r'}} \sim \frac{r\lambda_2}{r'\lambda_1} F_{2r,2r'}(\cdot)$, the critical region is given by

$$\left\{ \frac{S_{(r)}}{T_{(r')}} < k_2 \quad \text{and} \quad \frac{S_{(r)}}{T_{(r')}} > k_2' \right\},$$

where k_2 and k_2' are obtained such that

$$P\left\{ \frac{r'kS_{(r)}}{rT_{(r')}} < F_{2r,2r'} \cup \frac{r'kS_{(r)}}{rT_{(r')}} > F_{2r,2r'} \right\} = \nu.$$

Thus,

$$k_2 = \frac{r}{r'k} F_{2r,2r'} \left(1 - \frac{\nu}{2}\right) \quad \text{and} \quad k_2' = \frac{r}{r'k} F_{2r,2r'} \left(\frac{\nu}{2}\right).$$

2.3 Estimation and Testing Procedures Based on the Sampling Scheme of Bartholomew

Throughout this section, we assume that n items are put on a test and we terminate the life testing experiment at a preassigned time $t_\circ$. Suppose we carry out time-censored test where the items that fail are immediately replaced. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the failure times of n items under a test from (2.1.1), the test begins at time $X_{(0)} = 0$ and the system operates till $X_{(1)} = x_1$, when the first failure occurs. The failed item is replaced by a new one and the system operates till the second failure occurs at time $X_{(2)} = x_2$ and so on. The experiment is terminated at time $t_\circ$. Here, $X_{(i)}$ is the time until i^{th} failure measured from time 0.

Now, we provide an important lemma, which will be useful in deducing the main results of this section.

Lemma 2.3.1. *If* $N(t_\circ)$ *be the number of failures during the interval* $[0, t_\circ]$*. Then,* $N(t_\circ)$ *follows Poisson distribution.*

Proof. Let us make the transformations

$$\left.\begin{aligned} W_1 &= e^{x_1^\beta} - 1, \\ W_2 &= e^{x_2^\beta} - e^{x_1^\beta}, \\ &\vdots \\ W_n &= e^{x_n^\beta} - e^{x_{n-1}^\beta} \end{aligned}\right\} \tag{2.3.1}$$

The pdf of W_1 is

$$h(w_1) = n\lambda e^{-n\lambda w_1}.$$

Moreover, $W_2, W_3, ..., W_n$ are i.i.d. as W_1. Using the monotonicity property of $e^{x^\beta} - 1$, we get

$$\begin{aligned} P\{N(t_\circ) = r | t_\circ\} &= P[X_{(r)} \leq t_\circ] - P[X_{(r+1)} \leq t_\circ] \\ &= P\left[\left(e^{x_{(r)}^\beta} - 1\right) \leq \left(e^{t_\circ^\beta} - 1\right)\right] - P\left[\left(e^{x_{(r-1)}^\beta} - 1\right) \leq \left(e^{t_\circ^\beta} - 1\right)\right]. \end{aligned} \tag{2.3.2}$$

Using (2.3.1) and (2.3.2), we get

$$P\left\{N(t_\circ)=r|t_\circ\right\}=P\left[W_1+W_2+...+W_r\leq\left(e^{t_\circ^\beta}-1\right)\right]$$
$$-P\left[W_1+W_2+...+W_{r-1}\leq\left(e^{t_\circ^\beta}-1\right)\right]. \qquad (2.3.3)$$

From the additive property of exponentially distributed random variables (see Johnson and Kotz (1970), pp.170), $U=n\lambda\sum_{i=1}^{r}W_i$ follows gamma distribution with pdf:

$$h(u)=\frac{1}{\Gamma(r)}u^{r-1}e^{-u}, u>0. \qquad (2.3.4)$$

Using (2.3.4) and a result of Patel *et al.* (1976), we obtain from (2.3.3) that

$$\begin{aligned}P\left\{N(t_\circ)=r|t_\circ\right\}&=\frac{1}{\Gamma(r+1)}\int_{e^{t_\circ^\beta}-1}^{\infty}e^{-u}u^r du-\frac{1}{\Gamma(r)}\int_{e^{t_\circ^\beta}-1}^{\infty}e^{-u}u^{r-1}du\\&=exp\left(-n\lambda\left(e^{t_\circ^\beta}-1\right)\right)\left\{\sum_{j=0}^{r}\frac{\left[n\lambda\left(e^{t_\circ^\beta}-1\right)\right]^j}{j!}\right\}\\&-exp\left(-n\lambda\left(e^{t_\circ^\beta}-1\right)\right)\left\{\sum_{j=0}^{r-1}\frac{\left[n\lambda\left(e^{t_\circ^\beta}-1\right)\right]^j}{j!}\right\}. \qquad (2.3.5)\end{aligned}$$

Hence,

$$P[N(t_\circ)=r|t_\circ]=\frac{exp\left\{-n\lambda\left(e^{t_\circ^\beta}-1\right)\right\}\left\{n\lambda\left(e^{t_\circ^\beta}-1\right)\right\}^r}{r!}, \qquad (2.3.6)$$

and the lemma follows.

2.3.1 UMVU and ML Estimators of λ^q, $R(t)$, P and $h(t)$

In the following theorems, we provide the UMVU estimators of λ^q, $R(t)$, P and $h(t)$, based on the sampling scheme of Bartholomew (1963).

Theorem 2.3.2. *For positive integer q, the UMVU estimator of λ^q is given by*

$$\widetilde{\lambda}_I^q=\begin{cases}\frac{r!}{(r-q)!}\left[n\left(e^{t_\circ^\beta}-1\right)\right]^{-q} & : r-q>0\\ 0 & : otherwise.\end{cases}$$

Proof. It follows from Lemma 2.3.1 and Fisher-Neyman factorization theorem (see Rohatgi and Saleh (2012), pp. 347) that $N(t_\circ)$ is sufficient for λ. Moreover, since the distribution of $N(t_\circ)$ belongs to exponential family, it is also complete. The theorem now follows from the result that q^{th} factorial moment of the distribution of $N(t_\circ)$ is given by

$$E[N(t_\circ)(N(t_\circ)-1)(N(t_\circ)-2)...(N(t_\circ)-q+1)] = \left[n\lambda\left(e^{t_\circ^\beta}-1\right)\right]^q.$$

Corollary 2.3.2.1. *The UMVU estimator of $f(x;\lambda,\beta)$ at a specified point x is*

$$\widetilde{f}_I(x;\lambda,\beta) = \begin{cases} \frac{r\beta e^{x^\beta}x^{\beta-1}}{n(e^{t_\circ^\beta}-1)}\left(1-\frac{e^{x^\beta}-1}{n(e^{t_\circ^\beta}-1)}\right)^{r-1} & : e^{x^\beta}-1 < n(e^{t_\circ^\beta}-1) \\ 0 & : otherwise. \end{cases}$$

Proof. Let us write the pdf (2.1.1) as follow

$$\begin{aligned} f(x;\lambda,\beta) &= \lambda\beta e^{x^\beta}x^{\beta-1}e^{\lambda(1-e^{x^\beta})} \\ &= \lambda\beta e^{x^\beta}x^{\beta-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(e^{x^\beta}-1\right)^i\lambda^i. \end{aligned}$$

Then, the Corollary straight away follows from Theorem 2.3.2.

Theorem 2.3.3. *The UMVU estimator of $R(t)$ at a specified point t is given by*

$$\widetilde{R}(t)_I = \begin{cases} \left[1-\frac{e^{t^\beta}-1}{n(e^{t_\circ^\beta}-1)}\right]^r & : e^{t^\beta}-1 < n(e^{t_\circ^\beta}-1) \\ 0 & : otherwise. \end{cases}$$

Proof. Using Corollary 2.3.2.1,

$$\begin{aligned} \widetilde{R}(t)_I &= \int_t^\infty \widetilde{f}_I(x;\lambda,\beta)dx \\ &= \int_t^\infty \frac{r\beta e^{x^\beta}x^{\beta-1}}{n(e^{t_\circ^\beta}-1)}\left(1-\frac{e^{x^\beta}-1}{n(e^{t_\circ^\beta}-1)}\right)^{r-1}dx, \end{aligned}$$

and the result follows by substituting $\frac{e^{x^\beta}-1}{n(e^{t_\circ^\beta}-1)} = z$.

Let n items on X and m on Y be put on a life test, where X and Y follow distribution with pdf (2.2.4) and (2.2.5), respectively. Let $t_\circ$ and r be the termination times and number of failures before the termination time for X and $t_{\circ\circ}$

and r' be the corresponding figures for Y. Obviously, using Corollary 2.3.2.1, the UMVU estimators of $f_1(x;\lambda_1,\beta_1)$ and $f_2(y;\lambda_2,\beta_2)$ based on the sampling scheme of Bartholomew are given by

$$\widetilde{f}_{1I}(x;\lambda_1,\beta_1)=\frac{r\beta_1 e^{x^{\beta_1}}x^{\beta_1-1}}{n(e^{t_\circ^{\beta_1}}-1)}\left(1-\frac{e^{x^{\beta_1}}-1}{n(e^{t_\circ^{\beta_1}}-1)}\right)^{r-1};e^{x^{\beta_1}}-1<n(e^{t_\circ^{\beta_1}}-1),\quad (2.3.7)$$

$$\widetilde{f}_{2I}(y;\lambda_2,\beta_2)=\frac{r'\beta_2 e^{y^{\beta_2}}y^{\beta_2-1}}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\left(1-\frac{e^{y^{\beta_2}}-1}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\right)^{r'-1};e^{y^{\beta_2}}-1<m(e^{t_{\circ\circ}^{\beta_2}}-1).\quad (2.3.8)$$

Theorem 2.3.4. *The UMVU estimator of P is given by*

$$\widetilde{P}_I=\begin{cases}\int_{z=0}^{c}\left[1-\frac{exp\left\{ln(m(e^{t_{\circ\circ}}-1)z+1)\right\}^{\beta_2/\beta_1}-1}{n(e^{t_\circ^{\beta_1}}-1)}\right]^{r}r'(1-z)^{r'-1}dz;\\[2ex] \left[ln\left\{n(e^{t_\circ^{\beta_1}}-1)+1\right\}\right]^{1/\beta_1}\leq\left[ln\left\{m(e^{t_{\circ\circ}^{\beta_2}}-1)+1\right\}\right]^{1/\beta_2}\\[2ex] \int_{z=0}^{1}\left[1-\frac{exp\left\{ln(m(e^{t_{\circ\circ}}-1)z+1)\right\}^{\beta_2/\beta_1}-1}{n(e^{t_\circ^{\beta_1}}-1)}\right]^{r}r'(1-z)^{r'-1}dz;\\[2ex] \left[ln\left\{n(e^{t_\circ^{\beta_1}}-1)+1\right\}\right]^{1/\beta_1}>\left[ln\left\{m(e^{t_{\circ\circ}^{\beta_2}}-1)+1\right\}\right]^{1/\beta_2},\end{cases}$$

where $c=\left(exp\left[ln\left\{n(e^{t_\circ^{\beta_1}}-1)+1\right\}\right]^{\beta_2/\beta_1}-1\right)/\left(m(e^{t_{\circ\circ}^{\beta_2}}-1)\right)$.

Proof. It can be shown that the UMVU estimator of P is given by

$$\begin{aligned}\widetilde{P}_I&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}_{1I}(x;\lambda_1,\beta_1)\widetilde{f}_{2I}(y;\lambda_2,\beta_2)dxdy\\&=\int_{y=0}^{\infty}\widetilde{R}_I(y;\lambda_1,\beta_1)\widetilde{f}_{2I}(y;\lambda_2,\beta_2)dy,\end{aligned}$$

which on using (2.3.8) and Theorem 2.3.3 leads to

$$\widetilde{P}_I=\int_{y=0}^{\infty}\left[1-\frac{e^{y^{\beta_1}}-1}{n(e^{t_\circ^{\beta_1}}-1)}\right]^{r}\frac{r'\beta_2 e^{y^{\beta_2}}y^{\beta_2-1}}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\left(1-\frac{e^{y^{\beta_2}}-1}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\right)^{r'-1}dy;$$
$$e^{y^{\beta_1}}-1<n(e^{t_\circ^{\beta_1}}-1),\;\; e^{y^{\beta_2}}-1<m(e^{t_{\circ\circ}^{\beta_2}}-1)$$

$$= \int_{y=0}^{min\left[ln\left\{n(e^{t_{\circ}^{\beta_1}}-1)+1\right\}\right]^{1/\beta_1},\left[ln\left\{m(e^{t_{\circ\circ}^{\beta_2}}-1)+1\right\}\right]^{1/\beta_2}} \left[1-\frac{e^{y^{\beta_1}}-1}{n(e^{t_{\circ}^{\beta_1}}-1)}\right]^{r} \times\frac{r'\beta_2 e^{y^{\beta_2}}y^{\beta_2-1}}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\left(1-\frac{e^{y^{\beta_2}}-1}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\right)^{r'-1} dy. \tag{2.3.9}$$

The theorem now procceds by replacing $\frac{e^{y^{\beta_2}}-1}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}$ by z in the two cases.

Corollary 2.3.4.1. *The UMVU estimator of P when $\beta_1 = \beta_2$ and $t_\circ = t_{\circ\circ}$ is given by*

$$\widetilde{P_I} = \begin{cases} r' \sum_{i=0}^{r'-1}(-1)^i\binom{r'-1}{i}\left(\frac{n}{m}\right)^{i+1} B(i+1, r+1); n \le m \\ r' \sum_{j=0}^{r}(-1)^j\binom{r}{j}\left(\frac{m}{n}\right)^{j} B(j+1, r'); n > m. \end{cases}$$

Proof. From Theorem 2.3.4, when $n \le m$,

$$\widetilde{P_I} = r' \int_{z=0}^{n/m} \left[1-\frac{mz}{n}\right]^r (1-z)^{r'-1}dz$$
$$= r' \sum_{i=0}^{r'-1}(-1)^i\binom{r'-1}{i}\left(\frac{n}{m}\right)^{i+1}\int_{w=0}^{1} w^{i+1-1}(1-w)^{r+1-1}dw$$

and the first assertion follows. Similarly, one can prove the second assertion.

Theorem 2.3.5. *The UMVU estimator of $h(t)$ at specified point t is given by*

$$\widetilde{h}(t)_I = \frac{r}{n(e^{t_\circ^\beta}-1)}e^{t^\beta}\beta t^{\beta-1}.$$

Proof. Using 2.1.3,

$$\widetilde{h}(t)_I = \widetilde{\lambda}e^{t^\beta}\beta t^{\beta-1}, \tag{2.3.10}$$

and the result follows by using Theorem 2.3.2.

Theorem 2.3.6. *For $q \in (-\infty, \infty)$, the ML estimator of λ^q is given by:*

$$\widehat{\lambda}_I^q = \left(\frac{r}{n(e^{t_\circ^\beta}-1)}\right)^q.$$

Proof. We take natural log of both sides of (2.3.2) and differentiate it with respect to λ. On equating the differential with respect to zero, we get

$$\widehat{\lambda}_I = \frac{r}{n(e^{t_\circ^\beta}-1)}, \tag{2.3.11}$$

and hence the theorem follows.

Corollary 2.3.6.1. *The ML estimator of $f(x;\alpha,\beta)$ at a specified point x is*

$$\widehat{f}_I(x;\lambda,\beta) = exp\left\{\frac{r(1-e^{x^\beta})}{n(e^{t_\circ^\beta}-1)}\right\}\frac{r\beta e^{x^\beta}x^{\beta-1}}{n(e^{t_\circ^\beta}-1)}.$$

Proof. The ML estimator of $f(x)$ is given by

$$\widehat{f}(x)_{II} = \widehat{\lambda}\beta e^{x^\beta}x^{\beta-1}e^{\widehat{\lambda}(1-e^{x^\beta})}.$$

From (2.3.11) and invariance property of ML estimators, the result straightaway follows.

Theorem 2.3.7. *The ML estimator of $R(t)$ is given by*

$$\widehat{R}(t)_I = exp\left\{\frac{r(1-e^{t^\beta})}{n(e^{t_\circ^\beta}-1)}\right\}.$$

Proof. Using (2.1.2), the $R(t)$ at point t is given by

$$R(t) = exp\left\{\lambda(1-e^{t^\beta})\right\}. \tag{2.3.12}$$

From (2.3.11), (2.3.12) and the invariance property of ML estimators, the ML estimator of $R(t)$, based on the sampling scheme of Bartholomew (1963) is given by the above theorem.

Theorem 2.3.8. *The ML estimator of P is given by*

$$\widehat{P}_I = \int_{z=0}^{\infty} exp\left[\frac{-r\left\{exp\left(ln\left(\frac{m(e^{t_{\circ\circ}^{\beta_2}}-1)z+1}{s}\right)+1\right)^{\beta_1/\beta_2}-1\right\}}{n(e^{t_\circ^{\beta_1}}-1)}\right]e^{-z}dz.$$

Proof. The expression of the ML estimator of P is given by

$$\begin{aligned}\widehat{P}_I &= \int_{y=0}^{\infty}\int_{x=y}^{\infty}\widehat{f}_I(x;\lambda_1,\beta_1)\widehat{f}_I(y;\lambda_2,\beta_2)dxdy\\ &= \int_{y=0}^{\infty}\widehat{R}_I(y;\lambda_1,\beta_1)\widehat{f}_I(y;\lambda_2,\beta_2)dy,\end{aligned}$$

which on using Theorem 2.3.7 and Corollary 2.3.6.1 gives that

$$\widehat{P} = \int_{y=0}^{\infty} exp\left\{\frac{r(1-e^{y^{\beta_1}})}{n(e^{t_\circ^{\beta_1}}-1)}\right\}exp\left\{\frac{r'(1-e^{y^{\beta_2}})}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}\right\}\frac{r'\beta_2 e^{y^{\beta_2}}y^{\beta_2-1}}{m(e^{t_{\circ\circ}^{\beta_2}}-1)}dy.$$

The theorem now follows on putting $[r'(e^{y^{\beta_2}}-1]/[m(e^{t_{\circ\circ}^{\beta_2}}-1)] = z$.

Corollary 2.3.8.1. *The ML estimator of P when $\beta_1 = \beta_2$ and $t_\circ = t_{\circ\circ}$, is given by*

$$\widehat{P} = \frac{r'n}{r'n + rm}$$

Proof. The proof is on similar lines as that of Theorem 2.3.8.

Theorem 2.3.9. *The ML estimator of $h(t)$ at a specified point t is given by*

$$\widehat{h}(t)_I = \frac{r}{n(e^{t_\circ^\beta} - 1)} e^{t^\beta} \beta t^{\beta-1}.$$

Proof. Using 2.1.3,

$$\widehat{h}(t)_I = \widehat{\lambda} e^{t^\beta} \beta t^{\beta-1}.$$

From (2.3.11) and the invariance property of ML estimators, the ML estimator of $h(t)$, based on the sampling scheme of Bartholomew (1963) is given by the above theorem.

2.3.2 Hypothesis Testing

Under this section, we consider the hypothesis testing based on the sampling scheme of Bartholomew, for the following two cases:

1. Testing of $H_\circ : \lambda = \lambda_\circ$ against $H_1 : \lambda \neq \lambda_\circ$, when β is known.

2. Testing of $H_\circ : \lambda \leq \lambda.$ against $H_1 : \lambda > \lambda_\circ$, when β is known.

Proceeding in a similar manner as in Section 2.2.3 and using Lemma 2.3.1, it can be shown that, based on the sampling scheme of Bartholomew, the critical region for testing $H_\circ : \lambda = \lambda.$ against $H_1 : \lambda \neq \lambda_\circ$ is given by:

$$\left\{ r < k_1 \quad or \quad r > k_1' \right\}, \quad where \quad r \sim Poisson \left(n\lambda(e^{t_\circ^\beta} - 1) \right).$$

On the similar lines as in Section 2.3 and using Lemma 2.3.1, it can be shown that, based on the sampling scheme of Bartholomew, the uniformly most powerful critical region for testing $H_\circ : \lambda \leq \lambda.$ against $H_1 : \lambda > \lambda_\circ$ is given by:

$$\phi(r) = \begin{cases} 1 & ; r \geq K_1' \\ 0 & ; \text{otherwise.} \end{cases}$$

2.4 Simulation Study

In this section, we study the performance of our estimation and testing procedures through simulations. Throughout this section, comparisons are made on the basis of MSE of estimators and simulation experiments are conducted using the Monte Carlo simulation technique.

2.4.1 Simulation based on Estimation Procedures

First, we compare the performance of estimators of $\lambda^q, R(t), P$ and $h(t)$ based on Type II censoring scheme. For this purpose, we have generated 1000 random samples from (2.1.1) each of size $n = 50$ for (λ, β)=(0.5,0,5), (0.5,1), (0.5,2), (0.5,4). For each sample, we arranged the data in ascending order and considered a sample of first 'r' ($r \leq n$) observations.

For different values of r =10, 20, 30 and 50, we have computed average values of $\widetilde{\lambda}_{II}$ and $\widehat{\lambda}_{II}$ and their corresponding MSE and results are reported in Table 2.1. Similiarly, we obtain average length and coverage probability of interval estimates which are reported in Table 2.2.

From Table 2.1, it is observed that the MSE corresponding to UMVU estimator is much lower than the MSE of ML estimator. Thus, we can say that the performance of UMVU estimator of λ based on Type II censoring is much better than ML estimator. It can also be seen from Table 2.1 that, as r increases, the performance of both the estimators improve (as MSE is decreasing) and estimates come closer to each other. Also, from Table 2.2, we observe that, as the truncation number r increases, the length of confidence intervals decrease. This justifies the fact that as r moves closer to n, the precision of our estimate will increase .

Again for $r = 10$, 20, 30 and 50, we have computed the average values of $\widetilde{R(t)}$, $\widehat{R(t)}$ and their corresponding MSE and the results are reported in Table 2.3. Similarly, we obtain the average length and coverage probability of interval estimates which are reported in Table 2.4.

Comparing the estimates based on MSE, obtained in Table 2.3, it can be seen that, the UMVU estimate of $R(t)$ outperforms the ML estimator of $R(t)$ for various values of t. However, for $t = 1$ and beyond, the performance of ML estimator is better than UMVU estimator. Performance of both the estimators is quite similar in case of large values of r. From Table 2.3, it is also clear that, as r increases, the MSE corresponding to both the estimators decrease. Also from Table 2.4, we observe that as the truncation number r increases, the length of confidence intervals decreases. It establishes the improvement in the estimate of $R(t)$ for increasing values of r.

To evaluate the efficiency of the estimators of P, we have obtained 1000 random samples from each of the populations X and Y with sizes (n, m) from (2.1.1) with $\beta_1 = \beta_2 = 2$ and (λ_1, λ_2) = (0.5,0.5), (0.5,1), (0.5, 1.5) and (0.5,2). The samples corresponding to both the populations are arranged in ascending order and first (r, r') observations are considered. For (r, r')=(10,10), (20,20), (30,25) and (40,40), we have computed average values of $\widetilde{P}$ and $\widehat{P}$ and their corresponding MSE and the results are presented in Table 2.5. Similarly, we obtain the average length and coverage probability of interval estimates which are reported in Table 2.6.

From Table 2.5, it is clear that based on Type II censoring, for all values of (r, r'), ML estimator of P gives better results than UMVU estimator of P. Also from Table 2.6, we observe that, as truncation number (r, r') increases, the length of confidence intervals decrease. It establishes the improvement in estimate of P for increasing values of (r, r').

To compare the estimates of $h(t)$, we have plotted the hazard rates and their estimates against time t for $\lambda = 2$, $\beta = 0.8$, $n = 50$ and $r = 10, 20, 30$ and 40. In Figure 2.1, we have plotted the hazard rate and it's ML estimator and UMVU estimator for different values of r. We observe that, as r increases, the estimates come closer to the true values. For $r = 40$, the estimated values of hazard rate overlaps the plot. It establishes the consistency properties of estimators.

Now we compare the performance of estimators of λ^q, $R(t)$, P and $h(t)$ based on the sampling scheme of Bartholomew.

In order to obtain the point estimates of $R(t)$ based on the sampling scheme of Bartholomew, we have generated 1000 random samples each of size 100 from (2.1.1) with $\lambda = 0.5$ and $\beta = 2$. By fixing the termination time at $t_\circ$, and replacing the failure by operating one, values of r (number of failures before time $t_\circ$) are computed. For different termination times $t_\circ$=0.20, 0.45, 0.50, 0.65 and 0.80, we have computed average values of $\widetilde{R(t)}$ and $\widehat{R(t)}$, their corresponding MSE. For different values of t, results are presented in Table 2.7.

From Table 2.7, it is observed that based on the sampling scheme of Bartholomew, for small values of t and all values of $t_\circ$, UMVU estimator and ML estimator of $R(t)$ are equally efficient. For large values of t and small values of $t_\circ$, ML estimator is more efficient than UMVU estimator of $R(t)$. However, for large values of $t_\circ$, UMVU estimator becomes more efficient than ML estimator of $R(t)$. This result shows the importance of termination time $t_\circ$ in the sampling scheme of Bartholomew.

To test the performance estimators of P based on Bartholomew's sampling method, we have obtained 1000 random samples from each of the populations X and Y with sizes (n, m) from (2.1.1) with $\beta_1 = \beta_2 = 2$ and $(\lambda_1, \lambda_2) =$ (0.5,0.5), (0.5,1), (0.5,1.5) and (0.5,2). For each sample corresponding to both the populations by fixing the termination time at $t_\circ = t_{\circ\circ}$ and replacing the failure by operating one, values of r (number of failures before time $t_\circ$ in X) and values of s (number of failures before time $t_{\circ\circ}$ in Y) are computed. For $t_\circ = t_{\circ\circ}$=0.80, 1 and 1.5, we have computed average values of $\widetilde{P_I}$ and $\widehat{P_I}$ and their corresponding MSE for $n > m$ and $n < m$ and the results are presented in Tables 2.8 and 2.9 respectively.

From Table 2.8, for $n > m$, it is observed that for small m when $n = 50$, UMVU estimator of P gives better results than MLE of P. As m increases, both the estimators are equally efficient. From Table 2.9, for $n < m$, it is observed that for small n when $m = 50$, ML estimator of P gives better result than UMVU estimator of P. As m increases both the estimators are equally efficient.

To compare the estimates of $h(t)$ with the true value of $h(t)$, we have plotted

$h(t)$ and its estimates for $\beta = 0.5$, $\lambda = 0.8$ and $t_\circ = 0.15$ (Figure 2.2). From theory we observe that both UMVU estimator and ML estimator of $h(t)$ are same and hence we have plotted only one to represent both. Since the plot of estimate of $h(t)$ almost overlaps the plot of hazard rate, it establishes the consistency property of estimators.

2.4.2 Simulation based on Hypothesis Testing

In this section, we check the validity of hypotheses developed in sections 2.2.3 and 2.3.2, respectively. For this purpose, we first test the hypothesis, $H_\circ : \lambda = \lambda_\circ$ against $H_1 : \lambda \neq \lambda_\circ$ based on Type II censoring. We generate a sample of size $n = 50$ from (2.1.1) with $(\lambda_1 = 0.5, \beta_1 = 2)$ (Hereafter, denoted by Sample 1).

Now, using the chi-square table at $\nu = 5\%$ level of significance, for the above generated data set we obtained $k_\circ = 48.75756$ and $k_\circ^{'} = 95.02318$. As for $r = 35$, the value of $S_{35} = 85.5457$ is not lying in the critical region, thus we do not reject $H_\circ$ at 5% level of significance. Again considering Sample 1, for testing $H_\circ : \lambda \leq \lambda_\circ$ against $H_1 : \lambda > \lambda_\circ$ at 5% level of significance, we obtained $k_\circ^{''} = 51.1393$. As for $r = 35$, the value of $S_{(35)} = 85.5457$ is not lying in the critical region, thus we do not reject $H_\circ$ at 5% level of significance.

In order to test $H_\circ : P = 0.6667(P_\circ)$ against $H_1 : P \neq 0.6667(P_\circ)$ based on Type II censoring scheme, we generated another sample of size $m = 60$ from (2.1.1) with $(\lambda_2 = 1, \beta_2 = 2)$.
For $r' = 40$, from the second sample we get $T_{(40)} = 33.2952$. From these samples we get $S_{(35)}/T_{(40)} = 2.569314$. Now with the help of F-table at 5% level of significance, we obtained $k_2 = 1.4183$ and $k_2^{'} = 3.5386$. Hence, in this case we may accept $H_\circ$ at 5% level of significance.

The similar calculations may be done for testing the above hypotheses under the sampling scheme of Bartholomew.

2.5 Real Data Analysis

Now, we provide real data analysis based on Type II censoring, when all the parameters of the distribution are unknown, to see how the model works in practice.

We consider the first real data set (Data set I or X population) which was used by Yousaf *et al.* (2019) (initially taken from Lawless (2011)) to illustrate the proposed methodology. The data comprise of 50 observations, which represents the quantity of 1000s of cycles to failure for electrical appliances in a life test.

The second data set (Data set II or Y population) was used by Wang *et al.* (2015) (initially taken by Coetzeee (1996)). It represents the failure data of a 180 ton rear dump truck. The data shows the number of hours between 128 failures.

The KS test is used first to see whether the distribution given in (2.1.1) fits the X and Y populations. The following ML estimators of parameters of X population, (λ_1, β_1) and ML estimators of parameters of Y population, (λ_2, β_2) are obtained based on complete data sets.

$$(\lambda_1, \beta_1) = (2.3651, 0.9371)$$

and

$$(\lambda_2, \beta_2) = (3.5242, 0.6072).$$

With the help of these ML estimators we apply KS test which confirms that both the data observed for X $(KS = 0.0881; p = 0.7993)$ and $Y(KS = 0.0857; p = 0.3119)$ are drawn from (2.1.1). Figures 2.3 and 2.4 show that (2.1.1) is a good fit for these two data sets.

In order to obtain the ML estimator of $R(t)$ and P based on Type II censoring, we first consider $r = 30$ lifetimes from X population and rest 20 observations are considered as censored. Similarly, we consider first $r' = 90$ lifetimes from Y population and rest 38 observations are considered as censored. Considering Chen distribution as a lifetime model for X-population, for the first r observations, the ML estimator of λ_{1II} and β_{1II} comes out to be $\widehat{\lambda_{1II}} = 2.0992$ and $\widehat{\beta_{1II}} = 0.8702$. The MLE of $R(t)$ at time point $t = 0.15$ is given by $\widehat{R}(t)_{1II} = 0.6414$.

Similarly, by considering Chen distribution as a lifetime model for Y-population, for the first r' observations, the ML estimator of λ_{2II} and β_{2II} comes out to be $\widehat{\lambda_{2II}} = 4.1559$ and $\widehat{\beta_{2II}} = 0.6560$. The ML estimator of $R(t)$ at time point $t = 0.15$ is given by $\widehat{R}(t)_{2II} = 0.2497$

To evaluate ML estimator of P_{II}, we have considered first data set as X Population and second data set as Y population and obtain $\widehat{P}_{II} = 0.4331$. For X and Y populations and corresponding to different values of t, we have evaluated ML estimator of $h(t)$. Results are plotted in Figure 2.5.
In particular, for $t = 0.15$,

$$\widehat{h}_{1II}(t) = 2.8311$$

and

$$\widehat{h}_{2II}(t) = 6.1841.$$

From Figure 2.5, it is clear that the plot of $h(t)$ for both data sets have the bathtub shape.

2.6 Concluding Remarks

In this chapter, we have developed the estimation procedures for the Chen distribution based on Type II Censoring and Sampling scheme of Bartholomew. Both point and interval estimations are taken into account. For several parametric functions, hypotheses were formulated. The finite sample performance of the UMVU estimators and ML estimators of reliability functions and other parameters are investigated using extensive Monte Carlo simulation. For Type II censoring, the performance of UMVU estimator of λ^q is better than ML estimator. Furthermore, the performance of the UMVU estimator of $R(t)$ is superior than the performance of the ML estimate of $R(t)$ for different values of t. However, for $t = 1$ and beyond, the performance of ML estimator is better than UMVU estimator. Moreover, for all values of (r, r'), ML estimator of P gives better result than the UMVU estimator of P. In case of Sampling scheme of Bartholomew, for small values of t and all values of $t_\circ$, UMVU estimator and ML estimator of $R(t)$ are equally efficient. But for large values of t and small values of $t_\circ$, ML estimator is more efficient than UMVU estimator of $R(t)$. However, for large values of $t_\circ$, UMVU estimator becomes more efficient than ML estimator of $R(t)$, thus depicting the importance of termination time $t_\circ$ in this scheme. From the study of performance of P it has been observed that for small m when $n = 50$, UMVU estimator of P gives better result than ML estimator of P. On the other hand, for $n < m$, it is observed that for small n when $m = 50$, ML estimator of P gives better result than UMVU estimator of P. As n and m increases, both estimators become equally efficient. With the help of Figures 2.1 and 2.2, we have established the consistency of estimators of $h(t)$ under both censoring schemes. The real-life data examples demonstrate how the proposed estimators of two measures of reliability and confidence ellipsoids can be implemented in practice.

Table 2.1: **Average values of point estimates of λ and their MSE/ Variances, when β is known**

$r->$	**10**		**20**		**30**		**50**	
β	$\widetilde{\lambda}$	$\widehat{\lambda}$	$\widetilde{\lambda}$	$\widehat{\lambda}$	$\widetilde{\lambda}$	$\widehat{\lambda}$	$\widetilde{\lambda}$	$\widehat{\lambda}$
0.5	0.4987	0.5541	0.4978	0.524	0.5033	0.5207	0.4993	0.5095
	0.0265	0.0357	0.0142	0.0163	0.0094	0.0105	0.0052	0.0055
1	0.5065	0.5627	0.5015	0.5279	0.5044	0.5218	0.4992	0.5094
	0.0367	0.0492	0.0133	0.0155	0.0092	0.0103	0.0051	0.0054
2	0.5062	0.5624	0.5047	0.5313	0.4968	0.5139	0.5	0.5102
	0.0333	0.045	0.0147	0.0172	0.009	0.0099	0.0048	0.0051
4	0.4973	0.5526	0.5	0.5263	0.5033	0.5207	0.4981	0.5083
	0.0281	0.0375	0.0128	0.0149	0.009	0.01	0.0053	0.0056

Note: 1st and 2nd rows represent the average estimates and MSE of λ.

Table 2.2: **Average length and coverage probability of interval estimates**

$r->$	**10**		**20**		**30**		**50**	
β	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
0.5	0.6801	95.05	0.4562	95.2	0.3706	95.75	0.2817	95.05
1	0.6841	95	0.4555	95.7	0.3705	95.05	0.2825	95.7
2	0.6860	94.98	0.4621	95.06	0.3692	95.16	0.2832	95.15
4	0.6842	94.88	0.4575	95.32	0.3691	95.18	0.2822	95.6

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 2.3: **Average values of point estimates of $R(t)$ and their MSE/ Variances, when β is known**

$r->$		**10**		**20**		**30**		**50**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.2	0.9798	0.9795	0.9772	0.9798	0.97880	0.9797	0.97899	0.9799	0.9794
		5.3e-05	7.1e-05	2.1e-05	2.4e-05	1.5e-05	1.7e-05	8e-06	9e-06
0.5	0.8676	0.8681	0.8558	0.8688	0.8629	0.8683	0.8644	0.8667	0.8644
		0.0016	0.002	8e-04	9e-04	5e-04	5e-04	3e-04	3e-04
0.8	0.6388	0.6453	0.6231	0.639	0.6279	0.6401	0.6327	0.64	0.6355
		0.0089	0.0093	0.0044	0.0045	0.0025	0.0026	0.0016	0.0016
0.9	0.5358	0.5356	0.5132	0.5361	0.524705	0.5349	0.5272	0.5357	0.5311
		0.0125	0.0125	0.00575	0.0058	0.0038	0.0038	0.0023	0.0023
1	0.4235	0.4246	0.4052	0.4239	0.4138	0.4237	0.4169	0.4241	0.4199
		0.0141	0.0132	0.0068	0.0067	0.0045	0.0044	0.0027	0.0026
1.5	0.1998	0.201	0.196	0.2008	0.198	0.201	0.1991	0.1996	0.1984
		0.0108	0.0091	0.0051	0.0047	0.0035	0.0033	0.0021	0.002

Note: 1st and 2nd rows represent the average estimates and MSE of $R(t)$.

Table 2.4: **Average length and coverage probability of interval estimates**

$r->$	**10**		**20**		**30**		**50**	
t	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
0.2	0.027	95	0.0183	94.8	0.0148	94.6	0.0113	95.6
0.5	0.1608	95.1	0.111	95.04	0.0895	95.02	0.0689	94.76
0.8	0.3466	94.68	0.2483	095.07	0.2034	94.59	0.1582	094.98
0.9	0.4176	95	0.3068	94.98	0.2537	94.91	0.1987	95.26
1	0.4176	95	0.3068	94.98	0.2537	94.91	0.1987	95.26
1.5	0.1303	94.99	0.0748	95.22	0.0559	95.05	0.0396	95.03

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 2.5: **Average values of point estimates of P and their MSE/ Variances**

$\lambda_1->$	**0.5**		**0.5**		**0.5**		**0.5**	
$\lambda_2->$	**0.5**		**1**		**1.5**		**2**	
$P->$	**0.5**		**0.6666667**		**0.75**		**0.8**	
$(r, r') \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
(10,10)	0.5029	0.5028	0.6614	0.6545	0.7461	0.737	0.8039	0.7943
	0.013	0.0118	0.0112	0.0105	0.0084	0.0082	0.0055	0.0055
(20,20)	0.4994	0.4994	0.6688	0.6652	0.7506	0.7459	0.801	0.7961
	0.0063	0.006	0.0054	0.0052	0.0034	0.0034	0.0024	0.0024
(30,25)	0.6242	0.6229	0.6655	0.6635	0.7502	0.7474	0.8004	0.7971
	0.0042	0.0041	0.0036	0.0035	0.0027	0.0026	0.0018	0.0018
(40,40)	0.4991	0.4991	0.6666	0.6647	0.7511	0.7488	0.7998	0.7974
	0.0032	0.0032	0.0024	0.0024	0.0017	0.0017	0.0014	0.0014

Note: 1st and 2nd rows represent the average estimates and MSE.

Table 2.6: **Average length and coverage probability of interval estimates**

$\lambda_1->$	**0.5**		**0.5**		**0.5**		**0.5**	
$\lambda_2->$	**0.5**		**1**		**1.5**		**2**	
$P->$	**0.5**		**0.6666667**		**0.75**		**0.8**	
$(r, r') \downarrow$	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
(10,10)	0.4054	94.64	0.3716	94.63	0.3267	94.96	0.2879	94.78
(20,20)	0.2977	95.15	0.2687	94.95	0.232	95.09	0.2015	95.08
(30,25)	0.2575	94.75	0.232	94.76	0.1991	94.67	0.1723	95.11
(40,40)	0.2147	94.91	0.1927	94.82	0.1642	95.11	0.1415	95.07

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 2.7: **UMVU and ML Estimator of $R(t)$ based on the Sampling Scheme of Bartholomew**

$t_{\circ}->$		**0.20**		**0.45**		**0.50**		**0.65**		**0.80**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.25	0.9683	0.9704	0.9709	0.9693	0.9694	0.9699	0.9699	0.9717	0.9717	0.9735	0.9735
		9e-04	9e-04	2e-04	2e-04	0.00012	0.00012	7e-05	7e-05	5e-05	5e-05
0.45	0.8938	0.8945	0.8998	0.8974	0.8984	0.8991	0.8999	0.9031	0.9035	0.9107	0.9109
		0.0102	0.0093	0.0017	0.0017	0.0012	0.0012	6e-04	6e-04	6e-04	6e-04
0.8	0.6388	0.6428	0.6991	0.6499	0.6609	0.6597	0.6682	0.6678	0.6724	0.6843	0.6869
		0.0905	0.0713	0.0132	0.0129	0.0107	0.0107	0.0058	0.006	0.0047	0.0049
0.9	0.5358	0.5485	0.6382	0.5532	0.571	0.5504	0.5647	0.5685	0.5761	0.5924	0.5967
		0.1322	0.1028	0.0209	0.0209	0.0149	0.0149	0.0086	0.0089	0.0068	0.0072

Note: 1st and 2nd rows represent the average estimates and MSE.

Table 2.8: **UMVU and ML Estimator of P based on the Sampling Scheme of Bartholomew**

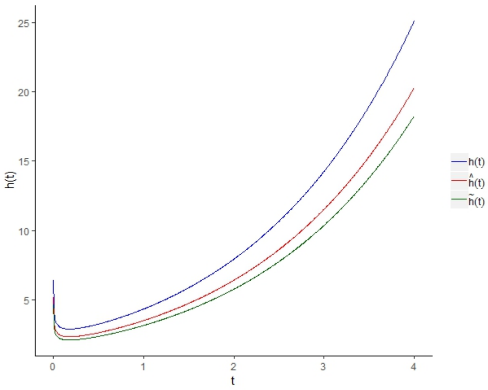
$\lambda_1 ->$	**0.5**		**0.5**		**0.5**		**0.5**	
$\lambda_2 ->$	**0.5**		**1**		**1.5**		**2**	
$P ->$	**0.5**		**0.6666667**		**0.75**		**0.8**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
				$(n = 50) > (m = 35)$				
0.80	0.506	0.5031	0.6357	0.6337	0.689	0.6874	0.7219	0.7206
	0.0062	0.0062	0.0043	0.0045	0.0061	0.0064	0.008	0.0082
1	0.5043	0.5025	0.6035	0.6021	0.6447	0.6435	0.6681	0.6671
	0.0024	0.0024	0.0055	0.0057	0.0125	0.0127	0.0186	0.0189
1.5	0.5009	0.5	0.5389	0.5381	0.5734	0.5726	0.6081	0.6074
	4e-04	4e-04	0.017	0.0172	0.0319	0.0322	0.0377	0.038
				$(n = 50) > (m = 45)$				
0.80	0.5022	0.5015	0.632	0.6315	0.6869	0.6865	0.7209	0.7206
	0.0052	0.0052	0.0043	0.0043	0.0062	0.0063	0.0079	0.008
1	0.5019	0.5014	0.6037	0.6034	0.6441	0.6438	0.6682	0.6679
	0.0021	0.0021	0.0055	0.0055	0.0124	0.0125	0.0185	0.0186
1.5	0.5006	0.5004	0.5374	0.5372	0.5742	0.574	0.6069	0.6067
	4e-04	4e-04	0.0172	0.0173	0.0315	0.0316	0.038	0.0381

Note: 1st and 2nd rows represent the average estimates and MSE.

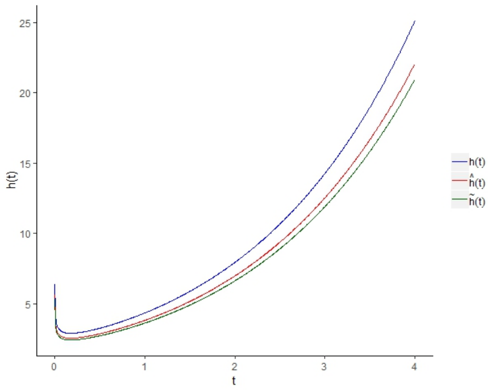
Table 2.9: **UMVU and ML Estimator of P based on the Sampling Scheme of Bartholomew**

$\lambda_1 - >$	**0.5**		**0.5**		**0.5**		**0.5**	
$\lambda_2 - >$	**0.5**		**1**		**1.5**		**2**	
$P - >$	**0.5**		**0.6666667**		**0.75**		**0.8**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
				$(n = 35) < (m = 50)$				
0.80	0.4971	0.5001	0.6273	0.6293	0.6891	0.6906	0.7217	0.723
	0.0063	0.0063	0.0052	0.0051	0.0067	0.0065	0.0088	0.0086
1	0.5015	0.5033	0.6024	0.6037	0.6438	0.645	0.6651	0.6662
	0.0025	0.0025	0.0061	0.0059	0.0129	0.0126	0.0196	0.0193
1.5	0.499	0.5	0.538	0.5389	0.5715	0.5723	0.6066	0.6073
	4e-04	4e-04	0.0171	0.0168	0.0325	0.0322	0.0381	0.0378
				$(n = 45) < (m = 50)$				
0.80	0.506	0.5031	0.6357	0.6337	0.689	0.6874	0.7219	0.7206
	0.0062	0.0062	0.0043	0.0045	0.0061	0.0064	0.008	0.0082
1	0.5043	0.5025	0.6035	0.6021	0.6447	0.6435	0.6681	0.6671
	0.0024	0.0024	0.0055	0.0057	0.0125	0.0127	0.0186	0.0189
1.5	0.5009	0.5	0.5389	0.5381	0.5734	0.5726	0.6081	0.6074
	4e-04	4e-04	0.017	0.0172	0.0319	0.0322	0.0377	0.038

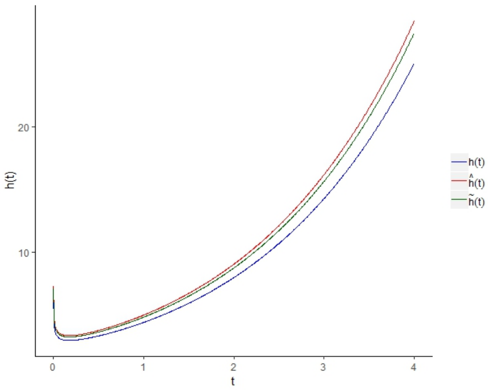
Note: 1st and 2nd rows represent the average estimates and MSE.



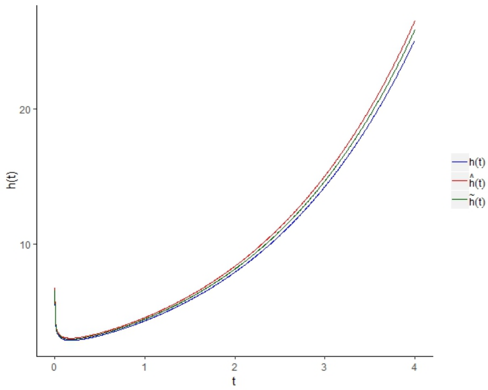
(a) **n=50, r=10**



(b) **n=50, r=20**



(c) **n=50, r=30**



(d) **n=50, r=40**

Figure 2.1: **Plots of $h(t)$ and it's estimates against time t**

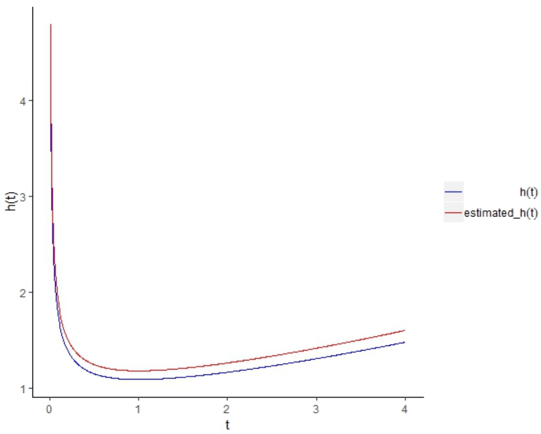


Figure 2.2: **Plots of h(t) and its estimates under Type I censoring**

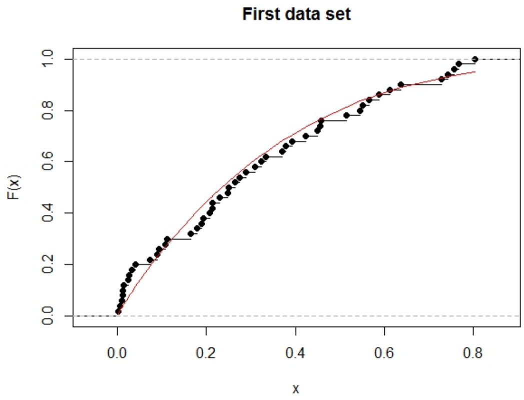


Figure 2.3: **The empirical and theoretical cdf of first data set**

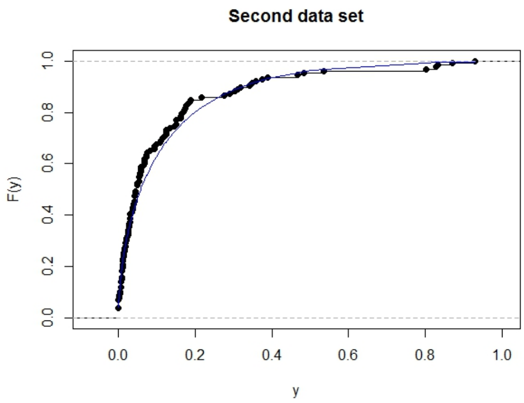
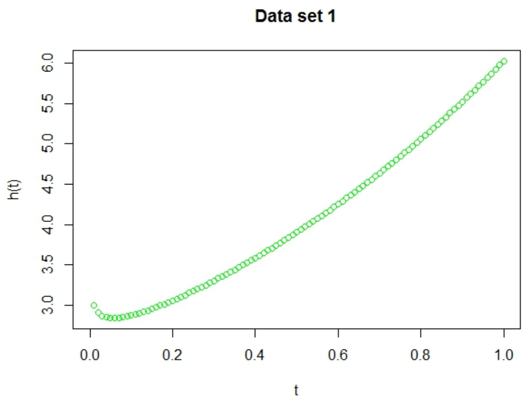
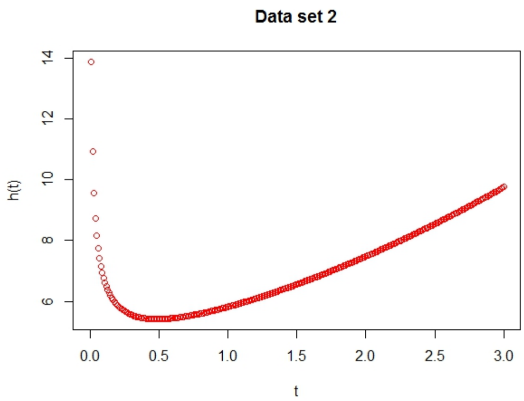


Figure 2.4: **The empirical and theoretical cdf of second data set**



(a) **The plot of h(t) against time t for first data set**



(b) **The plot of h(t) against time t for second data set**

Figure 2.5: **Plots of $h(t)$ for real life data sets**

Chapter 3

Classical Inference for the Reliability Functions of Kumaraswamy-G Family of Distributions Based on Censored Observations

3.1 Introduction

The Kumaraswamy (Kum) distribution is widely applied to model the random phenomenon having finite lower and upper bounds, e.g., the height of individuals, atmospheric temperatures, hydrological data such as daily rainfall, daily streamflow, etc. The distribution was first defined by Kumaraswamy (1976, 1978). Nadarajah (2008) demonstrated that the distribution might be viewed as a special case of three parameter Beta distribution. Several other unimodal distributions can also be approximated by Kumaraswamy's distribution [see Kumaraswamy (1980) and Ponnambalam *et al.* (2001)]. Garg (2009) studied the generalized order statistics from Kum distribution. Jones (2009) explored the background and genesis of the Kum distribution and demonstrated some similarities and differences between the beta and Kum distributions. He highlighted several advantages of the Kum distribution over the beta distribution. In hydrology and related areas, Kum distribution has received considerable interest [See, Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi *et al.* (2000), Ponnambalam *et al.* (2001) and Ganji *et al.* (2006)]. Sindhu *et al.* (2013) focused on Bayesian and non-Bayesian estimation for the shape parameter of the Kum distribution under Type-II censored samples.

Eldin *et al.* (2014) obtained the ML estimators and Bayes estimators for the parameters of the Kum distribution under general progressive Type II censoring. Mameli (2015) proposed a new generalization of the skew-normal distribution, referred as the Kum skew-normal distribution. He demonstrated that this new distribution is computationally more tractable than the Beta skew-normal distribution proposed by Mameli and Musio (2013). Kızılaslan and Nadar (2016) considered the Kum distribution, when the lower record values along with the number of observations following the record values (inter-record times) have been observed, and derived the maximum likelihood and Bayes estimators for estimating the parameters of the distribution as well as for the future record values prediction. Dey *et al.* (2017) focussed on Bayesian and non-Bayesian estimation of multicomponent stress–strength reliability when both step and strength follow Kum distribution with common shape parameter. Dey *et al.* (2018) considered and investigated performance of ten different frequentist approaches for the estimation of parameters of Kum distribution, namely, maximum likelihood estimators, moments estimators, L-moments estimators, percentile based estimators, least squares estimators, weighted least squares estimators, maximum product of spacings estimators, Cramér–von-Mises estimators, Anderson–Darling estimators and right tailed Anderson–Darling estimators.

In recent years, a large amount of literature has been developed regarding the generalization of classical distributions. For some of the citations, one may refer to Hassan *et al.* (2020) and the references therein. Cordeiro and Castro (2011) introduced a new Kumaraswamy generalized (Kum-G) family of distributions and discussed its basic statistical properties. They mentioned that the Kum-G family of densities has the ability of fitting skewed data and allows for greater flexibility of its tails. The distribution generalizes Kumaraswamy's modelling ability and can be used in a wide range of engineering and biological applications. Nadarajah *et al.* (2012) derived a simple representation for the Kum-G family of distributions as a linear combination of exponentiated distributions and studied its general properties. They obtained ML estimators of its parameters and discussed its bivariate extension as well. Tamandi and Nadarajah (2016) developed maximum spacing estimation procedure for the parameters of Kum-G distribution. Kundu and Chowdhary (2018) compared the minimums of two independent and heterogeneous samples each following Kum-G distribution with respect to usual stochastic ordering and hazard rate ordering. They also established likelihood ratio ordering between the minimum order statistics for heterogeneous multiple-outlier Kum-G random variables with the same parent distribution function. Kumari *et al.* (2019) provided characterization of Kum-G distribution based on record values and obtained point and interval estimates of the two measures of reliability based on records. They further developed procedures for testing the hypotheses related to various parametric functions. Chaturvedi and Bhatnagar (2020) developed classical and preliminary test estimators for the reliability functions of Kum-G distribution under progressive Type II censoring.

The purpose of this chapter is to extend the results of Kumari *et al.* (2019) for the cases of Type II censoring and the sampling scheme proposed by Bartholomew. Considering Kum-G distribution, we develop UMVU estimators and ML estimators for the reliability functions, $R(t)$ and P. For deriving UMVU estimators, we followed the approach proposed by Chaturvedi and Tomer (2003), which saves tedious and time-consuming calculation of stress–strength function. The chapter is organized as follows: In Section 3.2, we provide point estimators and exact confidence intervals for the q^{th} power of parameter α, for $q \in (-\infty, +\infty)$, and for functions $R(t)$ and P based on Type II censoring scheme. In Section 3.3, based on the sampling scheme proposed by Bartholomew (1963), the point estimators for the α^q, $R(t)$ and P are provided. In Section 3.4, we present findings of simulation studies followed by real data analysis in Section 3.5. We end with a brief set of conclusions in Section 3.6.

3.2 Estimation based on Type II Censoring Scheme

A random variable X is said to follow Kumaraswamy (1980) distribution if its pdf is given by

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}(1-x^{\beta})^{\alpha-1};\ 0<x<1,\ \alpha,\beta>0. \tag{3.2.1}$$

Considering the complete sample case, Nadar *et al.* (2014) have obtained the estimator of P for the distribution given in (3.2.1) assuming the parameter 'β' to be common for the two distributions.

A random variable X follows Kumaraswamy-G distributions [Cordeiro and Castro (2011)], if it's pdf and cdf are of the form

$$f(x;\alpha,\beta)=\alpha\beta g(x)G^{\beta-1}(x)[1-G^{\beta}(x)]^{\alpha-1};\ x>0,\ \alpha,\beta>0, \tag{3.2.2}$$

and

$$F(x;\alpha,\beta)=1-(1-G^{\beta}(x))^{\alpha} \tag{3.2.3}$$

respectively, where α and β are the shape parameters of the Kum-G distribution and $g(x)$ represents the pdf of $G(x)$.

It is to be noted that the distribution given in (3.2.2) reduces to Kum distribution when $G(x)=x$.

3.2.1 UMVU and ML Estimators of $\alpha^q, R(t)$ and P based on Type II Censoring

Suppose 'n' items are put on a test and the test is terminated after the first 'r' ordered observations are recorded. Let us denote by $0<X_{(1)}\leq X_{(2)}\leq ...\leq X_{(r)}$, $0<r<n$, the lifetimes of first r failures. Obviously $(n-r)$ items survived until $X_{(r)}$.

Here, we provide an important lemma, which will help prove the main results of this section.

Lemma 3.2.1. *Let*

$$S_{(r)} = -\left[\sum_{i=1}^{r} ln\left\{1 - G^{\beta}(x_i)\right\} + (n-r)ln\left\{1 - G^{\beta}(x_r)\right\}\right],$$

then, $S_{(r)}$ is complete and sufficient for the Kum-G distribution (3.2.2). Moreover, the pdf of $S_{(r)}$ is given by

$$g_{S_{(r)}}(s;\alpha) = \frac{1}{\Gamma(r)} s^{r-1}\alpha^{r} \exp\left\{-\alpha s\right\}, s > 0, \alpha > 0, r > 0, \tag{3.2.4}$$

where, $\Gamma(\cdot)$ denotes the Gamma function.

Proof. The pdf of Kum-G distribution can be rewritten as

$$f(x;\alpha,\beta) = \frac{\alpha\beta g(x)G^{\beta-1}(x)}{1 - G^{\beta}(x)} \exp\left[\alpha\, ln\left\{1 - G^{\beta}(x)\right\}\right]. \tag{3.2.5}$$

Using (3.2.5), the joint pdf of $0 < X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)} < \infty$ is given by

$$f^{*}(x_{(1)}, x_{(2)}, ..., x_{(n)}; \alpha, \beta) = n!(\alpha\beta)^{n} \prod_{i=1}^{n} \frac{g(x_{(i)})G^{\beta-1}(x_{(i)})}{1 - G^{\beta}(x_{(i)})} \times \exp\left[\alpha \sum_{i=1}^{n} ln(1 - G^{\beta}(x_{(i)}))\right]. \tag{3.2.6}$$

Integrating out $x_{(r+1)}, x_{(r+2)}, ..., x_{(n)}$ from (3.2.6) over the region $x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n)} < \infty$, the joint pdf of $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(r)}$ comes out to be

$$h(x_{(1)}, x_{(2)}, ..., x_{(r)}; \alpha, \beta) = \frac{n!}{(n-r)!}\alpha^{r}\beta^{r} \prod_{i=1}^{r} \frac{g(x_{(i)})G^{\beta-1}(x_{(i)})}{1 - G^{\beta}(x_{(i)})} \exp(-\alpha S_{(r)}). \tag{3.2.7}$$

It follows from (3.2.7) and Fisher-Neyman factorization theorem [See Rohatgi and Saleh (2012), pp. 361] that $S_{(r)}$ is sufficient for the Kum-G distribution. Moreover, if we consider the transformation $Z_i = (n - i + 1)\left\{U_{(i)} - U_{(i-1)}\right\}, i = 1, 2, ...r; U_0 = 0$, then $Z_i^{\prime}s$ are independent and identically distributed (i.i.d.) random variables, each having exponential distribution with mean life $1/\alpha$. It is easy to see that $\sum_{i=1}^{r} Z_i = S_{(r)}$. Lemma 3.2.1 now follows from the additive property of gamma distribution [see Johnson and Kotz (1970), pp. 170]. For known β, we can observe that the distribution of S_r belongs to one-parameter exponential family and hence, it is also complete [See Rohatgi and Saleh (2012), pp. 367].

Theorem 3.2.2. *For $q \in (-\infty, \infty)$, the UMVU estimator of α^q is given by:*

$$\widetilde{\alpha}^q_{II} = \begin{cases} \frac{\Gamma r}{\Gamma(r-q)} S^{-q}; & r - q > 0 \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. From (3.2.4),

$$E\left(S_{(r)}^{-q}\right) = \frac{\Gamma(r-q)}{\Gamma(r)}\alpha^q, r > q. \tag{3.2.8}$$

The Lehmann-Scheffe theorem can now be used to prove the statement of the theorem [see Rohatgi & Saleh (2012)].

Corollary 3.2.2.1. *At a given point x, the UMVU estimator of the sampled pdf is given by:*

$$\widetilde{f}_{II}(x; \alpha, \beta) = \begin{cases} \frac{\beta g(x) G^{\beta-1}(x)}{B(1,r-1) S_{(r)} (1-G^\beta(x))} \left(1 + \frac{ln(1-G^\beta(x))}{S_{(r)}}\right)^{r-2}; \\ \qquad\qquad -ln(1-G^\beta(x)) < S_{(r)} \\ 0; \qquad\qquad \textit{otherwise,} \end{cases}$$

where, $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

Proof. Let us write the pdf (3.2.2) as follows

$$f(x; \alpha, \beta) = \frac{\alpha\beta g(x) G^{\beta-1}(x)}{1 - G^\beta(x)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left\{-ln(1 - G^\beta(x))\right\}^i \alpha^i,$$

then the following Corollary straight away follows from Theorem 3.2.2.

Theorem 3.2.3. *The UMVU estimator of $R(t)$ at a specified point t is*

$$\widetilde{R}(t)_{II} = \begin{cases} \left[1 + \frac{ln(1-G^\beta(t))}{S_{(r)}}\right]^{r-1}; & -ln(1 - G^\beta(t)) < S_{(r)} \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. Using Corollary 3.2.2.1, we have

$$\widetilde{R}(t)_{II} = \int_t^\infty \frac{\beta g(x) G^{\beta-1}(x)}{B(1, r-1) S_{(r)} (1 - G^\beta(x))} \left(1 + \frac{ln(1 - G^\beta(x))}{S_{(r)}}\right)^{r-2} dx,$$

and the result follows by substituting $\frac{-ln(1-G^{\beta}(x))}{S_{(r)}} = v$.

Let X and Y be two independent random variables following the classes of distributions $f_1(x;\alpha_1,\beta_1)$ and $f_2(y;\alpha_2,\beta_2)$, respectively, where

$$f_1(x;\alpha_1,\beta_1) = \alpha_1\beta_1 g(x)G^{\beta_1-1}(x)(1-G^{\beta_1}(x))^{\alpha_1-1};\ x>0,\ \alpha_1,\beta_1>0, \quad (3.2.9)$$

and

$$f_2(y;\alpha_2,\beta_2) = \alpha_2\beta_2 h(y)H^{\beta_2-1}(y)(1-H^{\beta_2}(y))^{\alpha_2-1};\ y>0,\ \alpha_2,\beta_2>0. \quad (3.2.10)$$

Let n items on X and m items on Y are put on a life test and the termination numbers for X and Y are r and r', respectively. Let us define

$$S_{(r)} = -\left[\sum_{i=1}^{r} ln(1-G^{\beta_1}(x_i)) + (n-r)(ln(1-G^{\beta_1}(x_r))\right],$$

and

$$T_{(r')} = -\left[\sum_{j=1}^{r'} ln(1-H^{\beta_2}(y_j)) + (m-r')(ln(1-H^{\beta_2}(y_{r'}))\right].$$

In the following theorem, we obtain the UMVU estimator of P.

Theorem 3.2.4. *When X and Y come from different family of distributions, the UMVU estimator of P is as follows:*

$$\widetilde{P}_{II} = \begin{cases} \int_{z=0}^{c} \frac{1}{B(1,r'-1)}\left[1+\frac{ln\left\{1-G(H^{-1}(1-e^{-zT_{(r')}}))^{\beta_1/\beta_2}\right\}}{S_{(r)}}\right]^{r-1} \\ (1-z)^{r'-2}dz;\quad if\ G^{-1}\left\{(1-e^{-S_{(r)}})^{1/\beta_1}\right\} \le H^{-1}\left\{(1-e^{-T_{(r')}})^{1/\beta_2}\right\} \\ \\ \int_{z=0}^{1} \frac{1}{B(1,r'-1)}\left[1+\frac{ln\left\{1-G(H^{-1}(1-e^{-zT_{(r')}}))^{\beta_1/\beta_2}\right\}}{S_{(r)}}\right]^{r-1}(1-z)^{r'-2}dz; \\ if\ G^{-1}\left\{(1-e^{-S_{(r)}})^{1/\beta_1}\right\} > H^{-1}\left\{(1-e^{-T_{(r')}})^{1/\beta_2}\right\}, \end{cases}$$

where, $c = -T^{-1}ln\left[1-H\left\{G^{-1}(1-e^{-S_{(r)}})^{\beta_2/\beta_1}\right\}\right]$.

Proof. It follows from Corollary 3.2.2.1 that the UMVU estimators of $f_{1II}(x;\alpha_1,\beta_1)$ and $f_{2II}(y;\alpha_2,\beta_2)$ based on Type II censoring at specified point x and y, respectively, are given by

$$\widetilde{f}_{1II}(x;\alpha_1,\beta_1) = \frac{\beta_1 g(x)G^{\beta_1-1}(x)}{B(1,r-1)S_{(r)}(1-G^{\beta_1}(x))}\left(1+\frac{ln(1-G^{\beta_1}(x))}{S_{(r)}}\right)^{r-2}; \quad -ln(1-G^{\beta_1}(x)) < S_{(r)}, \quad (3.2.11)$$

and

$$\widetilde{f}_{2II}(y;\alpha_2,\beta_2)=\frac{\beta_2 h(y)H^{\beta_2-1}(y)}{B(1,r'-1)T_{(r')}(1-H^{\beta_2}(y))}\left(1+\frac{ln(1-H^{\beta_2}(y))}{T_{(r')}}\right)^{r'-2};$$
$$-ln(1-H^{\beta_2}(y))<T_{(r')}. \tag{3.2.12}$$

From the arguments similar to those adopted in proving Theorem 3.2.3, it can be shown that the UMVU estimator of P can be rewritten in terms of $\widetilde{R}(y;\alpha_1,\beta_1)_{II}$ as follows

$$\begin{aligned}\widetilde{P}_{II}&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}_{II}(x;\alpha_1,\beta_1)\widetilde{f}_{II}(y;\alpha_2,\beta_2)dxdy\\&=\int_{y=0}^{\infty}\widetilde{R}_{II}(y;\alpha_1,\beta_1)\widetilde{f}_{II}(y;\alpha_2,\beta_2)dy,\end{aligned}$$

which, on using Theorem 3.2.3 and (3.2.12) leads to

$$\widetilde{P}_{II}=\int_{y=0}^{\infty}\left[1+\frac{ln(1-G^{\beta_1}(y))}{S_{(r)}}\right]^{r-1}\frac{\beta_2 h(y)H^{\beta_2-1}(y)}{B(1,r'-1)T_{(r')}(1-H^{\beta_2}(y))}\times\left(1+\frac{ln(1-H^{\beta_2}(y))}{T_{(r')}}\right)^{r'-2}dy;$$
$$-ln(1-G^{\beta_1}(t))<S_{(r)},-ln(1-H^{\beta_2}(y))<T_{(r')}$$

$$=\int_{y=0}^{c'}\left[1+\frac{ln(1-G^{\beta_1}(y))}{S_{(r)}}\right]^{r-1}\frac{\beta_2 h(y)H^{\beta_2-1}(y)}{B(1,r'-1)T_s(1-H^{\beta_2}(y))}\times\left(1+\frac{ln(1-H^{\beta_2}(y))}{T_{(r')}}\right)^{r'-2}dy, \tag{3.2.13}$$

where $c^{'}=min\left[G^{-1}\left\{1-e^{-S_{(r)}}\right\}^{1/\beta_1},H^{-1}\left\{1-e^{-T_{(r')}}\right\}^{1/\beta_2}\right]$.

Now, from (3.2.13), for $G^{-1}\left\{1-e^{-S_{(r)}}\right\}^{1/\beta_1}\le H^{-1}\left\{1-e^{-T_{(r')}}\right\}^{1/\beta_2}$,

$$\widetilde{P}_{II}=\int_{y=0}^{G^{-1}\left\{1-e^{-S_{(r)}}\right\}^{1/\beta_1}}\left[1+\frac{ln(1-G^{\beta_1(y)})}{S_{(r)}}\right]^{r-1}\frac{\beta_2 h(y)H^{\beta_2-1}(y)}{B(1,r'-1)T_{(r')}(1-H^{\beta_2}(y))}\times\left(1+\frac{ln(1-H^{\beta_2}(y))}{T_{(r')}}\right)^{r'-2}dy, \tag{3.2.14}$$

and the first assertion follows by substituting $\frac{-ln(1-H^{\beta_2}(y))}{T_{(r')}} = z$.

Furthermore, for $G^{-1}\left\{1-e^{-S_{(r)}}\right\}^{1/\beta_1} > H^{-1}\left\{1-e^{-T_{(r')}}\right\}^{1/\beta_2}$,

$$\widetilde{P_{II}} = \int_{y=0}^{H^{-1}\left\{1-e^{-T_{(r')}}\right\}^{1/\beta_2}} \left[1+\frac{ln(1-G^{\beta_1(y))}}{S_{(r)}}\right]^{r-1} \frac{\beta_2 h(y) H^{\beta_2-1}(y)}{B(1,r'-1)T_{(r')}(1-H^{\beta_2}(y))} \times \left(1+\frac{ln(1-H^{\beta_2}(y))}{T_{(r')}}\right)^{r'-2} dy,$$

and the second assertion follows by substituting $\frac{-ln(1-H^{\beta_2}(y))}{T_{(r')}} = z$.

Along the lines of Theorem 3.2.4, we can easily prove the following Corollary.

Corollary 3.2.4.1. *When X and Y come from same family of distributions, i.e., when $G(\cdot) = H(\cdot)$ and $\beta_1 = \beta_2$, the UMVU estimator of P is as follows:*

$$\widetilde{P}_{II} = \begin{cases} \frac{1}{B(1,r'-1)} \sum_{i=0}^{r'-2} (-1)^i \binom{r'-2}{i} \left(\frac{S_{(r)}}{T_{(r')}}\right)^i B(i+1,r); & S_{(r)} \le T_{(r')} \\ \frac{1}{B(1,r'-1)} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \left(\frac{T_{(r')}}{S_{(r)}}\right)^j B(j+1,r'-1); & S_{(r)} > T_{(r')}. \end{cases}$$

Theorem 3.2.5. *For $q \in (-\infty, \infty)$, the ML estimator of α^q is given by:*

$$\widehat{\alpha}^q_{II} = \left(\frac{r}{S_{(r)}}\right)^q.$$

Proof. We take the natural log of both sides of (3.2.7) and differentiate it with respect to α. On equating the differential with respect to zero, we get

$$\widehat{\alpha}_{II} = \frac{r}{S_{(r)}}, \tag{3.2.15}$$

and hence the theorem follows.

Corollary 3.2.5.1. *The ML estimator of $f(x;\alpha,\beta)$ at a specified point x is given by*

$$\widehat{f}(x)_{II} = \frac{r}{S_r} \beta g(x) G^{\beta-1}(x) \left[1-G^{\beta}(x)\right]^{\frac{r}{S_r}-1}.$$

Proof. The ML estimator of $f(x)$ is given by

$$\widehat{f}(x)_{II} = \widehat{\alpha}\beta g(x) G^{\beta-1}(x) \left[1-G^{\beta}(x)\right]^{\widehat{\alpha}-1}.$$

From (3.2.15) and the invariance property of ML estimators, the result follows.

Theorem 3.2.6. *The ML estimator of $R(t)$ is given by*

$$\widehat{R}(t)_{II} = \left(1 - G^{\beta}(t)\right)^{\frac{r}{S_r}}.$$

Proof. From (3.2.3), the reliability function $R(t)$ for Kum-G distributions is given by

$$R(t) = (1 - G^{\beta}(x))^{\alpha}. \tag{3.2.16}$$

From (3.2.15), (3.2.16) and invariance property of ML estimator, the result follows.

Theorem 3.2.7. *When X and Y come from different family of distributions, the ML estimator of P is as follows:*

$$\widehat{P}_{II} = \int_{z=0}^{1} \left[1 - G^{\beta_1}\left\{H^{-1}(z^{1/\beta_2})\right\}\right]^{\frac{r}{S_r}} \frac{r'}{T_{r'}}(1-z)^{\frac{r'}{T_r}-1}.$$

Proof. We know that

$$\begin{aligned}\widehat{P}_{II} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widehat{f}_{II}(x;\alpha_1,\beta_1)\widehat{f}_{II}(y;\alpha_2,\beta_2)dxdy \\ &= \int_{y=0}^{\infty} \widehat{R}(y;\alpha_1,\beta_1)_{II}\widehat{f}_{II}(y;\alpha_2,\beta_2)dy,\end{aligned}$$

which, on using Corollary 3.2.5.1 and Theorem 3.2.6, gives

$$\widehat{P}_{II} = \int_{y=0}^{\infty} \left(1 - G^{\beta_1}(y)\right)^{\frac{r}{S_r}} \frac{r'}{T_{r'}}\beta h(y)H^{\beta_2-1}(y)\left[1 - H^{\beta_2}(y)\right]^{\frac{r'}{T_{r'}}-1},$$

and the result follows by substituting $H^{\beta_2}(y) = z$.

Corollary 3.2.7.1. *When X and Y come from same family of distributions, i.e., when $G(\cdot) = H(\cdot)$ and $\beta_1 = \beta_2$ the ML estimator of P is as follows:*

$$\widehat{P}_{II} = \frac{r'S_{(r)}}{r'S_{(r)} + rT_{(r')}}. \tag{3.2.17}$$

Proof. The proof is on similar lines as that of Theorem 3.2.7.

3.2.2 Exact Confidence Intervals for α, $R(t)$ and P based on Type II Censoring

First, we discuss the problem of constructing a two-sided confidence interval for α. The confidence interval is obtained by using pivotal quantity $2\alpha S_{(r)}$. If we define

$\chi^2(\nu)$ as the value of χ^2 such that

$$P(\chi^2 > \chi^2(\delta)) = \int_{\chi^2(\delta)}^{\infty} P(\chi^2) d\chi^2 = \delta, \tag{3.2.18}$$

where $P(\chi^2)$ is the pdf of χ^2 distribution with 2r degrees of freedom. Then by using the fact that $2\alpha S_{(r)} \sim \chi^2_{2r}$, the confidence interval is given by

$$P\left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}} \leq \alpha \leq \frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right) = 1-\delta, \tag{3.2.19}$$

where $\chi^2\left(\frac{\delta}{2}\right)$ and $\chi^2\left(1-\frac{\delta}{2}\right)$ are obtained by using (3.2.18). Thus, for known β, $100(1-\delta)\%$ confidence interval for α is given by

$$\left(\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}, \frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right).$$

The problem of obtaining the confidence interval for the reliability function $R(t) = (1-G^{\beta}(t))^{\alpha}$ can be solved by noting that $R(t_{\circ};\alpha)$ is a decreasing function of α. Thus $\Psi_1(x_1, x_2, ..., x_n) \leq (1-G^{\beta}(t_{\circ}))^{\alpha}$ is equivalent to $\alpha \leq ln\Psi_1(x_1, x_2, ..., x_n)/ln(1-G^{\beta}(t_{\circ}))$ and $\Psi_2(x_1, x_2, ..., x_n) \geq (1-G^{\beta}(t_{\circ}))^{\alpha}$ is equivalent to $\alpha \geq ln\Psi_2(x_1, x_2, ..., x_n)/ ln(1-G^{\beta}(t_{\circ}))$. Therefore, the expression

$$P\left(\Psi_1(x_1, x_2, ..., x_n) \leq (1-G^{\beta}(t_{\circ}))^{\alpha} \leq \Psi_2(x_1, x_2, ..., x_n)\right) = 1-\delta,$$

is equivalent to

$$P\left(\frac{ln\Psi_2(x_1, x_2, ..., x_n)}{ln(1-G^{\beta}(t_{\circ}))} \leq \alpha \leq \frac{ln\Psi_1(x_1, x_2, ..., x_n)}{ln(1-G^{\beta}(t_{\circ}))}\right) = 1-\delta. \tag{3.2.20}$$

Comparing (3.2.19) and (3.2.20), it immediately follows that $\chi^2\left(1-\frac{\delta}{2}\right)/2S_{(r)} = ln\Psi_2(x_1, x_2, ..., x_n)/ln(1-G^{\beta}(t_{\circ}))$ and $\chi^2\left(\frac{\delta}{2}\right)/2S_{(r)} = ln\Psi_1(x_1, x_2, ..., x_n)/ln(1-G^{\beta}(t_{\circ}))$. Therefore

$$\Psi_1 = exp\left[ln(1-G^{\beta}(t_{\circ}))\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right] \; and \; \Psi_2 = exp\left[ln(1-G^{\beta}(t_{\circ}))\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right].$$

Thus, for known β, $(1-\delta)100\%$ confidence interval for $R(t_{\circ}, \alpha)$ is given by

$$\left(exp\left[ln(1-G^{\beta}(t_{\circ}))\frac{\chi^2\left(\frac{\delta}{2}\right)}{2S_{(r)}}\right], exp\left[ln(1-G^{\beta}(t_{\circ}))\frac{\chi^2\left(1-\frac{\delta}{2}\right)}{2S_{(r)}}\right]\right).$$

In order to obtain the confidence interval for P, we utilize the fact that $\frac{2\alpha_1 S_{(r)}/2r}{2\alpha_2 T_{(r')}/2r'} \sim F_{2r,2r'}$. Thus, the confidence interval for P is given by

$$P\left[\left(\frac{rT_{(r')}F(\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1} \leq \frac{\alpha_2}{\alpha_1+\alpha_2} \leq \left(\frac{rT_{(r')}F(1-\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}\right] = 1-\delta.$$

Therefore, for known β, $(1-\delta)100\%$ confidence interval for P is given by

$$\left[\left(\frac{rT_{(r')}F(\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}, \left(\frac{rT_{(r')}F(1-\frac{\delta}{2})}{r'S_{(r)}}+1\right)^{-1}\right].$$

3.3 Estimation based on the Sampling Scheme of Bartholomew

Throughout this section, we assume that n items are put on a test and we terminate life-testing experiment at a preassigned time $t_\circ$. Suppose we carry out a time-censored test where the items that fail are immediately replaced. Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be the failure times of n items under a test from (3.2.2). The test begins at time $X_{(0)} = 0$ and the system operates till $X_{(1)} = x_1$, when the first failure occurs. The failed item is replaced by a new one and the system operates till the second failure occurs at time $X_{(2)} = x_2$ and so on. The experiment is terminated at time $t_\circ$. Here, $X_{(i)}$ is the time until i^{th} failure measured from time 0.

3.3.1 UMVU and ML Estimators of α^q, $R(t)$ and P, based on the Sampling Scheme of Bartholomew

We first, provide an important lemma, which will be utilized in deducing UMVU estimators and MLE estimators of α^q, $R(t)$ and P.

Lemma 3.3.1. *Let $N(t_\circ)$ be the number of failures during the interval $[0; t_\circ]$. Then, $N(t_\circ)$ follows Poisson distribution.*

Proof. Let us make the transformations

$$\left.\begin{aligned} W_1 &= -ln[1-G^{\beta}x_{(1)}], \\ W_2 &= -ln[1-G^{\beta}x_{(2)}] + ln[1-G^{\beta}x_{(1)}], \\ &\vdots \\ W_n &= -ln[1-G^{\beta}x_{(n)}] + ln[1-G^{\beta}x_{(n-1)}]. \end{aligned}\right\} \tag{3.3.1}$$

The pdf of W_1 is

$$h(w_1) = n\alpha \exp(-n\alpha w_1).$$

Also, $W_2, W_3, ..., W_n$ are i.i.d. as W_1.

Using the monotonicity property of $-ln\left[1-G^{\beta}(x)\right]$, we get

$$\begin{aligned} P\left\{N(t_\circ)=r|t_\circ\right\} &= P[X_{(r)} \le t_\circ] - P[X_{(r+1)} \le t_\circ] \\ &= P\left[-ln\left\{1-G^{\beta}(X_{(r)})\right\} \le -ln\left\{1-G^{\beta}(t_\circ)\right\}\right] \\ &\quad -P\left[-ln\left\{1-G^{\beta}(X_{(r+1)})\right\} \le -ln\left\{1-G^{\beta}(t_\circ)\right\}\right]. \end{aligned} \tag{3.3.2}$$

Using equations (3.3.1) and (3.3.2), we get

$$\begin{aligned} P\left\{N(t_\circ)=r|t_\circ\right\} &= P\left[W_1+W_2+...+W_r \le -ln\left\{1-G^{\beta}(t_\circ)\right\}\right] \\ &\quad -P\left[W_1+W_2+...+W_{r+1} \le -ln\left\{1-G^{\beta}(t_\circ)\right\}\right]. \end{aligned} \tag{3.3.3}$$

From the additive property of exponentially distribution r.v.s [see Johnson and Kotz (1970), pp.170], $U = n\alpha\sum_{i=1}^{r} W_i$ follows a gamma distribution with pdf

$$h(u) = \frac{1}{\Gamma(r)} u^{r-1} e^{-u}, u > 0. \tag{3.3.4}$$

Using equations (3.3.3) and (3.3.4) along with the result of Patel *et al.* (1976), we obtain

$$P\left\{N(t_\circ)=r|t_\circ\right\} = \frac{1}{\Gamma(r+1)}\int_{c'}^{\infty} e^{-u}u^{r}du - \frac{1}{\Gamma(r)}\int_{c'}^{\infty} e^{-u}u^{r-1}du$$

$$=exp\left(n\alpha\, ln(1-G^{\beta}(t_{\circ}))\right)\left\{\sum_{j=0}^{r}\frac{\left[-n\alpha\, ln\left(1-G^{\beta}(t_{\circ})\right)\right]^{j}}{j!}\right\}$$
$$-\, exp\left(n\alpha\, ln(1-G^{\beta}(t_{\circ}))\right)\left\{\sum_{j=0}^{r-1}\frac{\left[-n\alpha\, ln\left(1-G^{\beta}(t_{\circ})\right)\right]^{j}}{j!}\right\},$$

where, $c' = -n\alpha\, ln(1-G^{\beta}(t_{\circ}))$. Hence,

$$P[N(t_{\circ})=r|t_{\circ}] = \frac{[-n\alpha\, ln\,(1-G^{\beta}(t_{\circ}))]^{r}}{r!} exp\left\{n\alpha\, ln\,(1-G^{\beta}(t_{\circ}))\right\}. \qquad (3.3.5)$$

and the lemma follows.

In the following theorems, we provide the UMVU estimators of α^q, $R(t)$ and P, based on the sampling scheme of Bartholomew (1963).

Theorem 3.3.2. *For positive integer q, the UMVU estimator of α^q is given by*

$$\widetilde{\alpha}_I^q = \begin{cases} \frac{r!}{(r-q)!}[-n\, ln\left\{1-G^{\beta}(t_{\circ})\right\}]^{-q}; & r-q>0 \\ 0; & otherwise. \end{cases}$$

Proof. It follows from Lemma 3.3.1 and Fisher-Neyman factorization theorem [see Rohatgi and Saleh (2012), pp. 341] that r is sufficient for α. Moreover, since the distribution of r belongs to the exponential family, it is also complete [see Rohatgi and Saleh (2012), pp.347]. The theorem now follows from the result that the q^{th} factorial moment of the distribution of r is given by

$$E[r(r-1)(r-2)...(r-q+1)] = \left[-n\alpha\, ln\left\{1-G^{\beta}(t_{\circ})\right\}\right]^{q}.$$

Corollary 3.3.2.1. *The UMVU estimator of $f(x;\alpha,\beta)$ at a specified point x is*

$$\widetilde{f}_I(x;\alpha,\beta) = \begin{cases} \frac{r\beta g(x)G^{\beta-1}(x)}{\left[-n\, ln(1-G^{\beta}(t_{\circ}))\right](1-G^{\beta}(x))}\left(1-\frac{ln(1-G^{\beta}(x))}{n\, ln(1-G^{\beta}(t_{\circ}))}\right)^{r-1}; \\ \qquad ln(1-G^{\beta}(x)) < n\, ln(1-G^{\beta}(t_{\circ})) \\ 0; \qquad otherwise. \end{cases}$$

Proof. Let us write the pdf (3.2.2) as follow

$$f(x;\alpha,\beta) = \frac{\alpha\beta g(x)G^{\beta-1}(x)}{1-G^{\beta}(x)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{-ln(1-G^{\beta}(x))\right\}^{i}\alpha^{i}.$$

Then, the Corollary 3.3.2.1 straight away follows from Theorem 3.3.2.

Theorem 3.3.3. *The UMVU estimator of $R(t)$ at a specified point t is given by*

$$\widetilde{R}(t) = \begin{cases} \left[1 - \frac{ln(1-G^{\beta}(t))}{n\, ln(1-G^{\beta}(t_{\circ}))}\right]^{r}; & ln(1-G^{\beta}(t)) < n\, ln(1-G^{\beta}(t_{\circ})) \\ 0; & otherwise. \end{cases}$$

Proof. Using Corollary 3.3.2.1,

$$\widetilde{R}(t) = \int_{t}^{\infty} \frac{r\beta g(x) G^{\beta-1}(x)}{[-n\, ln(1-G^{\beta}(t_{\circ}))]\,(1-G^{\beta}(x))} \left(1 - \frac{ln(1-G^{\beta}(x))}{n\, ln(1-G^{\beta}(t_{\circ}))}\right)^{r-1} dx,$$

and the result follows by substituting $\frac{ln(1-G^{\beta}(x))}{n\, ln(1-G^{\beta}(t_{\circ}))} = z$.

Let n items on X and m on Y be put on a life test, where X and Y are distributed as in (3.2.3) and (3.2.4). Let $t_{\circ}$ and $t_{\circ\circ}$ be the termination times for X and Y, respectively and r and r' be the number of failures before $t_{\circ}$ and $t_{\circ\circ}$, respectively. Using Corollary 3.3.2.1, the UMVU estimators of $f_1(x;\alpha_1,\beta_1)$and $f_2(y;\alpha_2,\beta_2)$, based on the sampling scheme of Bartholomew are given by

$$\widetilde{f}_{1I}(x;\alpha_1,\beta_1) = \frac{r\beta_1 g(x) G^{\beta_1-1}(x)}{[-n\, ln(1-G^{\beta_1}(t_{\circ}))]\,(1-G^{\beta_1}(x))} \left(1 - \frac{ln(1-G^{\beta_1}(x))}{n\, ln(1-G^{\beta_1}(t_{\circ}))}\right)^{r-1}; \tag{3.3.6}$$

$$ln(1-G^{\beta_1}(x)) < n\;\; ln(1-G^{\beta_1}(t_{\circ})),$$

and

$$\widetilde{f}_{2I}(y;\alpha_2,\beta_2) = \frac{r'\beta_2 h(y) H^{\beta_2-1}(y)}{[-m\, ln(1-H^{\beta_2}(t_{\circ\circ}))]\,(1-H^{\beta_2}(y))} \left(1 - \frac{ln(1-H^{\beta_2}(y)}{m\, ln(1-H^{\beta_2}(t_{\circ\circ}))}\right)^{r'-1}; \tag{3.3.7}$$

$$ln(1-H^{\beta_2}(y)) < m\, ln(1-H^{\beta_2}(t_{\circ\circ})).$$

Theorem 3.3.4. *The UMVU estimator of P is given by*

$$\widetilde{P}_I = \begin{cases} r'\int_{z=0}^{c}\left[1 - \frac{ln\left\{1-G^{\beta_1}(H^{-1}(1-(1-H^{\beta_2}(t_{\circ\circ}))^{mz}))^{1/\beta_2}\right\}}{n\, ln\left\{1-G^{\beta_1}(t_{\circ})\right\}}\right](1-z)^{r'-1}dz; \\ G^{-1}\left\{1-(1-G^{\beta_1}(t_{\circ}))^{n}\right\}^{\frac{1}{\beta_1}} \leq H^{-1}\left\{1-(1-H^{\beta_2}(t_{\circ\circ}))^{m}\right\}^{\frac{1}{\beta_2}} \\ r'\int_{z=0}^{1}\left[1 - \frac{ln\left\{1-G^{\beta_1}(H^{-1}(1-(1-H^{\beta_2}(t_{\circ\circ}))^{mz}))^{1/\beta_2}\right\}}{n\, ln\left\{1-G^{\beta_1}(t_{\circ})\right\}}\right](1-z)^{r'-1}dz; \\ G^{-1}\left\{1-(1-G^{\beta_1}(t_{\circ}))^{n}\right\}^{\frac{1}{\beta_1}} > H^{-1}\left\{1-(1-H^{\beta_2}(t_{\circ\circ}))^{m}\right\}^{\frac{1}{\beta_2}}, \end{cases}$$

where, $c = \frac{ln\left[1-H^{\beta_2}\left\{G^{-1}(1-(1-G^{\beta_1}(t_{\circ}))^{n})^{1/\beta_1}\right\}\right]}{m\, ln\left\{1-G^{\beta_1}(t_{\circ})\right\}}$.

Proof. Using Corollary 3.3.2.1, the UMVU estimator of $f_2(y;\alpha_2,\beta_2)$, based on the sampling scheme of Bartholomew is given by

$$\widetilde{f}_{2I}(y;\alpha_2,\beta_2) = \frac{r'\beta_2 h(y)H^{\beta_2-1}(y)}{[-m\, ln(1-H^{\beta_2}(t_{\circ\circ}))]\,(1-H^{\beta_2}(y))}\left(1-\frac{ln(1-H^{\beta_2}(y)}{m\, ln(1-H^{\beta_2}(t_{\circ\circ}))}\right)^{r'-1};$$
$$ln(1-H^{\beta_2}(y)) < m\, ln(1-H^{\beta_2}(t_{\circ\circ})). \tag{3.3.8}$$

It can be shown that the UMVU estimator of P is given by

$$\begin{aligned}\widetilde{P}_I &= \int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}_{1I}(x;\alpha_1,\beta_1)\widetilde{f}_{2I}(y;\alpha_2,\beta_2)dxdy\\ &= \int_{y=0}^{\infty}\widetilde{R}_I(y;\alpha_1,\beta_1)\widetilde{f}_{2I}(y;\alpha_2,\beta_2)dy,\end{aligned}$$

which on using Theorem 3.3.3 and (3.3.8) gives that

$$\widetilde{P}_I = \int_{y=0}^{\infty}\left[1-\frac{ln(1-G^{\beta_1}(y))}{n\, ln(1-G^{\beta_1}(t_\circ))}\right]^r \frac{r'\beta_2 h(y)H^{\beta_2-1}(y)}{[-m\, ln(1-H^{\beta_2}(t_{\circ\circ}))]\,(1-H^{\beta_2}(y))}$$
$$\left(1-\frac{ln(1-H^{\beta_2}(y))}{m\, ln(1-H^{\beta_2}(t_{\circ\circ}))}\right)^{r'-1} dy;\;\; ln\left\{1-G^{\beta_1}(y)\right\} < n\, ln\left\{1-G^{\beta_1}(t_\circ)\right\},$$
$$ln\left\{1-H^{\beta_2}(y)\right\} < m\, ln\left\{1-H^{\beta_2}(t_{\circ\circ})\right\}$$

$$= \int_{y=0}^{min\left[G^{-1}\left\{1-(1-G^{\beta_1}(t_\circ))^n\right\}^{1/\beta_1},\,H^{-1}\left\{1-(1-H^{\beta_2}(t_{\circ\circ}))^m\right\}^{1/\beta_2}\right]}$$
$$\left[1-\frac{ln(1-G^{\beta_1}(y))}{n\, ln(1-G^{\beta_1}(t_\circ))}\right]^r \frac{r'\beta_2 h(y)H^{\beta_2-1}(y)}{[-m\, ln(1-H^{\beta_2}(t_{\circ\circ}))]\,(1-H^{\beta_2}(y))}\times \tag{3.3.9}$$
$$\left(1-\frac{ln(1-H^{\beta_2}(y))}{m\, ln(1-H^{\beta_2}(t_{\circ\circ}))}\right)^{r'-1} dy.$$

The theorem now moves on to examine the two situations by replacing $\frac{ln(1-H^{\beta_2}(y))}{m\, ln(1-H^{\beta_2}(t_{\circ\circ}))}$ with z.

Corollary 3.3.4.1. *When X and Y comes from the same family of distributions, i.e., when $G(\cdot)=H(\cdot)$ with $\beta_1=\beta_2$ and $t_\circ=t_{\circ\circ}$, the UMVU estimator of P is as follows:*

$$\widetilde{P}_I = \begin{cases} r'\sum_{i=0}^{r'-1}(-1)^i\binom{r'-1}{i}\left(\frac{n}{m}\right)^{i+1}B(i+1,r+1); & n\le m\\ r'\sum_{j=0}^{r}(-1)^j\binom{r}{j}\left(\frac{m}{n}\right)^{j}B(j+1,r'); & n>m.\end{cases}$$

Proof. The proof is similar to that of Theorem 3.3.4.

Theorem 3.3.5. *For $q \in (-\infty, \infty)$, the ML estimator of α^q is given by:*

$$\widehat{\alpha}_I^q = \left(\frac{-r}{n\, ln(1 - G^{\beta}(t_{\circ}))} \right)^q .$$

Proof. We take natural log of both sides of (3.3.5) and differentiate it with respect to α. On equating the differential with respect to zero, we get

$$\widehat{\alpha}_I = \frac{-r}{n\, ln(1 - G^{\beta}(t_{\circ}))}, \tag{3.3.10}$$

and hence the theorem follows.

Corollary 3.3.5.1. *The ML estimator of $f(x; \alpha, \beta)$ at a specified point x is*

$$\widehat{f}_I(x; \alpha, \beta) = \frac{-r}{n\, ln \left\{1 - G^{\beta}(t_{\circ})\right\}} g(x) G^{\beta-1}(x) [1 - G^{\beta}(x)]^{\frac{-r}{n\, ln\left\{1 - G^{\beta}(t_{\circ})\right\}} - 1} .$$

Proof. The ML estimator of $f(x)$ is given by

$$\widehat{f}(x)_I = \widehat{\alpha} \beta g(x) G^{\beta-1}(x) \left[1 - G^{\beta}(x)\right]^{\widehat{\alpha}-1} .$$

From (3.3.10) and the invariance property of ML estimators, the result follows.

Theorem 3.3.6. *The ML estimator of $R(t)$ is given by*

$$\widehat{R}(t)_I = [1 - G^{\beta}(t)]^{\frac{-r}{n\, ln(1 - G^{\beta}(t_{\circ}))}} .$$

Proof. From (3.2.3), the reliability function $R(t)$ for Kum-G distributions is given by

$$R(t) = (1 - G^{\beta}(x))^{\alpha}. \tag{3.3.11}$$

From (3.3.10), (3.3.11) and the invariance property of ML estimator, the result follows.

Theorem 3.3.7. *The ML estimator of P when X and Y belongs to the different family of distributions, is given by*

$$\widehat{P}_I = \int_{z=0}^{1} [1 - G^{\beta_1} \left\{H^{-1}(z^{1/\beta_2})\right\}]^{\frac{-r}{n ln(1 - G^{\beta_1}(t_{\circ}))}} \frac{-r'}{m\, ln(1 - H^{\beta_2}(t_{\circ\circ}))} \times$$
$$(1 - z)^{\frac{-r'}{m\, ln(1 - H^{\beta_2}(t_{\circ\circ}))}} dz.$$

Proof. We know that

$$\widehat{P}_I = \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widehat{f}_I(x;\alpha_1,\beta_1)\widehat{f}_I(y;\alpha_2,\beta_2)dxdy$$
$$= \int_{y=0}^{\infty} \widehat{R}_I(y;\alpha_1,\beta_1)\widehat{f}_I(y;\alpha_2,\beta_2)dy,$$

which on using Theorem 3.3.6 and Corollary 3.3.5.1 gives that

$$\widehat{P} = \int_{y=0}^{\infty} [1-G^{\beta_1}(y)]^{\frac{-r}{n\ ln(1-G^{\beta_1}(t_\circ))}} \frac{-r'}{m\ ln\left\{1-H^{\beta_2}(t_{\circ\circ})\right\}} \times$$
$$h(y)H^{\beta_2-1}(y)[1-H^{\beta_2}(y)]^{\frac{-r'}{m\ ln\left\{1-H^{\beta_2}(t_{\circ\circ})\right\}}-1} dy.$$

The theorem now follows on putting $H^{\beta_2}(y) = z$.

Corollary 3.3.7.1. *The ML estimator of P when X and Y belongs to the same family of distributions, i.e., $G(\cdot) = H(\cdot)$, $\beta_1 = \beta_2$ and $t_\circ = t_{\circ\circ}$, is given by*

$$\widehat{P} = \frac{sn}{sn + rm}.$$

Proof. The proof is on similar lines as that of Theorem 3.3.7.

3.4 Simulation Study

In order to validate the results obtained in Sections 3.2 and 3.3, we first consider Kum distribution as a particular case of the Kum-G distributions. The pdf and cdf of Kum distribution are given by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1-x^{\beta})^{\alpha-1};\ 0 < x < 1,\ \alpha,\beta > 0, \tag{3.4.1}$$

and

$$F(x;\alpha,\beta) = 1-(1-x^{\beta})^{\alpha}, \tag{3.4.2}$$

respectively.

3.4.1 Simulation Based on Type II Censoring

For comparing the performances of estimators of α based on Type II censoring scheme, we have generated 1000 random samples from (3.4.1) each of size $n = 50$

for (α, β)= (2,0.5), (2,1), (2,2). We arranged the data in ascending order for each sample and considered a sample of first $r(\leq n)$ observations. For different values of r =10, 20, 30 and 50 and $q = 1$, we have computed average values of $\widetilde{\alpha}_{II}$ and $\widehat{\alpha}_{II}$ and their corresponding MSE and results are reported in Table 3.1. Similarly, we obtain average length and coverage probability of interval estimates of α and the results are reported in Table 3.2.

It has been observed that MSE obtained corresponding to UMVU estimator is much lower than MSE obtained corresponding to ML estimator. Thus, the performance of UMVU estimator of α for $q = 1$ based on Type II censoring is much better than the performance of ML estimator of α. From Table 3.1, we observe that as r increases, the performance improves in the sense that their MSE decreases. It is also interesting to note that, with increasing r, the two estimators come close to each other. Further, from Table 3.2, we observe that, as the truncation number r increases, the length of confidence intervals decrease. This justifies the fact that as r moves closer to n, the precision of our estimate will increase.

For comparing the performance of ML estimator and UMVU estimator of reliability function $R(t)$, we have generated 1000 random samples from (3.4.1) each of size $n = 50$ for (α, β)=(0.5,2). We obtain the average values of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different values of r and t and their corresponding MSE and the results are presented in Table 3.3. We also obtain average length and coverage probability of interval estimates of $R(t)$ and the results are reported in Table 3.4. Comparing the estimates on the basis of MSE, we observe that the UMVU estimator of $R(t)$ performs slightly better than the ML estimator. As r increases, the performance of both the estimators improve and both estimators come close to each other. We also observe that, as the truncation number r increases, the length of confidence intervals decreases. This justifies the fact that as r moves closer to n, the precision of our estimate will increase.

In order to compare the performance of different estimators of P, we have obtained 1000 random samples from each of the populations X and Y with sizes (n, m) with $\beta_1 = \beta_2 = 2$ and (α_1, α_2) = (0.5,0.5), (0.5,1), (0.5, 1.5) and (1.5,2).

Samples corresponding to both the populations are arranged in ascending order and first (r, r') observations are considered. For (r, r')=(10,10), (20,20), (30,25), (40,40) and $(50, 50)$, we have computed average values of $\widetilde{P}$ and $\widehat{P}$ and their corresponding MSE and results are presented in Table 3.5. Similarly, we obtain the average length and coverage probability of interval estimates which are reported in Table 3.6. We observe that for all selected values of (r, r'), the ML estimator of P performs superior to the UMVU estimator of P in the sense that it has lower MSE. Further, as the truncation number r increases, the length of confidence intervals decreases. Hence, as r moves closer to n, the precision of our estimate will increase.

3.4.2 Simulation Based on Sampling Scheme of Bartholomew

In order to obtain point estimates of $R(t)$ based on the sampling scheme of Bartholomew, we have generated 1000 random samples each of size 50 from (3.4.1) with $\alpha = 2$ and $\beta = 0.9$. By fixing the termination time at $t_\circ$, and replacing the failure by operating one, values of r (number of failures before time $t_\circ$) is computed. For different termination time $t_\circ = 0.20, 0.50, 0.65, 0.80$ and 0.90, we have computed average values of $\widetilde{R(t)}$ and $\widehat{R(t)}$ and their corresponding MSE. For different values of t results are presented in Table 3.7. It has been observed that for small values of t and small values of $t_\circ$, ML estimator is slightly more efficient than UMVU estimator of $R(t)$. However, for large values of $t_\circ$, UMVU estimator becomes more efficient than ML estimator of $R(t)$. For large values of t and all values of $t_\circ$, both the estimators become equally efficient. These results show the importance of termination time $t_\circ$ in the sampling scheme of Bartholomew.

To test the performance of different estimators of P based on the sampling scheme of Bartholomew, we have obtained 1000 random samples from each of the X and Y populations with sizes (n, m) with $\beta_1 = \beta_2 = 2$ and $(\alpha_1, \alpha_2) =$ (0.5,0.75), (0.5,1), (0.5, 1.5) and (1.5,2.5). For each sample corresponding to both the population, fixing the termination time at $t_\circ = t_{\circ\circ}$ and replacing the failure by operating one, values of r (no. of failures before time $t_\circ$ in X) and values of r'

(no. of failures before time $t_{\circ\circ}$ in Y) are computed. For $t_\circ = t_{\circ\circ}$= 0.50, 0.70 and 0.80, we have computed average values of $\widetilde{P_I}$ and $\widehat{P_I}$ and their corresponding MSE for $n > m$ and $n < m$ and results are presented in Tables 3.8 and 3.9 respectively. From Table 3.8, for $n > m$, it is observed that for small m when $n = 50$, UMVU estimator of P performs superior than ML estimator of P. As m increases, both the estimators are equally efficient. However, for $n < m$, the results given in Table 3.9 show that for all n with $m = 50$, the ML estimator of P is superior than the UMVU estimator and, as n increases, both the estimators become equally efficient.

3.5 Real Data Study

In this section, to illustrate the usefulness of our procedure, we present real data analysis. We consider the real dataset used by Kumari *et al.* (2019), originally taken from Proschan (1963). The data corresponds to the period between air conditioning system failures (in hours) of a fleet of 13 Boeing 720 jet airplanes. According to Canavos and Tsokos (1971), the failure time distribution of the air conditioning system could be well represented by exponential distribution. We have considered the data on planes '7913' (Data set I or population X) and '7914' (Data set II or population Y) for our illustrative purposes. Before applying the Kolmogorov–Smirnov (KS) test, we transform the above given two data sets in the range of unit interval by using the transformation $X_i = \frac{X_i}{max(X_i)+1}$ and $Y_i = \frac{Y_i}{max(Y_i)+1}$.

The KS test is initially used to see if the Kum distribution fits the given data sets. We obtain the following ML estimates of the parameters of X population, (α_1, β_1) and ML estimates of the parameters of Y population, (α_2, β_2).

$$(\alpha_1, \beta_1)_{\text{complete data}} = (1.0728, 0.6022), \quad (\alpha_2, \beta_2)_{\text{complete data}} = (1.042, 0.6658).$$

We use these ML estimators to do a KS test, which verifies that both the data observed for X $(KS = 0.18226; p = 0.4026)$ and $Y(KS = 0.1289; p = 0.7604)$ are drawn from (3.4.1). Figure 3.1 further confirms the good fit of Kum distribution for these two data sets. In order to obtain the ML estimator of $R(t)$ and P based on Type II censoring, we first consider $r = 16$ lifetimes from X population and rest 8 observations are considered as censored. Similarly, we consider first $r' = 20$ lifetimes from Y population and the rest 7 observations are considered as censored.

Considering Kum distribution as a lifetime model for X-population, the ML estimators of α_{1II} and β_{1II} are obtained as $\widehat{\alpha_{1II}} = 1.272$ and $\widehat{\beta_{1II}} = 0.6659$. Similarly, considering Kum distribution as a lifetime model for Y -population, the ML estimators of α_{2II} and β_{2II} are $\widehat{\alpha_{2II}} = 1.4677$ and $\widehat{\beta_{2II}} = 0.8128$. To evaluate ML estimator of P_{II}, we have considered first data set as X Population and second data set as Y population. We get $\widehat{P}_{II} = 0.5847$. For X and Y populations and corresponding to different values of t, we have evaluated ML estimator of $R(t)$. Results are plotted in Figure 3.2. In particular, for $t = 0.8$, $\widehat{R_{1II}}(t) = 0.1081$ and $\widehat{R_{2II}}(t) = 0.127$.

From Figure 3.2, it is clear that the probability of survival is very high at initial time and as time increases probability of survival decreases.

3.6 Concluding Remarks

In this chapter, we have constructed the estimation techniques for the Kum-G family of distributions based on two censoring schemes. Considerations are given to both point and interval estimations. The finite sample performance of the UMVU estiators and ML estimators of reliability functions and other parameters are investigated using an extensive Monte Carlo experiment. The comparisons are made on the basis of MSE of the estimators. The main conclusions of the simulation experiments are as follows:

For Type II Censoring, for all values of n, the UMVU estimator of α^q performs better than ML estimator of α^q. Similarly, for all selected values of t, the performance of the UMVU estimator of $R(t)$ is marginally better than the performance of the ML estimator of $R(t)$. However, the performance of both estimators is very similar for large values of r. Further, as r increases, MSE corresponding to both the estimator decreases. On the contrary, for estimating P, the ML estimator performs superior than the UMVU estimator.

For the sampling scheme of Bartholomew, for small values of t and $t_\circ$, ML estimator is more efficient than UMVU estimator of $R(t)$. However, for large values of $t_\circ$, UMVU estimator becomes more efficient than ML estimator of $R(t)$. For large values of t and all values of $t_\circ$, both the estimators are almost equally efficient. These results show the importance of termination time $t_\circ$ in the sampling scheme of Bartholomew. For comparing the performance of ML estimator and UMVU estimator of P, we observe that, when $n = 50$ and $m < n$, UMVU estimator outperforms ML estimator. As m increases both the estimators become equally efficient. On the contrary, for $n < m$ and $m = 50$ for small n, ML estimator of P gives better performance than UMVU estimator. But as m increases both the estimators become almost equally efficient.

Table 3.1: **Average values of point estimates of α and their MSE/ Variances based on Type II Censoring, when β is known**

$r->$	**10**		**20**		**30**		**50**	
β	$\widetilde{\alpha}$	$\widehat{\alpha}$	$\widetilde{\alpha}$	$\widehat{\alpha}$	$\widetilde{\alpha}$	$\widehat{\alpha}$	$\widetilde{\alpha}$	$\widehat{\alpha}$
0.5	2.0374	2.2638	2.0153	2.1214	2.0136	2.083	1.998	2.0388
	0.5007	0.686	0.2287	0.2679	0.1554	0.173	0.0845	0.0895
1	2.0183	2.2425	2.0079	2.1136	2.0127	2.0821	1.9999	2.0407
	0.5733	0.7661	0.2212	0.2579	0.147	0.1639	0.0845	0.0897
2	2.0028	2.2254	2.0186	2.1248	1.9962	2.065	2.0149	2.056
	0.4796	0.6429	0.2259	0.2655	0.1414	0.1555	0.0845	0.0909

Note: 1st and 2nd rows represent the average estimates and MSE of α.

Table 3.2: **Average length and coverage probability of interval estimates of α based on Type II Censoring, when β is known**

$r->$	**10**		**20**		**30**		**50**	
β	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
0.5	2.7021	95.9	1.8433	96.2	1.4799	95.5	1.1254	95.3
1	2.7127	95.5	1.8381	95.6	1.4627	95.1	1.1225	95.5
2	2.6951	95.1	1.8352	95.4	1.4825	96.1	1.1292	95

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 3.3: **Average values of point estimates of $R(t)$ and their MSE/ Variances based on Type II Censoring, when β is known**

$r->$		**10**		**20**		**30**		**50**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.25	0.9682	0.9683	0.9649	0.9682	0.9666	0.9683	0.9672	0.9682	0.9676
		1e-04	2e-04	1e-04	1e-04	0	0	0	0
0.5	0.893	0.8931	0.8828	0.8928	0.8878	0.8929	0.8897	0.8929	0.891
		0.0012	0.0015	6e-04	6e-04	4e-04	4e-04	2e-04	2e-04
0.65	0.7599	0.7605	0.7417	0.7598	0.7506	0.7601	0.754	0.7598	0.7562
		0.005	0.0057	0.0023	0.0025	0.0015	0.0016	9e-04	9e-04
0.75	0.6614	0.6599	0.6379	0.6609	0.65	0.6613	0.654	0.6614	0.6571
		0.0087	0.0095	0.0041	0.0043	0.0026	0.0027	0.0015	0.0015
0.85	0.5268	0.5278	0.5056	0.5273	0.516	0.527	0.5195	0.5272	0.5227
		0.0125	0.0126	0.0059	0.0059	0.0039	0.004	0.0023	0.0023

Note: 1st and 2nd rows represent the average estimates and MSE of $R(t)$.

Table 3.4: **Average length and coverage probability of interval estimates of $R(t)$ based on Type II Censoring**

$r->$	**10**		**20**		**30**		**50**	
$t\downarrow$	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
0.25	0.0423	0.9477	0.0286	0.9579	0.0229	0.9544	0.0177	0.9512
0.5	0.1627	0.9503	0.1122	0.9489	0.0907	0.947	0.0697	0.9522
0.65	0.2618	0.9524	0.1843	0.9477	0.1504	0.9507	0.1161	0.9519
0.75	0.3326	0.9507	0.2382	0.9512	0.1949	0.9494	0.1513	0.9478
0.85	0.3961	0.9506	0.2881	0.9505	0.2373	0.9535	0.1852	0.9476

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 3.5: **Average values of point estimates of P and their MSE/ Variances based on Type II Censoring, when β is known**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**0.5**	
$\alpha_2->$	**0.5**		**1**		**1.5**		**2**	
$P->$	**0.5**		**0.6666667**		**0.75**		**0.8**	
$(r,r')\downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
(10,10)	0.4917	0.492	0.6673	0.6601	0.7521	0.7428	0.8021	0.7926
	0.0133	0.0121	0.0102	0.0095	0.0072	0.007	0.0057	0.0057
(20,20)	0.5004	0.5004	0.6643	0.6607	0.7485	0.7439	0.7993	0.7945
	0.0062	0.0059	0.0053	0.0052	0.0037	0.0037	0.0029	0.0029
(30,25)	0.5	0.5008	0.666	0.6641	0.7492	0.7464	0.8016	0.7986
	0.0048	0.0046	0.0035	0.0034	0.0027	0.0027	0.002	0.002
(40,40)	0.5028	0.5027	0.6695	0.6677	0.7499	0.7475	0.7981	0.7957
	0.003	0.0029	0.0025	0.0024	0.0017	0.0017	0.0013	0.0013
(50,50)	0.4998	0.4998	0.6653	0.6638	0.7502	0.7483	0.8006	0.7987
	0.0024	0.0024	0.002	0.002	0.0014	0.0014	0.001	0.001

Note: The 1st and 2nd row represents the average estimates and MSE of $R(t)$

Table 3.6: **Average length and coverage probability of interval estimates of P based on Type II Censoring**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**0.5**	
$\alpha_2->$	**0.5**		**1**		**1.5**		**2**	
$P->$	**0.5**		**0.6667**		**0.75**		**0.8**	
$(r,r')\downarrow$	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*	*A.L.*	*C.P.*
(10,10)	0.4056	94.67	0.3721	95.10	0.3251	94.95	0.2862	94.68
(20,20)	0.2974	94.79	0.2687	94.95	0.2317	94.83	0.2006	94.67
(30,25)	0.2575	95.10	0.2319	94.5	0.1987	95.10	0.1720	95.09
(40,40)	0.2145	94.91	0.1925	94.90	0.1643	94.98	0.1414	95.27

Note: A.L.: Average Length, C.P.: Coverage Probability

Table 3.7: **UMVU and ML Estimators of $R(t)$ based on the Sampling Scheme of Bartholomew**

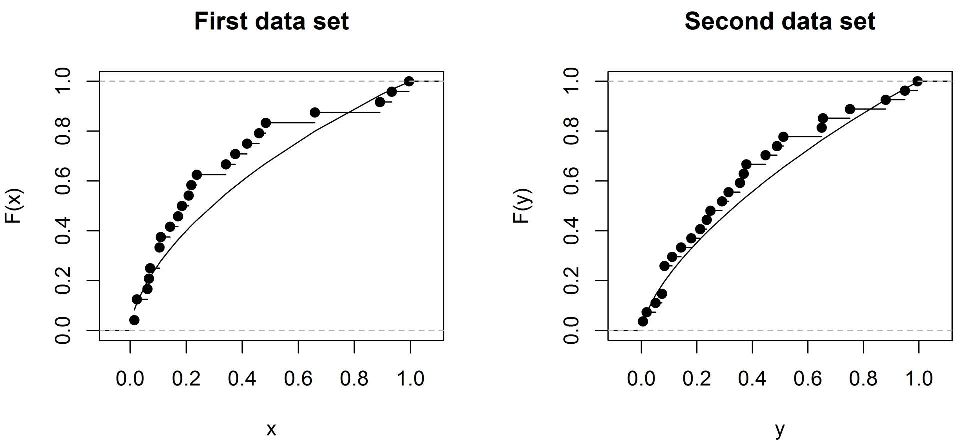
$t_\circ->$		**0.20**		**0.50**		**0.65**		**0.80**		**0.90**	
$t\downarrow$	$R(t)\downarrow$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$	$\widetilde{R(t)}$	$\widehat{R(t)}$
0.25	0.9682	0.9676	0.9678	0.9702	0.9702	0.9715	0.9715	0.9742	0.9742	0.9773	0.9773
		5e-04	5e-04	1e-04	1e-04	0	0	0	0	1e-04	1e-04
0.5	0.866	0.866	0.8703	0.8724	0.873	0.8798	0.8801	0.8899	0.8901	0.9023	0.9024
		0.0075	0.007	9e-04	9e-04	6e-04	6e-04	7e-04	7e-04	0.0014	0.0014
0.65	0.7599	0.7604	0.774	0.7706	0.7725	0.781	0.7819	0.7995	0.8	0.8236	0.8238
		0.0209	0.0191	0.0028	0.0028	0.0017	0.0017	0.0021	0.0021	0.0043	0.0043
0.75	0.6614	0.6591	0.71	0.6807	0.6881	0.6901	0.6939	0.7159	0.7179	0.745	0.7461
		0.0782	0.0621	0.0094	0.0095	0.0055	0.0056	0.0047	0.0049	0.0078	0.0079
0.85	0.5268	0.5352	0.6277	0.5448	0.5593	0.5584	0.566	0.5933	0.5972	0.6322	0.6344
		0.1403	0.1077	0.0149	0.0151	0.0079	0.0083	0.0078	0.0083	0.0126	0.0131

Note: The 1st and 2nd row represents the average estimates and MSE of $R(t)$

Table 3.8: **UMVU and ML Estimators of P when $n > m$ based on the Sampling Scheme of Bartholomew**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**1.5**	
$\alpha_2->$	**0.75**		**1**		**1.5**		**2.5**	
$P->$	**0.6**		**0.6666667**		**0.75**		**0.625**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
$(n = 50) > (m = 35)$								
0.50	0.6011	0.5944	0.6535	0.6482	0.7314	0.7278	0.6001	0.5978
	0.0166	0.0169	0.0123	0.0128	0.0077	0.0079	0.0048	0.0049
0.70	0.5915	0.5885	0.6408	0.6383	0.7028	0.701	0.5763	0.575
	0.0058	0.0059	0.0054	0.0055	0.0054	0.0056	0.0038	0.004
0.80	0.5781	0.5759	0.6235	0.6217	0.6784	0.6769	0.5602	0.5591
	0.0039	0.0041	0.0046	0.0048	0.0072	0.0074	0.0051	0.0053
$(n = 50) > (m = 45)$								
0.50	0.5962	0.5945	0.6564	0.655	0.7261	0.7252	0.6012	0.6006
	0.013	0.0131	0.012	0.012	0.0079	0.008	0.0044	0.0044
0.70	0.5893	0.5886	0.639	0.6383	0.7029	0.7024	0.5757	0.5753
	0.0056	0.0056	0.0053	0.0053	0.0052	0.0052	0.0038	0.0038
0.80	0.5781	0.5776	0.626	0.6255	0.6801	0.6797	0.5598	0.5596
	0.0038	0.0038	0.0042	0.0042	0.0068	0.0068	0.0051	0.0052

Note: The 1st and 2nd row represents the average estimates and MSE of P



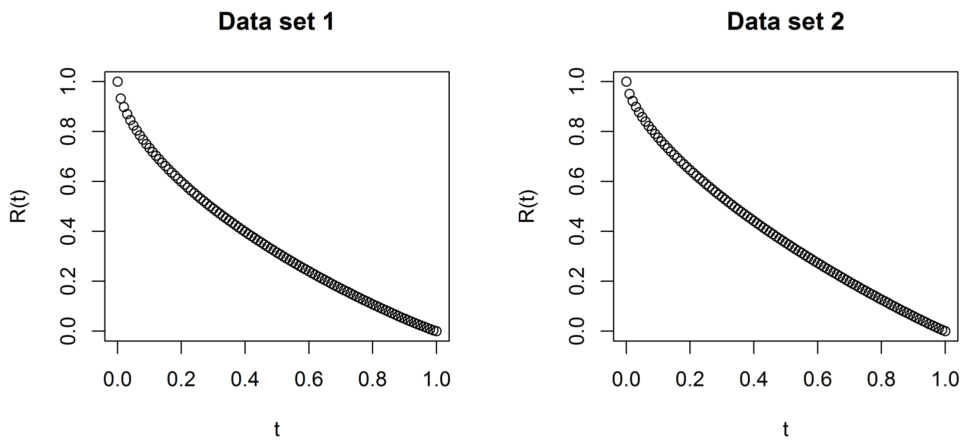
(a) **Empirical and theoretical cdf of first data set** (b) **Empirical and theoretical cdf of second data set**

Figure 3.1: **Plots of empirical and theoretical cdf**

Table 3.9: **UMVU and ML Estimators of P when $n < m$ based on the Sampling Scheme of Bartholomew**

$\alpha_1->$	**0.5**		**0.5**		**0.5**		**1.5**	
$\alpha_2->$	**0.75**		**1**		**1.5**		**2.5**	
$P->$	**0.6**		**0.6666667**		**0.75**		**0.625**	
$t_\circ = t_{\circ\circ} \downarrow$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$	$\widetilde{P}$	$\widehat{P}$
				$(n = 35) < (m = 50)$				
0.50	0.5958	0.6024	0.6516	0.6568	0.734	0.7375	0.6031	0.6055
	0.0165	0.0164	0.0137	0.0135	0.009	0.0088	0.0054	0.0053
0.70	0.5859	0.5889	0.6374	0.6399	0.7034	0.7052	0.576	0.5773
	0.007	0.0069	0.0064	0.0062	0.0062	0.006	0.004	0.0039
0.80	0.5758	0.578	0.6241	0.626	0.678	0.6795	0.5567	0.5578
	0.0043	0.0041	0.0054	0.0053	0.0078	0.0076	0.0056	0.0055
				$(n = 45) < (m = 50)$				
0.50	0.5949	0.5966	0.6594	0.6607	0.7375	0.7384	0.601	0.6016
	0.0153	0.0153	0.0117	0.0116	0.0082	0.0081	0.0043	0.0043
0.70	0.5761	0.5764	0.6374	0.638	0.7013	0.7018	0.5769	0.5773
	0.0039	0.0039	0.0056	0.0056	0.0055	0.0055	0.0036	0.0036
0.80	0.577	0.5775	0.6254	0.6258	0.6791	0.6795	0.559	0.5593
	0.004	0.004	0.0044	0.0044	0.0072	0.0071	0.0053	0.0053

Note: The 1st and 2nd row represents the average estimates and MSE of P.



(a) **Plot of ML Estimator of R(t) of first data set** (b) **Plot of ML Estimator of R(t) of second data set**

Figure 3.2: **Plots of MLE of R(t)**

Chapter 4

Classical and Bayesian Estimation in Kumaraswamy Distribution using Randomly censored data

4.1 Introduction

In life testing experiments, the data are frequently censored. Censoring occurs in a life testing experiment when exact lifespan are only known for a fraction of test items and the remaining lifetimes are known to surpass specific values. In life testing studies, there are various censoring schemes available in the literature, most popular ones being Type I and Type II censorings. But these censoring schemes do not allow intermittent removals from the life testing experiments. In the last two decades, two censoring schemes, progressive and progressive first failure censoring schemes, have become popular as they allow intermittent removals. See, for example, Krishna and Kumar (2013), Kumar *et al.* (2015), Dube *et al.* (2016), Kumar *et al.* (2017), Krishna *et al.* (2017).

Random censoring is a sort of censoring in which the item under study is lost or withdrawn from the experiment at random before it fails. In other words, at the conclusion of the experiment, some of the subjects had not experienced the event of interest. For example, in a clinical trial or a medical study, some patients may still be untreated and leave the course of treatment before its completion. In a social study, some subjects are lost for the follow-up in the middle of the survey. In reliability engineering, an electrical or electronic device such as bulb on test may break before its failure. Since the actual survival time of these subjects is unclear, in such instances, they are referred to as randomly censored observations. Gilbert (1962) pioneered the use of random censoring in literature. Following that, Breslow and Crowley(1974), Koziol and Green(1976), and many others conducted early research on random censoring. Recently, several authors investigated statistical inference in different lifetime models under random censoring like Ghitany and Al-Awadhi (2002), Kumar and Garg (2014), Krishna *et al.* (2015), Garg *et al.* (2016), Krishna and Goel (2017), Kumar (2018), Kumar and Kumar (2020) etc.

The Kumaraswamy (Kum) distribution was proposed by Kumaraswamy (1980) for double-bounded random processes with hydrological applications. For more information and references on Kum distribution, one can refer to previous chapter. In this chapter, the classical and Bayesian estimation procedures in Kum under random censoring scheme are considered. The rest of the chapter is organized as follows: the Kumaraswamy distribution is discussed in Section 4.2. Also, mathematical formulation is given for random censoring with failure and censoring time distributions. Section 4.3 deals with the maximum likelihood estimation and asymptotic confidence intervals of the parameters. Section 4.4 deals with the Expected Time on Test of items. Section 4.5 describes the formulation of Bayes estimation procedure using importance sampling procedure under squared error loss function based on non-informative and gamma informative priors and Gibbs sampling method. Highest posterior density (HPD) credible intervals for the parameters are obrained using MCMC method. Section 4.6 deals with a Monte Carlo simulation study to investigate the performance of various estimates proposed in this chapter. A real data set is analyzed for illustration purposes in Section 4.7. Finally conclusive remarks are given in section 4.8. Also, it is essential to mention that the statistical software R is used for computation purposes throughout the chapter.

4.2 Random Censoring Model

Suppose the failure times $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables with pdf $f_X(x)$, $x > 0$ and cdf $F_X(x)$, $x > 0$. Associated with these failure times $T_1, T_2, \ldots, T_n$ are i.i.d. censoring times with pdf $f_T(t)$, $t > 0$ and cdf $F_T(t)$, $t > 0$. Further, let $X_i's$ and $T_i's$ be mutually independent. We observe the minimum of failure and censoring time $Y_i = min(X_i, T_i)$; $i = 1, 2, \ldots, n$ and the corresponding censor indicators $D_i = 1(0)$ if failure (censoring) occurs. Since, X_i and T_i are independent, so will be Y_i and D_i, $i = 1, 2, \ldots, n$. This censoring scheme includes, as special cases, complete sample case when $T_i = \infty$ for all $i = 1, 2, \ldots, n$ and Type I censoring when $T_i = t_\circ$ for all $i = 1, 2, \ldots, n$ where, $t_\circ$ is the pre-fixed study period. The joint pdf of Y and D is obtained by

$$f_{Y,D}(y, d) = \{f_X(y)(1 - F_T(y))\}^d \{f_T(y)(1 - F_X(y))\}^{1-d}. \quad (4.2.1)$$

The marginal distributions of Y and D are derived by using

$$f_Y(y) = f_X(y)(1 - F_T(y)) + f_T(y)(1 - F_X(y)), \ y > 0$$

and

$$P[D = d] = p^d(1 - p)^{1-d}, \ d = 0, 1,$$

respectively, where, p is the probability of observing a failure and is given by

$$p = P[X \leq T] = \int_0^\infty (1 - F_T(y)) f_X(y) dy.$$

The pdf and cdf of Kum distribution are given by

$$f(x; \alpha, \beta) = \alpha\beta x^{\beta-1}(1 - x^\beta)^{\alpha-1}; \ 0 < x < 1, \ \alpha, \beta > 0, \quad (4.2.2)$$

and

$$F(x; \alpha, \beta) = 1 - (1 - x^\beta)^\alpha, \quad (4.2.3)$$

respectively. Then, the corresponding reliability function, $R(t)$, failure rate function, $h(t)$ and Mean Time to System Failure (MTSF) are given, respectively, by

$$R(t) = (1 - t^{\beta})^{\alpha};\ t > 0, \tag{4.2.4}$$

$$h(t) = \frac{\alpha\beta t^{\beta-1}(1 - t^{\beta})^{\alpha-1}}{(1 - t^{\beta})^{\alpha}} = \frac{\alpha\beta t^{\beta-1}}{1 - t^{\beta}}, \tag{4.2.5}$$

and

$$MTSF = \alpha B\left(\alpha, 1 + \frac{1}{\beta}\right), \tag{4.2.6}$$

where, $\Gamma(a)$ is the gamma function and $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function. The present chapter considers that the failure time X and the censoring time T follow Kum distribution with common shape parameter β. Let X follow $Kum(\alpha_1, \beta)$ and T follow $Kum(\alpha_2, \beta)$, then, their pdfs are given by,

$$f_X(x, \alpha_1, \beta) = \alpha_1\beta x^{\beta-1}(1 - x^{\beta})^{\alpha_1-1};\ 0 < x < 1,\ \alpha_1, \beta > 0, \tag{4.2.7}$$

and

$$f_T(t, \alpha_2, \beta) = \alpha_2\beta t^{\beta-1}(1 - t^{\beta})^{\alpha_2-1};\ 0 < t < 1,\ \alpha_2, \beta > 0, \tag{4.2.8}$$

respectively. The joint pdf of Y and D can now be determined using (4.2.1),(4.2.7), and (4.2.8), as shown below:

$$f_{Y,D}(y_i, d_i, \alpha_1, \alpha_2, \beta) = \alpha_1^{d_i}\alpha_2^{1-d_i}\beta y_i^{\beta-1}(1 - y_i^{\beta})^{\alpha_1+\alpha_2-1};\ d = 0, 1,\ 0 < y_i < 1. \tag{4.2.9}$$

Thus, from (4.2.9),the marginal distribution of Y_i is

$$\begin{aligned} f_Y(y) &= \sum_{d=0}^{1} f_{Y,D}(y, d) \\ &= \alpha_1\beta y_i^{\beta-1}(1 - y_i^{\beta})^{\alpha_1+\alpha_2-1} + \alpha_2\beta y_i^{\beta-1}(1 - y_i^{\beta})^{\alpha_1+\alpha_2-1} \\ &= (\alpha_1 + \alpha_2)\beta y_i^{\beta-1}(1 - y_i^{\beta})^{\alpha_1+\alpha_2-1};\ 0 < y_i < 1 \end{aligned} \tag{4.2.10}$$

Hence, Y follows $Kum(\alpha_1 + \alpha_2, \beta)$. The marginal distribution of D_i is given by

$$\begin{aligned} P(D_i = d_i) &= \int_{y=0}^{1} f_{Y,D}(y, d)dy \\ &= \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^{d_i}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)^{1-d_i};\ d_i = 0, 1 \\ &= p^{d_i}(1 - p)^{1-d_i}, \end{aligned} \tag{4.2.11}$$

where, $p = P[X_i \le T_i] = \frac{\alpha_1}{\alpha_1+\alpha_2}$.

4.3 Maximum Likelihood Estimation

In this section, we obtain ML estimators for the unknown parameters of Kum distribution using randomly censored data. Let $(\underset{\sim}{y}, \underset{\sim}{d}) = (y_1, d_1), (y_2, d_2), \ldots, (y_n, d_n)$ be the randomly censored sample of size n generated from (4.2.9). Then, the likelihood function for this randomly censored sample $(\underset{\sim}{y}, \underset{\sim}{d})$ is given by

$$L(\underset{\sim}{y}, \underset{\sim}{d}, \alpha_1, \alpha_2, \beta) = \prod_{i=1}^{n} f_{Y,D}(y_i, d_i)$$
$$= \alpha_1^{\sum d_i} \alpha_2^{n-\sum d_i} \beta^n \prod_{i=1}^{n} y_i^{\beta-1} \prod_{i=1}^{n} (1 - y_i^{\beta})^{\alpha_1+\alpha_2-1} \tag{4.3.1}$$

Taking log on both the sides, we have

$$\log L = \sum_{i=1}^{n} d_i \log(\alpha_1) + (n - \sum_{i=1}^{n} d_i) \log(\alpha_2) + n \log(\beta) + (\beta - 1) \sum_{i=1}^{n} \log(y_i) + (\alpha_1 + \alpha_2 - 1) \sum_{i=1}^{n} \log(1 - y_i^{\beta}) \tag{4.3.2}$$

Differentiating the equation (4.3.2) with respect to unknown model parameters α_1, α_2 and β and equating them to zero, we have the normal equations as

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{\sum_{i=1}^{n} d_i}{\alpha_1} + \sum_{i=1}^{n} \log(1 - y_i^{\beta}) = 0$$

$$\implies \alpha_1 = \frac{-\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \log(1 - y_i^{\beta})} \tag{4.3.3}$$

$$\frac{\partial \log L}{\partial \alpha_2} = \frac{\sum_{i=1}^{n} (n - d_i)}{\alpha_2} + \sum_{i=1}^{n} \log(1 - y_i^{\beta}) = 0$$

$$\implies \alpha_2 = \frac{-\sum_{i=1}^{n} (n - d_i)}{\sum_{i=1}^{n} \log(1 - y_i^{\beta})} \tag{4.3.4}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(y_i) - (\alpha_1 + \alpha_2 - 1) \sum_{i=1}^{n} \frac{y_i^{\beta} \log(y_i)}{1 - y_i^{\beta}} = 0 \tag{4.3.5}$$

Substituting (4.3.3) and (4.3.4) in (4.3.5), we obtain

$$\frac{n}{\beta} + \sum_{i=1}^{n} \log(y_i) + \left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \log(1 - y_i^{\beta})} + \frac{\sum_{i=1}^{n} (n - d_i)}{\sum_{i=1}^{n} \log(1 - y_i^{\beta})} - 1 \right) \times \sum_{i=1}^{n} \frac{y_i^{\beta} \log(y_i)}{1 - y_i^{\beta}} = 0 \tag{4.3.6}$$

Since, it is not easy to solve equation (4.3.6), we need an iterative method to solve it. Let $\widehat{\beta}$ be the ML estimator of β then the ML estimator of α_1 and α_2, respectively, are given by

$$\widehat{\alpha}_1 = \frac{-\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} \log(1 - y_i^{\widehat{\beta}})},$$

and

$$\widehat{\alpha}_2 = \frac{-\sum_{i=1}^{n} (n - d_i)}{\sum_{i=1}^{n} \log(1 - y_i^{\widehat{\beta}})}.$$

Then, by the invariance property of ML estimators, the ML estimator of $R(t)$, $h(t)$ and MTSF are respectively, given by

$$\widehat{R}(t) = (1 - t^{\widehat{\beta}})^{\widehat{\alpha}_1};\ t > 0, \tag{4.3.7}$$

,

$$\widehat{h}(t) = \frac{\widehat{\alpha}_1 \widehat{\beta} t^{\widehat{\beta}-1}}{(1 - t^{\widehat{\beta}})}, \tag{4.3.8}$$

and

$$\widehat{MTSF} = \widehat{\alpha}_1 B\left(\widehat{\alpha}_1, 1 + \frac{1}{\widehat{\beta}}\right). \tag{4.3.9}$$

4.3.1 Asymptotic Confidence Interval (ACI)

In this subsection, in order to obtain ACI, we first obtain Fisher's Information Matrix. The observed Fisher information matrix is given to evaluate the estimated variance of the $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ and $\widehat{\beta}$. The observed Fisher information matrix is given by

$$I(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\beta}) = - \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha_1^2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta} \\ \\ \frac{\partial^2 \log L}{\alpha_2 \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \alpha_2^2} & \frac{\partial^2 \log L}{\partial \alpha_2 \partial \beta} \\ \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha_1} & \frac{\partial^2 \log L}{\partial \beta \partial \alpha_2} & \frac{\partial^2 \log L}{\partial \beta^2} \end{pmatrix}_{\alpha_1=\widehat{\alpha}_1, \alpha_2=\widehat{\alpha}_2, \beta=\widehat{\beta}} \tag{4.3.10}$$

The second order partial deriatives are

$$\frac{\partial^2 \log L}{\partial \alpha_1^2} = -\frac{\sum_{i=1}^n d_i}{\alpha_1^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha_2^2} = -\frac{(n - \sum_{i=1}^n d_i)}{\alpha_2^2}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = \frac{-n}{\beta^2} - (\alpha_1 + \alpha_2 - 1) \sum_{i=1}^{n} \frac{y_i^\beta (\log(y_i))^2}{(1 - y_i^\beta)^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \log L}{\partial \alpha_2 \partial \alpha_1} = 0$$

$$\frac{\partial^2 \log L}{\partial \alpha_1 \partial \beta} = -\frac{y_i^\beta (\log(y_i))}{(1 - y_i^\beta)} = \frac{\partial^2 \log L}{\partial \alpha_2 \partial \beta} = \frac{\partial^2 \log L}{\partial \beta \partial \alpha_1} = \frac{\partial^2 \log L}{\partial \beta \partial \alpha_2}$$

The estimated variance of $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ and $\widehat{\beta}$ are the diagonal terms of $I^{-1}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\beta})$, where $I^{-1}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\beta})$ is the inverse of the observed Fisher's Information Matrix. The $(1 - \alpha)100\%$ ACI for α_1, α_2 and β are given by

$$\left\{ \widehat{\alpha}_1 - Z_{\alpha/2} \sqrt{var(\widehat{\alpha}_1)}\ ,\ \ \widehat{\alpha}_1 + Z_{\alpha/2} \sqrt{var(\widehat{\alpha}_1)} \right\},$$

$$\left\{ \widehat{\alpha}_2 - Z_{\alpha/2} \sqrt{var(\widehat{\alpha}_2)}\ ,\ \ \widehat{\alpha}_2 + Z_{\alpha/2} \sqrt{var(\widehat{\alpha}_2)} \right\},$$

and

$$\left\{ \widehat{\beta} - Z_{\alpha/2} \sqrt{var(\widehat{\beta})}\ ,\ \ \widehat{\beta} + Z_{\alpha/2} \sqrt{var(\widehat{\beta})} \right\},$$

respectively.

4.4 Expected Time on Test

Expected Time on Test (ETT) is analyzed in this section. Considering time is directly related to cost, it is therefore beneficial to have an estimate about the expected time of the experiment. First time in literature, the ETT for randomly censored data was introduced by Krishna *et al.* (2015). Let $Y_i = min(X_i, T_i)$; $i = 1, 2, \ldots, n$ be the random sample of size n generated from $Kum(\alpha_1 + \alpha_2, \beta)$. Let $Y_{(n)} = max(Y_1, Y_2, \ldots, Y_n)$ be the n^{th} ordered statistic. Then, the cdf of Y_n is given by

$$F_{Y_{(n)}} = P[Y_n \leq y] = [1 - (1 - y^\beta)^{\alpha_1 + \alpha_2}]^n$$

Thus, for randomly censored Kum data, the ETT is given by

$$\begin{aligned} ETT_{RC} &= E(Y_{(n)}) \\ &= \int_0^1 [1 - F_{Y_{(n)}}]dy \\ &= \int_0^1 [1 - (1 - y^{\beta})^{\alpha_1+\alpha_2}]^n dy. \end{aligned} \tag{4.4.1}$$

Then, the ML estimator of ETT_{RC} is given by

$$\widehat{ETT}_{RC} = \int_0^1 [1 - (1 - y^{\widehat{\beta}})^{\widehat{\alpha}_1+\widehat{\alpha}_2}]^n dy \tag{4.4.2}$$

Further, the Observed Time on Test (OBTT) is the maximum ordered statistic in $Y_1, Y_2, \ldots, Y_n$, i.e.,

$$OBTT_{RC} = Y_{(n)}. \tag{4.4.3}$$

Similarly, let $X_{(n)}$ denote the n^{th} order statistic in the case of complete sample. Then, ETT in case of complete sample is given by

$$\begin{aligned} ETT_{CS} &= E(X_{(n)}) \\ &= \int_0^1 [1 - F_{X_{(n)}}]dx \\ &= \int_0^1 [1 - (1 - x^{\beta})^{\alpha_1}]^n dx \end{aligned} \tag{4.4.4}$$

Equations (4.4.2) and (4.4.4) can now be solved for different combinations of α_1, α_2, β and n with the help of *integral* function in R software.

4.5 Bayesian Estimation

Here, we develop the Bayes estimates of the unknown model parameters of the randomly censored Kumaraswamy distribution. In order to compute the Bayes estimates, let α_1, α_2 and β independently follow the gamma priors with hyper-parameters (a_1, b_1), (a_2, b_2), and (a_3, b_3), respectively, with their respective pdf's

$$g(\alpha_1, a_1, b_1) = \frac{a_1^{b_1}}{\Gamma(b_1)} e^{-a_1\alpha_1} \alpha_1^{b_1-1}; \quad a_1, b_1 > 0,$$

$$g(\alpha_2, a_2, b_2) = \frac{a_2^{b_2}}{\Gamma(b_2)} e^{-a_2\alpha_2} \alpha_2^{b_2-1}; \quad a_2, b_2 > 0,$$

$$g(\beta, a_3, b_3) = \frac{a_3^{b_3}}{\Gamma(b_3)} e^{-a_3\beta}\beta^{b_3-1}; \quad a_3, b_3 > 0.$$

Thus, the joint prior distribution of α_1, α_2 and β is

$$g(\alpha_1, \alpha_2, \beta) \propto e^{-(a_1\alpha_1 + a_2\alpha_2 + a_3\beta)} \alpha_1^{b_1-1}\alpha_2^{b_2-1}\beta^{b_3-1} \tag{4.5.1}$$

Now using the likelihood function given in equation (4.3.1) and the joint prior given in equation (4.5.1), the joint posterior distribution of the parameters α_1, α_2 and β is given by

$$\pi(\alpha_1, \alpha_2, \beta|data) = \frac{L(\underset{\sim}{y}, \underset{\sim}{d}, \alpha_1, \alpha_2, \beta)g(\alpha_1, \alpha_2, \beta)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\underset{\sim}{y}, \underset{\sim}{d}, \alpha_1, \alpha_2, \beta)g(\alpha_1, \alpha_2, \beta)d\alpha_1 d\alpha_2 d\beta}$$

$$\begin{aligned}
\implies \pi(\alpha_1, \alpha_2, \beta) \propto & \alpha_1^{\sum d_i}\alpha_2^{n-\sum d_i}\beta^n \prod_{i=1}^n y_i^{\beta-1} \prod_{i=1}^n (1 - y_i^{\beta-1})^{\alpha_1+\alpha_2-1} \times \alpha_1^{b_1-1}\alpha_2^{b_2-1}\beta^{b_3-1} \times \\
& e^{-a_2\alpha_2} e^{-a_3\beta} \\
= & \alpha_1^{b_1+\sum d_i-1} e^{-a_1\alpha_1} \alpha_2^{n-\sum d_i+b_2-1} e^{-a_2\alpha_2} \beta^{n+b_3-1} e^{-a_3\beta} e^{(\alpha_1+\alpha_2-1)\sum_{i=1}^n log(1-y_i^\beta)} \times \\
& e^{(\beta-1)\sum_{i=1}^n log(y_i)}
\end{aligned}$$

$$\begin{aligned}
\implies \pi(\alpha_1, \alpha_2, \beta) \propto & \alpha_1^{b_1+\sum d_i-1} e^{-\alpha_1\left(a_1-\sum_{i=1}^n log(1-y_i^\beta)\right)} \alpha_2^{(n+b_2-\sum d_i-1)} e^{-\alpha_2\left(a_2-\sum_{i=1}^n log(1-y_i^\beta)\right)} \times \\
& \beta^{n+b_3-1} e^{-\beta\left(a_3-\sum_{i=1}^n log(y_i)\right)} e^{-\sum_{i=1}^n log(1-y_i^\beta)}
\end{aligned} \tag{4.5.2}$$

Now, we compute the Bayes estimates under the squared error loss function (SELF). Let $\phi(\alpha_1, \alpha_2, \beta)$ be any function of the parameters α_1, α_2 and β, then the Bayes estimate of $\phi(\alpha_1, \alpha_2, \beta)$ under SELF is given by

$$\begin{aligned}
\phi^* =& E(\phi(\alpha_1, \alpha_2, \beta)|data) \\
=& \int_0^\infty \int_0^\infty \int_0^\infty \phi(\alpha_1, \alpha_2, \beta)\pi(\alpha_1, \alpha_2, \beta|data)d\alpha_1 d\alpha_2 d\beta
\end{aligned} \tag{4.5.3}$$

From the equation (4.5.3), we observe that the closed form solution of the equation (4.5.3) is not possible. Therefore, two approximation methods, (i) Importance Sampling Technique, and (ii) Gibbs Sampling Method, are used to derive Bayes estimates.

4.5.1 Importance Sampling Technique

The importance sampling procedure is adopted in this subsection to calculate Bayes estimates of the parameters and reliability characteristics. The posterior distribution given in (4.5.2) can be rewritten as

$$\pi(\alpha_1, \alpha_2, \beta) \propto gamma(A_1, B_1) \times gamma(A_2, B_2) \times gamma(A_3, B_3) \times W(\alpha_1, \alpha_2, \beta),$$

where, $A_1 = a_1 - \sum_{i=1}^{n} log(1 - y_i^{\beta})$, $B_1 = b_1 + \sum_{i=1}^{n} d_1$, $A_2 = a_2 - \sum_{i=1}^{n} log(1 - y_i^{\beta})$, $B_2 = b_2 + n - \sum_{i=1}^{n} d_1$, $A_3 = a_3 - \sum_{i=1}^{n} log(y_i)$, $B_3 = n + b_3$, and $W(\alpha_1, \alpha_2, \beta) = \frac{e^{-\sum_{i=1}^{n} log(1-y_i^{\beta})}}{A_1^{B_1} A_2^{B_2}}$.

Now, for the computation of Bayes estimates using the Importance sampling technique, following steps are used:

Step 1. Generate $\beta^{(1)}$ from gamma(A_3, B_3).

Step 2. Generate $\alpha_1^{(1)}$ from gamma(A_1, B_1) using $\beta^{(1)}$ generated in Step 1.

Step 3. Generate $\alpha_2^{(1)}$ from gamma(A_2, B_2) using $\beta^{(1)}$ generated in Step 1.

Step 4. Compute $W\left(\alpha_1^{(1)}, \alpha_2^{(1)}, \beta^{(1)}\right)$.

Step 5. Repeat steps 1 to 4, $(M - 1)$ times to obtain importance samples.

Now, the approximate Bayes estimates of the parameters and reliability characteristics under SELF are given by

$$\alpha_{1IS}^* = \frac{\sum_{i=1}^{M} \alpha_1^{(i)} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)},$$

$$\alpha_{2IS}^* = \frac{\sum_{i=1}^{M} \alpha_2^{(i)} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)},$$

$$\beta_{IS}^* = \frac{\sum_{i=1}^{M} \beta^{(i)} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)},$$

$$R_{IS}^*(t) = \frac{\sum_{i=1}^{M} \left(1 - t^{\beta^{(i)}}\right)^{\alpha_1^{(i)}} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)},$$

$$h_{IS}^*(t) = \frac{\sum_{i=1}^{M} \frac{\alpha_1^{(i)} \beta^{(i)} t^{\beta^{(i)}-1}}{1-t^{\beta^{(i)}}} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}$$

, and

$$MTSF^* = \frac{\sum_{i=1}^{M} \alpha_1^{(i)} B\left(\alpha_1^{(i)}, 1 + \frac{1}{\beta^{(i)}}\right) W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}{\sum_{i=1}^{M} W\left(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}\right)}.$$

4.5.2 Gibbs Sampling Technique

Here, we consider Marko Chain Monte Carlo (MCMC) technique to compute Bayes estimates and the corresponding highest posterior density credible intervals. The Gibbs sampling is a particular type of MCMC method [see Smith and Roberts (1993)]. The full posterior conditional distributions for the parameters α_1, α_2 and β, respectively, are given by

$$\pi_1(\alpha_1|\beta, data) \propto gamma(A_1, B_1), \tag{4.5.4}$$

$$\pi_2(\alpha_2|\beta, data) \propto gamma(A_2, B_2), \tag{4.5.5}$$

and

$$\pi_3(\beta|\alpha_1, \alpha_2, data) \propto \beta^{n+b_3-1} e^{-\beta(a_3 - \sum_{i=1}^{n} log(y_i))} e^{(\alpha_1+\alpha_2-1)\sum_{i=1}^{n} log(1-y_i^{\beta})}. \tag{4.5.6}$$

From the above equations (4.5.4) and (4.5.5), it can be seen that the posterior samples of α_1 and α_2 can be easily generated using gamma distributions, but the posterior sample of β cannot be generated directly. For the generation of the sample of β, we shall use the Metropolis-Hasting algorithm, see Metropolis *et al.* (1953) and Hasting (1970). Thus, we use the following steps to generate samples from the full conditional posterior densities given in equations (4.5.4), (4.5.5) and (4.5.6), respectively:

Step 1. Start with an initial guess of α_1, α_2 and β, say $\alpha_1^{(0)}$, $\alpha_2^{(0)}$ and $\beta^{(0)}$.

Step 2. Set $j = 1$.

Step 3. Generate $\beta^{(j)}$ from $\pi_3(\beta|\alpha_1^{(j-1)}, \alpha_2^{(j-1)}, data)$ using MH algorithm with normal proposal distribution as

(i) Generate a candidate point β_c^j from the proposal distribution $N(\beta^{(j-1)}, 1)$.

(ii) Generate u from Uniform(0,1).

(iii) Calculate $\eta = min\left(1, \frac{\pi_3(\beta_c^{(j)}|\alpha_1^{(j-1)}, \alpha_2^{(j-1)}, data)}{\pi_3(\beta^{(j-1)}|\alpha_1^{(j-1)}, \alpha_2^{(j-1)}, data)}\right)$.

(iv) If $u \leq \eta$, set $\beta^{(j)} = \beta_c^{(j)}$ with acceptance probability η, otherwise $\beta^{(j)} = \beta^{(j-1)}$.

Step 4. Generate $\alpha_1^{(j)}$ from $gamma(A_1, B_1)$ using $\beta^{(j)}$ generated in Step 3.

Step 5. Generate $\alpha_2^{(j)}$ from $gamma(A_2, B_2)$ using $\beta^{(j)}$ generated in Step 3.

Step 6. Set j=j+1.

Step 7. Repeat step 3 to step 6, N times, to obtain the sequence of the parameters as $\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \beta^{(1)}$, $\alpha_1^{(2)}, \alpha_2^{(2)}, \ldots, \beta^{(2)}, \ldots$, $\alpha_1^{(N)}, \alpha_2^{(N)} \ldots, \beta^{(N)}$.

We discard first $N_\circ = 20\%$ of the N of the generated values of the parameters as the burn-in-period to obtain independent samples from the stationary distribution of the Markov chain which are typically the posterior distributions. Thus, the Bayes estimates of the parameters α_1, α_2, β and reliability characteristics $R(t)$, $h(t)$ and MTSF under SELF, respectively, are given by

$$\alpha^*_{1GS} = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} \alpha_1^{(j)},$$

$$\alpha^*_{2GS} = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} \alpha_2^{(j)},$$

$$\beta^*_{GS} = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} \beta^{(j)},$$

$$R^*_{GS}(t) = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} (1 - t^{\beta^{(j)}})^{\alpha_1^{(j)}},$$

$$h^*_{GS}(t) = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} \frac{\alpha_1^{(j)} \beta^{(j)} t^{\beta^{(j)}-1}}{1 - t^{\beta^{(j)}}},$$

and

$$MTSF^*_{GS} = \frac{1}{(N - N_\circ)} \sum_{j=N_\circ+1}^{N} \alpha_1^{(j)} B\left(\alpha_1^{(j)}, 1 + \frac{1}{\beta^{(j)}}\right).$$

4.5.3 Highest Posterior Density (HPD) Credible Interval

In this sub-section, we construct HPD credible intervals of the unknown parameters α_1, α_2 and β, respectively, by using the algorithm proposed by Chen and Shao (1999). Let $\alpha_{1(1)} < \alpha_{1(2)} < \ldots < \alpha_{1(N-N_\circ)}$ denote the ordered form of the MCMC sample of α_1 generated in the previous sub-section. Thus, the $100(1-\delta)\%$, where $0<\delta<1$, HPD credible interval for α_1 is given by

$$(\alpha_{1(j)}, \alpha_{1(j+[(1-\delta)M])}),$$

where j is chosen such that

$$\alpha_{1(j+[(1-\delta)M])} - \alpha_{1(j)} = \underset{1\le i\le \delta M}{min} \left(\alpha_{1(i+[(1-\delta)M])} - \alpha_{1(i)}\right);\ j = 1, 2, \ldots, M,$$

here, $[x]$ is the largest integer less than or equal to x. Similarly, we can construct the $100(1-\delta)\%$ HPD credible intervals for α_2 and β, respectively.

4.6 Simulation Study

In this Section, a Monte Carlo simulation study is carried out to estimate the performance and efficiency of the different estimation methods. For this we have used an algorithm to generate a randomly censored sample from Kumaraswamy distribution. The algorithm is as follow:

Step 1. Generate a random sample $u_1, u_2, \ldots, u_n$ from standard Uniform distribution, i.e., $U(0,1)$.

Step 2. Make a transformation to obtain failure observations x_i, $i = 1, 2, \ldots, n$,

$$x_i = (1 - (1 - u_i)^{1/\alpha_1})^{1/\beta}.$$

Step 3. Generate another random sample $v_1, v_2, \ldots, v_n$ from U(0,1).

Step 4. Make another transformation to obtain censoring observations t_i, $i = 1, 2, \ldots, n$,

$$t_i = (1 - (1 - v_i)^{1/\alpha_2})^{1/\beta}.$$

Step 5. Now obtain y_i and d_i by using the condition that if $x_i < t_i$, then $y_i = x_i$ and $d_i = 1$, else $y_i = t_i$ and $d_i = 0$. Hence, we obtain a randomly censored sample (y_i, d_i) of n observations from Kum distribution.

In order to study the behaviour of different estimates, we have generated 10,000 randomly censored samples by using the above algorithm for different values of sample sizes $n = 30(5)100$ and for different values of parameters α_1, α_2 and β.

We have computed the average values of ML estimators according to the method discussed in Section 4.3. We have also computed the average length of the 95% asymptotic confidence intervals and their corresponding Coverage Probabilities (C.P.). Further, we have obtained the Bayes estimates using Importance Sampling technique and Gibbs Sampling method and 95% HPD credible intervals of the parameters by taking $M = 10,000$ as discussed in Section 4.5. Hyperparameters are chosen in such a way that the mean of the prior distribution is equal to the true value of the parameter.

The average ML estimates and the average Bayes estimates of α_1 with hyperparameters $a_1 = 2$, $a_2 = 2$, $a_3 = 3$, $b_1 = 1$, $b_2 = 3$, $b_3 = 6$ and their corresponding MSE for true values of parameters $\alpha_1 = 0.5$, $\alpha_2 = 1.5$ and $\beta = 2$ are computed and the results are reported in Table 4.1. Comparing the estimates on the basis of MSE, from Table 4.1 we observe that the Bayes estimate based on Importance Sampling performs the best and ML estimate performs the least. Bayes estimate based on Gibbs sampling method lies between the two. Further, as n increases, the performance of all the estimators improve and the three estimators come close to each other. Similiarly, the average ML estimate and the average Bayes estimates for α_2 and their corresponding MSE are computed and the results are reported in Table 4.2. From Table 4.2 we can observe that the Bayes estimate of α_2 based on the Importance Sampling performs better than the Bayes estimate based on Gibbs Sampling Method and ML estimate. However, as n increases, their MSE decreases and all the estimates become almost equally efficient. In case of β, for $n < 60$, Bayes estimate based on Importance sampling performs the best, ML estimate performs the least and Bayes estimate based on Gibbs sampling lies in between the two. However, for $n \geq 60$, Bayes estimate based on Gibbs sampling performs the best, ML estimate performs the least and the Bayes estimate based on Importance sampling lies between the two. Further, for $n \geq 85$, Bayes estimate based on Gibbs sampling performs the best, Bayes estimate based on Importance sampling performs the least and MLE lies between the two. Also, as sample size increases, all the estimates become almost equally efficient. The average length and coverage probabilities of the 95% asymptotic confidence intervals and HPD credible intervals of the three parameters are obtained for different values of n and the results are presented in Table 4.4. From Table 4.4 we observe that the length of both the intervals for α_1, α_2 and β decreases as n increases, which shows the improvement in precision of the estimates as sample size increases. We also observe that the coverage probabilities of the interval estimates are much close to nominal values even for small values of n. The average length of asymptotic C.I. of α_1 is smaller than average length of HPD credible interval, while, coverage probability of HPD is higher than asymptotic C.I.. However, for α_2 and β, the average length of HPD is smaller than the average length of asymptotic C.I. while, coverage probability of asymptotic C.I. is higher than HPD.

For comparing the performance of ML estimate and Bayes estimate of reliability function $R(t)$, we obtain the average estimates of MLE and Bayes estimate and their MSE for $\alpha_1 = 0.5$, $\alpha_2 = 1.5$ and $\beta = 2$ and different values of n and $t = 0.5$ and the results are reported in Table 4.5. Comparing the estimates on the basis of MSE, we observe that the Bayes estimate of $R(t)$ based on Gibbs sampling method performs better than the ML estimate and Bayes estimate based on Importance sampling method. Also, as n increases, MSE of the three estimates decreases and the estimates become almost equally efficient. Along the similiar lines, we obtain the average value of ML estimate and Bayes estimate and their MSE of the failure rate function $h(t)$ and the results are reported in Table 4.6. From Table 4.6, on the basis of MSE, we can conclude that the Bayes estmate of $h(t)$ based on Gibbs sampling performs better than the ML estimate and the Byes estimate based on Importance sampling method and the estimates improve and come close to each other as n increases. Similiarly, for investigating the performance of MTSF, we have obtained average estimates of ML estimate and Bayes estimate and their MSE and the results are presented in Table 4.7. From Table 4.7 we conclude that Bayes estimate of MTSF based on Gibbs sampling performs better than the Bayes estimate based on Importance sampling technique and MLE. Also, as n increases, MSE of the three estimates decreases and the estimates become almost equally efficient.

We have also calculated ETT based on complete sample and randomly generated sample and OBTT based on randomly generated sample as discussed in Section 4.4 for different sample sizes and the results are presented in Table 4.8. From Table 4.8, we observe that the random censoring reduces the expected time on test and as n increases, ETT increases.

4.7 Real Data Analysis

In this section, we perform a real data analysis study. Here we consider the data set which contains the ordered survival times (in months) of 24 patients with Dukes'C colorectal cancer. This data set was originally studied by McIllmurray and Turkie (1987). Also, this data set has been analyzed by Danish and Aslam (2014) for randomly censored Weibull distribution. First of all, without loss of generality, we divide all observations with 50 for computational ease and the transformed data are as follow:

0.06+, 0.12, 0.12, 0.12, 0.12, 0.16, 0.16, 0.24, 0.24, 0.24+, 0.30+, 0.32+, 0.36+, 0.36+, 0.40, 0.44+, 0.48, 0.56+, 0.56+, 0.56+, 0.60, 0.60+, 0.66+, 0.84.

The + sign denotes the censoring times. Now, we compare the fitted Kum distribution with some other well-known survival models, like, exponential, Rayleigh, and Weibull distributions for Dukes'C colorectal cancer data. The ML estimators of the parameters of these distributions under random censorship model are obtained. These estimates can be used to calculate the negative log likelihood function $-ln\ L$, as well as the Akaike information criterion ($AIC = 2 \times k - 2ln\ L$), introduced by Akaike (1974) and Bayesian information criterion ($BIC = k \times kln\ (n) - 2ln\ L$) introduced by Schwarz (1978), where k is the number of parameters in the reliability model, n is the number of observations in the given data set, L is the maximized value of the likelihood function for the estimated model and Kolmogorov Smirnov (K-S) statistic with its p-value. The lowest $-ln\ L$, AIC, BIC and K-S statistic, as well as the highest p value indicate the ideal distribution and these values for different distributions are listed in Table 4.9. Table 4.9 shows that the considered Kum is the best choice among the competing distributions. Also, we consider the empirical cdf for fitting the randomly censored data through the graphs. Figure 4.1 shows the graph of the empirical cdf estimator as well as cdf estimates for the randomly censored exponential, Rayleigh, Weibull, and Kum distributions. From this figure, we conclude that the cdf estimate for Kum is quite close to the one given by the ecdf estimator. Thus, the ecdf also supports the choice of Kum to represent the Dukes' C colorectal cancer data.

Now, we consider the estimation of the parameters of randomly censored Kum distribution for this data set. Here, $n = 24$ and the effective sample size $m = 12$. Also, we take mission time $t = 0.34$, the median of the data. Since we don't have any prior information, therefore, we obtain the Bayes estimates of the parameters using non-informative priors under SELF. For non-informative priors, the hyper-parameters are taken as $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$. The Bayes estimates are obtained using importance sampling technique and Gibbs sampling method. For importance sampling method we take $M = 10000$ and for Gibbs sampling method we take $N = 50000$ with burn-in-period $N_\circ = 10000$. Also, the 95% asymptotic confidence and HPD credible intervals for the parameters are computed. All results of the real data set are reported in Tables 4.10 and 4.11.

4.8 Concluding Remarks

In this chapter, we have proposed the estimation techniques for the randomly censored Kum distribution. Both point and interval estimations of the parameters are discussed. The finite sample performance of the ML estimates and the Bayes estimates based on Importance sampling technique and Gibbs sampling method of the parameters, reliability function, $R(t)$, failure rate function, $h(t)$ and MTSF are investigated using extensive Monte Carlo experiment. The comparisons are made on the basis of MSE of the estimators. On the basis of simulation studies, we conclude that for α_1, the Bayes estimate based on Importance Sampling performs the best and ML estimator performs the least. However, for α_2, Bayes estimate based on the Importance Sampling performs better than the Bayes estimate based on Gibbs Sampling Method and ML estimator. In case of β, for $n < 60$, Bayes estimate based on Importance sampling performs better than Bayes estimate based on Gibbs sampling and ML estimator. However, for $n \geq 60$, Bayes estimate based on Gibbs sampling performs better than Bayes estimate based on Importance sampling and MLE. Further, as n increases, the performance of all the three estimators improve and they come close to each other. Also, as n increases, the length of C.I. and HPD decreases, which shows the improvement in precision of the estimates as sample size increases. We also conclude that the Bayes estimates of $R(t)$, $h(t)$ and MTSF based on the Gibbs sampling performs better than the Bayes estimates based on Importance sampling and the ML estimator and the estimates improve and come close to each other as n increases. Further, we observe that the random censoring reduces the expected time on test and as n increases, ETT increases. Real life data set is also studied to illustrate the proposed estimation methods.

Table 4.1: **Different estimates of** α_1

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	$\widehat{\alpha_1}$		α^*_{1IS}		α^*_{1GS}	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	0.562	0.0708	0.5001	0.0287	0.5255	0.0335
35	0.5437	0.0568	0.4888	0.0276	0.5173	0.0318
40	0.5394	0.0444	0.4882	0.0236	0.5196	0.027
45	0.5443	0.0408	0.4851	0.0222	0.5249	0.0259
50	0.5365	0.0355	0.4777	0.0185	0.5209	0.0241
55	0.5207	0.029	0.4653	0.0178	0.5097	0.0207
60	0.5317	0.0294	0.4734	0.0174	0.5208	0.0216
65	0.5221	0.0241	0.4641	0.0154	0.5126	0.0179
70	0.5335	0.0237	0.4694	0.015	0.5235	0.018
75	0.527	0.022	0.4662	0.014	0.5188	0.0169
80	0.5166	0.018	0.4527	0.014	0.5099	0.0143
85	0.5239	0.0181	0.4559	0.0124	0.5172	0.0144
90	0.5217	0.0164	0.4584	0.012	0.5161	0.0134
95	0.5274	0.0167	0.4518	0.0114	0.5215	0.0137
100	0.5158	0.0144	0.4465	0.0118	0.5113	0.0121

Table 4.2: **Different estimates of** α_2

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	$\widehat{\alpha_2}$		α^*_{2IS}		α^*_{2GS}	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	1.6933	0.3529	1.5097	0.0894	1.5839	0.1267
35	1.652	0.2577	1.4865	0.0767	1.5726	0.1145
40	1.6147	0.2137	1.466	0.0767	1.5549	0.1058
45	1.6405	0.1983	1.4701	0.0688	1.5814	0.1043
50	1.6092	0.1598	1.4432	0.0564	1.5634	0.091
55	1.5871	0.1387	1.4199	0.0548	1.552	0.0843
60	1.5715	0.1129	1.4036	0.0497	1.5421	0.0724
65	1.5886	0.1215	1.4038	0.0503	1.5586	0.079
70	1.5872	0.1241	1.3953	0.0531	1.5572	0.0831
75	1.574	0.0976	1.3845	0.0477	1.5502	0.0673
80	1.5686	0.0874	1.3701	0.046	1.5474	0.0619
85	1.5688	0.0858	1.3661	0.0462	1.5485	0.0619
90	1.5485	0.0739	1.3491	0.0508	1.5326	0.0549
95	1.5513	0.0683	1.3465	0.0503	1.5354	0.0515
100	1.5585	0.0678	1.3482	0.0464	1.5442	0.052

Table 4.3: **Different estimates of** β

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	$\widehat{\boldsymbol{\beta}}$		$\boldsymbol{\beta^*_{IS}}$		$\boldsymbol{\beta^*_{GS}}$	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	2.1514	0.2132	1.9838	0.0772	2.0722	0.1016
35	2.112	0.1709	1.9497	0.0672	2.052	0.0891
40	2.0838	0.1555	1.9219	0.0708	2.0359	0.0882
45	2.0884	0.1403	1.9018	0.0641	2.0415	0.0824
50	2.0799	0.1187	1.8928	0.0634	2.0405	0.0737
55	2.0594	0.1004	1.8677	0.0642	2.0291	0.0656
60	2.0533	0.091	1.8464	0.0614	2.0255	0.0612
65	2.0547	0.0864	1.8362	0.0623	2.028	0.059
70	2.0671	0.0847	1.8362	0.0657	2.0399	0.0589
75	2.0516	0.0717	1.8202	0.0628	2.029	0.0513
80	2.0591	0.0775	1.8131	0.0686	2.0377	0.0571
85	2.0473	0.0649	1.7925	0.0719	2.0271	0.0484
90	2.0386	0.0608	1.7794	0.0747	2.0216	0.0459
95	2.047	0.0555	1.7765	0.0741	2.0295	0.0426
100	2.0383	0.0513	1.7649	0.0793	2.0234	0.04

Table 4.4: **The 95% asymptotic confidence and HPD credible intervals**

	ML Estimate		Bayes Estimate		ML Estimate		Bayes Estimate		ML Estimate		Bayes Estimate	
n	$\widehat{\boldsymbol{\alpha}}_{\mathbf{1}}$		α_1^*		$\widehat{\boldsymbol{\alpha}}_{\mathbf{2}}$		α_2^*		$\widehat{\boldsymbol{\beta}}$		β^*	
	AL	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**
30	0.93	0.93	2.0129	0.966	1.6847	0.964	0.7295	0.951	1.4964	0.971	1.2905	0.969
35	0.8334	0.929	1.8013	0.963	1.5356	0.963	0.6795	0.935	1.4054	0.979	1.2178	0.975
40	0.7692	0.943	1.6421	0.955	1.4225	0.952	0.6453	0.95	1.3231	0.962	1.1547	0.955
45	0.7347	0.939	1.5747	0.959	1.3397	0.945	0.6246	0.948	1.2891	0.959	1.1044	0.948
50	0.6845	0.945	1.4577	0.959	1.2684	0.951	0.5928	0.952	1.222	0.966	1.0639	0.959
55	0.635	0.936	1.3645	0.955	1.2016	0.956	0.5597	0.944	1.1666	0.965	1.0234	0.963
60	0.6157	0.935	1.2947	0.959	1.1475	0.951	0.5472	0.932	1.1229	0.961	0.9891	0.96
65	0.5859	0.951	1.2561	0.954	1.1024	0.951	0.5251	0.952	1.0962	0.963	0.9558	0.957
70	0.5735	0.944	1.2126	0.938	1.0681	0.945	0.5165	0.948	1.0661	0.939	0.933	0.945
75	0.5475	0.949	1.1575	0.956	1.0247	0.951	0.4975	0.951	1.0279	0.957	0.9023	0.953
80	0.5216	0.942	1.1142	0.961	0.9974	0.93	0.4774	0.95	0.9974	0.964	0.883	0.932
85	0.5114	0.96	1.0823	0.955	0.9611	0.952	0.4699	0.959	0.9744	0.96	0.8578	0.949
90	0.4933	0.946	1.0372	0.957	0.9328	0.944	0.4557	0.952	0.9417	0.963	0.8363	0.951
95	0.4842	0.944	1.0116	0.964	0.91	0.958	0.4486	0.95	0.9211	0.969	0.8177	0.952
100	0.4649	0.952	0.9886	0.962	0.8837	0.948	0.4329	0.953	0.9044	0.966	0.7972	0.951

NOTE: AL represents the Average Length and CP represents the Coverage Probability of the confidence intervals.

Table 4.5: **Different estimates of $R(t)$ for $t = 0.5$**

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	$\widehat{R}(t)$		$R^*_{IS}(t)$		$R^*_{GS}(t)$	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	0.8694	0.0022	0.8651	0.0019	0.8686	0.0017
35	0.8693	0.002	0.8647	0.0018	0.8687	0.0016
40	0.8663	0.0019	0.8614	0.0018	0.8659	0.0015
45	0.8659	0.0015	0.8605	0.0016	0.8656	0.0013
50	0.8665	0.0015	0.8609	0.0016	0.8662	0.0013
55	0.868	0.0013	0.8619	0.0014	0.8677	0.0011
60	0.8651	0.0012	0.8578	0.0014	0.865	0.0011
65	0.8672	0.001	0.859	0.0013	0.8669	9.00e-04
70	0.8656	0.001	0.8575	0.0013	0.8654	9.00e-04
75	0.8658	9.00e-04	0.8567	0.0012	0.8656	8.00e-04
80	0.8684	9.00e-04	0.8596	0.0011	0.8681	8.00e-04
85	0.8657	8.00e-04	0.8564	0.0011	0.8655	7.00e-04
90	0.8653	7.00e-04	0.8545	0.001	0.8652	7.00e-04
95	0.8646	8.00e-04	0.8556	0.001	0.8645	7.00e-04
100	0.8665	7.00e-04	0.8562	0.001	0.8664	7.00e-04

Table 4.6: **Different estimates of $h(t)$ for $t = 0.5$**

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	$\widehat{h}(t)$		$h^*_{IS}(t)$		$h^*_{GS}(t)$	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	0.682	0.0649	0.6634	0.0479	0.6648	0.0451
35	0.674	0.0587	0.6585	0.0462	0.6606	0.0433
40	0.681	0.0496	0.6675	0.0423	0.6701	0.0384
45	0.6857	0.0436	0.6689	0.0378	0.6748	0.0343
50	0.681	0.0416	0.6633	0.0347	0.6715	0.0338
55	0.6681	0.0351	0.6526	0.0309	0.661	0.0292
60	0.6835	0.0343	0.6695	0.0308	0.6757	0.0288
65	0.672	0.0286	0.6604	0.0282	0.6655	0.0243
70	0.6838	0.0282	0.6677	0.0269	0.6767	0.0242
75	0.6799	0.0255	0.668	0.025	0.6737	0.022
80	0.6657	0.0223	0.6509	0.0233	0.6609	0.0197
85	0.6785	0.0218	0.6615	0.0213	0.6733	0.0192
90	0.6787	0.02	0.6684	0.0203	0.6739	0.0178
95	0.6844	0.0209	0.6609	0.0195	0.6794	0.0186
100	0.6722	0.0186	0.6555	0.0184	0.6681	0.0168

Table 4.7: **Different estimates of MTSF**

	ML Estimate		**Bayes Estimate (Importance Sampling)**		**Bayes Estimate (Gibbs Sampling)**	
n	**MTSF**		$MTSF^*_{IS}$		$MTSF^*_{GS}$	
	AE	**MSE**	**AE**	**MSE**	**AE**	**MSE**
30	0.7873	0.003	0.7924	0.0025	0.7928	0.0023
35	0.7886	0.0027	0.7927	0.0024	0.7931	0.0022
40	0.7857	0.0024	0.7895	0.0022	0.7899	0.0019
45	0.7845	0.002	0.7884	0.002	0.7884	0.0017
50	0.7855	0.002	0.789	0.0019	0.789	0.0017
55	0.7878	0.0017	0.7908	0.0017	0.7908	0.0015
60	0.7843	0.0017	0.7866	0.0016	0.7873	0.0014
65	0.7865	0.0014	0.7882	0.0015	0.7891	0.0012
70	0.7839	0.0013	0.7861	0.0014	0.7865	0.0012
75	0.7845	0.0012	0.7856	0.0013	0.787	0.0011
80	0.7874	0.0011	0.7892	0.0013	0.7895	0.001
85	0.7845	0.001	0.7862	0.0012	0.7866	0.0009
90	0.7843	0.001	0.7842	0.0011	0.7863	0.0009
95	0.7831	0.001	0.7857	0.0011	0.7851	0.0009
100	0.7857	9.00e-04	0.7867	0.001	0.7874	0.0009

Table 4.8: **The estimation of ETT or OBTT**

n	ETT_{CS}	ETT_{RC}	$\widehat{\boldsymbol{ETT}}_{\boldsymbol{RC}}$	$OBTT_{RC}$
30	0.9989	0.9154	0.9068	0.9148
40	0.9994	0.9271	0.9177	0.9246
50	0.9996	0.9351	0.9304	0.936
60	0.9997	0.9409	0.9363	0.9403
70	0.9998	0.9454	0.9414	0.9438
80	0.9998	0.949	0.9452	0.948
90	0.9999	0.952	0.949	0.9525
100	0.9999	0.9545	0.9513	0.9535

Table 4.9: **Goodness-of-fit tests for the Dukes' C colorectal cancer data**

MODEL	**ML Estimate**	**-ln L**	**AIC**	**BIC**	**K-S TEST**	
					Statistic	**P-value**
$X \sim Exp(\theta)$, $T \sim Exp(\lambda)$	$\widehat{\theta} = 1.3921$ $\widehat{\lambda} = 1.3921$	16.0603	36.1206	38.4767	0.2424	0.1193
$X \sim Rayleigh(\theta)$, $T \sim Rayleigh(\lambda)$	$\widehat{\theta} = 5.8213$ $\widehat{\lambda} = 5.8213$	11.1734	26.3468	28.7029	0.1532	0.6262
$X \sim Weibull(\alpha, \theta)$, $T \sim Weibull(\alpha, \lambda)$	$\widehat{\alpha} = 1.8095$ $\widehat{\theta} = 2.5673$ $\widehat{\lambda} = 2.5673$	10.9778	27.9557	31.4898	0.1261	0.8401
$X \sim Kum(\alpha_1, \beta)$, $T \sim Kum(\alpha_2, \beta)$	$\widehat{\alpha}_1 = 1.5904$ $\widehat{\alpha}_2 = 1.5904$ $\widehat{\beta} = 1.5267$	10.6754	27.3508	30.885	0.1104	0.9318

Table 4.10: **The ML and Bayes estimates of the parameters and reliability characteristics for the Dukes' C colorectal cancer data set. Here, mission time $t = 0.34$ (Median of the data), $M = 10000$, $N = 50000$, $N_\circ = 10000$**

Parameter	**ML Estimates**	**Bayes Estimates**	
		Importance Sampling	**Gibbs Sampling Method**
α_1	1.5904	1.7945	1.5704
α_2	1.5904	1.6402	1.5741
β	1.5267	1.4648	1.4931
$R(t)$	0.7116	0.6765	0.7132
$h(t)$	1.7038	1.9197	1.6630
MTSF	0.5039	0.4841	0.5150

Table 4.11: **The 95% asymptotic confidence and HPD credible intervals of the parameters for the Dukes' C colorectal cancer data set**

Parameter	**95% ACI**	**95% HPD CI**
α_1	(0.3741, 2.8067)	(0.5544, 2.8096)
α_2	(0.3741, 2.8067)	(0.5220, 2.7813)
β	(0.9102, 2.1432)	(0.8984, 2.0625)

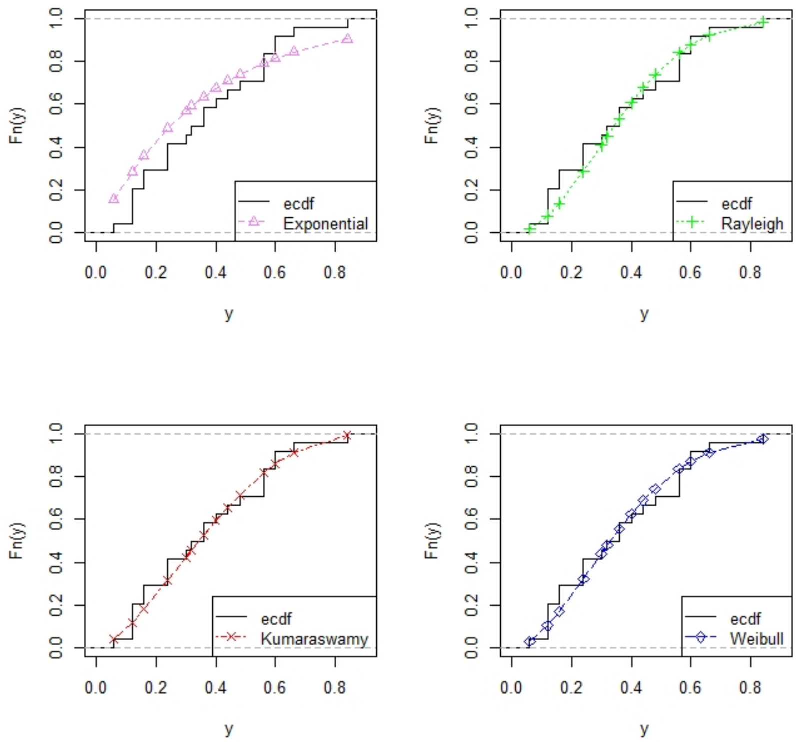


Figure 4.1: **The empirical and fitted cdfs of different competing models for Dukes'C colorectal cancer data**

Chapter 5

A Generalized Positive Exponential Family of Distributions and the Estimation of Reliability Characteristics

5.1 Introduction

In literature, various univariate statistical distributions such as normal, exponential, Weibull, gamma, etc., exist for modelling datasets. However, due to the increasing complexity of data, it is often difficult to model using these classical distributions as they do not give an appropriate fit. In order to increase flexibility in modelling data, researchers are continuously developing new distributions, which are generalizations of existing ones. Some recent generalized families of distributions introduced in literature are Beta-Normal distribution by Eugene *et al.* (2002), Kumaraswamy-G (Kw-G) family by Cordeiro and Castro (2012), Weibull-G family of distributions by Bourguignon *et al.* (2014), Lomax generator of distributions by Cordeiro *et al.* (2014), logistic-X family of distributions by Tahir *et al.* (2016) and Odd Burr-G family by Alizadeh *et al.* (2017).

Liang (2008) proposed a family of lifetime distributions, which he named as

Positive Exponential family of distributions. He studied empirical Bayes estimation of reliability of this family with three distributions, exponential, Weibull and Gamma as special cases. Chaturvedi and Malhotra (2018) developed estimation procedures for the reliability characteristics of this family of distributions.

In this chapter, we generalize the positive exponential family of distributions proposed by Liang (2008). The family covers as many as ten distributions to be particular cases. We discuss some important properties and derive UMVU estimators, ML estimators and MM estimators of the reliability characteristics. The chapter is structured as follows:

In Section 5.2, we propose the generalized exponential family of distributions and study its properties. In Section 5.3, we derive UMVU estimators, ML estimators and MM estimators of the q^{th} power of unknown parameter θ, when other parameters α, β and ν are known. In Section 5.4, we derive ML estimators when all the parameters are unknown. In Section 5.5, we present simulation studies and in Section 5.6, we present two examples of real data. Finally, in Section 5.7, we give the concluding remarks.

5.2 The Generalized Positive Exponential Family of Distributions and it's properties

A random variable X is said to follow the generalized positive exponential family of distributions (GPEFD) if its probability density function is given by

$$f\left(x;\alpha,\beta,\nu,\theta\right)=\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma(\nu)}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right);x>0,\alpha,\beta,\nu,\theta>0. \quad (5.2.1)$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x)=\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}, \quad (5.2.2)$$

where $\gamma(x,a)=\int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

We note that for given β, this family covers the following distributions as special cases:

1. For $\alpha = \nu = \beta = 1$, we get one parameter exponential distribution (Johnson and Kotz, 1970, pp. 197).

2. For $\alpha = \beta = 1$, it gives a gamma distribution. Further, for integral values of α, it gives an Erlang distribution (Johnson and Kotz, 1970, pp. 166).

3. For $\beta = 1$, it leads to a generalized gamma distribution (Johnson and Kotz, 1970, pp. 197).

4. For $\beta = \nu = 1$, it turns out to be a Weibull distribution (Johnson and Kotz, 1970, pp. 250).

5. For $\nu = \frac{1}{2}, \beta = 1, \alpha = 2$, it is known as half normal distribution (Davis, 1952).

6. For $\nu = \frac{m}{2}$, $\alpha = 2$, $\beta = \frac{1}{2}, m > 0$ we get a chi distribution (Patel *et al.*, 1976, pp. 173) and for $m = 3$ we get a Maxwell distribution (Tyagi and Bhattacharya, 1989).

7. For $\alpha = 2$, $\nu = 1$, $\beta = 1$, we get a Rayleigh distribution (Sinha, 1986, pp. 200).

8. For $\alpha = 2$, $\beta = 1$, $\nu = k+1$; $k \geq 0$ we get a generalized Rayleigh distribution of Voda (1978).

9. For $\nu = \beta$ and $\alpha = 2$, $\nu > o$, $\beta > 0$ we get the Nakagami (1960) distribution.

10. For $\beta = 1$, we get the positive exponential family of distribution (Liang, 2008 and Chaturvedi and Malhotra, 2018).

Figure 5.1 shows some of the possible shapes of pdfs covered by this family for different values of parameters.
From these graphs, we conclude that:

1. For $\alpha = \beta = 1$, $\nu \leq 1$ and different values of θ, it has decreasing shape.

2. For $\alpha = \beta = 1$, $\nu > 1$ and different values of θ , the pdf is either unimodal or upside down bathtub shaped.

3. For $\beta = \nu = 1$, $\alpha \leq 1$ and different values of θ it has decreasing shape.

4. For $\beta = \nu = 1$, $\alpha > 1$ and different values of θ, the pdf is either unimodal or upside down bathtub shaped.

5. For $\alpha = \nu = 1$ and all values of β and θ it has decreasing shape.

The hazard function and survival function of this family of distributions are given by

$$\begin{aligned} h(x;\alpha,\beta,\nu,\theta) &= \frac{f_X(x)}{1-F_X(x)} \\ &= \frac{\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma\nu}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right)}{1-\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}} \end{aligned}$$

and

$$\begin{aligned} s(x;\alpha,\beta,\nu,\theta) &= 1-F_X(x), \\ &= 1-\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}, \end{aligned}$$

respectively. The hazard function can take different shapes like increasing, decreasing, unimodal and upside-down bathtub shapes. Figure 5.2 provides possible shapes of hazard functions of this family of distributions for different values of α, β, ν and θ.

We have made few observations graphically, which are stated below:

1. For $\alpha = \beta = \nu = 1$ and different values of θ, we get constant hazard rate.

2. For $\alpha = \beta = 1$, $\nu < 1$ and different values of θ it has decreasing shape.

3. For $\alpha = \beta = 1$, $\nu > 1$ and different values of θ , the hazard rate is either unimodal or upside down bathtub shaped.

4. For $\beta = \nu = 1$, $\alpha < 1$ and different values of θ it has decreasing shape.

5. For $\beta = \nu = 1$, $\alpha > 1$ and different values of θ, it has increasing shape.

5.2.1 Distribution Properties

Here we discuss some important statistical properties and reliability characteristics of the Generalized Positive Exponential Family of Distributions.

1. **Moments**

   The rth raw moment (about the origin) of this family of distributions is given by
   $$\mu_r^{'} = \left(\frac{\theta}{\beta}\right)^{r/\alpha} \frac{1}{\Gamma(\nu)} \Gamma\left(\nu + \frac{r}{\alpha}\right).$$
   In particular, mean and variance of $GPEFD(\alpha, \beta, \nu, \theta)$ are respectively given by
   $$\mu = \left(\frac{\theta}{\beta}\right)^{1/\alpha} \frac{\Gamma\left(\nu + \frac{1}{\alpha}\right)}{\Gamma(\nu)},$$
   and
   $$\sigma^2 = \left(\frac{\theta}{\beta}\right)^{2/\alpha} \frac{1}{\Gamma(\nu)} \left[\Gamma\left(\nu + \frac{2}{\alpha}\right) - \frac{[\left(\Gamma\left(\nu + \frac{1}{\alpha}\right)\right)^2}{\Gamma(\nu)}\right].$$

2. **Mode**

   Mode is the value of x for which $f(x)$ is maximum. The mode of the distribution is given by
   $$Mode = \left(\frac{\theta}{\beta} \frac{(\alpha\nu - 1)}{\alpha}\right)^{\left(\frac{1}{\alpha}\right)}.$$

3. **Median**

   Median is the solution of the following equation:
   $$F(Md) = 0.5$$
   $$\Rightarrow \gamma\left(\nu, \frac{\beta(Md)^\alpha}{\theta}\right) - 0.5 = 0.$$

4. **Quantiles**

   The q^{th} quantile x_q can be obtained by solving the equation
   $$q = F_X(x_q; \theta)$$
   $$\Longrightarrow q = \frac{\gamma\left(\nu, \frac{\beta x_q^\alpha}{\theta}\right)}{\Gamma(\nu)}$$
   $$\Longrightarrow x_q = \left(\frac{\theta}{\beta} \mathcal{G}^{-1}(\nu, q)\right)^{\frac{1}{\alpha}},$$

where $\mathcal{G}^{-1}(\nu, q)$ is an inverse gamma regularized function and can be approximated by using the following series expansion

$$\mathcal{G}^{-1}(a,z) = (-(z-1)\Gamma(a+1))^{1/a} + \frac{\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^2}{a+1} + \frac{(3a+5)\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^3}{2(a+1)^2(a+2)} + \mathcal{O}((z-1)^{4/a}),$$

where $\mathcal{O}(\cdot)$ represents higher order terms.

5. **Coefficients of Skewness and Kurtosis**

The coefficients of skewness (β_1) and kurtosis (β_2) can now be obtained as

$$\beta_1 = \frac{\left[\Gamma\left(\nu+\frac{3}{\alpha}\right)(\Gamma(\nu))^2 - 3\Gamma\left(\nu+\frac{2}{\alpha}\right)\Gamma\left(\nu+\frac{1}{\alpha}\right)\Gamma(\nu) + 2\left(\Gamma\left(\nu+\frac{1}{\alpha}\right)\right)^3\right]^2}{\left[\Gamma\left(\nu+\frac{2}{\alpha}\right)\Gamma(\nu) - \left(\Gamma\left(\nu+\frac{1}{\alpha}\right)\right)^2\right]^3}$$

and

$$\beta_2 = \frac{(\Gamma(\nu))^3\Gamma\left(\nu+\frac{4}{\alpha}\right) - 4(\Gamma(\nu))^2\Gamma\left(\nu+\frac{1}{\alpha}\right)\Gamma\left(\nu+\frac{3}{\alpha}\right) + 6\Gamma(\nu)\Gamma\left(\nu+\frac{2}{\alpha}\right)\left(\Gamma\left(\nu+\frac{1}{\alpha}\right)\right)^2 - 3\left(\Gamma\left(\nu+\frac{1}{\alpha}\right)\right)^4}{\left[\Gamma(\nu)\Gamma\left(\nu+\frac{2}{\alpha}\right) - \left(\Gamma\left(\nu+\frac{2}{\alpha}\right)\right)^2\right]^2},$$

respectively. We can note that β_1 and β_2 are functions of just ν and α and are independent of θ and β. From Figure 5.3, we observe that GPEFD is positively skewed as $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} > 0$ for different values of α and ν. Figure 5.4(a) shows that for $\nu \leq 1$ and $\alpha > 1$, the family exhibits the shapes higher than normal curve as $\beta_2 > 3$, i.e., the family is leptokurtic. However, for a fixed value of ν, $\nu > 1$, [See Figure 5.4(b)] the value of β_2 is either greater than or less than 3 and tends to 3 as value of α increases, i.e., the family becomes mesokurtic as the value of α increases for fixed value of ν.

6. **Moment Generating Function**

The moment generating function of random variable X is given by

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\frac{\theta}{\beta}\right)^{j/\alpha}\frac{\Gamma\left(\nu+\frac{j}{\alpha}\right)}{\Gamma(\nu)}.$$

Consequently, the characteristic function $\phi_X(t) = E(e^{tX})$ is given by

$$\phi_X(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}\left(\frac{\theta}{\beta}\right)^{j/\alpha}\frac{\Gamma\left(\nu+\frac{j}{\alpha}\right)}{\Gamma(\nu)}.$$

7. **Conditional Moment and Moment Generating Function**

   Let X be a random variable following the WGPEFD, then the conditional moment $E(X^r|X > t)$ and the conditional moment generating function $E(e^{tX}|X > x_\circ)$ are respectively given by

$$E(X^r|X > t) = \frac{\left(\frac{\theta}{\beta}\right)^{r/\alpha}\left[\Gamma\left(\nu + \frac{r}{\alpha}\right) - \gamma\left(\nu, \frac{\beta t^\alpha}{\theta}\right)\right]}{\Gamma(\nu) - \gamma\left(\nu, \frac{\beta t^\alpha}{\theta}\right)},$$

   and

$$E(e^{tX}|X > x_\circ) = \frac{\sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{\theta}{\beta}\right)^{i/\alpha}\left[\Gamma\left(\nu + \frac{i}{\alpha}\right) - \gamma\left(\nu, \frac{\beta x_\circ^\alpha}{\theta}\right)\right]}{\Gamma(\nu) - \gamma\left(\nu, \frac{\beta x_\circ^\alpha}{\theta}\right)}.$$

8. **Stochastic Ordering**

   A random variable X is said to be stochastically greater than Y, i.e., $Y \leq_{st} X$, if, $F_Y(t) \geq F_X(t)$ for all t. Further, X is said to be greater than Y in the

   (a) hazard rate order, $Y \leq_{hr} X$, if $h_Y(t) \geq h_X(t)$ for all t.

   (b) mean residual life order, $Y \leq_{mrl} X$ if $m_Y(t) \geq m_X(t)$ for all t.

   (c) likelihood ratio order, $Y \leq_{lr} X$ if $\frac{f_X(t)}{f_Y(t)}$ decreases in t.

   Shaked and Shantikumar (1994) gave a result regarding stochastic ordering, which shows that the existence of likelihood ratio ordering implies the existence of all the orderings mentioned above.

   Let $X \sim GPEFD(\alpha_1, \beta_1, \nu_1, \theta_1)$ and $Y \sim GPEFD(\alpha_2, \beta_2, \nu_2, \theta_2)$. Then, likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1}{\alpha_2}\left(\frac{\beta_1}{\beta_2}\right)\left(\frac{\theta_2}{\theta_1}\right)\left(\frac{\Gamma(\nu_2)}{\Gamma(\nu_1)}\right)x^{\alpha_1\nu_1-\alpha_2\nu_2}exp\left[-\left(\frac{\beta_1\theta_2 x^{\alpha_1} - \beta_2\theta_1 x^{\alpha_2}}{\theta_1\theta_2}\right)\right]$$

$$\implies \frac{d}{dx}\frac{f_X(x)}{f_Y(x)} = \frac{f_X(x)}{f_Y(x)}\left[\frac{\alpha_1\nu_1 - \alpha_2\nu_2}{x} - \frac{\alpha_1\beta_1\theta_2 x^{\alpha_1-1} - \alpha_2\beta_2\theta_1 x^{\alpha_2-1}}{\theta_1\theta_2}\right]. \tag{5.2.3}$$

   From (5.2.3), we can observe that $\frac{d}{dx}\frac{f_X(x)}{f_Y(x)}$ is decreasing in x, if $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$ and $\theta_2 < \theta_1$, $\forall x$, $0 < \alpha_1, \alpha_2 < 1$. Hence, $Y \leq_{lr} X$ when $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$, $\theta_2 < \theta_1$ and $0 < \alpha_1, \alpha_2 < 1$. Therefore

$$\begin{array}{c}(Y \leq_{lr} X) \implies (Y \leq_{hr} X) \implies (Y \leq_{mrl} X)\\ \Downarrow \\ (Y \leq_{st} X).\end{array}$$

9. **Mean time to system failure**

   MTSF for this family of distributions is given by

$$MTSF = \frac{\left(\frac{\theta}{\beta}\right)^{\left(\frac{1}{\alpha}\right)}}{\Gamma(\nu)}\Gamma\left(\frac{1}{\alpha}+\nu\right).$$

10. **Mean Residual Life**

    The MRL is given by

$$\mu(t) = \frac{\left[\int_t^\infty 1 - \frac{\gamma\left(\nu, \frac{\beta u^\alpha}{\theta}\right)}{\Gamma(\nu)}\right] du}{\left[1 - \frac{\gamma\left(\nu, \frac{\beta t^\alpha}{\theta}\right)}{\Gamma(\nu)}\right]}.$$

5.3 UMVU, ML and MM Estimators of $\theta^q, R(t)$ and P when α, β and ν are known

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution given in (5.2.1). Then, assuming ν, β and α to be known, the likelihood function of the parameter θ given the sample observations $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ is:

$$L(\theta \mid \underset{\sim}{x}) = \left(\frac{\alpha}{\Gamma(\nu)}\right)^n \left(\frac{\beta}{\theta}\right)^{n\nu} e^{-\frac{\beta}{\theta}\sum_{i=1}^{n} x_i^\alpha} \prod_{i=1}^{n} x_i^{\alpha\nu-1}. \tag{5.3.1}$$

The following theorem provides UMVU estimator of powers of θ.

Theorem 5.3.1. *For $q \in (-\infty, \infty)$, the UMVU estimator of θ^q is given by:*

$$\widetilde{\theta}^q = \begin{cases} \left\{\dfrac{\Gamma(n\nu)}{\Gamma(n\nu+q)}\right\} S^q; & n\nu + q > 0 \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. It follows from (5.3.1) and factorization theorem [see Rohtagi and Saleh, 2012, pp. 367] that S is sufficient statistic for θ and the pdf of S is

$$f_s(s \mid \theta) = \frac{s^{n\nu-1}}{\Gamma(n\nu)\theta^{n\nu}} \exp\left(-\frac{s}{\theta}\right); \quad \nu > 0,\ \theta > 0,\ s \geq 0. \tag{5.3.2}$$

From (5.3.1), since the distribution of S belongs to the exponential family, it is also complete. Now it follows from (5.3.2) that

$$E[S^q] = \left\{\frac{\Gamma(n\nu + q)}{\Gamma(n\nu)}\right\}\theta^q, \tag{5.3.3}$$

and the theorem follows.

The following theorem provides the UMVU estimator of the sampled pdf.

Theorem 5.3.2. *At a given point x, the UMVU estimator of the sampled pdf is given by*

$$\widetilde{f}(x;\theta) = \begin{cases} \frac{\alpha}{\beta(\nu,(n-1)\nu)}\left(\frac{\beta}{S}\right)^{\nu} x^{\alpha\nu-1}\left[1 - \frac{\beta x^{\alpha}}{S}\right]^{(n-1)\nu-1}; & \beta x^{\alpha} < S \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. We can write

$$f(x;\theta) = \alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma(\nu)}x^{\alpha\nu-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\beta x^{\alpha}}{\theta}\right)^i.$$

Applying Theorem 5.3.1,

$$\begin{aligned}\widetilde{f}(x;\theta) &= \frac{\alpha\beta^{\nu}x^{\alpha\nu-1}}{\Gamma(\nu)}\sum_{i=0}^{\infty}\frac{(-1)^i\,(\beta x^{\alpha})^i}{i!}(\widetilde{\theta})^{(-(\nu+i))} \\ &= \frac{\alpha\left(\frac{\beta}{S}\right)^{\nu}x^{(\alpha\nu-1)}}{\beta\left(\nu,(n-1)\nu\right)}\sum_{i=0}^{(n-1)\nu-1}(-1)^i\binom{(n-1)\nu-1}{i}\left(\frac{\beta x^{\alpha}}{S}\right)^i\end{aligned}$$

and the result follows.

The following theorem provides UMVU estimator of the reliability function $R(t)$.

Theorem 5.3.3. *The UMVU estimator of $R(t)$ is:*

$$\widetilde{R}(t) = \begin{cases} 1 - I_{\frac{\beta t^{\alpha}}{S}}(\nu,(n-1)\nu); & \beta t^{\alpha} < S \\ 0; & \textit{otherwise,} \end{cases}$$

where $I_x(p,q) = \frac{1}{\beta(p,q)}\int_0^x y^{p-1}(1-y)^{q-1}dy;\ 0 \leq y \leq 1,\ x < 1,\ p,q > 0$ *is the incomplete beta function.*

Proof. We note that the expectation of $\int_t^{\infty} \widetilde{f}(x;\theta)dx$ with respect to S is $R(t)$. Thus, applying Theorem 5.3.2,

$$\begin{aligned}\widetilde{R}(t) &= \int_t^{\infty} \widetilde{f}(x;\theta)dx \\ &= \frac{\alpha}{\beta(\nu,(n-1)\nu)}\left(\frac{\beta}{S}\right)^{\nu}\int_t^{\infty} x^{\alpha\nu-1}\left[1-\frac{\beta x^{\alpha}}{S}\right]^{(n-1)\nu-1} dx,\end{aligned}$$

and the result follows by substituting $\frac{\beta x^{\alpha}}{S} = z$.

Let X and Y be two independent random variables with respective pdf:

$$f\left(x;\theta_1\right) = \alpha_1\left(\frac{\beta_1}{\theta_1}\right)^{\nu_1}\frac{1}{\Gamma\nu_1}x^{\alpha_1\nu_1-1}\exp\left(\frac{-\beta_1 x^{\alpha_1}}{\theta_1}\right)$$

and

$$f\left(y;\theta_2\right) = \alpha_2\left(\frac{\beta_2}{\theta_2}\right)^{\nu_2}\frac{1}{\Gamma\nu_2}y^{\alpha_2\nu_2-1}\exp\left(\frac{-\beta_2 y^{\alpha_2}}{\theta_2}\right).$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $f(x;\theta_1)$ and $Y_1, Y_2, \ldots, Y_m$ be a random sample of size m from $f(y;\theta_2)$. Define, $S = \sum_{i=1}^{n}\beta_1 X_i^{\alpha_1}$ and $T = \sum_{i=1}^{m}\beta_2 Y_i^{\alpha_2}$. Now the UMVU estimator of P is given in the following theorem.

Theorem 5.3.4. *The UMVU estimator of P is*

$$\tilde{P} = \begin{cases} \int_{z=0}^{1}\frac{1}{\beta\{\nu_1,(n-1)\nu_1\}}z^{\nu_1-1}\left(1-z\right)^{(n-1)\nu_1-1} I_{\left\{\frac{\beta_2\left(\frac{Sz}{\beta_1}\right)^{\left(\frac{\alpha_2}{\alpha_1}\right)}}{T}\right\}}\left(\nu_2,(m-1)\nu_2\right) \\ \text{where}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} \leq \left(\frac{T}{\beta_2}\right)^{1/\alpha_2} \\ 1-\frac{1}{\beta\{\nu_2,(m-1)\nu_2\}\beta\{\nu_1,(n-1)\nu_1\}}\int_{z=0}^{1}z^{\nu_2-1}\left(1-z\right)^{(m-1)\nu_2-1}\int_{w=0}^{(zT/\beta_2)^{\alpha_1/\alpha_2}} w^{\nu_1-1} \\ \times\left(1-w\right)^{(n-1)\nu_1-1}dw \\ \text{where}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} > \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}. \end{cases}$$

Proof. It follows from Theorem 5.3.2 that

$$\widetilde{f}(x;\theta_1) = \begin{cases} \frac{\alpha_1}{\beta(\nu_1,(n-1)\nu_1)}\left(\frac{\beta_1}{S}\right)^{\nu_1}x^{\alpha_1\nu_1-1}\left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1} & ;\beta_1 x^{\alpha_1} < S, \\ 0 & ;otherwise. \end{cases}$$

and

$$\widetilde{f}(y;\theta_2)=\begin{cases}\frac{\alpha_2}{\beta(\nu_2,(n-1)\nu_2)}\left(\frac{\beta_2}{T}\right)^{\nu_2}y^{\alpha_2\nu_2-1}\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1} & ;\beta_2 y^{\alpha_2}<T,\\ 0 & ;otherwise.\end{cases}$$

From the arguments similar to those used in Theorem 5.3.3,

$$\begin{aligned}\widetilde{P}&=\int_{x=0}^{\infty}\int_{y=0}^{x}\widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\ dy\\ &=\int_{x=0}^{\min\left(\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}},\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\right)}\frac{\alpha_1 x^{\alpha_1\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}\left(\frac{\beta_1}{S}\right)^{\nu_1}\left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1}\\ &\quad\times I_{\frac{\beta_2 x^{\alpha_2}}{T}}(\nu_2,(m-1)\nu_2)dx. \end{aligned}\tag{5.3.4}$$

When $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}\leq\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, the UMVU estimator of P is given by

$$\widetilde{P}=\int_{x=0}^{\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}}\frac{\alpha_1 x^{\alpha_1\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}\left(\frac{\beta_1}{S}\right)^{\nu_1}\left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1}\times \tag{5.3.5}$$

$$I_{\frac{\beta_2 x^{\alpha_2}}{T}}(\nu_2,(m-1)\nu_2)dx, \tag{5.3.6}$$

and the first assertion follows by substituting $\frac{\beta_1 x^{\alpha_1}}{S}=z$.
When $\left(\frac{S}{\beta_1}\right)^{1/\alpha_1}>\left(\frac{T}{\beta_2}\right)^{1/\alpha_2}$, the UMVU estimator of P is given by

$$\begin{aligned}\widetilde{P}&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\ dy\\ &=\int_{y=0}^{\infty}\widetilde{R}(y;\theta_1)\widetilde{f}(y;\theta_2)dy\end{aligned}$$

$$\widetilde{P}=\int_{y=0}^{\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}}\frac{\alpha_2}{\beta\left(\nu_2,(n-1)\nu_2\right)}\left(\frac{\beta_2}{T}\right)^{\nu_2}y^{\alpha_2\nu_2-1}\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1}\times \left[1-I_{\frac{\beta_1 y^{\alpha_1}}{S}}\left(\nu_1,(n-1)\nu_1\right)\right]\ dy, \tag{5.3.7}$$

and the second assertion follows by substituting $\frac{\beta_2 y^{\alpha_2}}{T}=z$.

Corollary 5.3.4.1. *The UMVU estimator of P when the shape parameters ν_1 and*

ν_2 *are integers is:*

$$\widetilde{P} = \begin{cases} \frac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)} \sum_{i=0}^{(m-1)\nu_2-1} \frac{(-1)^i}{\nu_2+i}\binom{(m-1)\nu_2-1}{i} \\ \cdot \int_0^1 z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}\left[\frac{\beta_2\left(\frac{zS}{\beta_1}\right)^{\frac{\alpha_2}{\alpha_1}}}{T}\right]^{\nu_2+i} dz; \qquad \left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} \le \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}} \\ \\ 1-\frac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)} \sum_{i=0}^{(n-1)\nu_1-1} \frac{(-1)^i}{\nu_1+i}\binom{(n-1)\nu_1-1}{i} \\ \cdot \int_0^1 z^{\nu_2-1}(1-z)^{(m-1)\nu_2-1}\left[\frac{\beta_1\left(\frac{zT}{\beta_2}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right]^{\nu_1+i} dz; \qquad \left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} > \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}. \end{cases}$$

Proof. From Theorem 5.3.4, for $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} \le \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$,

$$\widetilde{P} = \int_{z=0}^{1} \frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)} \times \int_{w=0}^{\frac{\beta_2}{T}\left(\frac{zS}{\beta_1}\right)^{\frac{\alpha_2}{\alpha_1}}} w^{\nu_2-1}(1-w)^{(m-1)\nu_2-1} dw dz,$$

and the first assertion follows by Binomial expansion of $(1-w)^{(m-1)\nu_2-1}$. For $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} > \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, we consider

$$\begin{aligned} \widetilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\ dy \\ &= \int_{y=0}^{\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}} \frac{\alpha_2 y^{\alpha_2\nu_2-1}}{\beta(\nu_2,(m-1)\nu_2)}\left(\frac{\beta_2}{T}\right)^{\nu_2}\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1} \times \\ &\quad [1-I_{\frac{\beta_1 y^{\alpha_1}}{S}}(\nu_1,(n-1)\nu_1)]dy, \end{aligned}$$

and the second assertion follows on substituting $\frac{\beta_2 y^{\alpha_2}}{T} = z$.

It is interesting to note that on putting $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$, we get the UMVU estimator of $P(X > Y)$ obtained by Constantine *et al.* (1986). Hence, we were able to obtain another generalized expression of UMVU estimator of $P(X > Y)$ by a different yet simpler approach when the shape parameters ν_1

and ν_2 are assumed to be integers.

Now, we provide ML estimator of $R(t)$ in the following theorem:

Theorem 5.3.5. *The ML estimator of $R(t)$ is given by:*

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\nu, \frac{n\nu\beta t^{\alpha}}{S}\right)}{\Gamma(\nu)},$$

where $\gamma(a,r) = \int_0^r y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

Proof. It can be easily seen from (5.3.1) that the ML estimator of θ^q is $\widehat{\theta^q} = \left(\frac{S}{n\nu}\right)^q$, where $S = \beta\Sigma {x_i}^{\alpha}$. Now from the invariance property of ML estimators, the ML estimator of sampled pdf is:

$$\widehat{f}(x;\theta) = \frac{\alpha x^{\alpha\nu-1}}{\Gamma(\nu)}\left(\frac{n\nu\beta}{S}\right)^{\nu}\exp\left\{\frac{-n\nu\beta x^{\alpha}}{S}\right\}.$$

Thus, $\widehat{R}(t) = \int_t^{\infty}\widehat{f}(x;\theta)dx$ and the theorem follows.

The ML estimator of P is given in the following theorem:

Theorem 5.3.6. *The ML estimator of P is:*

$$\widehat{P} = 1 - \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{z=0}^{\infty} z^{\nu_2-1}e^{-z}\gamma\left(\nu_1, \frac{n\nu_1\beta_1\left(\frac{zT}{m\nu_2\beta_2}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right)dz.$$

Proof. We have,

$$\begin{aligned}\widehat{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty}\widehat{f}(x;\theta_1)\widehat{f}(y;\theta_2)dx\, dy\\ &= \int_{y=0}^{\infty}\widehat{R}_X(y)\widehat{f}(y;\theta_2)dy\\ &= \int_{y=0}^{\infty}\left[1 - \frac{\gamma\left(\nu_1, \frac{n\nu_1\beta_1 y^{\alpha_1}}{S}\right)}{\Gamma(\nu_1)}\right]\frac{\alpha_2 y^{\alpha_2\nu_2-1}}{\Gamma(\nu_2)}\left(\frac{m\nu_2\beta_2}{T}\right)^{\nu_2}\exp\left\{\frac{-m\nu_2\beta_2 y^{\alpha_2}}{T}\right\}dy,\end{aligned}$$

and the theorem follows on substituting $\frac{m\nu_2\beta_2 y^{\alpha_2}}{T} = z$.

Next, we derive the MM estimator of θ.

Theorem 5.3.7. *The MM estimator of θ^q is given by*

$$\widehat{\theta^q_M} = \left(\left(\frac{\Gamma(\nu)}{\Gamma\left(\frac{1}{\alpha}+\nu\right)}\right)^{\alpha}\overline{X}^{\alpha}\beta\right)^{q}.$$

Proof. From (5.2.1), we obtain the rth moment as:

$$E(X^r) = \int_0^\infty \alpha \left(\frac{\beta}{\theta}\right)^\nu \frac{1}{\Gamma\nu} x^{r+\alpha\nu-1} \exp\left(\frac{-\beta x^\alpha}{\theta}\right) dx. \tag{5.3.8}$$

On substituting $\frac{\beta x^\alpha}{\theta} = y$ in (5.3.7) and making necessary transformations we obtain

$$E(X^r) = \frac{\left(\frac{\theta}{\beta}\right)^{\frac{r}{\alpha}}}{\Gamma(\nu)} a_r, \tag{5.3.9}$$

where $a_r = \Gamma\left(\frac{r}{\alpha} + \nu\right)$.

For $r = 1$ and denoting $E(X^r)$ by $\overline{X}^r$, we obtain the following equation:

$$\Gamma(\nu)\overline{X} - a_1 \left(\frac{\theta}{\beta}\right)^{\frac{1}{\alpha}} = 0 \tag{5.3.10}$$

and hence the theorem follows.

5.4 ML Estimators when all the parameters are unknown

Now we discuss the case when the three parameters α, ν and θ are unknown. For ML estimators, the log-likelihood function of the parameters α, ν and θ given the sample observations $\underset{\sim}{x}$ and β is:

$$l(\alpha, \nu, \theta \mid \underset{\sim}{x}, \beta) = n\log(\alpha) - n\log(\Gamma(\nu)) + n\nu\log(\beta) - n\nu\log(\theta)$$
$$- \frac{\beta}{\theta}\sum_{i=1}^{n} x_i^\alpha + (\alpha\nu - 1)\sum_{i=1}^{n}\log(x_i).$$

The ML estimators of α, ν and θ are given by the simultaneous solution of the following three equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \frac{\beta}{\theta}\sum_{i=1}^{n} x_i^\alpha \log(x_i) + \nu\sum_{i=1}^{n}\log(x_i) = 0 \tag{5.4.1}$$

$$\frac{\partial l}{\partial \nu} = \frac{-n}{\Gamma(\nu)}\frac{d\Gamma(\nu)}{d\nu} - n\log(\theta) + n\log(\beta) + \alpha\sum_{i=1}^{n}\log(x_i) = 0 \tag{5.4.2}$$

$$\frac{\partial l}{\partial \theta} = \frac{-n\nu}{\theta} + \frac{\beta\sum\limits_{i=1}^{n} x_i^\alpha}{\theta^2} = 0 \tag{5.4.3}$$

Since these non-linear equations don't have a closed form solution, therefore we apply the Newton Raphson algorithm to compute ML estimators of α, ν and θ.

Further, using the invariance property of ML estimators, the ML estimator of $R(t)$ is given as:

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\widehat{\nu}, \frac{\beta x^{\widehat{\alpha}}}{\widehat{\theta}}\right)}{\Gamma\left(\widehat{\nu}\right)}, \tag{5.4.4}$$

where $\widehat{\alpha}$, $\widehat{\nu}$ and $\widehat{\theta}$ are the ML estimators of α, ν and θ respectively and the ML estimator of P is given as:

$$\widehat{P} = 1 - \frac{1}{\Gamma\left(\widehat{\nu}_1\right)\Gamma\left(\widehat{\nu}_2\right)} \int_{z=0}^{\infty} z^{\widehat{\nu}_2 - 1} e^{-z} \times \gamma\left(\widehat{\nu}_1, \frac{\beta_1}{\widehat{\theta}_1}\left(\frac{\widehat{\theta}_2 z}{\beta_2}\right)^{\frac{\widehat{\alpha}_1}{\widehat{\alpha}_2}}\right) dz. \tag{5.4.5}$$

5.5 Simulation Studies

Firstly, we conduct Monte Carlo simulation studies to compare the performance of $\widetilde{\theta}^q$, $\widehat{\theta_M{}^q}$ and $\widehat{\theta}^q$ for different sample sizes and powers of parameter θ. For $\alpha = 3$ and $\beta = \nu = 2$, we generate 10,000 samples each of size n from (5.2.1) and repeat this procedure for several values of θ.

Figure 5.5 shows the mean square error (MSE) of the UMVU estimator, MM estimator and ML estimator of θ^q. From these figures, we note that for smaller sample sizes and for $q = 2$, the ML estimator performs the best and the MM estimator performs the worst. The performance of UMVU estimator is in between the two. As the sample size increases the three curves come closer to each other.

On similar lines, we perform simulation studies to compare the performance of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different sample sizes. For $t = 7$ and $\alpha = \beta = \nu = 2$, we generate 10,000 samples each of size n from the generalization of positive exponential family of distributions and repeat this procedure for several values of $R(t)$. Figure 5.6 shows the MSE of the UMVU estimator and ML estimator of $R(t)$. From these figures, we note that the MSE of the UMVU estimator of $R(t)$ is always greater than that of the ML estimator, however for large sample sizes these estimators of $R(t)$ are better and almost equally efficient.

Now, we compare the performance of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. By Monte Carlo simulation, for $\alpha_1 = 1$ and $\beta_1 = \nu_1 = 2$ and $\alpha_2 = 2$ and $\beta_2 = \nu_2 = 3$, we generate 10,000 samples each of size n and m from GPEFD and repeat this procedure for several values of P. Figure 5.7 shows the MSE of the UMVU estimator and ML estimator of P. From these figures we note that the MSE of the UMVU estimator of P is always greater than that of the ML estimator, however as sample size increases, the performance of both the estimators improve and both the estimators become almost equally efficient.

Figure 5.8 shows the estimation of pdf in equation (5.2.1) based on ML estimator and UMVU estimators.

5.6 Real Life Data Examples

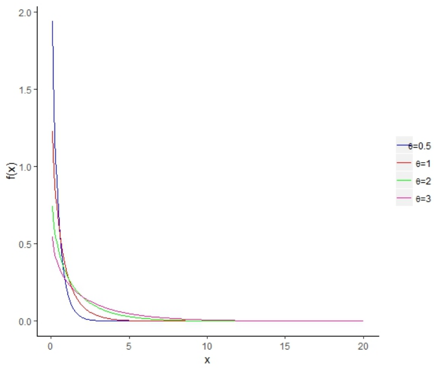
This section deals with the example of real data to illustrate the proposed estimation methods.

This data set was considered by Ghitany *et al.* (2009) [also see Basheer (2019)], for illustrative purpose. The data represent the waiting times (in minutes) before customer service in two different banks. The data sets represents the waiting time (in minutes) before customer service in bank A (Data set I or Population X) and bank B (Data set II or Population Y), respectively.

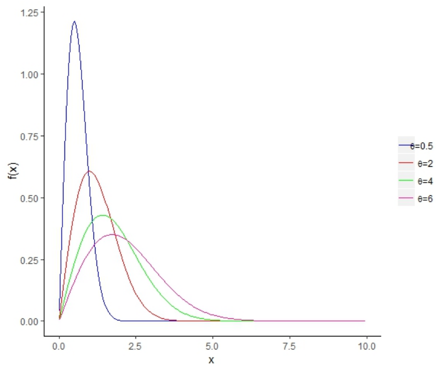
We first apply the KS test to check whether this family fits the given two data sets. Let us assign the random variable $X \sim f(x; \alpha_1, \beta_1, \nu_1, \theta_1)$ to Data set I and the random variable $Y \sim f(y; \alpha_2, \beta_2, \nu_2, \theta_2)$ to Data set II. According to the KS test, we do not reject the null hypothesis that both the data observed for X $(KS = 0.0364; p = 0.9994)$ and $Y(KS = 0.0631; p = 0.9705)$ are drawn from (5.2.1). For these two data sets, the figure demonstrates that (5.2.1) is a good fit. Now, for the above two data sets we obtain various estimators of the parameters, $R(t)$ and P and the results are presented in Tables 5.1 and 5.2. The ML estimator and UMVU estimator of P are given by 0.6532 and 0.6614, respectively.

5.7 Concluding Remarks

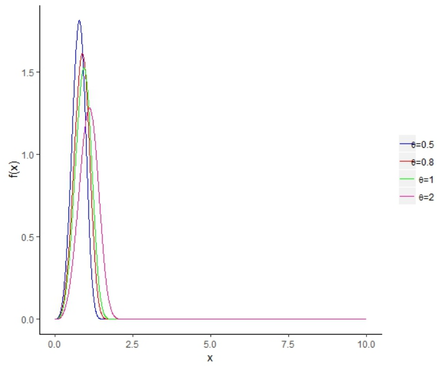
In the present chapter, we have generalized the results of Chaturvedi and Malhotra(2018) to a family of distributions which we name as generalized exponential family of distributions. This family of distribution covers as many as ten distributions as particular cases. Various important statistical properties and reliability characteristics are discussed. UMVU estimators, ML estimators and MM estimators are developed for the powers of parameters, $R(t)$ and P. Efficiency comparison of the three methods of estimation is done.



(a) $\alpha = 1,\ \beta = 1,\ \nu = 0.8,$
$\theta = 0.5, 1, 2, 3$

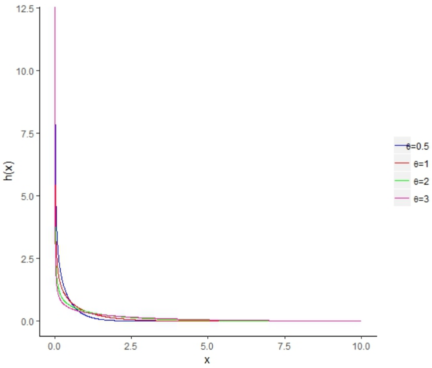


(b) $\alpha = 2,\ \beta = 1,\ \nu = 1,$
$\theta = 0.5, 2, 4, 6$



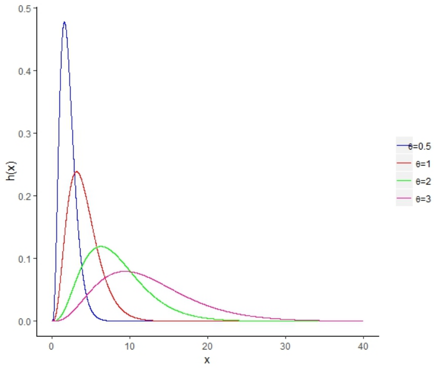
(c) $\alpha = 4, \beta = 1,\ \nu = 1,$
$\theta = 0.5, 0.8, 1, 2$

Figure 5.1: **Probability Density function plots for different values of parameters**



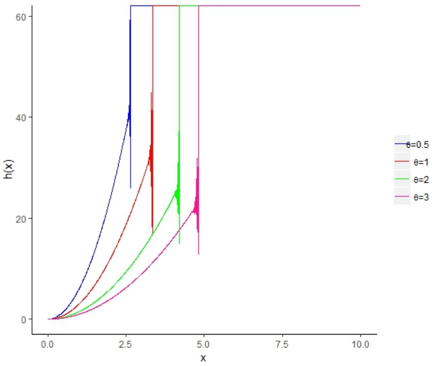
(a) $\alpha = 1,\ \beta = 1,\ \nu = 0.5,$

$\theta = 0.5, 1, 2, 3$



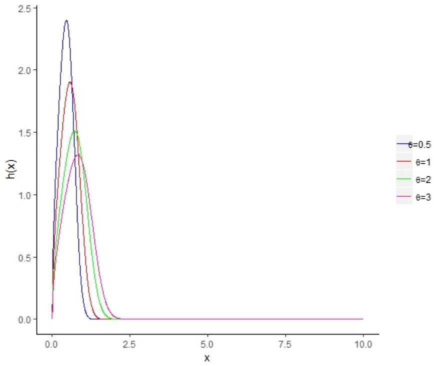
(b) $\alpha = 1,\ \beta = 1,\ \nu = 4,$

$\theta = 0.5, 1, 2, 3$



(c) $\alpha = 3, \beta = 1,\ \nu = 1,$

$\theta = 0.5, 1, 2, 3$



(d) $\alpha = 3, \beta = 2,\ \nu = 0.5,$

$\theta = 0.5, 1, 2, 3$

Figure 5.2: **Hazard function plots for different values of parameters**

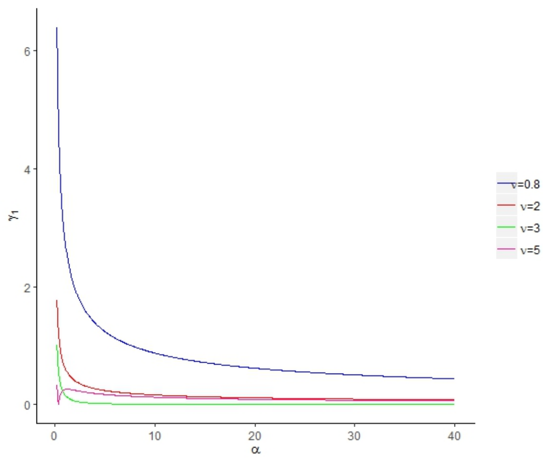
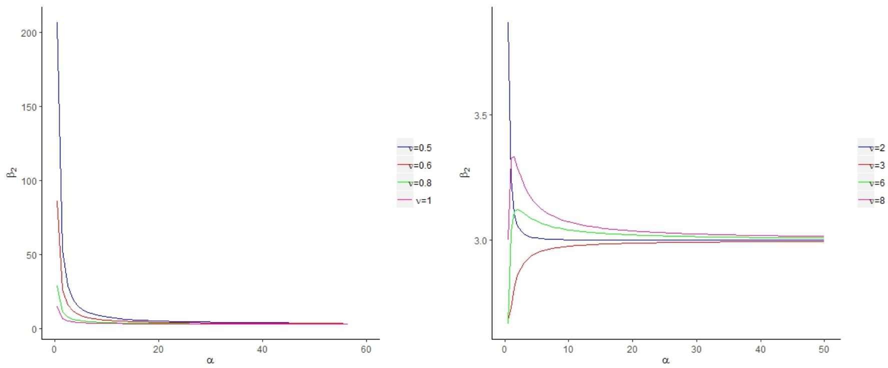


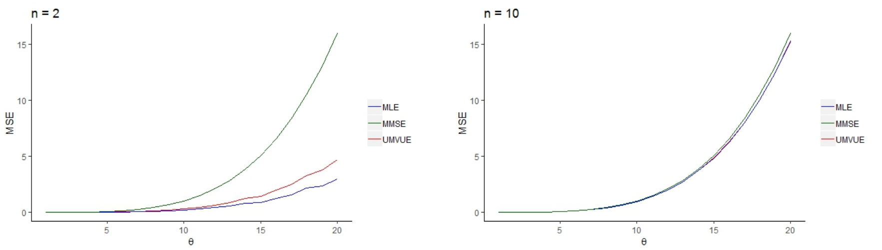
Figure 5.3: **Skewness plot for different values of α and ν**



(a) $\alpha = 1,\ \beta = 1,\ \nu = 0.5,$
$\theta = 0.5, 1, 2, 3$

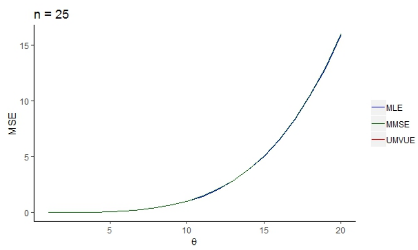
(b) $\alpha = 1,\ \beta = 1,\ \nu = 4,$
$\theta = 0.5, 1, 2, 3$

Figure 5.4: **Kurtosis plots for different values of α and ν**



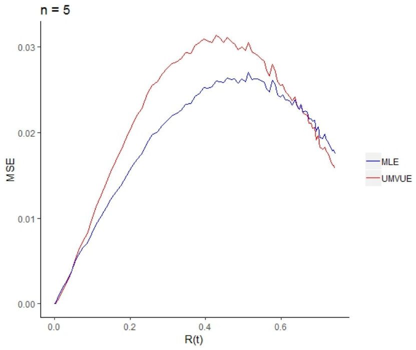
(a) MSE of the UMVU, ML and MM Estimators of θ^q for n=2

(b) MSE of the UMVU, ML and MM Estimator of θ^q for n=10

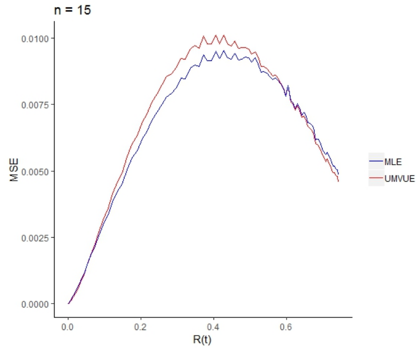


(c) MSE of the UMVU, ML and MM Estimator of θ^q for n=25

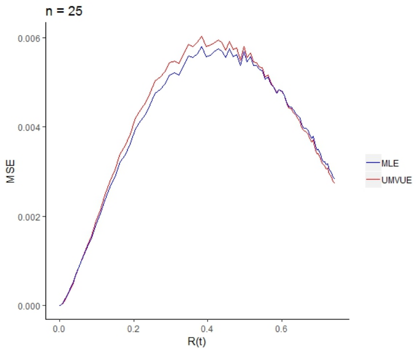
Figure 5.5: **MSE of the UMVU, ML and MM Estimator of θ^q for different sample sizes**



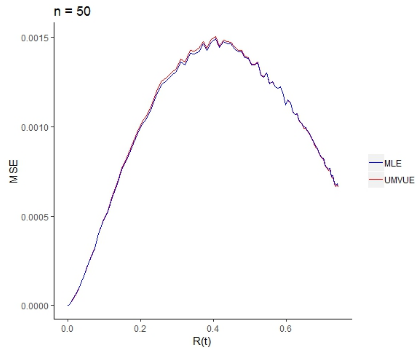
(a) MSE of the UMVU and ML Estimator of $R(t)$ for n=5



(b) MSE of the UMVU and ML Estimator of $R(t)$ for n=15

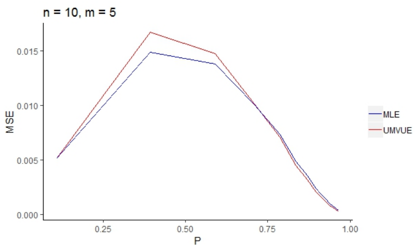


(c) MSE of the UMVU and ML Estimator of $R(t)$ for n=25

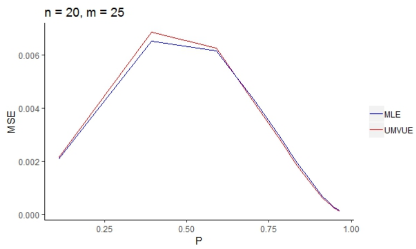


(d) MSE of the UMVU and ML Estimator of $R(t)$ for n=50

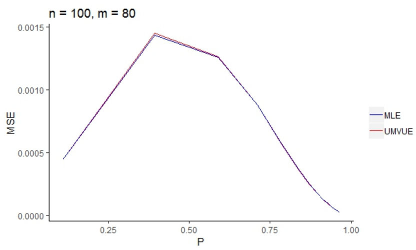
Figure 5.6: **MSE of the UMVU and ML Estimator of $R(t)$ for different sample sizes**



(a) MSE of the UMVU and ML Estimator of P for n=10 and m=5



(b) MSE of the UMVU and ML Estimator of P for n=20 and m=25



(c) MSE of the UMVU and ML Estimator of P for n=100 and m=80

Figure 5.7: **MSE of the UMVU and ML Estimator of P for different sample sizes**

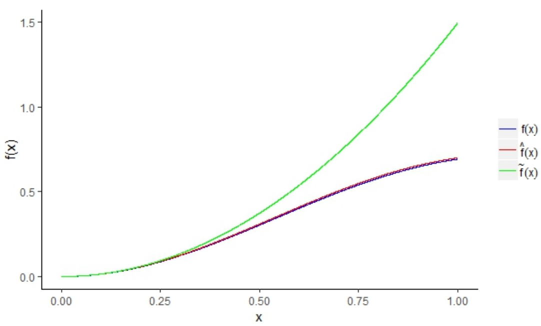


Figure 5.8: **ML and UMVU Estimator of sampled pdf**

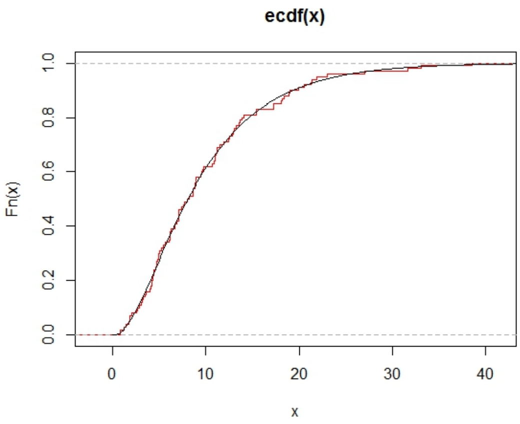


Figure 5.9: **The empirical and theoretical cdf of** $f(x;\alpha_1,\beta_1,\nu_1,\theta_1)$

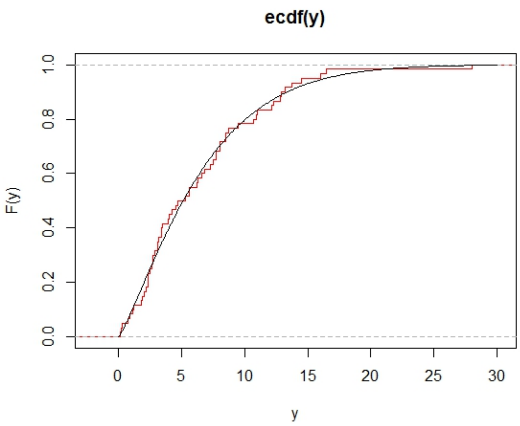


Figure 5.10: **The empirical and theoretical cdf of** $f(y;\alpha_2,\beta_2,\nu_2,\theta_2)$ **model**

Table 5.1: **The ML, UMVU and MM Estimator of parameters of $f(x;\alpha_1,\beta_1,\nu_1,\theta_1)$ and its corresponding reliability function $R_X(t)$ for time $t=2$ based on Data set I**

$\widehat{\alpha}_1$	$\widehat{\beta}_1$	$\widehat{\nu}_1$	$\widehat{\theta}_1$	$\widetilde{\theta}_1$	$\widehat{\theta}_{1M}$	$\widehat{R}_X(t)$	$\widetilde{R}_X(t)$
0.7181	1.2938	3.6514	1.7473	1.7471	1.7472	0.9438	0.9445

Table 5.2: **The ML, UMVU and MM Estimator of parameters of $f(y;\alpha_2,\beta_2,\nu_2,\theta_2)$ and its corresponding reliability function $R_X(t)$ for time $t=2$ based on Data set I**

$\widehat{\alpha}_2$	$\widehat{\beta}_2$	$\widehat{\nu}_2$	$\widehat{\theta}_2$	$\widetilde{\theta}_2$	$\widehat{\theta}_{2M}$	$\widehat{R}_Y(t)$	$\widetilde{R}_Y(t)$
1.3548	0.1546	0.8600	2.5271	2.5272	2.5284	0.8008	0.8034

Chapter 6

Estimation of the Reliability Characteristics of Weighted Generalized Positive Exponential Family of Distributions

6.1 Introduction

The concept of weighted distributions was introduced by Fisher (1934) and first applied by Rao (1965) to model statistical data which standard distributions could not formulate. Moreover, weighted distribution theory gives a unified approach to dealing with model specification and data interpretation problems. Weighted distributions frequently occur in studies related to reliability, survival analysis, analysis of family data, biomedicine, ecology and several other areas.

Suppose X is a non-negative continuous random variable (r.v.) with probability density function (pdf) $f(x)$. The pdf of the weighted r.v. X_w is given by:

$$f_w(x) = \frac{w(x)\, f(x)}{\mu_w}, \;\; x > 0, \qquad (6.1.1)$$

where $w(x)$ is a non-negative weight function and $\mu_w = E[W(X)] < \infty$.

For the weight function $w(x) = x^c$ in equation (6.1.1), the resultant distribution is named size biased distribution. For $c = 1$, the weight function depends on the length of units of interest and the resulting distribution is called length biased distribution while, for $c = 2$, the resulting distribution is called area biased distribution. Gupta and Kundu (2009) introduced a new weighted exponential model which has the pdf whose shape is very close to the shape of Weibull, gamma or generalized exponential distributions and hence can be used as their alternative. Das and Roy (2011a, b) considered length-biased weighted generalized Rayleigh distribution and length-biased weighted Weibull distribution. They explored various properties of these distributions and demonstrated their relationship with several well known distributions. Kilany (2016) considered weighted Lomax distribution and studied its properties. Fallah and Kazemi (2020) revisited the generalized weighted exponential distribution. They developed some new distributional results for this distribution and provided its closed form expressions and related characteristics.

Kumar and Chaturvedi (2020) generalized the family proposed by Liang (2008), which covers as many as ten distributions to be particular cases. They explored the properties and different methods of estimation of the parameters and the reliability characteristics. Chaturvedi *et al.* (2020) have developed sequential and two-stage procedures for the parameter of this generalized positive exponential family of distributions.

The main goal of the present chapter is to provide an extension of the generalized positive exponential family of distributions. The size biased distribution is proposed to increase the flexibility of modelling data. The advantage of this size biased positive exponential family of distributions is that for different values of β, the size biased form of the distributions described by Kumar and Chaturvedi (2020) are special cases of this family of distributions. The present investigation aims to study some structural properties of the proposed weighted generalization of the positive exponential family of distributions. The rest of the chapter is organized as follows: In Section 6.2, the weighted generalization of the positive exponential family of distributions is proposed and the associated properties are investigated. In Section 6.3, we derive UMVU estimators, ML estimators and MM estimators of the q^{th} power of the parameter θ of the proposed weighted family of distributions, when other parameters are known. We also derive UMVU estimators and ML estimators of the reliability functions. In section 6.4, we derive ML estimators when all the parameters are unknown. Section 6.5 of our paper comprises of an extensive simulation study followed by real data illustrations in Section 6.6. We end with a brief set of conclusions in Section 6.7.

6.2 The Weighted Generalized Positive Exponential Family of Distributions and its properties

A random variable X is said to follow Generalized Positive Exponential Family of Distributions if its pdf and cdf are respectively given by

$$f\left(x;\alpha,\beta,\nu,\theta\right)=\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma\nu}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right);x>0,\alpha,\beta,\nu,\theta>0 \quad (6.2.1)$$

and

$$F(x)=\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma\nu}, \quad (6.2.2)$$

where $\gamma(x,a)=\int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.
We construct a weighted family of distributions by taking $w(x)=x^c$, and hence size biased Weighted Generalized Positive Exponential Family of Distributions is given by

$$g\left(x;\alpha,\beta,\nu,\theta,c\right)=\frac{\alpha}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\left(\frac{\beta}{\theta}\right)^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right); \quad x>0,\alpha,\beta,\nu,\theta,c>0. \quad (6.2.3)$$

We denote it by $WGPEFD(c,\alpha,\beta,\nu,\theta)$. The pdfs of length biased (LGPEFD) and area biased Generalized Positive Exponential Family of Distributions (AGPEFD) distributions can be, respectively, obtained by substituting $c=1$ and $c=2$ in (6.2.3). The corresponding cdf and hazard function are respectively given by

$$G(x;\alpha,\beta,\nu,\theta,c) = \frac{\gamma\left(\nu+\frac{c}{\alpha},\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)} \tag{6.2.4}$$

and

$$h(x) = \frac{\alpha\left(\frac{\beta}{\theta}\right)^{\nu+\frac{c}{\alpha}} x^{\alpha\nu+c-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)-\gamma\left(\nu+\frac{c}{\alpha},\frac{\beta x^{\alpha}}{\theta}\right)}. \tag{6.2.5}$$

Figures 6.1 and 6.2 provide possible shapes of probability density functions and hazard functions of this family of distributions for different values of α, β, ν, c and θ.

For different values of β, this family covers the following distributions as special cases:

1. For $\alpha=\nu=\beta=1$, we get one parameter size biased exponential distribution.
2. For $\alpha=\beta=1$, it gives size biased gamma distribution. Further, for integral values of α, it gives size biased Erlang distribution.
3. For $\beta=1$, it leads to size biased generalized gamma distribution.
4. For $\beta=\nu=1$, it turns out to be size biased Weibull distribution.
5. For $\nu=\frac{1}{2},\beta=1,\alpha=2$, it is known as size biased half normal distribution.
6. For $\nu=\frac{m}{2}$, $\alpha=2$, $\beta=\frac{1}{2},m>0$ we get size biased chi distribution and for m = 3 we get size biased Maxwell distribution (Sharma *et al.*, 2017).
7. For $\alpha=2$,$\nu=1$,$\beta=1$, we get a size biased Rayleigh distribution.
8. For $\alpha=2$,$\beta=1$,$\nu=k+1$; $k\geq 0$ we get a size biased generalized Rayleigh distribution of Voda (1978).
9. For $\nu=\beta$ and $\alpha=2$, $\nu>o$, $\beta>0$ we get the size biased Nakagami distribution (Mudsir and Ahmad, 2018).

The various distributional properties and reliability characteristics related to this family of distributions are stated below:

1. **Moments**

   The rth raw moment (about the origin) of this family of distributions is given by

$$\mu_r^{'} = \left(\frac{\theta}{\beta}\right)^{r/\alpha} \frac{1}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right).$$

   In particular, mean and variance of $WGPEFD(c, \alpha, \beta, \gamma, \nu, \theta)$ are respectively given by

$$\mu = \left(\frac{\theta}{\beta}\right)^{1/\alpha} \frac{\Gamma\left(\nu + \frac{c}{\alpha} + \frac{1}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \tag{6.2.6}$$

   and

$$\sigma^2 = \left(\frac{\theta}{\beta}\right)^{2/\alpha} \frac{1}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \left[\Gamma\left(\nu + \frac{c}{\alpha} + \frac{2}{\alpha}\right) - \frac{\left[\left(\Gamma\left(\nu + \frac{c}{\alpha} + \frac{1}{\alpha}\right)\right)^2\right.}{\Gamma\left(\nu + \frac{c}{\alpha}\right)}\right]. \tag{6.2.7}$$

2. **Skewness and Kurtosis**

   The coefficients of skewness (β_1) and kurtosis (β_2) can be obtained as

$$\beta_1 = \frac{[a_3 a_0^2 - 3a_2 a_1 a_0 + 2a_1^3]^2}{[a_2 a_0 - a_1^2]^3}$$

   and

$$\beta_2 = \frac{a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_2 a_1^2 - 3a_1^4}{[a_0 a_2 - a_1^2]^2},$$

   respectively, where $a_r = \Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right)$. From Figure 6.3, we observe that WGPEFD is positively skewed as $\beta_1 > 0$ for different values of parameters. Figure 6.4 clearly indicates that this family exhibits the shapes higher than normal curve as β_2 is larger than 3 for the given c and different values of θ and hence this family is leptokurtic.

3. **Mode**

   Mode of the distribution is given by

$$X_{mode} = \left(\frac{\alpha\nu + c - 1}{\alpha}\left(\frac{\theta}{\beta}\right)\right)^{1/\alpha}.$$

   The pdf of this family is uni-modal for given α, β, ν, c and θ.

4. **Median**

Median is the solution of the following equation:

$$F(Md) = 0.5$$

$$\Rightarrow \frac{\gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta(Md)^{\alpha}}{\theta}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} - 0.5 = 0$$

.

5. **Quantiles**

The q^{th} quantile x_q can be obtained by solving the equation

$$\begin{aligned} q =& F_X(x_q;\theta) \\ \Longrightarrow q =& \frac{\gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta x_q^{\alpha}}{\theta}\right)}{\Gamma(\nu + \frac{c}{\alpha})} \\ \Longrightarrow x_q =& \left(\frac{\theta}{\beta}\mathcal{G}^{-1}\left(\nu + \frac{c}{\alpha}, q\right)\right)^{\frac{1}{\alpha}}, \end{aligned}$$

where $\mathcal{G}^{-1}\left(\nu + \frac{c}{\alpha}, q\right)$ is an inverse gamma regularized function and can be approximated by using the following series expansion

$$\mathcal{G}^{-1}(a,z) = (-(z-1)\Gamma(a+1))^{1/a} + \frac{\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^2}{a+1} + \frac{(3a+5)\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^3}{2(a+1)^2(a+2)} + \mathcal{O}((z-1)^{4/a}),$$

where $\mathcal{O}(\cdot)$ represents higher order terms.

6. **Moment Generating function and Characteristic function**

The moment generating function of X is given by

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\frac{\theta}{\beta}\right)^{j/\alpha} \frac{\Gamma\left(\nu + \frac{c}{\alpha} + \frac{j}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)}.$$

Consequently, the characteristic function $\phi_X(t) = E(e^{tX})$ is given by

$$\sum_{j=0}^{\infty} \frac{(it)^j}{j!}\left(\frac{\theta}{\beta}\right)^{j/\alpha} \frac{\Gamma\left(\nu + \frac{c}{\alpha} + \frac{j}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)}.$$

7. **Conditional Moments and Conditional Moment Generating Function**

Let X be a random variable following the WGPEFD, then the conditional moment $E(X^r|X > t)$ and the conditional moment generating function $E(e^{tX}|X > x_\circ)$ are respectively given by

$$E(X^r|X > t) = \frac{\left(\frac{\theta}{\beta}\right)^{r/\alpha}\left[\Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta t^\alpha}{\theta}\right)\right]}{\Gamma\left(\nu + \frac{c}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta t^\alpha}{\theta}\right)}$$

and

$$E(e^{tX}|X > x_\circ) = \frac{\sum_{i=0}^{\infty} \frac{t^i}{i!}\left(\frac{\theta}{\beta}\right)^{i/\alpha}\Gamma\left(\nu + \frac{c}{\alpha} + \frac{i}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta x_\circ^\alpha}{\theta}\right)}.$$

8. **Stochastic Ordering**

A random variable X is said to be stochastically greater than Y, i.e., $Y \leq_{st} X$, if $F_Y(t) \leq F_X(t)$ for all t. Further, X is said to be greater than Y in the

(a) hazard rate order, $Y \leq_{hr} X$, if $h_Y(t) \geq h_X(t)$ for all t.

(b) mean residual life order, $Y \leq_{mrl} X$ if $m_Y(t) \geq m_X(t)$ for all t.

(c) likelihood ratio order, $Y \leq_{lr} X$ if $\frac{f_X(t)}{f_Y(t)}$ decreases in t.

Shaked and Shantikumar (1994) gave a result regarding stochastic ordering which shows that the existence of likelihood ratio ordering implies the existence of all the orderings mentioned above.

Let $X \sim WGPEFD(\alpha_1, \beta_1, \nu_1, \theta_1, c_1)$ and $Y \sim WGPEFD(\alpha_2, \beta_2, \nu_2, \theta_2, c_2)$. Then, the likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1}{\alpha_2}\left(\frac{\beta_1^{\nu_1+\frac{c_1}{\alpha_1}}}{\beta_2^{\nu_2+\frac{c_2}{\alpha_2}}}\right)\left(\frac{\theta_2^{\nu_2+\frac{c_2}{\alpha_2}}}{\theta_1^{\nu_1+\frac{c_1}{\alpha_1}}}\right)\left(\frac{\Gamma\left(\nu_2 + \frac{c_2}{\alpha_2}\right)}{\Gamma\left(\nu_1 + \frac{c_1}{\alpha_1}\right)}\right) x^{\alpha_1\nu_1 - \alpha_2\nu_2 + c_1 - c_2} \times$$

$$exp\left[-\left(\frac{\beta_1\theta_2 x^{\alpha_1} - \beta_2\theta_1 x^{\alpha_2}}{\theta_1\theta_2}\right)\right]$$

$$\implies \frac{d}{dx}\frac{f_X(x)}{f_Y(x)} = \frac{f_X(x)}{f_Y(x)}\left[\frac{\alpha_1\nu_1 - \alpha_2\nu_2 + c_1 - c_2}{x} - \frac{\alpha_1\beta_1\theta_2 x^{\alpha_1-1} - \alpha_2\beta_2\theta_1 x^{\alpha_2-1}}{\theta_1\theta_2}\right]. \tag{6.2.8}$$

From (6.2.8), we can observe that $\frac{d}{dx}\frac{f_X(x)}{f_Y(x)}$ is decreasing in x, if $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$, $\theta_2 < \theta_1$ and $c_1 < c_2$, $\forall x$, $0 < \alpha_1, \alpha_2 < 1$.

Hence, $Y \leq_{lr} X$ when $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$, $\theta_2 < \theta_1$, $c_1 < c_2$ and $0 < \alpha_1, \alpha_2 < 1$ and hence,

$$(Y \leq_{lr} X) \implies (Y \leq_{hr} X) \implies (Y \leq_{mrl} X)$$
$$\Downarrow$$
$$(Y \leq_{st} X)$$

9. **Mean Residual Life Function**

   The Mean Residual Life function is given by

$$\mu_(t) = \frac{\left[\int_t^{\infty} 1 - \frac{\gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta u^{\alpha}}{\theta}\right)}{\Gamma(\nu)}\right] du}{\left[1 - \frac{\gamma\left(\nu, \frac{\beta t^{\alpha}}{\theta}\right)}{\Gamma(\nu + \frac{c}{\alpha})}\right]}.$$

10. **Mean Time to System Failure**

    Mean time to system failure of this family of distributions is given by

$$MTSF = \frac{\left(\frac{\theta}{\beta}\right)^{\left(\frac{1}{\alpha}\right)}}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \Gamma\left(\nu + \frac{c}{\alpha} + \frac{1}{\alpha}\right)$$

All the properties described above can be derived for LGPEFD and AGPEFD by taking $c = 1$ and $c = 2$ respectively.

6.3 UMVU and ML Estimators of powers of parameter θ, $R(t)$ and P

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the WGPEFD. Then, assuming α, β, ν and c to be known, the likelihood function of the parameter θ given the sample observations $\underline{x} = (x_1, x_2, \ldots, x_n)$ is given by:

$$L(\theta \mid \underline{x}) = \left(\frac{\alpha}{\Gamma(\nu + \frac{c}{\alpha})}\right)^n \left(\frac{\beta}{\theta}\right)^{n\left(\nu + \frac{c}{\alpha}\right)} e^{-\frac{\beta}{\theta}\sum_{i=1}^{n} x_i^{\alpha}} \prod_{i=1}^{n} x_i^{\alpha\nu + c - 1}. \quad (6.3.1)$$

The following theorem provides UMVU estimator of powers of θ.

Theorem 6.3.1. *For $q \in (-\infty, \infty)$, the UMVU estimator of θ^q is given by:*

$$\widetilde{\theta}^q = \begin{cases} \left\{ \dfrac{\Gamma\left(n\left(\nu + \frac{c}{\alpha}\right)\right)}{\Gamma(n\left(\nu + \frac{c}{\alpha}\right) + q)} \right\} S^q; & n\left(\nu + \frac{c}{\alpha}\right) + q > 0 \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. It follows from (6.3.1) and factorization theorem [see Rohtagi and Saleh (2012), pp. 367] that S is sufficient statistic for θ and the pdf of S is

$$f_s(s \mid \theta) = \frac{s^{n\left(\nu + \frac{c}{\alpha}\right) - 1}}{\Gamma\left(n\left(\nu + \frac{c}{\alpha}\right)\right) \theta^{n\left(\nu + \frac{c}{\alpha}\right)}} \exp\left(-\frac{s}{\theta}\right); \quad \nu,\ \alpha,\ \beta,\ \theta,\ c > 0,\ s \geq 0. \quad (6.3.2)$$

From (6.3.1), since the distribution of S belongs to exponential family, it is also complete. Now it follows from (6.3.2) that

$$E[S^q] = \left\{ \frac{\Gamma\left(n\left(\nu + \frac{c}{\alpha}\right) + q\right)}{\Gamma\left(n\left(\nu + \frac{c}{\alpha}\right)\right)} \right\} \theta^q, \quad (6.3.3)$$

and the theorem follows.

In the following theorem, we obtain UMVU estimator of the sampled pdf at a specified point x.

Theorem 6.3.2. *The UMVU estimator of the sampled pdf at a specified point x is:*

$$\widetilde{g}(x; \alpha, \beta, \nu, \theta, c) = \begin{cases} \frac{\alpha}{\beta\left(\nu + \frac{c}{\alpha}, (n-1)\left(\nu + \frac{c}{\alpha}\right)\right)} \left(\frac{\beta}{S}\right)^{\nu + \frac{c}{\alpha}} x^{\alpha\nu + c - 1} \left[1 - \frac{\beta x^\alpha}{S}\right]^{(n-1)\left(\nu + \frac{c}{\alpha}\right) - 1}; & \beta x^\alpha < S \\ 0; & \textit{otherwise.} \end{cases}$$

Proof. We can write the pdf given in (6.2.1) as

$$\begin{aligned} g(x; \alpha, \beta, \nu, \theta, c) &= \frac{\alpha}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \left(\frac{\beta}{\theta}\right)^{\nu + \frac{c}{\alpha}} x^{\alpha\nu + c - 1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\beta x^\alpha}{\theta}\right)^i \\ &= \frac{\alpha \beta^{\nu + \frac{c}{\alpha}} x^{\alpha\nu + c - 1}}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \sum_{i=0}^{\infty} \frac{(-1)^i \left(\beta x^\alpha\right)^i}{i!} \theta^{-\left(\nu + \frac{c}{\alpha} + i\right)}. \end{aligned}$$

Applying Theorem 6.3.1,

$$\begin{aligned} \widetilde{g}(x; \alpha, \beta, \nu, \theta, c) &= \frac{\alpha}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \beta^{\nu + \frac{c}{\alpha}} x^{\alpha\nu + c - 1} \sum_{i=0}^{\infty} \frac{(-1)^i \left(\beta x^\alpha\right)^i}{i!} (\widetilde{\theta})^{\left(-(\nu + \frac{c}{\alpha} + i)\right)} \\ &= \frac{\alpha \left(\frac{\beta}{S}\right)^{\nu + \frac{c}{\alpha}} x^{(\alpha\nu + c - 1)}}{\beta\left(\nu + \frac{c}{\alpha}, (n-1)\left(\nu + \frac{c}{\alpha}\right)\right)} \sum_{i=0}^{(n-1)\left(\nu + \frac{c}{\alpha}\right) - 1} (-1)^i \binom{(n-1)\left(\nu + \frac{c}{\alpha}\right) - 1}{i} \times \\ &\quad \left(\frac{\beta x^\alpha}{S}\right)^i, \end{aligned}$$

and the result follows.

The following theorem provides UMVU estimator of the reliability function $R(t)$.

Theorem 6.3.3. *The UMVU estimator of $R(t)$ is:*

$$\widetilde{R}(t) = \begin{cases} 1 - I_{\frac{\beta t^\alpha}{S}}\left(\nu + \frac{c}{\alpha}, (n-1)\left(\nu + \frac{c}{\alpha}\right)\right); & \beta t^\alpha < S \\ 0 & ; \textit{otherwise}, \end{cases}$$

where $I_x(p,q) = \frac{1}{\beta(p,q)} \int_0^x y^{p-1}(1-y)^{q-1}dy;\ 0 \leq y \leq 1,\ x < 1,\ p,q > 0$ *is the incomplete beta function.*

Proof. On applying Theorem 6.3.2, the UMVU estimator of $R(t)$ is given by

$$\begin{aligned} \widetilde{R}(t)_{II} =& \int_t^\infty \widetilde{g}(x;\alpha,\beta,\nu,\theta,c)dx \\ =& \frac{\alpha}{\beta\left(\nu + \frac{c}{\alpha}, (n-1)\left(\nu + \frac{c}{\alpha}\right)\right)} \left(\frac{\beta}{S}\right)^{\nu + \frac{c}{\alpha}} \times \\ & \int_t^\infty x^{\alpha\nu + c - 1} \left[1 - \frac{\beta x^\alpha}{S}\right]^{(n-1)\left(\nu + \frac{c}{\alpha}\right) - 1} dx, \end{aligned}$$

and the result follows by substituting $\frac{\beta x^\alpha}{S} = z$.

Let X and Y be two independent random variables with respective pdf:

$$g\left(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1\right) = \frac{\alpha_1}{\Gamma\left(\nu_1 + \frac{c_1}{\alpha_1}\right)} \left(\frac{\beta_1}{\theta_1}\right)^{\nu_1 + \frac{c_1}{\alpha_1}} x^{\alpha - 1\nu_1 + c_1 - 1} \exp\left(\frac{-\beta_1 x^{\alpha_1}}{\theta_1}\right);$$

$$x > 0, \alpha_1, \beta_1, \nu_1, \theta_1, c_1 > 0$$

and

$$g\left(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2\right) = \frac{\alpha_2}{\Gamma\left(\nu_2 + \frac{c_2}{\alpha_2}\right)} \left(\frac{\beta_2}{\theta_2}\right)^{\nu_2 + \frac{c_2}{\alpha_2}} y^{\alpha_2\nu_2 + c_2 - 1} \exp\left(\frac{-\beta_2 y^{\alpha_2}}{\theta_2}\right);$$

$$y > 0, \alpha_2, \beta_2, \nu_2, \theta_2, c_2 > 0.$$

Now the UMVU estimator of P is given in the following theorem.

Theorem 6.3.4. *The UMVU estimator of P is*

$$\tilde{P} = \begin{cases} \int_{z=0}^{1} \frac{1}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)} z^{\nu_1+\frac{c_1}{\alpha_1}-1}(1-z)^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1} \times \\ I_{\left\{\frac{\beta_2\left(\frac{Sz}{\beta_1}\right)^{\left(\frac{\alpha_2}{\alpha_1}\right)}}{T}\right\}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right); for \left(\frac{S}{\beta_1}\right)^{1/\alpha_1} \le \left(\frac{T}{\beta_2}\right)^{1/\alpha_2} \\ 1-\frac{1}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)} \int_{z=0}^{1} z^{\nu_2+\frac{c_2}{\alpha_2}-1}(1-z)^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1} \times \\ I_{\left\{\frac{\beta_1\left(\frac{Tz}{\beta_2}\right)^{\left(\frac{\alpha_1}{\alpha_2}\right)}}{S}\right\}}\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right); for \left(\frac{S}{\beta_1}\right)^{1/\alpha_1} > \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}. \end{cases}$$

Proof. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $g(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)$ and $Y_1, Y_2, \ldots, Y_m$ be a random sample of size m from $g(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)$. Further, let $S=\sum_{i=1}^{n}\beta_1 X_i^{\alpha_1}$ and $T=\sum_{i=1}^{m}\beta_2 Y_i^{\alpha_2}$. It follows from Theorem 6.3.2 that

$$\widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1) = \frac{\alpha_1}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}\left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}} x^{\alpha_1\nu_1+c_1-1}\times \left[1-\frac{\beta_1 y^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1} ;\beta_1 x^{\alpha_1}<S \tag{6.3.4}$$

and

$$\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2) = \frac{\alpha_2}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)}\left(\frac{\beta_2}{T}\right)^{\nu_2+\frac{c_2}{\alpha_2}} y^{\alpha_2\nu_2+c_2-1}\times \left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1} ;\beta_2 y^{\alpha_2}<T. \tag{6.3.5}$$

The UMVU estimator of P can be obtained as

$$\widetilde{P} = \int_{x=0}^{\infty}\int_{y=0}^{x}\widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)\,dx\,dy$$

$$= \int_{x=0}^{\min\left(\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}},\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\right)} \frac{\alpha_1 x^{\alpha_1\nu_1+c_1-1}}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}\left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}}\times \left[1-\frac{x^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1} I_{\frac{\beta_2 x^{\alpha_2}}{T}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)\,dx.$$

When $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} \leq \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, the UMVU estimator of P is given by

$$\widetilde{P} = \int_{x=0}^{\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}} \frac{\alpha_1 x^{\alpha_1\nu_1+c_1-1}}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}\left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}}\left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1} \times$$
$$I_{\frac{\beta_2 x^{\alpha_2}}{T}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)\, dx, \tag{6.3.6}$$

and the first assertion follows by substituting $\frac{\beta_1 x^{\alpha_1}}{S} = z$.
When $\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} > \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}$, the UMVU estimator of P is given by

$$\begin{aligned}\widetilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dx\, dy\\ &= \int_{y=0}^{\infty} \widetilde{R}(y;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dy\\ &= \frac{\alpha_2}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)}\left(\frac{\beta_2}{T}\right)^{\nu_2+\frac{c_2}{\alpha_2}} y^{\alpha_2\nu_2+c_2-1}\times\\ &\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1}\left[1-I_{\frac{\beta_1 y^{\alpha_1}}{S}}\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)\right] dy,\end{aligned} \tag{6.3.7}$$

and the second assertion follows by substituting $\frac{\beta_2 y^{\alpha_2}}{T} = z$.

Now we provide ML estimator of $R(t)$ in the following theorem:

Theorem 6.3.5. *The ML estimator of $R(t)$ is given by:*

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\nu+\frac{c}{\alpha},\frac{n\beta t^\alpha}{S}\left(\nu+\frac{c}{\alpha}\right)\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)},$$

where $\gamma(a,r) = \int_0^r y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

Proof. It can be easily seen from (6.3.1) that the ML estimator of θ^q is given by

$$\widehat{\theta^q} = \left(\frac{S}{n\left(\nu+\frac{c}{\alpha}\right)}\right)^q, \tag{6.3.8}$$

where, $S = \beta\Sigma {x_i}^\alpha$.

Using (6.2.4), the expression of $R(t)$ at a point t is given by

$$R(t) = 1 - \frac{\gamma\left(\nu+\frac{c}{\alpha},\frac{\beta t^\alpha}{\theta}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}. \tag{6.3.9}$$

Now from (6.3.8), (6.3.9) and the invariance property of ML estimators, we obtain the required result.

The ML estimator of P is given in the following theorem:

Theorem 6.3.6. *The ML estimator of P is given by*

$$\widehat{P} = 1 - \frac{1}{\Gamma\left(\nu_1 + \frac{c_1}{\alpha_1}\right)\Gamma\left(\nu_2 + \frac{c_2}{\alpha_2}\right)} \int_{z=0}^{\infty} z^{\nu_2 + \frac{c_2}{\alpha_2} - 1} e^{-z} \times$$
$$\gamma\left(\nu_1 + \frac{c_1}{\alpha_1}, \frac{n\beta_1\left(\nu_1 + \frac{c_1}{\alpha_1}\right)\left(\frac{zT}{m\beta_2}\left(\nu_2 + \frac{c_2}{\alpha_2}\right)^{-1}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right) dz.$$

Proof. We have,

$$\begin{aligned}
\widehat{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widehat{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widehat{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dx\ dy \\
&= \int_{y=0}^{\infty} \widehat{R}(y;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widehat{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dy.
\end{aligned}$$

Now from the invariance property of ML estimators, the ML estimator of sampled pdf is:

$$\widehat{g}(x;\alpha,\beta,\nu,\theta,c) = \frac{\alpha x^{\alpha\nu+c-1}}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\left(\frac{n\beta\left(\nu+\frac{c}{\alpha}\right)}{S}\right)^{\nu+\frac{c}{\alpha}} \exp\left\{\frac{-n\beta x^{\alpha}}{S}\left(\nu+\frac{c}{\alpha}\right)\right\}. \tag{6.3.10}$$

Hence, ML estimator of P is given by

$$\begin{aligned}
\widehat{P} = \int_{y=0}^{\infty} &\left[1 - \frac{\gamma\left(\nu_1+\frac{c_1}{\alpha_1}, \frac{n\beta_1 y^{\alpha_1}}{S}\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}{\Gamma\left(\nu_1+\frac{c_1}{\alpha_1}\right)}\right]\frac{\alpha_2 y^{\alpha_2\nu_2+c_2-1}}{\Gamma\left(\nu_2+\frac{c_2}{\alpha_2}\right)}\times \\
&\left(\frac{m\beta_2\left(\nu_2+\frac{c_2}{\alpha_2}\right)}{T}\right)^{\nu_2+\frac{c_2}{\alpha_2}} \exp\left\{\frac{-m\beta_2 y^{\alpha_2}}{T}\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right\},
\end{aligned} \tag{6.3.11}$$

and the theorem follows on substituting $\frac{m\beta_2 y^{\alpha_2}}{T}\left(\nu_2+\frac{c_2}{\alpha_2}\right) = z$.

Next, we derive the MM estimator of θ.

Theorem 6.3.7. *The MM estimator of θ^q is given by*

$$\widehat{\theta^q_M} = \left(\left(\frac{\Gamma\left(\nu+\frac{c}{\alpha}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}+\frac{1}{\alpha}\right)}\right)^{\alpha}\overline{X}^{\alpha}\beta\right)^{q}.$$

Proof. From equation (6.2.1), we obtain the rth moment as:

$$E(X^r) = \int_0^\infty \frac{\alpha}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \left(\frac{\beta}{\theta}\right)^{\frac{\nu}{\alpha}} x^{\alpha\nu+c+r-1} \exp\left(\frac{-\beta x^\alpha}{\theta}\right) dx$$
$$= \left(\frac{\theta}{\beta}\right)^{r/\alpha} \frac{1}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} a_r,$$

where $a_r = \Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right)$.

For $r = 1$ and denoting $E(X^r)$ by $\overline{X}^r$, we obtain the following equation:

$$\Gamma\Big(\nu + \frac{c}{\alpha}\Big)\overline{X} - a_1 \left(\frac{\theta}{\beta}\right)^{\frac{1}{\alpha}} = 0, \tag{6.3.12}$$

and hence the theorem follows.

6.4 ML Estimators when all the parameters are unknown

Now we discuss the case when all the parameters α, ν, θ and c are unknown. The log-likelihood function of the parameters α, ν, θ and c given the sample observations $\underline{x}$ and different values of β is:

$$l(\alpha, \nu, \theta, c \mid \underline{x}, \beta) = n\log(\alpha) - n\log\left(\Gamma\left(\nu + \frac{c}{\alpha}\right)\right) + n\nu\log(\beta) + \frac{nc}{\alpha}\log(\beta) - n\nu\log(\theta) - \frac{nc}{\alpha}\log(\theta) - \frac{\beta}{\theta}\sum_{i=1}^n x_i^\alpha + (\alpha\nu + c - 1)\sum_{i=1}^n \log(x_i).$$

The MLEs of α, ν and c are given by the simultaneous solution of the following four equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \frac{n}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \frac{d\Gamma\left(\nu + \frac{c}{\alpha}\right)}{d\alpha} \frac{c}{\alpha^2} - \frac{nc}{\alpha^2} log(\beta) + \frac{nc}{\alpha^2} log(\theta) - \frac{\beta}{\theta}\sum_{i=1}^n x_i^\alpha \log(x_i) + \nu\sum_{i=1}^n \log(x_i) = 0 \tag{6.4.1}$$

$$\frac{\partial l}{\partial \nu} = \frac{-n}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \frac{d\Gamma\left(\nu + \frac{c}{\alpha}\right)}{d\nu} + nlog(\beta) - n\log(\theta) + n\log(\beta) + \alpha\sum_{i=1}^n \log(x_i) = 0 \tag{6.4.2}$$

$$\frac{\partial l}{\partial c} = \frac{-n}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} \frac{d\Gamma\left(\nu + \frac{c}{\alpha}\right)}{dc} \left(\frac{1}{\alpha}\right) + \frac{n}{\alpha} log(\beta) - \frac{n}{\alpha} log(\theta) + \sum_{i=1}^{n} \log(x_i) = 0 \quad (6.4.3)$$

$$\frac{\partial l}{\partial \theta} = -\frac{n\nu}{\theta} - \frac{nc}{\alpha\theta} + \frac{\beta \sum_{i=1}^{n} x_i^{\alpha}}{\theta^2} = 0 \quad (6.4.4)$$

Since, these non-linear equations don't have a closed form solution, we apply Newton Raphson algorithm to compute ML estimators of α, ν, c and θ .

Further, using the invariance property of ML estimators, the ML estimator of $R(t)$ is given as:

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\widehat{\nu} + \frac{\widehat{c}}{\widehat{\alpha}}, \frac{\beta x^{\widehat{\alpha}}}{\widehat{\theta}}\right)}{\Gamma\left(\widehat{\nu} + \frac{\widehat{c}}{\widehat{\alpha}}\right)}, \quad (6.4.5)$$

where $\widehat{\alpha}$, $\widehat{\nu}$, $\widehat{\theta}$ and $\widehat{c}$ are the ML estimators of α, ν, θ and c respectively, and the ML estimator of P is given as:

$$\widehat{P} = 1 - \frac{1}{\Gamma\left(\widehat{\nu}_1 + \frac{\widehat{c}_1}{\widehat{\alpha}_1}\right)\Gamma\left(\widehat{\nu}_2 + \frac{\widehat{c}_2}{\widehat{\alpha}_2}\right)} \int_{z=0}^{\infty} z^{\widehat{\nu}_2 + \frac{\widehat{c}_2}{\widehat{\alpha}_2} - 1} e^{-z} \times \gamma\left(\widehat{\nu}_1 + \frac{\widehat{c}_1}{\widehat{\alpha}_1}, \frac{\beta_1}{\widehat{\theta}_1}\left(\frac{\widehat{\theta}_2 z}{\beta_2}\right)^{\frac{\widehat{\alpha}_1}{\widehat{\alpha}_2}}\right) dz. \quad (6.4.6)$$

6.5 Simulation Studies

In order to compare the performance of $\widetilde{\theta^q}$, $\widehat{\theta^q_M}$ and $\widehat{\theta^q}$ for different sample sizes, we conduct a Monte Carlo simulation study. For $\alpha = 0.5$, $\beta = 2, \nu = 3, c = 0.8$, we generate 10,000 samples each of size n from WGPEFD and repeat this procedure for several values of θ. Figure 6.5 shows the Mean Square Error (MSE) of the UMVU estimator, MM estimator and ML estimator of θ^q. From these figures, we note that for smaller sample sizes and for $q = 2$, the ML estimator performs the best and the MM estimator performs the worst. The performance of UMVU estimator is in between the two. As the sample size increases, the three curves come close to each other.

Along the similar lines, we perform the simulation studies to compare the performance of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different sample sizes. For $t = 10$ and $\alpha = 0.5, \beta = 2, \nu = 1.5, c = 0.8$, we generate 10,000 samples each of size n from the

WGPEFD and repeat this procedure for several values of θ. Figure 6.6 shows the MSE of the UMVU estimator and ML estimator of $R(t)$. From these figures, we note that the MSE of the ML estimator of $R(t)$ is always less than that of the UMVU estimator and hence ML estimator of $R(t)$ performs better than UMVU estimator of $R(t)$. However, for large sample sizes these estimators of $R(t)$ are almost equally efficient.

Now, we compare the performance of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. By Monte Carlo simulation, for $\alpha_1 = 0.4$, $\beta_1 = 0.6$, $\nu_1 = 4$, $c_1 = 0.5$ and $\alpha_2 = 0.1$, $\beta_2 = 0.3$, $\nu_2 = 1$, $c_2 = 0.5$, we generate 10,000 samples each of size n and m from WGPEFD and repeat this procedure for several values of θ_1 and $\theta_2 = 0.8$. Figure 6.7 shows the MSE of the UMVU estimator and ML estimator of P. From these figures, we note that the MSE of the UMVU estimator of P is always greater than that of the ML estimator, however, for large sample sizes these estimators of P are almost equally efficient.

6.5.1 ML Estimation of the reliability functions when all parameters are unknown

We consider the special case of weighted generalized gamma distribution (WGGD) which is given by taking $\beta = 1$ in (6.2.3). In order to compute the ML estimates of the reliability functions, when all the parameters are unknown, we have first generated 1000 random samples of size $n = 50$ from the WGGD (say, X population or random strength X) with $\alpha_1 = 0.2$, $\nu_1 = 2$, $\theta_1 = 2.5$ and $c_1 = 2$. We generated ML estimators of these parameters for these 1000 samples and obtained the mean of estimates of them. The ML estimators of α_1, ν_1, θ_1 and c_1 comes out to be $\widehat{\alpha}_1 = 0.3311$, $\widehat{\nu}_1 = 1.6521$, $\widehat{\theta}_1 = 2.3548$ and $\widehat{c}_1 = 1.5833$ respectively. Also for $t = 50$, actual $R(t) = 0.9871$ and $\widehat{R(t)} = 0.9159$.

Now, to obtain the estimates of P, when all the parameters are unknown, we have generated 1000 random samples of size $m = 60$ from the WGGD (say, Y population or random stress Y) with $\alpha_2 = 0.18$, $\nu_2 = 1.6$, $\theta_2 = 2.6$ and $c_2 = 1.8$. We generated ML estimators of these parameters for these 1000 samples and obtained mean of estimates of them. The ML estimators of α_2, ν_2, θ_2 and c_2 are obtained as $\widehat{\alpha}_2 = 0.1752$, $\widehat{\nu}_2 = 1.5798$, $\widehat{\theta}_2 = 2.5697$ and $\widehat{c}_2 = 1.7456$ respectively. Here actual value of $P = 0.9941$ and the MLE of P comes out to be $\widehat{P} = 0.9946$.

The ML estimators obtained here are close to the true parameter values, though we observe that all the parameters are slightly underestimated except P, which is slightly overestimated. These findings verify the validity of our theoretical results.

6.6 Real life data examples

This section deals with examples of real data to establish the superiority of Weighted Gamma distribution (WGD) over Gamma and Weighted Generalized Gamma distribution (WGGD) which are special cases of Weighted Generalized Positive Exponential Family of distributions and to illustrate the proposed estimation methods.

Data set I (representing Population X) was reported by Stablein *et al.* (1981) [see Chaturvedi *et al.* (2019)]. It corresponds to the survival times in days from a clinical trial on a locally advanced, non-resectable gastric carcinoma, involving 90 patients randomized to either chemotherapy alone or a combination of chemotherapy and radiation. The plot of empirical and theoretical cdfs of Weighted Gamma distribution given in Figure 6.8 shows that it fits well to this data.

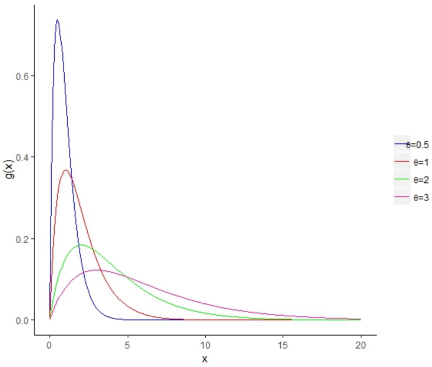
For comparing WGD with Gamma and WGGD, we use the concept of Akaike Information Criterion (AIC), KS distance and corresponding P-value. The best model is the one that has the least values of AIC and comparatively high P-value. For establishing the superiority of WGD, the calculated values of AIC, KS distance and P-values are reported in Table 6.1.

Since AIC is the least and P-value is reasonably good for WGD, we choose it to fit the data. Let the population $X \sim WG(x; \nu_1, \theta_1, c_1)$. We obtain ML estimators of the parameters ν_1, θ_1 and c_1 and then assuming ν_1 and c_1 to be known, we obtain UMVU estimator and moment estimator of θ_1. We also obtain UMVU estimator and ML estimator of reliability function $R_X(t)$ at $t = 10$. All the estimated values are listed in Table 6.2.

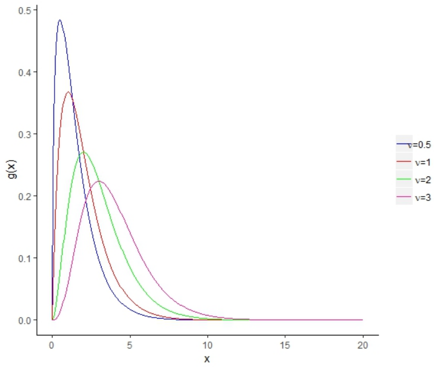
Data set II (representing Population Y), reported by Efron (1988) [see Shankar *et al.*(2016) also], represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT). The plot of empirical and theoretical cdfs of Weighted Gamma distribution given in Figure 6.9 shows that it fits well to this data.
For comparison purposes, the calculated values of AIC, KS distance and P values are reported in Table 6.3.
Since, AIC is least and P-value is reasonably good for WGD we choose it to fit the data. Let the population $Y \sim WG(y; \nu_2, \theta_2, c_2)$. We obtain ML estimator of the prameters ν_2, θ_2 and c_2 and then assuming ν_2 and c_2 to be known, we obtain UMVU estimator and moment estimator of θ_2. We also obtain UMVU estimator and ML estimator of reliability function $R_Y(t)$ at $t = 10$. All the estimated values are listed in Table 6.4.
Now, for the above two data sets, we obtain estimators of $P = P(X > Y)$. The ML estimator and UMVU estimator of P are obtained as 0.7374 and 0.7225, respectively.

6.7 Concluding Remarks

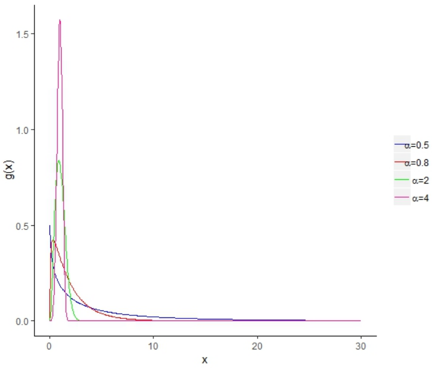
In this article, we have considered weighted family of the Generalization of Positive Exponential Family of Distributions developed by Kumar and Chaturvedi (2020). We have considered size biased family of distributions by taking weight $w(x) = x^c$. UMVU, ML and MM estimators are developed for the powers of parameters, $R(t)$ and P. All the estimates can be obtained for length biased and area biased Generalization of Positive Exponential Family of distributions by taking $c = 1$ and $c = 2$, respectively. Efficiency comparison of the three methods of estimation through Monte Carlo Simulation studies is done. Real life data sets are studied to show the superiority of Weighted Gamma distribution, which is special case of this family of distributions and to illustrate the proposed estimation methods.



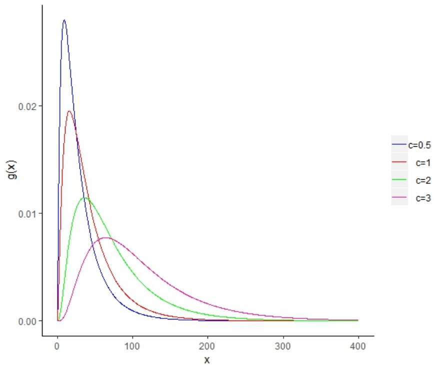
(a) $\alpha = \beta = \nu = c = 1$ and $\theta = 0.5, 1, 2, 3$



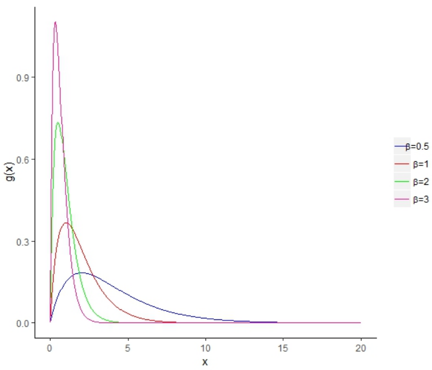
(b) $\alpha = \beta = \theta = c = 1$ and $\nu = 0.5, 1, 2, 3$



(c) $\beta = \nu = \theta = 1$, $c = 0.5$ and $\alpha = 0.5, 0.8, 2, 4$

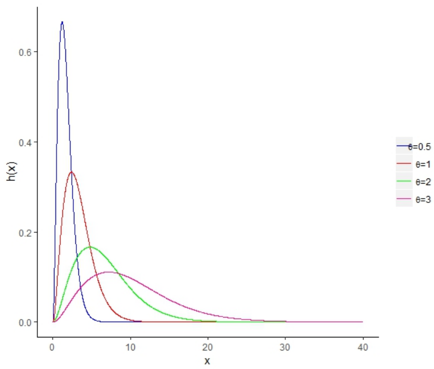


(d) $\alpha = \theta = \nu = \beta = 1$ and $c = 0.5, 1, 2, 3$

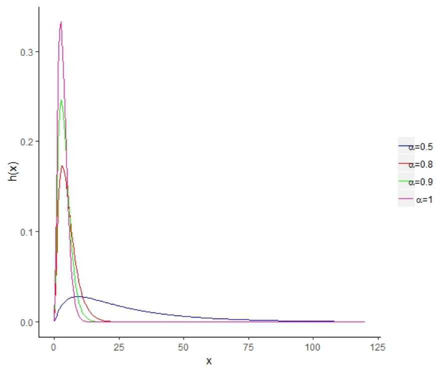


(e) $\alpha = \theta = \nu = c = 1$ and $\beta = 0.5, 1, 2, 3$

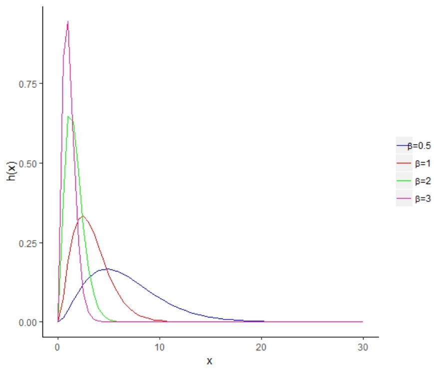
Figure 6.1: **Probability Density function plots for different values of parameters**



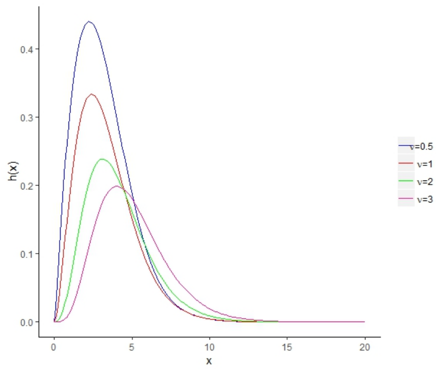
(a) $\alpha = \beta = \nu = 1$, $c = 2$ and $\theta = 0.5, 1, 2, 3$



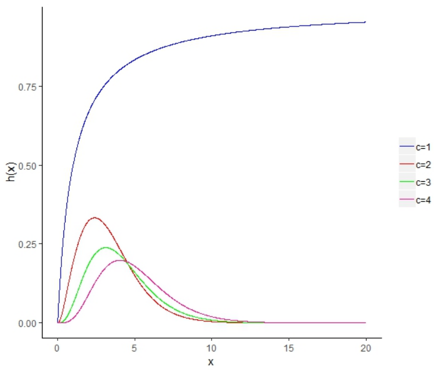
(b) $\nu = \beta = \theta = 1$, $c = 2$ and $\alpha = 0.5, 0.8, 0.9, 1$



(c) $\alpha = \nu = \theta = 1$, $c = 2$ and $\beta = 0.5, 1, 2, 3$

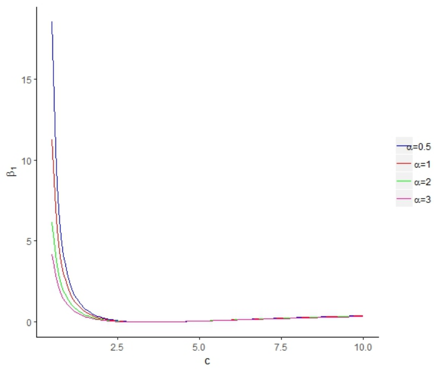


(d) $\alpha = \theta = \beta = 1$, $c = 2$ and $\nu = 0.5, 1, 2, 3$

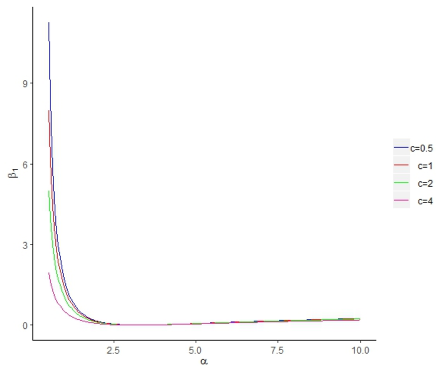


(e) $\alpha = \theta = \nu = \beta = 1$ and $c = 1, 2, 3, 4$

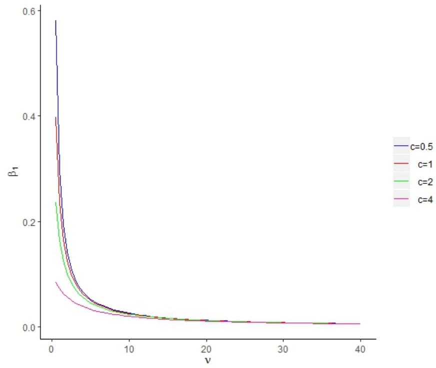
Figure 6.2: **Hazard rate function plots for different values of parameters**



(a) Skewness plot along c for different values of α

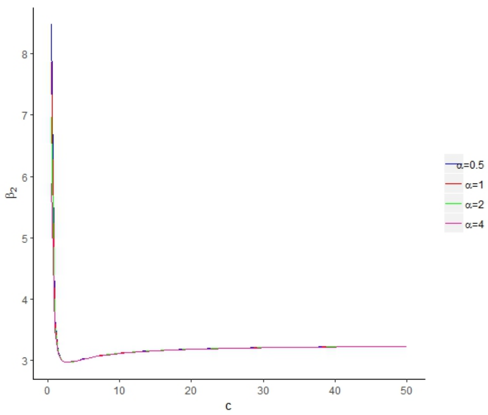


(b) Skewness plot along α for different values of c

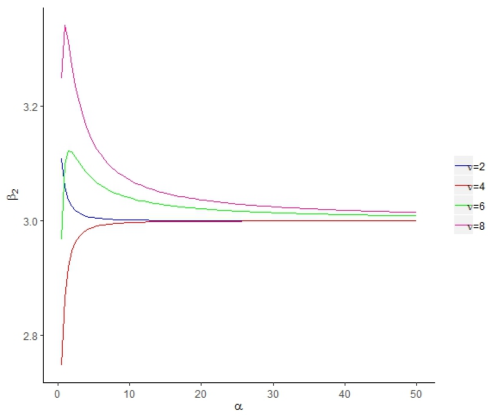


(c) Skewness plot along ν for different values of c

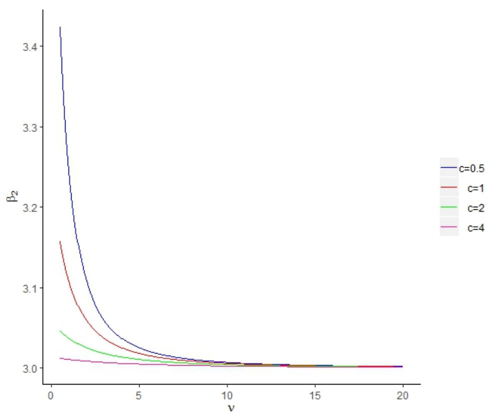
Figure 6.3: **Skewness plots for different values of parameters**



(a) Kurtosis plot along c for different values of α

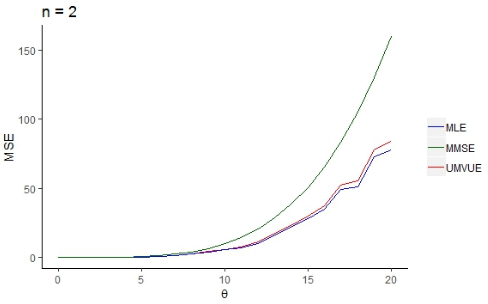


(b) Kurtosis plot along α for different values of ν

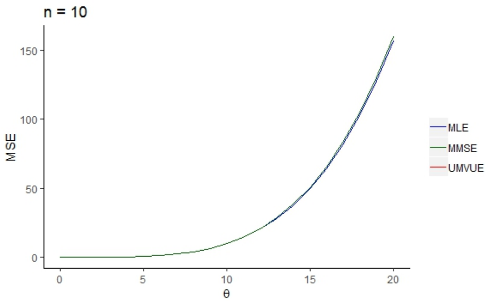


(c) Kurtosis plot along ν for different values of c

Figure 6.4: **Kurtosis plots for different values of parameters**

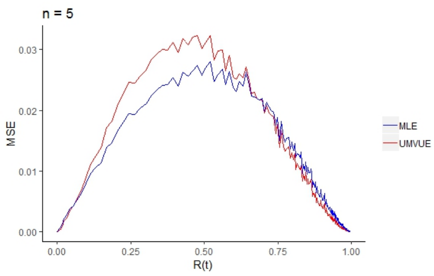


(a) MSE of the UMVU, ML and MM Estimator of θ^q for n=2

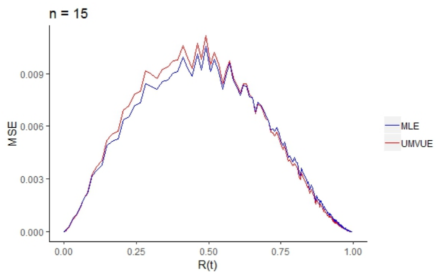


(b) MSE of the UMVU, ML and MM Estimator of θ^q for n=10

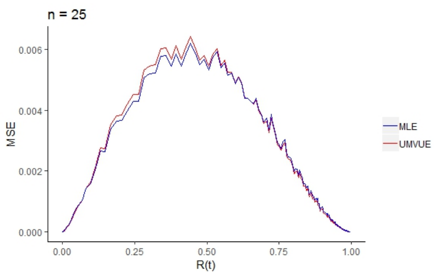
Figure 6.5: **MSE of the UMVU, ML and MM Estimator of θ^q for different sample sizes**



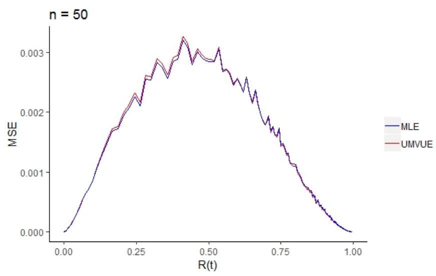
(a) MSE of the UMVU and ML Estimator of $R(t)$ for n=5



(b) MSE of the UMVU and ML Estimator of $R(t)$ for n=15

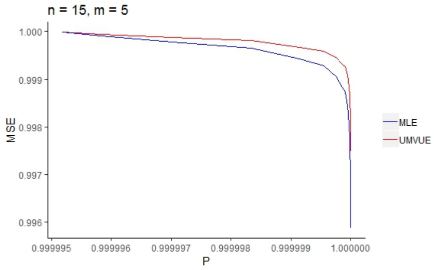


(c) MSE of the UMVU and ML Estimator of $R(t)$ for n=25

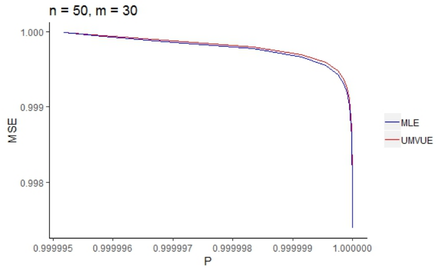


(d) MSE of the UMVU and ML Estimator of $R(t)$ for n=50

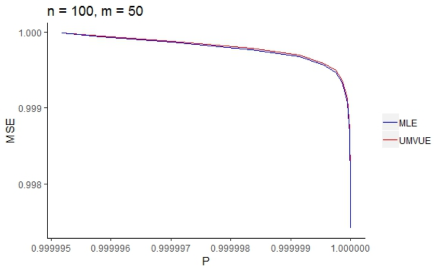
Figure 6.6: **MSE of the UMVU and ML Estimator of $R(t)$ for different sample sizes**



(a) MSE of the UMVU and ML Estimator of P for n=15 and m=5



(b) MSE of the UMVU and ML Estimator of P for n=50 and m=30



(c) MSE of the UMVU and ML Estimator of P for n=100 and m=50

Figure 6.7: **MSE of the UMVU and ML Estimator of P for different sample sizes**

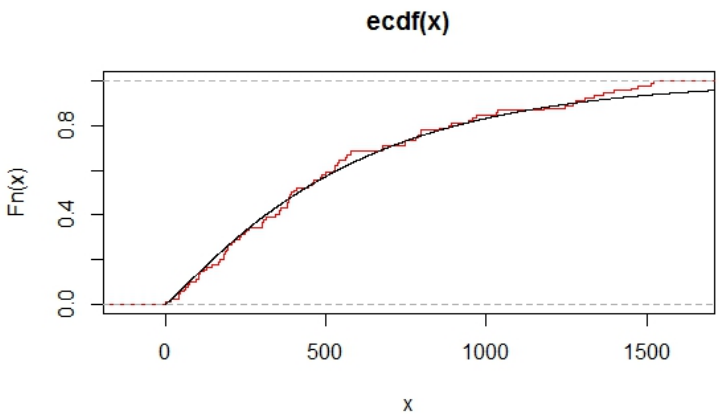


Figure 6.8: **The empirical and theoretical cdf of WGD(ν_1, θ_1, c_1) model**

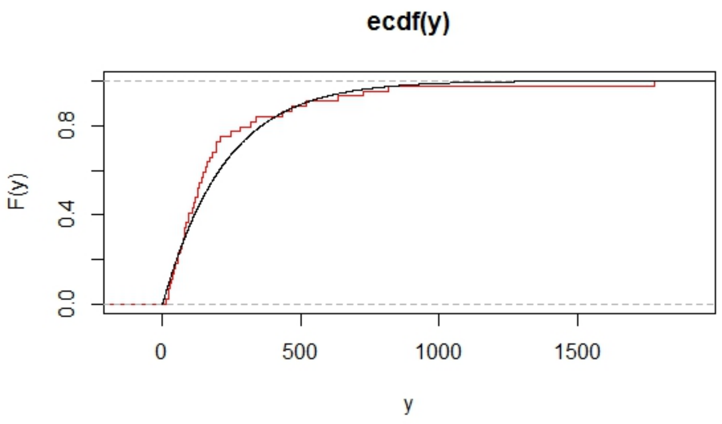


Figure 6.9: **The empirical and theoretical cdf of WGD(ν_2, θ_2, c_2) model**

Table 6.1: **Comparison of different distributions based on AIC, KS-Distance and P values for data set I**

CRITERIA	AIC	KS-Distance	KS P-value
WGD	111.5418	0.1034	0.6837
Gamma	396.3332	0.1132	0.6811
WGGD	401.7463	0.0716	0.8664

Table 6.2: **Estimates Based on Data Set I**

$\widehat{\nu}_1$	$\widehat{c}_1$	$\widehat{\theta}_1$	$\widetilde{\theta}_1$	$\widehat{\theta}_{iMM}$	$\widehat{R}_X(t)$	$\widetilde{R}_X(t)$
0.5701	0.5701	500.1502	465.6973	465.6972	0.9876	0.9853

Table 6.3: **Comparison of different distributions based on AIC, KS-Distance and P values for data set II**

CRITERIA	**AIC**	**KS-Distance**	**KS P-value**
WGD	144.0705	0.1473	0.668
Gamma	566.1502	0.1573	0.268
WGGD	566.4392	0.09923	0.7419

Table 6.4: **Estimates Based on Data Set II**

$\widehat{\nu_2}$	$\widehat{c_2}$	$\widehat{\theta_2}$	$\widetilde{\theta_2}$	$\widehat{\theta}_{2MM}$	$\widehat{R}_Y(t)$	$\widetilde{R}_Y(t)$
0.5118	0.5118	218.3114	218.3116	218.3116	0.9583	0.9597

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